New developments for increased performance of the SBP-SAT finite difference technique

Jan Nordström¹ and Peter Eliasson²

¹ Department of Mathematics, Computational Mathematics, University of Linköping, SE-581 83 Linköping, Sweden,

`jan.nordstrom@liu.se`,

² Department of Aeronautics and Autonomous Systems, FOI, The Swedish Defense Research Agency, SE-164 90 Stockholm, Sweden,

`peter.eliasson@foi.se`

Abstract. In this article, recent developments for increased performance of the high order and stable SBP-SAT finite difference technique is described. In particular we discuss the use of weak boundary conditions and dual consistent formulations. The use of weak boundary conditions focus on increased convergence to steady state, and hence efficiency. Dual consistent schemes produces superconvergent functionals and increases accuracy.

Keywords: High order accuracy, stability, finite difference, summation-by-parts, weak boundary conditions, convergence to steady state, dual consistency, super-convergence

1 Introduction

We briefly describe two different new contributions to the theory for high order stable SBP-SAT finite difference schemes. The complete description of the weak boundary procedures for convergence to steady state is given in [1] while the development of dual consistence schemes is presented in [2],[3],[4]. The reader is referred to articles mentioned above for potentially missing details.

2 Weak solid wall boundary conditions and steady state

The formulations of solid wall boundary conditions for the Navier-Stokes equations and the related slip condition for the Euler equations are well known. Less well known is the relation between these two formulations with a weak implementation. One of the more striking features is the fact that with a weak implementation of the boundary condition, the velocity at the wall becomes zero only for very fine meshes.

The main reason to use weak boundary procedures stems from the fact that together with summation-by-parts operators they lead to provable stable schemes. For application of this technique to finite difference methods, node-centered finite volume methods, spectral domain methods and various hybrid

methods see the references in the original article [1] In this section we will consider a new effect of using weak boundary procedures, namely that it in many cases (all that we tried) speeds up the convergence to steady-state.

Most of the work on convergence acceleration to steady-state is done for the discrete problem. Techniques such as local time-stepping, multigrid, residual smoothing, etc are used to enhance the convergence to steady-state. In the majority of the investigations, the discrete problem including boundary conditions, is formulated first, and the numerical convergence acceleration technique is more or less independently added on afterwards. We will not consider this type of numerical convergence acceleration techniques but rather focus on more fundamental aspects related directly to the governing equations and the numerical scheme.

We will consider the basic requirement for convergence to steady-state, namely the position of the eigenvalues (the spectrum) of the Initial Boundary Value Problem (IBVP). The eigenvalues of the corresponding semi-discrete Initial Value Problem (IVP) problem obtained by using the method of lines are equally important and will also be discussed. We start by formulating the relevant IBVP and the data requirements for the existence of a steady-state solution. Next we make sure that we have a well posed procedure (such that it is possible to reach the steady-state) and finally quantify the speed of convergence using the utmost right lying eigenvalue in the spectrum. That is repeated with minor technical modifications (such as replacing the concept well-posedness by stability, investigating numerical eigenvalues, etc.) for the IVP.

Finally we apply the general theory to the specific case of solid wall boundary conditions. Our basic theoretical tools will be the classical ones, namely the energy method, the Laplace transform technique and the matrix exponential. Our basic computational tool is a node vertex edge based flow solver for unstructured grids, the Edge code developed by FOI.

2.1 The formulation of the IBVP for steady-state calculations

Consider the following time-dependent one-dimensional model problem.

$$\begin{aligned} u_t + \mathcal{A}u &= F(x, t), & x \geq 0, & \quad t \geq 0 \\ Lu &= g(t), & x = 0, & \quad t \geq 0 \\ u &= f(x), & x \geq 0, & \quad t = 0, \end{aligned} \tag{1}$$

where $u(x, t) = (u^1, u^2, \dots, u^m)^T$ is the solution vector with m components, and $\mathcal{A} = \mathcal{A}(\partial/\partial x)$ is the differential operator. The forcing function F , the boundary data g and the initial function f are the data of the problem. For simplicity we disregard the influence of the right boundary.

Assume now that we want to use (1) and compute a steady solution v that satisfies

$$\begin{aligned} \mathcal{A}v &= \tilde{F}(x), & x \geq 0 \\ Lv &= \tilde{g}, & x = 0. \end{aligned} \tag{2}$$

The difference problem that describes the possible convergence to steady-state is obtained by subtracting (2) from (1). The deviation $e = u - v$ from steady-state satisfies

$$\begin{aligned} e_t + \mathcal{A}e &= dF(x, t), & x \geq 0, & t \geq 0 \\ Le &= dg(t), & x = 0, & t \geq 0 \\ e &= df(x), & x \geq 0, & t = 0, \end{aligned} \quad (3)$$

where $dF(x, t) = F - \tilde{F}$, $dg(t) = g - \tilde{g}$ and $df(x) = f - v$. Note that we have used the fact that $v_t = 0$ to arrive at (3).

For a steady-state solution to exist, no time-dependent data are allowed. This means that dF and dg must vanish as $t \rightarrow \infty$. Furthermore, we cannot expect to be able to guess the initial condition that makes $df = 0$. The task is therefore to develop techniques for the IBVP (3) with zero forcing function, zero boundary data and non-zero initial function such that $e \rightarrow 0$ as $t \rightarrow \infty$.

For later reference, the final version of the steady-state problem under investigation is

$$\begin{aligned} e_t + \mathcal{A}e &= 0, & x \geq 0, & t \geq 0 \\ Le &= 0, & x = 0, & t \geq 0 \\ e &= f, & x \geq 0, & t = 0. \end{aligned} \quad (4)$$

Our ambition is to reduce e from its initial value f and reach zero fast.

2.2 Convergence to steady-state of solutions to the IBVP

We discuss the requirements for obtaining steady-state solutions ($e = 0$) in (4). In almost all practical cases, the spatial operator $\mathcal{A}$ is given. The speed at which $e \rightarrow 0$, the convergence rate to steady-state, can therefore only be manipulated by the boundary condition $Le = 0$. Different choices of the boundary operator L leads to different convergence rates.

The Laplace-transform method for the convergence rate The convergence rate will be obtained by using the so called Laplace-transform technique, see the references in [1]. Assume that a suitable boundary operator L has been determined. The Laplace transformed version of (4) is

$$\begin{aligned} (sI + \mathcal{A})\hat{e} &= f, & x \geq 0 \\ L\hat{e} &= 0, & x = 0, \end{aligned} \quad (5)$$

where $s = \eta + i\xi$ is the dual variable to time and

$$\hat{e}(x, s) = \mathcal{L}e = \int_0^\infty e(x, t) \exp(-st) dt. \quad (6)$$

The integral in (6) is well defined if the time-growth of the solution e in (4) is bounded, i.e. if (4) is well posed and η is sufficiently large and positive.

To solve (5) we make the ansatz $\hat{e}_h = \psi \exp(\kappa x)$ for the homogeneous solution. The particular solution $\hat{e}_p$ which depends on the initial data f is assumed known. That leads to a generalized eigenvalue problem for $\kappa(s)$ of the form

$$(sI + \mathcal{A}(\kappa))\psi = 0, \quad |sI + \mathcal{A}(\kappa)| = 0. \quad (7)$$

The first equation in (7) has a non-trivial solution $\psi \neq 0$ if and only if there are κ such that the second relation is satisfied. $\mathcal{A}(\kappa)$ is a polynomial in κ with matrix coefficients. As an example, $\mathcal{A} = A\partial/\partial x + B\partial^2/\partial x^2$ leads to $\mathcal{A}(\kappa) = A\kappa + B\kappa^2$ where A and B are matrices. Note that $\psi = \psi(\kappa(s), s)$.

The homogeneous solution in the absence of multiple generalized eigenvalues κ , is

$$\hat{e}_h = \sum_i \sigma_i \psi_i \exp(\kappa_i x) = \Psi \bar{X}(x) \sigma, \quad (8)$$

where $\bar{X}(x) = \text{diag}[e^{\kappa_1 x}, e^{\kappa_2 x}, \dots]$, $\bar{X}(x) = \text{diag}[e^{\kappa_1 x}, e^{\kappa_2 x}, \dots]$, $\Psi = [\psi_1, \psi_2, \dots]$ and $\sigma = [\sigma_1, \sigma_2, \dots]^T$. The coefficients σ_i will be determined by the boundary conditions. The total solution is given by $\hat{e} = \hat{e}_h + \hat{e}_p$. The boundary conditions in (5) lead to $L\hat{e}_h = \hat{g}$ where $\hat{g} = -L\hat{e}_p$. By using (8), the final equation for the coefficients σ_i becomes

$$E(s)\sigma = \hat{g}, \quad E(s) = L\Psi\bar{X}(0). \quad (9)$$

$E(s)$ is a matrix with the structure given by the boundary operator L , the eigenvectors ψ_i and the generalized eigenvalues κ_i . The right-hand side $\hat{g}$ is known and depend on the particular solution and it's gradients on the boundary.

A unique solution σ is obtained if E in (9) is non-singular. With s such that $\eta > \eta^*$ where all the possible singularities (or eigenvalues) in E lies to the left in the complex plane, we can solve for σ and formally transform back to time domain by

$$e(x, t) = \mathcal{L}^{-1}\hat{e} = \exp(\eta^* t) \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{e}(x, \eta^* + i\xi) \exp(i\xi t) d\xi \right). \quad (10)$$

Convergence to steady-state is obtained if $\eta^* < 0$. The way to increase the convergence rate (η^*) to steady-state is to choose the boundary operator L such that η^* lies as far as possible to the left in the complex plane. For later reference we denote $\eta^* =$ the continuous decay rate.

Remark: With multiple roots κ to the first equation in (7), the ansatz $\hat{e}_h = \psi \exp(\kappa x)$ must be reformulated. Instead of a constant vector, ψ is now of the form $\psi = \psi_0 + x\psi_1 + \dots + x^n\psi_n$, where $n+1$ is the multiplicity of the root κ . That complicates the derivation above technically, but in principle it remains the same. Also the conclusion that we have convergence to steady-state for $\eta^* < 0$ remains the same.

The energy method for deriving boundary conditions In the discussion above, the boundary operator was assumed given. To derive suitable boundary operators L such that $e \rightarrow 0$ fast as $t \rightarrow \infty$ we use the energy method. By choosing dissipative boundary operators L we hope to push the spectrum (the utmost right lying eigenvalues) as far left as possible in the left half plane. Multiply (4) with e^T , add the transpose of the equation, and integrate over the domain. That leads to

$$\|e\|_t^2 = - \int_0^\infty e^T (\mathcal{A} + \mathcal{A}^T) e dx = BT(e, e_x)_{x=0} - \int_0^\infty R(e, e_x) dx. \quad (11)$$

For (11) to be well posed, the right-hand side (RHS) in (11) must be bounded by $const. \|e\|^2$.

Remark: In (11) we have assumed that the operator $\mathcal{A}$ may include second derivatives (integration-by-parts yield boundary and volume terms with first derivatives). All important hyperbolic (Euler, Maxwells, wave equations) and parabolic (heat, stress equations) as well as incompletely parabolic problems (Navier-Stokes equations) are thereby included.

For fast energy decay, the RHS in (11) should be made as negative as possible. In almost all practical cases, the spatial operator $\mathcal{A}$ is given and hence also the original form of the RHS. There is no possibility to modify the volume term R (which must be negative semi-definite). The rate at which the norm is decreasing (or increasing) can only be manipulated by the boundary condition $Le = 0$. Different choices of L leads to different sizes and signs of $BT(e, e_x)_{x=0}$ and different convergence rates.

Remark: No direct method for constructing boundary conditions that lead to fast convergence to steady-state exist. In this paper we assume that forcing the energy to decay fast, will lead to fast (a negative η^* with large magnitude) convergence to steady-state.

2.3 The formulation of the IVP for steady-state calculations

We discretize the IBVP (4) in space and leave time continuous, i.e. use the method of lines. The focus is on the matrix properties of the resulting system of ordinary differential equations. The semi-discrete version of (4) can formally be written

$$\begin{aligned} \mathbf{e}_t + \bar{A}\mathbf{e} &= 0, & t \geq 0 \\ \mathbf{e} &= f, & t = 0. \end{aligned} \tag{12}$$

where $\mathbf{e} = (e_0, e_1, \dots, e_N)^T$, $e_j = (e^1, e^2, \dots, e^m)_j^T$ is the discrete version of the deviation e from steady-state. $\bar{A} = A - \Sigma$ is a modified version of A , where A is the $(N+1)m \times (N+1)m$ discretization matrix approximating the operator $\mathcal{A}$ and $N+1$ is the number of grid points. When the scheme is adjusted to include the numerical treatment of the boundary conditions (Σ) we obtain $\bar{A}$.

There are two distinctly different ways to prescribe boundary conditions for node vertex solvers where the unknowns are located on the boundary. One can use a weak or a strong boundary procedure. In a weak boundary procedure, the quantities at the boundaries, even though they are known, are updated in time. The boundary value typically deviates slightly from the prescribed value but the deviation is reduced as the grid is refined. With a strong boundary procedure, on the other hand, the specified boundary value is injected into the dependent variable on the boundary. The boundary quantity is no longer an unknown, and there is hence no need for an update.

In the case of a weak boundary procedure we have $\bar{A} = A - \Sigma$ where the matrix A correspond to the internal discretization (often on so called summation-by-parts (SBP) form). The incorporation of the continuous boundary conditions $Le = 0$ is done weakly by using the penalty term $\Sigma\mathbf{e}$. The form of the penalty

matrix Σ depend on the boundary operator L , i.e. $\Sigma = \Sigma(L)$. Roughly speaking, the penalty term must be accurate and lead to stability.

A strong implementation of the continuous boundary conditions is not clearly separable from the internal discretization (as is the weak boundary procedure). It essentially amounts to modifying the internal operator A directly in the boundary region and turn it into $\bar{A}$. To facilitate the comparison with weak boundary conditions we reformulate the strong implementation such that it directly mimics the weak one. That means that we add and subtract terms to obtain the matrix relation $\bar{A} = A - \Sigma$ where A is the same for both weak and strong boundary procedures.

2.4 Convergence to steady-state of solutions to the IVP

We discuss the requirements for obtaining steady-state solutions $\mathbf{e} = 0$ in (12). The sign and size of the real part of the eigenvalues to $\bar{A}$ is the crucial factor and they depend on the discretization of the spatial operator as well as on the boundary procedure. We will focus on the influence of the boundary procedure and keep the discretization of the spatial operator fixed.

The matrix exponential method for obtaining the convergence rate

Once we have the IVP on the form (12) we can write down the solution as

$$\mathbf{e} = \exp(-\bar{A}t)f, \quad t \geq 0, \quad (13)$$

using the definition of the matrix exponential. Next we assume that $\bar{A}$ is diagonalizable (the algebraic multiplicity of the eigenvalues λ_i is equal to the geometric multiplicity) such that $\bar{A} = X\Lambda X^{-1}$ where $\Lambda = \text{diag}(\lambda_i)$. That leads to

$$\mathbf{e} = X \exp(-\Lambda t) X^{-1} f, \quad t \geq 0, \quad (14)$$

and convergence to steady-state if all $\Re(\lambda_i) > 0$. The way to increase the convergence rate ($\min_i \Re(\lambda_i)$) to steady-state is to choose a boundary procedure such that $\min_i \Re(\lambda_i)$ lies as far as possible to the right in the complex plane. The sign convention is such that $\min_i \Re(\lambda_i) \rightarrow -\eta^*$ as the mesh is refined.

Remark: If there are multiple roots λ_i such that $\bar{A}$ is *not* diagonalizable (the algebraic multiplicity of the eigenvalues λ_i is larger than the geometric multiplicity), then we can use the Jordan decomposition of $\bar{A}$. The derivation above remains almost the same and the conclusion that we have convergence to steady-state for $\min_i \Re(\lambda_i) > 0$ remains exactly the same.

The energy method for deriving boundary procedures To derive a suitable discretization, i.e. a matrix $\bar{A}$ with eigenvalues such that $\mathbf{e} \rightarrow 0$ as $t \rightarrow \infty$ as was discussed above, we use the energy method. Multiply (12) with $\mathbf{e}^T \bar{P}$ from the left, where $\bar{P} \equiv (P \otimes I_m)$. I_m is the $m \times m$ identity matrix and the

$(N+1) \times (N+1)$ matrix P is symmetric and positive definite. The same holds for $\bar{P}$, which thus is a valid norm $\|\mathbf{e}\|_{\bar{P}}^2 = \mathbf{e}^T \bar{P} \mathbf{e}$. This leads to

$$(\|\mathbf{e}\|_{\bar{P}}^2)_t = -\mathbf{e}^T (\bar{P}\bar{A} + (\bar{P}\bar{A})^T) \mathbf{e}. \quad (15)$$

Recall that $\bar{A}$ includes both the interior approximation and the boundary procedure. For (15) to be energy stable, the right-hand side (RHS) must be bounded by *const.* $(\|\mathbf{e}\|_{\bar{P}}^2)$.

For a steady state to exist, we know from the previous section that the eigenvalues of $\bar{A}$ must have strictly positive real parts. We also know that if $\min_i \Re(\lambda_i)$ is large, the convergence will be fast. To manipulate the eigenvalues of $\bar{A}$ directly is almost impossible. However, by using the energy method, see (15), we can modify the matrix $(\bar{P}\bar{A})^S = (\bar{P}\bar{A} + (\bar{P}\bar{A})^T)/2$ in such a way that it modifies $\bar{A}$ and most likely enhances the convergence rate.

In most cases, the internal part of the discretization (A) is given by the class of method chosen. The choice of boundary approximation (Σ), on the other hand is crucial and will be considered in detail. As was mentioned above ($\Sigma = \Sigma(L)$) and it is therefore highly dependent on the boundary conditions in (4) but also on the specific numerical implementation technique. In this paper we focus on the latter.

Let us assume that we have chosen a particular boundary approximation such that $(\bar{P}\bar{A})^S$ is positive definite and leads to an energy decay in (15). Let λ and x be an eigenvalue and eigenvector to $\bar{A}$. By multiplying the relation $\bar{A}x = \lambda x$ from the left with $x^* \bar{P}$ and rearranging we get

$$\lambda = \Re(\lambda) + i\Im(\lambda), \quad \Re(\lambda) = \frac{x^* (\bar{P}\bar{A})^S x}{x^* \bar{P} x}, \quad \Im(\lambda) = \frac{x^* (\bar{P}\bar{A})^{AS} x}{x^* \bar{P} x}, \quad (16)$$

where $(\bar{P}\bar{A})^{AS} = (\bar{P}\bar{A} - (\bar{P}\bar{A})^T)/2$. Consequently, we can guarantee convergence to steady-state if we have an energy decay in (15). We can also see from (16) that by making $(\bar{P}\bar{A})^S = (\bar{P}\bar{A} + (\bar{P}\bar{A})^T)/2$ more positive definite, there is a possibility that we can increase the convergence rate $\min_i \Re(\lambda_i)$. For later reference we denote $\min_i \Re(\lambda_i) =$ the discrete decay rate.

The strategy in this paper for studying the influence of weak and strong solid wall boundary procedures can be summarized as follows. First we make sure that the continuous solid wall boundary conditions are well posed and lead to convergence to steady-state (otherwise further investigation is meaningless). Next we discretize and derive the different forms of $\bar{A} = A - \Sigma$. The inner scheme A is the same for all cases but Σ varies depending on the form of boundary implementation we use. We will study three types, the weak, the strong and an intermediate form of implementation. After that we use the energy method and try to determine if the schemes are stable. Finally we compute the eigenvalues of $\bar{A}$ and see which procedure yields the largest value of $\min_i \Re(\lambda_i)$.

2.5 Numerical results using a fully nonlinear finite volume solver

The analysis of our model problem, see details in the full article [1], indicate that the weak scheme lead to faster convergence to steady-state. To investigate if this

result carries over to the fully nonlinear Navier-Stokes equations, we use the finite volume solver Edge, which is applicable on both structured and unstructured grids. The governing equations are integrated explicitly with a multistage local time-stepping Runge-Kutta scheme to steady-state and acceleration by FAS agglomeration multigrid can be used. There are numerous boundary conditions in Edge for walls, external boundaries and periodic boundaries. In all calculations we have used the same boundary conditions on all boundaries that are not solid walls. These boundary conditions are imposed weakly.

Remark: The eigenvalues must be located in the right-half plane also for local time-stepping schemes. Consider an IVP on the form $V_t + \bar{A}V = F$. Diagonalizing the IVP yields a system of scalar equations of the form $(w_j)_t + \lambda_j w_j = \tilde{f}_j$. The scalar equations all require eigenvalues λ_j with positive real part in order not to grow. This requirement is independent of whether one uses a time-accurate or local time-stepping scheme.

The flow conditions for the NACA0012 airfoil are as follows: Mach number $M = 0.8$, Reynolds number $Re = 500$ and $\alpha = 0^\circ$ as the angle of attack. These parameters yield a subsonic, low Reynolds flow which is assumed laminar.

Fig. 1. The convergence of the density residual. NACA0012 at Mach number $M = 0.8$, Reynolds number $Re = 500$ and $\alpha = 0^\circ$ as the angle of attack.

Figure 1 displays the convergence to steady-state using the density residual. A 3-stage first order accurate Runge-Kutta time integrator with $CFL=1.25$ is used in combination with local time steps. No artificial dissipation is used. The weak solution converges faster than the solutions obtained by the other two schemes. The difference is more pronounced for the coarsest grid, which is what is expected from the linear results. Next we investigate the flow over a flat plate.

For $x < 0$, where no plate exist, a symmetry condition is used, and for $x \geq 0$ the no-slip solid wall condition is imposed. The flat plate is 7 m long and has a Reynolds number of 10.5×10^6 based on that length. The flow is modelled as fully turbulent with an Explicit Algebraic Reynolds Stress Model (EARSM) formulation based on a two-equation $\kappa - \omega$ model.

Fig. 2. The convergence of the density residual. Flat plate at Mach number $M = 0.07$ and Reynolds number $Re = 10.5 \cdot 10^6$. Single grid calculation.

The convergence of the density residual is shown in Figure 2. The weak scheme converges slightly faster than the mixed scheme, while the strong scheme does not converge at all. This behavior is also consistent with the linear analysis which showed that the strong scheme had an eigenvalue passing zero for the coarsest mesh.

Next we investigate the influence of solid wall boundary procedures on the multigrid acceleration technique. We consider the same flow field (the flat plate) again. Now neither the mixed nor the strong boundary procedure converges, see Figure 3. This is probably due to the fact that eigenvalues close to or equal to zero (for our analogous linear problem in the full article) gives a non-decaying energy.

Remark: The differences in steady state convergence between the boundary conditions are most evident without numerical dissipation. In other calculations the differences are smaller but still in favour of the weak boundary conditions.

Fig. 3. The convergence of the density residual. Flat plate, $M = 0.07$, Reynolds number $Re = 10.5 \cdot 10^6$. Multigrid with three grid levels.

3 Functionals and dual problems

The solution of the governing equations might not be the output of primary interest in many CFD applications. Of equal, or even greater, importance is the computation of functionals from the solution. In general, a functional is defined as any map from a vector space V into the underlying scalar field $\mathcal{K}$. Every vector space has an associated vector space called its dual (or adjoint) space. The dual space is denoted by V^* and is defined as the space of all linear functionals $V \rightarrow \mathcal{K}$.

The adjoint, or dual, operator L^* of a linear operator L is the (unique) operator satisfying

$$(v, Lu)_V = (L^*v, u)_V, \quad (17)$$

where $(\cdot, \cdot)_V$ denotes the inner product on the space V . The study of linear functionals and dual spaces is the topic of functional analysis and additional preliminaries can be found in any functional analysis textbook.

In this section, we consider initial boundary value problems of the form

$$\begin{aligned} u_t + \mathcal{L}(u) &= F, \quad x \in \Omega, \\ \mathcal{B}(u) &= g_\Gamma, \quad x \in \Gamma \subseteq \partial\Omega, \\ u &= f, \quad t = 0. \end{aligned} \quad (18)$$

For applications in CFD, a linear functional of interest usually represents the lift or drag on a solid body in a fluid, which is computed in terms of an integral of the solution of (18). The functional can be represented in terms of an integral

inner product as

$$J(u) = (g, u) = \int_{\Omega} g^T u d\Omega, \quad (19)$$

where g is a weight function. A main complication in CFD is that no physically relevant solutions have compact support in the computational domain. The dual operator is obtained through integration by parts which will introduce boundary terms that must be removed. The dual PDE has thus to be supplied with dual boundary conditions to close the system.

The associated dual problem has been extensively studied and used in the context of error control and adaptive mesh refinement as well as within optimization and control problems. In error control and mesh adaptation, the dual problem is derived and treated as a variational problem. In optimization and control problems, the dual problem is derived and treated as a sensitivity problem with respect to design parameters. In the end, the two different formulations yield the same dual problem. A similarity for the different areas of applications is that most of them are based on unstructured methods.

3.1 Quadrature accuracy

Only recently was the study of duality introduced to structured methods, such as the SBP-SAT technique. Recall that the SBP operator was constructed to satisfy

$$(v_h, D_1 u_h)_h = u_h^T (E_N - E_0) v_h - (D_1 v_h, u_h)_h, \quad (20)$$

which mimics an integration property, rather than a differentiation property. While the differentiation properties of the SBP operator has been extensively studied and used, the integration properties of the matrix P have been much less explored. The integration properties of P was thoroughly investigated by Hicken and Zingg [5]. It was shown that the requirements on P to obtain an accurate SBP operator include, and extend, the Gregory formulas for quadrature rules using equidistant points. Two main results were proven and are restated here for convenience. The first theorem establishes the accuracy of P as an integration operator;

Theorem 1. *Let P be a full, restricted-full, or diagonal mass matrix from an SBP first-derivative operator $D_1 = P^{-1}Q$, which is a $2p$ -order accurate approximation to the first derivative in the interior. Then the mass matrix P constitutes a $2p$ -order accurate quadrature for integrands $u \in C^{2p}(\Omega)$.*

The second theorem extends the results to include discrete integrands computed from an SBP differentiation;

Theorem 2. *Let $D_1 = P^{-1}Q$ be a an SBP first derivative operator with a diagonal mass matrix P and $2p$ -order interior accuracy. Then $(v_h, D_1 u_h)_h$ is a $2p$ -order accurate approximation of (v, u_x) .*

These theorems proved in summary that it is possible to retain the full order of accuracy when computing integrals from an SBP discretization, even with a diagonal P .

3.2 Dual consistency

For initial boundary value problems (IBVPs), it is not sufficient to integrate the solution obtained by an SBP-SAT discretization using P to obtain a functional of $2p$ -order accuracy. It was shown in [6] that an additional property of the discretization was required—the so called dual consistency property. The main result in [6] extends the results in [5] to include SBP-SAT solutions to IBVPs. Even though the solution u_h to an IBVP using SBP-SAT is accurate of order $p + 1$ when using a diagonal P , any linear functional of u_h is accurate of order $2p$ when integrated using P , if the discretization is dual consistent.

As suggested by the name, dual consistency requires that the discretization of the primal problem is also a consistent approximation of the dual problem. In order to construct a dual consistent discretization, one first have to derive the dual problem and work with both the primal and dual problems simultaneously. To obtain the dual differential operator we consider the linear, or linearized, Cauchy problem,

$$\begin{aligned} u_t + Lu &= f, & x \in \Omega, \\ u &= 0, & t = 0, \\ J(u) &= (g, u) \end{aligned} \quad (21)$$

where $J(u)$ is a linear functional of interest. We seek a function θ , in some appropriate function space, such that

$$\int_0^T J(u) dt = \int_0^T (\theta, f) dt. \quad (22)$$

Using integration by parts, we can write

$$\begin{aligned} \int_0^T J(u) dt &= \int_0^T J(u) dt - \int_0^T (\theta, u_t + Lu - f) dt \\ &= \int_0^T (\theta_t - L^* \theta + g, u) dt - [(\theta, u)]_{t=0}^T + \int_0^T (\theta, f) dt \end{aligned} \quad (23)$$

and it is clear that $\theta = 0$ at $t = T$ is needed, and that θ has to satisfy the dual equation $-\theta_t + L^* \theta = g$. The time transform $\tau = T - t$ is usually introduced, and the dual Cauchy problem becomes

$$\begin{aligned} \theta_\tau + L^* \theta &= g, & x \in \Omega, \\ \theta &= 0, & \tau = 0. \end{aligned} \quad (24)$$

The situation is more complicated for IBVPs. Since the primal equation does not have compact support in general, the boundary terms resulting from the integration by parts procedure has to be properly taken care of by the homogeneous primal boundary conditions. The dual boundary conditions are defined as the minimal set of homogeneous conditions such that the boundary terms vanish after the homogeneous primal boundary conditions have been applied. Still, one needs to investigate the well-posedness of the dual equation with the

resulting dual boundary conditions. A well-posed set of boundary conditions for the primal problem does not necessary lead to a well-posed dual problem.

A discretization of a problem with a functional of interest can be written as

$$\begin{aligned}\frac{d}{dt}u_h + L_h u_h &= f, \\ J_h(u_h) &= (g, u_h)_h,\end{aligned}\tag{25}$$

where the entire spatial discretization, including the boundary conditions, has been collected into the discrete operator L_h . Recall that the inner product is defined as

$$(v_h, u_h)_h = v_h^T P u_h \tag{26}$$

in an SBP-SAT framework. The discrete adjoint operator L_h^* is defined, analogously to (17), as the unique operator satisfying

$$(v_h, L_h u_h)_h = (L_h^* v_h, u_h)_h. \tag{27}$$

The discrete adjoint operator can hence be explicitly computed, using (26) and (27), as

$$L_h^* = P^{-1} L_h^T P. \tag{28}$$

The discrete dual problem is obtained analogously to the continuous case by finding θ_h such that $\int_0^T J_h(u_h) dt = \int_0^T (\theta_h, f) dt$. Integration by parts and (28) gives

$$\begin{aligned}\int_0^T J_h(u_h) dt &= \int_0^T (g, u_h)_h dt - \int_0^T (\theta_h, \frac{d}{dt}u_h + L_h u_h - f)_h dt \\ &= \int_0^T (\frac{d}{dt}\theta_h - L_h^* \theta_h + g, u_h)_h dt - [(\theta_h, u_h)_h]_{t=T} + \int_0^T (\theta_h, f)_h dt\end{aligned}\tag{29}$$

and hence the θ_h has to satisfy the discrete dual problem

$$\begin{aligned}\frac{d}{d\tau}\theta_h + L_h^* \theta_h &= g, \\ \theta_h &= 0, \tau = 0,\end{aligned}\tag{30}$$

where $\tau = T - t$. Dual consistency can now be defined in terms of L_h^* and L^* ;

Definition 1. *A discretization is called dual consistent if L_h^* is a consistent approximation of L^* and the continuous dual boundary conditions.*

The above definition is not specific for SBP-SAT discretizations. Any discretization which can be written in the form (25) is applicable. The SBP-SAT technique is particularly well-suited for this framework because of the well-defined inner product and operator form.

It is common, in optimization for example, that continuous and discrete adjoint methods are distinguished. This is because the discrete adjoint operator does not approximate the continuous adjoint operator and boundary conditions in general. In the SBP-SAT framework, the dual consistency property can allow for very efficient use of adjoint based techniques due to the unification of the continuous and discrete adjoints. SBP-SAT is not the only method which

offers consistency with the dual equations. It was shown that, for example, the discontinuous Galerkin method can also exhibit this property.

The dual consistency property can be easily exemplified using the model problem (31). Dual consistency does not depend on any data of the problem but only the differential operator and the form of the boundary conditions. We hence consider the inhomogeneous problem with homogeneous boundary and initial conditions,

$$\begin{aligned} u_t + \bar{u}u_x &= f, & 0 \leq x \leq 1 \\ u(0, t) &= 0, \\ u(x, 0) &= 0, \\ J(u) &= (g, u) \end{aligned} \quad (31)$$

where $J(u)$ is a linear functional of interest. We seek a function θ so that $\int_0^T J(u)dt = \int_0^T (\theta, f)dt$ and integration by parts gives

$$\begin{aligned} \int_0^T J(u)dt &= \int_0^T J(u)dt - \int_0^T (\theta, u_t + \bar{u}u_x - f)dt \\ &= \int_0^T (\theta_t + \bar{u}\theta_x + g, u)dt - \int_0^1 [\theta u]_{t=T}dx - \int_0^T [\bar{u}\theta u]_{x=1}dt + \int_0^T (\theta, f)dt. \end{aligned} \quad (32)$$

It is clear that θ has to satisfy the dual problem

$$\begin{aligned} \theta_\tau - \bar{u}\theta_x &= g, & 0 \leq x \leq 1, \\ \theta(1, \tau) &= 0, \\ \theta(x, 0) &= 0, \end{aligned} \quad (33)$$

where we have introduced the time transform $\tau = T - t$.

The model problem (31) can be discretized as

$$\begin{aligned} \frac{d}{dt}u_h + \bar{u}D_1u_h &= \sigma P^{-1}(e_0^T u_h - 0)e_0 + f, \\ J_h(u_h) &= (g, u_h)_h, \end{aligned} \quad (34)$$

and the parameter σ has to be determined so that the scheme is not only stable, but also a consistent approximation of the dual problem (33). It is convenient to rewrite (34) in operator form as

$$\frac{d}{dt}u_h + L_h u_h = f, \quad (35)$$

where the spatial discretization, including the boundary condition, is included in the operator

$$L_h = \bar{u}D_1 - \sigma P^{-1}E_0. \quad (36)$$

The discrete dual operator can be directly computed as

$$L_h^* = P^{-1}L_h^T P = -\bar{u}D_1 + \bar{u}P^{-1}E_N - (\sigma + \bar{u})P^{-1}E_0, \quad (37)$$

and it is seen that L_h^* imposes a boundary condition at $x = 0$, due to the last term in (37), unless $\sigma = -\bar{u}$. With $\sigma = -\bar{u}$, the discrete dual problem becomes

$$\frac{d}{d\tau}\theta_h - \bar{u}D_1\theta_h = -\bar{u}P^{-1}E_N\theta_h + g, \quad (38)$$

which is a consistent approximation of the dual problem (33). Since $\sigma = -\bar{u}$ does not contradict the stability condition ($\sigma \leq -\bar{u}/2$), the scheme is both stable and dual consistent. In Table 1 we show the convergence rates q for the solution and the functionals, together with the functional error, using the dual inconsistent and consistent schemes. As we can see from Table 1, the convergence rate for

Table 1. Convergence rates q , and functional errors for the dual inconsistent and consistent schemes

5th-order ($2p = 8$)						
	$\sigma = -1/2$			$\sigma = -1$		
N	$q(u_h)$	$q(J_h(u_h))$	Error	$q(u_h)$	$q(J_h(u_h))$	Error
96	4.58	4.51	1.87e-05	5.14	8.20	7.54e-09
128	4.87	4.80	3.02e-06	5.34	7.96	2.71e-10
160	4.97	4.91	7.58e-07	5.41	8.02	2.74e-11
192	5.02	4.97	2.53e-07	5.44	8.06	4.58e-12
224	5.05	5.01	1.02e-07	5.46	8.21	1.05e-12
256	5.06	5.04	4.72e-08	5.46	8.62	2.97e-13

the linear functional increases from $p + 1$ to $2p$ when using the dual consistent discretization. Also notice that dual consistency is merely a choice of parameters. The solution of the dual problem is never required and hence the increased rate of convergence for linear functionals comes at no extra computational cost.

3.3 Boundary conditions

The theory of dual consistency is not only useful for deriving schemes with superconvergent integral functionals. By simultaneously considering the primal and dual equals, new boundary conditions can be derived. As an example, we will consider the linearized and symmetrized compressible Navier–Stokes and Euler equals in two space dimensions:

$$U_t + AU_x + BU_y = \varepsilon((C_{11}U_x + C_{12}U_y)_x + (C_{21}U_x + C_{22}U_y)_y). \quad (39)$$

The Euler equations are obtained by setting $\varepsilon = 0$. This seemingly small change, to have $\varepsilon = 0$ or not, have a huge impact on the boundary conditions. Lets consider the unit square for simplicity.

A commonly used far-field boundary condition is of the form

$$\begin{aligned} H_W U - \varepsilon(C_{11}U_x + C_{12}U_y) &= 0, & H_E U + \varepsilon(C_{11}U_x + C_{12}U_y) &= 0, \\ H_S U - \varepsilon(C_{21}U_x + C_{22}U_y) &= 0, & H_N U + \varepsilon(C_{21}U_x + C_{22}U_y) &= 0, \end{aligned} \quad (40)$$

where the matrices $H_{W,E,S,N}$ (W,E,S,N refers to the west, east, south, and north boundaries, respectively) have to be construed for well-posedness. A well-known problem is that for subsonic outflow boundaries, matrices that give a well-posed problem for one of the equations, give an ill-posed problem for the other. One

can hence not switch between the Navier–Stokes and Euler equations by simply putting $\varepsilon = 0$ or not. When attempting to remedy this problem, one has to put up general matrices with general coefficients and try to determine them to get the properties one wants. In two dimensions, the matrices are 4×4 , each with 16 undetermined coefficients, and the parameter space simply becomes too large.

More equations are needed to deal with the amount of undetermined parameters. By considering not only well-posedness of the primal equations, but also of the dual equations, we get exactly what we need. To derive the dual Navier–Stokes equations we consider (39) in the form

$$\begin{aligned} U_t + LU &= F, & (x, y) &\in \Omega, \\ BU &= 0, & (x, y) &\in \Gamma \subseteq \partial\Omega, \\ U &= 0, & t &= 0, \\ J(U) &= (G, U). \end{aligned} \quad (41)$$

In (41), $J(U)$ is a linear integral functional with a weight function G and B implements the boundary conditions in (40). The right-hand side F may be identically zero, but a symbol is needed to perform integration by parts when deriving the dual problem. The differential operator L is given by

$$L = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} - \varepsilon \left(\left(C_{11} \frac{\partial}{\partial x} + C_{12} \frac{\partial}{\partial y} \right) \frac{\partial}{\partial x} + \left(C_{21} \frac{\partial}{\partial x} + C_{22} \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} \right). \quad (42)$$

We seek a function $\Theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T$ such that $\int_0^T J(U) dt = \int_0^T (\Theta, F)$, and we get by the Gauss–Green formula

$$\begin{aligned} \int_0^T J(U) dt &= \int_0^T (G, U) - \int_0^T (\Theta, U_t + LU - F) dt \\ &= \int_0^T (\Theta, F) dt + \int_0^T (\Theta_t - L^* \Theta + G, U) dt \\ &\quad + \int_0^T \int_W \Theta^T (AU - \varepsilon(C_{11}U_x + C_{12}U_y)) dy dt + \varepsilon \int_0^T \int_W (\Theta_x^T C_{11} + \Theta_y^T C_{12}) U dy dt \\ &\quad - \int_0^T \int_E \Theta^T (AU - \varepsilon(C_{11}U_x + C_{12}U_y)) dy dt - \varepsilon \int_0^T \int_E (\Theta_x^T C_{11} + \Theta_y^T C_{12}) U dy dt \\ &\quad + \int_0^T \int_S \Theta^T (BU - \varepsilon(C_{21}U_x + C_{22}U_y)) dx dt + \varepsilon \int_0^T \int_S (\Theta_x^T C_{21} + \Theta_y^T C_{22}) U dx dt \\ &\quad - \int_0^T \int_N \Theta^T (BU - \varepsilon(C_{21}U_x + C_{22}U_y)) dx dt - \varepsilon \int_0^T \int_N (\Theta_x^T C_{21} + \Theta_y^T C_{22}) U dx dt \\ &\quad - \int_\Omega [\Theta^T U]_{t=0}^{t=T} d\Omega, \end{aligned} \quad (43)$$

where $\int_{W,E,S,N}$ denotes integration over the west, east, south, and north boundary, respectively. The dual operator, L^* , is given by

$$L^* = -A \frac{\partial}{\partial x} - B \frac{\partial}{\partial y} - \varepsilon \left(\left(C_{11} \frac{\partial}{\partial x} + C_{12} \frac{\partial}{\partial y} \right) \frac{\partial}{\partial x} + \left(C_{21} \frac{\partial}{\partial x} + C_{22} \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} \right), \quad (44)$$

and we obtain the dual boundary conditions by applying the homogeneous primal boundary conditions to the boundary integral terms.

By using (40), we can write (43) as

$$\begin{aligned}
\int_0^T J(U)dt &= \int_0^T (\Theta, F)dt + \int_0^T (\Theta_t - L^* \Theta + G, U)dt \\
&+ \int_0^T \int_W U^T ((A - H_W^T) \Theta + \varepsilon(C_{11} \Theta_x + C_{12} \Theta_y)) dy dt \\
&- \int_0^T \int_E U^T ((A + H_E^T) \Theta + \varepsilon(C_{11} \Theta_x + C_{12} \Theta_y)) dy dt \\
&+ \int_0^T \int_S U^T ((B - H_S^T) \Theta + \varepsilon(C_{21} \Theta_x + C_{22} \Theta_y)) dx dt \\
&- \int_0^T \int_N U^T ((B + H_N^T) \Theta + \varepsilon(C_{21} \Theta_x + C_{22} \Theta_y)) dx dt.
\end{aligned} \tag{45}$$

We introduce the dual time variable, $\tau = T - t$, and the function Θ has to satisfy the dual Navier–Stokes equations

$$\Theta_\tau - A \Theta_x - B \Theta_y = \varepsilon((C_{11} \Theta_x + C_{12} \Theta_y)_x + (C_{21} \Theta_x + C_{22} \Theta_y)_y) + G \tag{46}$$

with the dual boundary conditions

$$\begin{aligned}
(A - H_W^T) \Theta + \varepsilon(C_{11} \Theta_x + C_{12} \Theta_y) &= 0, \quad (A + H_E^T) \Theta + \varepsilon(C_{11} \Theta_x + C_{12} \Theta_y) = 0, \\
(B - H_S^T) \Theta + \varepsilon(C_{21} \Theta_x + C_{22} \Theta_y) &= 0, \quad (B + H_N^T) \Theta + \varepsilon(C_{21} \Theta_x + C_{22} \Theta_y) = 0,
\end{aligned} \tag{47}$$

together with a homogeneous initial condition at $\tau = 0$.

An interesting property is that the primal equations (39) with the primal boundary conditions (40), and the dual equations (46) with the dual boundary conditions (47), share the same energy estimate. Let Φ be either the primal variable U or the dual variable Θ . Then

$$\begin{aligned}
&\|\Phi\|_{t,\tau}^2 + 2\varepsilon \int_\Omega (\nabla \Phi^T) C (\nabla \Phi)^T d\Omega = \\
&- \int_W \Phi^T (-A + H_W + H_W^T) \Phi dy - \int_E \Phi^T (A + H_E + H_E^T) \Phi dy \\
&- \int_S \Phi^T (-B + H_S + H_S^T) \Phi dx - \int_N \Phi^T (B + H_N + H_N^T) \Phi dx,
\end{aligned} \tag{48}$$

where C is a positive semi-definite matrix. It is clear that $H_{W,E,S,N}$ must be chosen such that

$$\begin{aligned}
-A + H_W + H_W^T &\geq 0, \quad A + H_E + H_E^T \geq 0, \\
-B + H_S + H_S^T &\geq 0, \quad B + H_N + H_N^T \geq 0,
\end{aligned} \tag{49}$$

in order to obtain a bounded energy growth and hence an energy estimate for both the primal and dual Navier–Stokes and Euler equations. Energy estimates are, however, not sufficient. It is also required that the correct number of boundary conditions are imposed to get well-posed problems, see Table 2. An operator

Table 2. Number of boundary conditions required for the primal and dual Navier–Stokes and Euler equations under subsonic conditions with positive velocity components

Boundary	Number of b.c.			
	West	East	South	North
Primal Navier–Stokes	4	3	4	3
Dual Navier–Stokes	3	4	3	4
Primal Euler	3	1	3	1
Dual Euler	1	3	1	3

which have an energy estimate with a minimal number of boundary conditions, such that existence is guaranteed, is called maximally semi-bounded and leads directly to well-posedness.

For subsonic outflow boundaries, we chose to construct H_E such that the pressure is specified for the primal Euler equations. In this case, H_E has to satisfy:

1. The top row of H_E is zero
2. The top row of $A + H_E^T$ is non-zero
3. $\text{rank}(H_E) = 1$
4. $\text{rank}(A + H_E^T) = 3$
5. $A + H_E + H_E^T \geq 0$

Requirements (1) and (2) set the correct number of boundary conditions for the primal and dual Navier-Stokes equations, requirements (3) and (4) set the correct number of boundary conditions for the primal and dual Euler equations, and requirement (5) gives energy estimates of both the primal and dual Navier-Stokes and Euler equations, respectively. The above requirements make the matrix H_E uniquely determined. From potentially 16 undetermined parameters, the requirements of primal and dual well-posedness gives unique solutions to all parameters. The result is summarized in

Theorem 3. *Let the matrix H_E be given by*

$$H_E = \frac{\bar{u}^2 - \bar{c}^2}{\bar{c}\sqrt{\gamma}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \sqrt{\gamma - 1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (50)$$

Then the boundary conditions

$$H_E \Phi + \varepsilon(C_{11}\Phi_x + C_{12}\Phi_y) = 0 \quad (51)$$

are well-posed subsonic outflow conditions for the primal and dual Navier–Stokes and, for $\varepsilon = 0$, the primal and dual Euler equations.

The details of the proof, and the remaining boundaries together with the dual consistent SBP-SAT discretizations, can be found in [4].

4 Multigrid for higher order accurate finite difference schemes

Multigrid is well established for industrial second order accurate CFD codes like Edge and other codes for structured grids [7]. For higher order accurate finite difference CFD codes, however, multigrid has so far not been used for convergence acceleration. We have taken the first steps towards using multigrid in a higher order accurate finite difference solver. We have analyzed the eigenvalues to the iteration matrix using multigrid for a model problem of the linearized one-dimensional Navier-Stokes equations. The spatial operators considered are exactly the operators based on the SBP-SAT technique.

The Navier-Stokes equations in one dimension can be written as

$$u_t + F_x = 0; F = F^I - \epsilon F^V \quad (52)$$

where superscripts I, V denotes the inviscid and viscous fluxes, respectively and u is the vector of the three conservative variables. In order to study the eigenvalues of the spectra we consider the non-dimensional and linearized Navier-Stokes equations

$$w_t + A_w w_x - B w_{xx} = 0 \quad (53)$$

where w contains the primitive variables. Matrices A_w, B are constant matrices and contain averaged values from which the linearization is made.

Multigrid is applied to accelerate the convergence to steady state of a linear problem formulated as

$$w_t + Lw = 0 \quad (54)$$

where the spatial operator L contains the entire discretization in one space dimension x including boundary conditions,

$$L(w) = (\xi_x \otimes I_3) ((D_\xi \otimes A_w)w + (D_\xi \otimes I_3)(\xi_x \otimes I_3)(D_\xi \otimes B)w - Pen), \quad (55)$$

where ξ_x is the metric operator, I_3 is a 3×3 identity matrix, $D_\xi = P^{-1}Q$ is the first difference SBP operator and Pen contains the boundary penalty terms.

In non-linear multigrid the same equations are solved and iterated on coarser grids but with a forcing term on the right hand side. We assume $N_{grid} \geq 1$ multigrid levels and introduce index l to denote the current grid level and denote $l = 1$ the finest level with the original grid and $l = N_{grid}$ the coarsest grid. Then the following equations are solved on the different grid levels:

$$\begin{aligned} \text{Level } 1 & : w_{lt} + L_l w_l = 0 \\ \text{Level } l, 1 < l \leq N_{grid} & : w_{lt} + L_l w_l = F_l = L_l r_{l-1}^l w_{l-1} - r_{l-1}^l L_{l-1} w_{l-1} \end{aligned} \quad (56)$$

where F_l is the forcing function and is constant when iterating on the coarse grids. r_{l-1}^l is the restriction operator used to transfer information from fine to coarse grid between levels $l - 1, l$. The coarse grid solution is iterated one time level after which the solution difference is interpolated, or prolonged, back to the finer level using a prolongation operator p_l^{l-1} .

The solution is iterated in time using a time integrator which is denoted a smoother. Here we consider an explicit m -stage Runge-Kutta time integrator which integrates (54) as

$$\begin{aligned} w_l^0 &= w_l^n \\ w_l^1 &= w_l^0 - \alpha_1 \Delta t (L_l w_l^0 - F_l) \\ &\vdots \\ w_l^m &= w_l^0 - \alpha_m \Delta t (L_l w_l^{m-1} - F_l) \\ w_l^{n+1} &= w_l^m \end{aligned} \quad (57)$$

where $F_1 = 0$ and Δt the time step. The smoother S_l can then be formulated as

$$w_l^{n+1} = S_l w_l^n + (I - S_l) L_l^{-1} F_l \quad (58)$$

where w_l^n, w_l^{n+1} are the solution vectors at two consecutive time levels $n, n+1$ and where

$$S_l = I + \beta_1 \Delta t L_l + \beta_2 \Delta t^2 L_l^2 + \beta_3 \Delta t^3 L_l^3 + \dots \quad (59)$$

and $(\beta_1, \beta_2, \beta_3, \dots) = (\alpha_m, \alpha_m \alpha_{m-1}, \alpha_m \alpha_{m-1} \alpha_{m-2}, \dots)$.

We want to compare the eigenvalues of the fine grid iteration matrix M_1 , sometimes denoted amplification matrix, where

$$w_1^{n+1} = M_1 w_1^n. \quad (60)$$

Stability requires that all eigenvalues satisfy $|M_1| \leq 1$. The iteration matrix for multigrid can be written as

$$\begin{aligned} \text{Level } l, 1 \leq l < N_{grid} : M_l &= S_l^{\nu_1} (I - p_{l+1}^l (I - M_{l+1}) L_{l+1}^{-1} r_l^{l+1}) S_l^{\nu_2} \\ \text{Level } N_{grid} : M_{N_{grid}} &= S_{N_{grid}} \end{aligned} \quad (61)$$

where ν_1, ν_2 are the number of pre- and post smoothing steps. A single smoothing step is performed on the coarsest grid.

In the example below we apply wall boundary conditions at one end ($x = 0$) and characteristic outflow conditions at the other end ($x = 1$) with corresponding penalty terms with stable values of penalty parameters. We use a first order accurate 3-stage Runge-Kutta time integrator with $(\alpha_1, \alpha_2, \alpha_3) = (2/3, 2/3, 1)$, and we choose inviscid flow conditions, $\epsilon = 0$, at a low Mach number $M_\infty = 0.1$. The grid is stretched towards the wall reflected in the metric operator ξ_x in (55).

In Figures 4 and 5 the eigenvalues corresponding to a second and third order (fourth order in the interior and second order on the boundaries) accurate discretization are shown using 1 – 3 grid levels. The eigenvalues are within the stability limits with both discretizations. The stability depends on the choice of grid transfer operators (restriction and prolongation) though which requires further analysis investigations.

5 Conclusions

Two completely new developments for increased performance of the high order and stable SBP-SAT finite difference technique has been described. The use of

Fig. 4. Eigenvalues of iteration matrix M_1 using a second order SBP discretization

Fig. 5. Eigenvalues of iteration matrix M_1 using a third order SBP discretization

weak boundary conditions leads to increased convergence rate to steady state both for single and multigrid calculations. Dual consistent schemes produces superconvergent functionals such as lift and drag and increases accuracy. The dual consistency comes at no extra cost.

The first steps have been taken to introduce multigrid in a higher order accurate finite difference scheme by analyzing eigenvalues of a one-dimensional model problem of the linearized Euler equations. The results look promising and there is no indication that multigrid should not work as convergence accelerator in a higher order accurate finite difference code, further investigations are required.

Finally we mention that the obvious benefits shown in this chapter of using weak boundary procedures and dual consistent schemes are also valid for other computational techniques techniques such discontinuous Galerkin, finite continuous finite elements, spectral elements etc. Only technical implementation issues differ, but not the fundamental gains that can be obtained.

References

1. Nordström, J., Eriksson, S., Eliasson, P.: Weak and Strong Wall Boundary Procedures and Convergence to Steady-State of the Navier–Stokes Equations. *Journal of Computational Physics*, 231, 4867–4884 (2012).
2. Berg, J., Nordström, J.: Superconvergent functional output for time-dependent problems using finite differences on summation-by-parts form. *Journal of Computational Physics*, 231, 6846–6860 (2012).
3. Berg, J., Nordström, J.: On the impact of boundary conditions on dual consistent finite difference discretizations. *Journal of Computational Physics*, 236, 41–55, (2013).
4. Berg, J., Nordström, J.: Duality based boundary conditions and dual consistent finite difference discretizations of the Navier–Stokes and Euler equations. *Journal of Computational Physics*, 259, 135–153 (2014).
5. Hicken, J. E., Zingg, D. W.: Summation-by-parts operators and high-order quadrature. *Journal of Computational and Applied Mathematics* 237, 111–125 (2013)
6. Hicken, J. E., Zingg, D. W.: Superconvergent Functional Estimates from Summation-By-Parts Finite-Difference Discretizations. *SIAM Journal on Scientific Computing*, 33, 893–922, (2011).
7. Wesseling, P., Oosterlee, C. W.: Geometric multigrid with applications to computational fluid dynamics. *Journal of Computational and Applied Mathematics*, 128, 311–334, (2001).