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SUCCESSIVE ENCODING OF CORRELATED SOURCES

by

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ABSTRACT

The encoding of a discrete memoryless multiple source $\{(X_i, Y_i)\}_{i=1}^{\infty}$ for reconstruction of a sequence $\{Z_i\}_{i=1}^{\infty}$ with $Z_i = F(X_i, Y_i)$; $i = 1, 2, \dots$ is considered. We require that the encoding should be such that $\{X_i\}_{i=1}^{\infty}$ is encoded first without any consideration of $\{Y_i\}_{i=1}^{\infty}$, while in a second part of the encoding this latter sequence is encoded based on knowledge of the outcome of the first encoding. The resulting scheme is called successive encoding. We find general outer and inner bounds for the corresponding set of achievable rates along with a complete single letter characterization for the special case $H(X_i | Z_i, Y_i) = 0$. Comparisons with the Slepian-Wolf problem [3] and the Ahlswede-Körner-Wyner side information problem [2], [9] are carried out.

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I. INTRODUCTION

Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of independent pairs of correlated random variables ranging over the finite sets X, Y ; let $F: X \times Y \rightarrow Z$ be a given function from the product set $X \times Y$ to another finite set Z ; and let the random sequence $\{Z_i\}_{i=1}^{\infty}$ be defined by $Z_i \triangleq F(X_i, Y_i)$; $i = 1, 2, \dots$. Our problem is to encode the sequence $\{(X_i, Y_i)\}_{i=1}^{\infty}$ in such a way that the sequence $\{Z_i\}_{i=1}^{\infty}$ can be reconstructed essentially without errors. The special feature of this problem is that we impose a certain restrictive structure on the encoder. We require the encoding to be carried out in such a way that the sequence $\{X_i\}_{i=1}^{\infty}$ is encoded first without any consideration of the sequence $\{Y_i\}_{i=1}^{\infty}$. Then this second sequence is encoded taking into account the coded version of the first sequence. The resulting scheme will be called successive encoding. Our interest is to find the set of rates for which error free decoding can be obtained from an encoding of this form. We will obtain outer and inner bounds for this set of achievable rates and we will show that in a certain interesting class of cases these bounds coincide, thus establishing the true rate region for our problem.

Coding of correlated sources with certain structure restrictions on the encoder was first studied in the by now classical paper by Slepian and Wolf [3]. The restriction imposed by them is that the two subsources $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ should be encoded separately and the aim of the encoding is to guarantee the reconstruction of the whole sequence $\{(X_i, Y_i)\}_{i=1}^{\infty}$ at, possibly, low rates. In our notation the case they studied is $Z_i = F(X_i, Y_i) = (X_i, Y_i)$; $i = 1, 2, \dots$. The result of Slepian and Wolf was extended independently by Ahlswede-Körner [2] and Wyner [4], [9], who solved the case $Z_i = F(X_i, Y_i) = X_i$; $i = 1, 2, \dots$ under the same kind of restriction on the encoder.

The unrestricted problem, i.e. the encoding of $\{(X_i, Y_i)\}_{i=1}^{\infty}$ for reconstruction of $\{F(X_i, Y_i)\}_{i=1}^{\infty}$ without any restriction on the

encoder, is of course the conventional noiseless source coding problem solved already by Shannon [13]. The interesting fact about the Slepian-Wolf result is that within certain limits (on the rates) the restriction to separate encoding of the subsources causes no degradation in terms of required total rate. This is generally regarded as a rather surprising result. The extension by Ahlswede-Körner and Wyner indicate that the Slepian-Wolf case is in this respect rather exceptional. In general more rate is required if the subsources are encoded individually.

Very little is known about separate encoding in the case of a general function $F: X \times Y \rightarrow Z$.^{*)} One of the few results available is for the modulo-2 sum problem of Körner and Marton [8]. (However, the original result of Slepian-Wolf could be extended in various other directions, cf. e.g. [7], [10], [11].)

Our problem is in between the cases of joint and separate encoding. Besides its obvious relation to the above mentioned problems it is also somewhat related to a seemingly different problem on "two-way communication" recently studied by Ericson and Ingemarsson [5]. That two-way problem can be reformulated as a source coding problem for a pair of individually encoded sources, where one of the encoders is being fed with information from the other. However, in contrast to our present problem the information from the first encoder to the second one is in that case not the same as the information sent from the first encoder to the decoder. A problem of that kind has also been recently studied by Kaspi and Berger [14].

The present paper is organized as follows. A more precise mathematical formulation of the problem is provided in section II. Outer and inner bounds in terms of a converse and a coding theorem are presented in sections III and IV. Both are concerned with the general case as described above and both are in a non-computable "product" form in

^{*)} We prefer not to mention some wrong results that appeared lately in Z. f. Wahrscheinlichkeits. verw. Geb.

as much as they involve functions of random sequences of all possible lengths. In section V a so called single letter characterization is obtained for the case when $H(X_i|Z_i, Y_i) = 0$; $i = 1, 2, \dots$. This restriction means that the function $F(\cdot, y)$ - regarded as a mapping from X to Z with $y \in Y$ as a parameter - is essentially invertible for all $y \in Y$. It will be shown that the outer and inner bounds coincide in this case, and consequently we will have obtained a computable characterization of the achievable rate region. In section VI some special cases are considered in more detail and comparisons are made with the Slepian-Wolf and the Ahlswede-Körner-Wyner problems mentioned above.

II. NOTATION AND BASIC DEFINITIONS

The problem at hand is illustrated in fig.1. We consider a discrete memoryless multiple source (DMMS) $\{(X_i, Y_i, Z_i)\}_{i=1}^{\infty}$ such that $Z_i = F(X_i, Y_i)$; $i = 1, 2, \dots$. By this we understand that $\{(X_i, Y_i, Z_i)\}_{i=1}^{\infty}$ is a sequence of independent triples of random variables X_i, Y_i, Z_i ranging over the finite sets X, Y and Z respectively. The probability distribution of the random triple (X_i, Y_i, Z_i) over the product set $X \times Y \times Z$ is the same for all indices $i = 1, 2, \dots$, and is arbitrary, subject only to the restriction that $Z_i = F(X_i, Y_i)$; $i = 1, 2, \dots$ for some fixed function $F: X \times Y \rightarrow Z$.

We will be interested in finding conditions under which block-encodings of the sequence $\{(X_i, Y_i)\}_{i=1}^{\infty}$ can be found such that the sequence $\{Z_i\}_{i=1}^{\infty}$ can be reconstructed essentially without errors. For this purpose we need the following concepts. An n-length block code is a triple (f_n, g_n, φ_n) of functions $f_n: X^n \rightarrow M_1$; $g_n: Y^n \times M_1 \rightarrow M_2$; $\varphi_n: M_1 \times M_2 \rightarrow Z^n$; where X^n, Y^n and Z^n denote the n-th Cartesian powers of the sets X, Y , and Z respectively and where $M_i = \{1, 2, \dots, M_i\}$; $i = 1, 2$. We will write X_m^n for the subsequence $\{X_i\}_{i=m}^n$ (with corresponding notations for other sequences). Usually the lower index will be $m = 1$, and in those cases it will be omitted, i.e. we will write X^n rather than X_1^n .

We define the random sequence $\{\hat{Z}_i\}_{i=1}^{\infty}$ as follows:

$$\hat{Z}_{ni-n+1}^{ni} \triangleq \varphi_n[f_n(X_{ni-n+1}^{ni}), g_n(Y_{ni-n+1}^{ni}, f_n(X_{ni-n+1}^{ni}))]; i = 1, 2, \dots$$

The functions f_n, g_n will be called encoders while φ_n will be called a decoder. The engineering interpretation is obvious. The pair (f_n, g_n) of encoders of the specific form indicated above will be said to yield a successive encoding of the source $\{(X_i, Y_i)\}_{i=1}^{\infty}$. The sequence $\{\hat{Z}_i\}_{i=1}^{\infty}$ is the output from the decoder when the encoders are fed with the processes $\{X_i\}_{i=1}^{\infty}, \{Y_i\}_{i=1}^{\infty}$ respectively.

We will use X as a generic variable for the process $\{X_i\}_{i=1}^{\infty}$. By this we mean that X is a random variable over X with the same distribution as any one of the variables X_i ; $i = 1, 2, \dots$. A similar convention applies to all other memoryless stationary sources.

We say that a pair (R_1, R_2) of non-negative real numbers is an achievable rate pair if there exists a sequence $\{(f_n, g_n, \varphi_n)\}_{n=1}^{\infty}$ of codes such that

$$i) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log ||f_n|| \leq R_1$$

$$ii) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log ||g_n|| \leq R_2$$

$$iii) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \Pr\{\hat{Z}_i \neq Z_i\} = 0.$$

Here we use $||f||$ as a notation for the finite cardinality of the range of the function f , i.e. $||f_n|| = M_1$; $||g_n|| = M_2$.

Our main problem is the characterization of the set R of achievable rate pairs.

For the basic concepts and notations the reader is referred to the book of Csiszár-Körner [1].

III. A CONVERSE

We start our investigation by proving the following technical lemma, which provides us with some necessary conditions fulfilled by any pair $(R_1, R_2) \in R$. A result of this kind is usually referred to as a converse.

Lemma 1:

For any achievable rate pair (R_1, R_2) and any $\delta > 0$ there is - for large enough m - some function ψ over X^m such that

$$\frac{1}{m} H(\psi(X^m)) + \frac{1}{m} H(Z^m | \psi(X^m), Y^m) < R_1 + \delta$$

$$\frac{1}{m} H(Z^m | \psi(X^m)) < R_2 + \delta.$$

Proof:

Let $\delta > 0$ and $(R_1, R_2) \in R$ be given and let $\{(f_m, g_m, \varphi_m)\}_{m=1}^{\infty}$ be a sequence of block-codes generating (R_1, R_2) . By construction and a well-known property of the entropy function we have

$$\frac{1}{m} H(Z^m | f_m(X^m), Y^m) \leq \frac{1}{m} H(Z^m | f_m(X^m), g_m(Y^m, f_m(X^m))) \leq \frac{1}{m} H(Z^m | \hat{Z}^m). \quad (1)$$

Now recall that by assumption the symbol error probability of the code (f_m, g_m, φ_m) tends to zero with increasing m . Hence by Fano's inequality ([1], p.54) the last term in (1) will also tend to zero with increasing m . We next observe the following inequalities:

$$\frac{1}{m} H(f_m(X^m)) \leq \frac{1}{n} \log \|f_m\|;$$

$$\begin{aligned} \frac{1}{m} H(Z^m | f_m(X^m)) &\leq \frac{1}{m} H(Z^m, g_m(Y^m, f_m(X^m)) | f_m(X^m)) \leq \\ &\leq \frac{1}{m} H(g_m(Y^m, f_m(X^m)) | f_m(X^m)) + \frac{1}{m} H(Z^m | f_m(X^m), g_m(Y^m, f_m(X^m))) \leq \\ &\leq \frac{1}{m} \log \|g_m\| + \frac{1}{m} H(Z^m | f_m(X^m), g_m(Y^m, f_m(X^m))). \end{aligned}$$

The lemma follows as we can now obviously choose $\psi = f_m$ for any m large enough for all the inequalities

$$\frac{1}{m} H(Z^m | \hat{Z}^m) < \delta/2$$

$$\frac{1}{m} \log \| f_m \| < R_1 + \delta/2$$

$$\frac{1}{m} \log \| g_m \| < R_2 + \delta/2$$

to be fulfilled simultaneously.

IV. A CODING THEOREM

Our next aim is to prove a coding theorem which will provide us with sufficient conditions for a point (R_1, R_2) to belong to R . However, to reach this end we will first need a simpler coding result for a slightly more complicated network. The situation is depicted in fig.2. Here our starting point is a DMMS $\{(U_i, X_i, Y_i, Z_i)\}_{i=1}^{\infty}$ with $Z_i = F(X_i, Y_i)$; $i = 1, 2, \dots$. A code is a quadruple $(f_n, g_n, h_n, \varphi_n)$ of functions $f_n: U^n \rightarrow M_0$; $g_n: X^n \times M_0 \rightarrow M_1$; $h_n: Y^n \times M_0 \times M_1 \rightarrow M_2$; $\varphi_n: M_0 \times M_1 \times M_2 \rightarrow Z^n$; where U, X, Y, Z are the ranges of the random variables in the various subprocesses and where we take $M_i \triangleq \{1, 2, \dots, M_i\}$; $i = 0, 1, 2$. The triple (f_n, g_n, h_n) constitutes a successive encoder for the DMMS $\{(U_i, X_i, Y_i)\}_{i=1}^{\infty}$ and φ_n is the corresponding decoder. If now we use $\{\hat{Z}_i\}_{i=1}^{\infty}$ as our notation for the output of the decoder when the encoder is fed with the DMMS $\{(U_i, X_i, Y_i)\}_{i=1}^{\infty}$ we can define the symbol error probability analogously to the previous case, namely as

$$e(f_n, g_n, h_n, \varphi_n) \triangleq \frac{1}{n} \sum_{i=1}^n \Pr\{\hat{Z}_i \neq Z_i\}.$$

Finally we say that (R_0, R_1, R_2) is an achievable rate triple for the present case if there exists a sequence $\{(f_n, g_n, h_n, \varphi_n)\}_{n=1}^{\infty}$ of codes with error probability tending to zero and with the appropriate limiting rates upperbounded by R_0, R_1, R_2 respectively. The result we will need is the following.

Lemma 2:

$[H(U), H(X|U, Y), H(Z|U)]$ is an achievable rate triple for the network in fig.2.

Proof:

It is enough to find for any $\delta > 0$ a sequence $\{(f_n, g_n, h_n, \varphi_n)\}_{n=1}^{\infty}$ of codes such that all of the inequalities

$$e(f_n, g_n, h_n, \varphi_n) < \delta \quad (2a)$$

$$\frac{1}{n} \log \| f_n \| < H(U) + \delta/2 \quad (2b)$$

$$\frac{1}{n} \log \| g_n \| < H(X|U, Y) + \delta/2 \quad (2c)$$

$$\frac{1}{n} \log \| h_n \| < H(Z|U) + \delta/2 \quad (2d)$$

are satisfied for all large enough n . However, the existence of such sequences is a simple consequence of the Slepian-Wolf result [3] in combination with the conventional noiseless source coding theorem. The argument is as follows. First choose an f_n such that (2b) is satisfied for all large enough n and so that U^n can be decoded from observation of $f_n(U^n)$ with an error probability less than $\delta/3$. This is possible by the noiseless source coding theorem [1], p.15. Notice that this means that $\{U_i\}_{i=1}^{\infty}$ is now essentially available at both of the other encoders as well as at the decoder. Now applying the Slepian-Wolf result [3] we choose g_n such that (2c) is satisfied for all large enough n and such that X^n can be decoded given (U^n, Y^n) with an error probability less than $\delta/3$. As the third encoder has Y^n available along with coded versions of U^n and X^n he will - with a probability exceeding $(1 - 2\delta/3)$ - be able to determine (U^n, X^n) and consequently compute Z^n using the relation $Z_i = F(X_i, Y_i)$; $i = 1, 2, \dots, n$. Once again applying the Slepian-Wolf result we finally choose h_n so as to satisfy (2d) and such that Z^n can be reconstructed with an error probability less than $\delta/3$. It is clear that φ_n can be chosen so as to reproduce Z^n with a probability exceeding $1 - \delta$. As the symbol error probability cannot exceed the codeword error probability this fact proves our lemma.

We now return to our original problem. With the aid of the above lemma we can now easily prove the following coding theorem for the source network in fig.1.

Theorem 1:

For any $\delta > 0$, any positive integer m and any function ψ over X^m there is some pair $(R_1, R_2) \in \mathcal{R}$ such that

$$R_1 < \frac{1}{m} H(\psi(X^m)) + \frac{1}{m} H(X^m | \psi(X^m), Y^m) + \delta$$

$$R_2 < \frac{1}{m} H(Z^m | \psi(X^m)) + \delta.$$

Let us consider an arbitrary function ψ with domain X^m . Let its range be M and let us define the DMMS $\{(\tilde{U}_i, \tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)\}_{i=1}^\infty$ as a "super source" according to the following

$$\tilde{U}_i \triangleq \psi(X_{mi-i+1}^m); \quad \tilde{X}_i \triangleq X_{mi-i+1}^m$$

$$\tilde{Y}_i \triangleq Y_{mi-m+1}^m; \quad \tilde{Z}_i \triangleq Z_{mi-i+1}^m$$

$$i = 1, 2, \dots$$

According to lemma 2 we can find a sequence $\{(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k, \tilde{\varphi}_k)\}_{k=1}^\infty$ of codes for this "correlated supersource" such that

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\tilde{f}_k\| \leq H(\psi(X^m))$$

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\tilde{g}_k\| \leq H(X^m | \psi(X^m), Y^m)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\tilde{h}_k\| \leq H(Z^m | \psi(X^m))$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \Pr\{\hat{\tilde{Z}}_i \neq \tilde{Z}_i\} = 0$$

where of course $\{\hat{\tilde{Z}}_i\}_{i=1}^\infty$ is the output from decoder $\tilde{\varphi}_k$ when the encoder $(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k)$ is being fed with the DMMS $\{(\tilde{U}_i, \tilde{X}_i, \tilde{Y}_i)\}_{i=1}^\infty$.

Now let $\{(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k, \tilde{\varphi}_k)\}_{k=1}^{\infty}$ be as before and let us introduce the following notation:

$$\underline{x} = \underline{x}_1, \underline{x}_2, \dots, \underline{x}_k \in X^{km}; \quad \underline{x}_i \in X^m$$

$$\underline{y} = \underline{y}_1, \underline{y}_2, \dots, \underline{y}_k \in Y^{km}; \quad \underline{y}_i \in Y^m$$

$$i = 1, 2, \dots, k,$$

$$\underline{u} \triangleq \psi(\underline{x}_1) \psi(\underline{x}_2) \dots \psi(\underline{x}_k) .$$

Furthermore, let $\tilde{M}_0, \tilde{M}_1, \tilde{M}_2$ be the ranges of $\tilde{f}_k, \tilde{g}_k, \tilde{h}_k$ (in order to simplify the notation we suppress the dependence on k in $\tilde{M}_0, \tilde{M}_1, \tilde{M}_2$) and let θ be a bijection from $\tilde{M}_0 \times \tilde{M}_1$ into

$$M_1 \triangleq \{1, 2, \dots, |\tilde{M}_0 \times \tilde{M}_1|\} .$$

$$\text{Finally let } M_2 \triangleq \tilde{M}_2 .$$

For $n = km; k = 1, 2, \dots$ we define the codes (f_n, g_n, φ_n) according to the following:

$$f_{km}(\underline{x}) \triangleq \theta[\tilde{f}_k(\underline{u}), \tilde{g}_k(\underline{x}, \underline{u})]; \quad \underline{x} \in X^{km}$$

$$g_{km}(\underline{x}) \triangleq \tilde{h}_k(\underline{y}, \tilde{f}_k(\underline{u}), \tilde{g}_k(\underline{x}, \underline{u})); \quad (\underline{x}, \underline{y}) \in X^{km} \times Y^{km}$$

$$\varphi_{km}(i, j) \triangleq \tilde{\varphi}_k(\theta^{-1}(i), j); \quad (i, j) \in M_1 \times M_2 .$$

Clearly the sequence $\{(f_{km}, g_{km}, \varphi_{km})\}_{k=1}^{\infty}$ satisfies

$$\limsup_{k \rightarrow \infty} \frac{1}{km} \log \|f_{km}\| \leq \frac{1}{m} H(\psi(X^m)) + \frac{1}{m} H(X^m | \psi(X^m), Y^m)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{km} \log \|g_{km}\| \leq \frac{1}{m} H(Z^m | \psi(X^m))$$

$$\lim_{k \rightarrow \infty} \frac{1}{km} \sum_{i=1}^{km} \Pr\{\hat{Z}_i \neq Z_i\} = 0$$

where again $\{\hat{z}_i\}_{i=1}^{\infty}$ is the output from φ_{km} . It is fairly obvious that the previous definition can easily be extended so as to yield a sequence $\{(f_n, g_n, \varphi_n)\}_{n=1}^{\infty}$ of codes defined for all n and still satisfying the previous inequalities. This proves the theorem.

V. A SINGLE LETTER CHARACTERIZATION

Our converse (Lemma 1) and coding theorem (Theorem 1) do of course provide a certain characterization of the set R of achievable rates in as much as they indicate outer and inner bounds to R . A direct comparison reveals that the main difference between them is the presence of the term $H(Z^m | \psi(X^m), Y^m)$ in the converse in contrast with the term $H(X^m | \psi(X^m), Y^m)$ at the corresponding place in the coding theorem. Obviously the first term is always less than or equal to the second one. In fact, as in general strict inequality prevails it follows that our outer and inner bounds are in general different.

However, they coincide whenever $H(X|Z, Y) = 0$. To see this we notice that

$$\begin{aligned} H(X^m | \psi(X^m), Y^m) &\leq H(X^m, Z^m | \psi(X^m), Y^m) = \\ &= H(Z^m | \psi(X^m), Y^m) + H(X^m | \psi(X^m), Z^m, Y^m) \end{aligned} \quad (3)$$

However, now the last term is zero by assumption. It follows that for the corresponding class of functions F our results provide a complete characterization of R . Still that characterization is not very useful as it is not computable. That defect will now be removed by the following single-letter characterization.

Theorem 2:

For sources satisfying the condition $H(X|Z, Y) = 0$ the set R of achievable rate pairs (R_1, R_2) equals the set of pairs (R_1, R_2) satisfying

$$\left. \begin{aligned} R_1 &\geq H(X, Y) - H(Y|U) - t \\ R_2 &\geq H(Z|U) + t \end{aligned} \right\} \quad (4)$$

for some t with $0 \leq t \leq \min [I(U \wedge Y), I(U \wedge Z)]$ and some random variable U ranging over a set $\mathcal{U}$ such that

$$|\mathcal{U}| \leq |X| + 2$$

and satisfying the condition of being conditionally independent of Y given X .

Remark:

The last condition of course means that U should satisfy a certain Markov chain condition which we indicate by the notation [1]

$$U \rightarrow X \rightarrow Y .$$

Proof:

We first observe the following identities:

$$\begin{aligned} H(\psi(X^m)) + H(X^m | \psi(X^m), Y^m) &= \\ &= H(\psi(X^m)) + H(X^m, Y^m | \psi(X^m)) - H(Y^m | \psi(X^m)) = \\ &= H(X^m, Y^m, \psi(X^m)) - H(Y^m | \psi(X^m)) = \\ &= m H(X, Y) - H(Y^m | \psi(X^m)) . \end{aligned}$$

It follows from Lemma 1 and Theorem 1 in combination with the inequality (3) that R is the closure of the set of pairs (R_1, R_2) such that for some function ψ over X^m

$$R_1 \geq H(X, Y) - \frac{1}{m} H(Y^m | \psi(X^m))$$

$$R_2 \geq \frac{1}{m} H(Z^m | \psi(X^m)) .$$

The result follows now by a direct application of Theorem 3.20, p. 342 in Csiszár-Körner [1].

VI. FOUR SPECIAL CASES

As an illustration we now consider four special cases where the results are more explicit and where interesting comparisons with other source coding problems can be made. Of the four cases below only D represents a new result of the authors.

A — — Z = (X,Y) (Slepian-Wolf [1]).

In this case the condition $H(X|Z,Y) = 0$ is obviously satisfied, so Theorem 2 applies and gives us the true characterization of R . Observing the relations

$$H(X,Y) - H(Y|U) - I(U \wedge Y) = H(X|Y)$$

$$H(X,Y|U) = H(X|U) + H(Y|U,X) \geq H(Y|X)$$

$$H(X,Y) - H(Y|U) + H(X,Y|U) \geq H(X,Y)$$

we realize that any point in R must satisfy all of the inequalities

$$R_1 \geq H(X|Y)$$

$$R_2 \geq H(Y|X)$$

$$R_1 + R_2 \geq H(X,Y) \quad .$$

To get the corresponding direct result, choose $U \equiv X$.

In conclusion, we have obtained the result of Slepian-Wolf [1] showing that in this case the extra link between the encoders is of no help; at least not in an asymptotic sense.

B — — — Z = X

Also in this case $H(X|Z,Y) = 0$. Exactly as before we have $R_1 \geq H(X|Y)$. For R_2 we have the obvious bound $R_2 \geq 0$ and for the

sum we have in this case

$$R_1 + R_2 \geq H(X) + H(Y|X) - H(Y|U) + H(X|U) .$$

But $H(Y|X) = H(Y|UX)$ by the Markov chain condition, and hence we have

$$\begin{aligned} H(Y|X) - H(Y|U) + H(X|U) &= \\ &= H(X,Y|U) - H(Y|U) \geq 0 . \end{aligned}$$

Consequently we realize that all points $(R_1, R_2) \in \mathcal{R}$ must satisfy

$$R_1 \geq H(X|Y)$$

$$R_2 \geq 0$$

$$R_1 + R_2 \geq H(X)$$

Choosing $U \equiv X$ we see that all these points are achievable.

It is interesting here to make a comparison with the Ahlswede-Körner-Wyner problem, i.e. the problem that arises if we delete the link from f_n to g_n . From Ahlswede-Körner [2] and Wyner [9] we know that the true rate region in that case is the set of pairs (R_1, R_2) satisfying

$$R_1 \geq H(X|U)$$

$$R_2 \geq I(U \wedge Y)$$

for some random variable U ranging over an alphabet $\mathcal{U}$ such that $|\mathcal{U}| \leq |\mathcal{Y}| + 2$ and being such that U is conditionally independent of X given Y (i.e. satisfies $U \perp\!\!\!\perp X \mid Y$). That rate region - called the A-K-W-region - is depicted in fig.3. along with our rate region $\mathcal{R}$. As can be seen the presence of the link from f_n to g_n does matter in this case. It is interesting to note that the minimum value of R_1

is $H(X|Y)$ in both cases while the corresponding minimum value of R_2 is $I(X \wedge Y)$ in our case as compared with $H(Y)$ in the Ahlswede-Körner-Wyner case.

C — — — Z = Y (Gray-Wyner [16], Ahlswede-Körner [2])

In this case the condition $H(X|Z,Y) = 0$ does not apply and neither does Theorem 2. Gray-Wyner [16] and Ahlswede-Körner [2] have shown that the achievable rate region is the set of pairs

$$R_1 \geq I(U \wedge X)$$

$$R_2 \geq H(Y|U)$$

where U is as in Theorem 2. Furthermore the achievable rate region does not decrease if the link between the encoders is removed, c.f. [2], [9]. This case is illustrated in fig. 4 showing the true rate region as well as the bound obtained from Theorem 2.

D — — — Z = X ⊕ Y

In this case we assume $X = Y = Z = \{0,1\}$ and that Z is the modulo-two sum of X and Y . (We use " $\oplus$ " as a notation for modulo-two addition). This problem without the link between the encoders was first studied by Körner and Marton [8], who solved it for "symmetric sources" i.e. sources satisfying

$$\Pr\{X=Y=0\} = \Pr\{X=Y=1\}$$

$$\Pr\{X=0,Y=1\} = \Pr\{X=1,Y=0\},$$

c.f. also Körner [12] pp. 197-200 and Csiszár-Körner [1] pp. 399-401.

It is interesting to note that our Theorem 2 implies an outer bound for this modulo-two adder problem. We first notice that the condition $H(X|Z,Y) = 0$ is met here, as well as the condition $H(Y|Z,X) = 0$. It follows that Theorem 2 applies and that the corresponding rate region R

is an outer bound in the case without the link. Also by symmetry the region $\tilde{R}$ obtained upon reversing the roles of X and Y is a valid outer bound. Finally the intersection $R \cap \tilde{R}$ of these two regions is clearly also a valid outer bound. The interest of this bound is that, to our knowledge, no better outer bound is available for any particular joint distribution of (X,Y) . Combining this observation with previously known results we obtain the following

Theorem 3:

Any achievable rate (R_1, R_2) for the modulo-two adder problem satisfies the following conditions.

a) If $H(Z) < \min [H(X), H(Y)]$, then

$$\begin{aligned} R_1 &\geq \max [H(X,Y) - H(Y|U), H(Z|V)] \\ R_2 &\geq \max [H(Z|U), H(X,Y) - H(X|V)] \end{aligned}$$

for some random variables U, V taking at most 4 values each and satisfying the Markov chain conditions $U \rightarrow X \rightarrow Y$; $V \rightarrow Y \rightarrow X$.

b) If $H(Z) \geq \min [H(X), H(Y)]$, then

$$\begin{aligned} R_1 &\geq H(X|Y); \quad R_2 \geq H(Y|X); \\ R_1 + R_2 &\geq H(X,Y) . \end{aligned}$$

Further, the bound in b) is tight.

Proof:

The result of b) is due to Körner, cf. [1], p. 400. The result of a) follows from Theorem 2. To see this, notice that the inequality $H(Z) < H(Y)$ implies that the channel matrix $P_{Z|X}$ is a degraded version of the channel matrix $P_{Y|X}$, and thus the parameter t can be omitted in Theorem 2, cf. [1], Problem 3.3.2 on p. 343.

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Fig. 1. A source network with successive encoding.

Fig. 2. The source network appearing in Lemma 2.

Fig. 3. A comparison of the rate regions for the cases of successive and separate encoding.

Fig. 4. Rate regions for the case $Z = Y$.

a) General; b) The case $H(X|Y) = 0$.