SOME ESTIMATES OF THE MEAN CURVATURE OF GRAPHS OVER DOMAINS IN $\mathbb{R}^n$

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1. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $P, Q \subset \overline{\Omega}$ be closed disjoint subsets. The conformal capacity of the capacitor $(P, Q; \Omega)$ (see [1], Chapter II, §3) is defined to be

$$\text{cap}(P, Q; \Omega) = \inf \int_{\Omega} \|\nabla \varphi(x)\|^n \, dx,$$  \hspace{1cm} dx = dx_1 \cdots dx_n,$$

where the infimum is over all possible functions $\varphi(x)$ that are locally Lipschitz on $\Omega$, continuous on $\overline{\Omega}$, and equal to 0 on $Q$ and 1 on $P$. A compact set $P$ is said to have zero capacity if there exists a closed set $Q$ with $Q \cap P = \emptyset$ such that $\mathbb{R}^n \setminus Q$ is bounded and $\text{cap}(P, Q; \mathbb{R}^n) = 0$. A closed set $P$ has capacity zero if every compact subset of it does.

Let $H(t), \; t \in \mathbb{R}$, be a nondecreasing continuous function, and let $f(x) = f(x_1, \ldots, x_n)$ be a $C^2$-solution of the equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left( \frac{f x_i}{\sqrt{1 + |\nabla f|^2}} \right) = n H(f(x)),$$

in the domain $\Omega$. The solutions of this equation are graphs $x_{n+1} = f(x)$ with mean curvature the given function of the coordinate $x_{n+1}$. For $H(t) = a + bt$, $b > 0$, the solutions of (2) describe the phenomenon of capillarity in a column of liquid with cross-section $\Omega$, and has been treated in [2] and [3].

We have the following estimate of the integral mean curvature $H(f(x))$ of the graph $F$ for a solution $x_{n+1} = f(x)$ of (2).

THEOREM 1. Let $f(x)$ be a solution of (2), and let $P \subset \Omega$ be an arbitrary closed set. Then

$$\int_P |H(f(x))|^n \, dx \leq \text{cap}(P, \partial \Omega; \Omega).$$

The idea of the proof consists in the following. We introduce the notation $t^{(n)} = t \cdot |t|^{n-1}$ and fix a function $\varphi(x)$ that is admissible in the variational problem (1) for the capacitor $(P, \partial \Omega; \Omega)$. We consider an arbitrary $C^1$-smooth function $H_1(t)$ with $H_1'(t) \geq 0$ such that

$$|H_1(t)| \leq |H(t)|, \quad H(t) \cdot H_1(t) \geq 0.$$
Using (2), we arrive at the inequality

\[ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ H_1^{(n-1)}(f(x)) \frac{f_{x_i}}{\sqrt{1 + |\nabla|^2}} \phi^n(x) \frac{f_{x_i}}{\sqrt{1 + |\nabla|^2}} \right] \geq n H_1^{(n-1)}(f(x)) H(f(x)) \phi^n(x) + n H_1^{(n-1)}(f(x)) \phi^{n-1}(x) \sum_{i=1}^{n} \frac{\phi_{x_i} f_{x_i}}{\sqrt{1 + |\nabla|^2}}. \]

First using (4), we integrate the last inequality and use the Cauchy formula. As a result,

\[ \int_{\Omega} \phi^n(x) |H_1(f(x))|^n \, dx \leq \int_{\Omega} ||\nabla \phi||^n \, dx. \]

Using the condition \( \phi \equiv 1 \) on \( \phi(x) \) for \( x \in P \), we pass to the infimum with respect to \( \phi(x) \) in the inequality just obtained:

\[ \int_{P} |H_1(f(x))|^n \, dx \leq \text{cap}(P, \partial \Omega; \Omega). \]

Finally, approximating \( H(t) \) by functions \( H_1(t) \) satisfying (4), we arrive at the required estimate (3).

**Corollary 1.** Let \( f(x) \) be the solution of (2), which is defined everywhere in \( \mathbb{R}^d \) except perhaps for a closed set \( P \) of capacity zero. Then the mean curvature satisfies \( H(f(x)) \equiv 0 \) everywhere in \( \mathbb{R}^d \setminus P \). In particular:

a) If \( 2 \leq n \leq 7 \), then \( f(x) \) is a linear function.

b) If \( H(t) \) is strictly monotone, then \( f(x) \equiv \text{const} \) for \( n \geq 2 \).

In the two-dimensional case, if \( H(t) \) is of constant sign, \( H'(t) \geq 0 \), and the set \( P \) is empty, assertion a) is due to Cheng and Yau [4].

**2. Notation.** \( B(x, R) \) is the ball of radius \( R > 0 \) about \( x \in \mathbb{R}^n \), and \( \text{dist}(x, E) \) is the distance from a point \( x \) to a set \( E \).

**Theorem 2.** Let \( f(x) \) be a solution of equation (2) in the domain \( \Omega \subset \mathbb{R}^n \). Then

\[ \sup_{x \in \Omega} |H(f(x))| \cdot \text{dist}(x, \partial \Omega) \leq 1. \]

Equality is attained in (5) in the case when \( \Omega \) is a ball and \( f(x) \) describes a hemisphere over \( \Omega \).

**Corollary 2.** Let \( f(x) \) be a solution of (2) in the ball \( B(0, R) \). Then

\[ |H(f(0))| \leq 1/R. \]

This assertion was obtained by Bernstein [5] and Finn [3] in the cases of mean curvature that is constant or bounded away from zero, respectively.

It is not hard to see that there exists a bounded radially symmetric \( C^2 \)-solution \( f(x) \) of equation (2) with right-hand side

\[ H(f(x)) = \frac{n-1}{n} (R^n - ||x||^n)^{-1/n}, \quad x \in B(0; R), \]

i.e., the mean curvature \( H(f(x)) \) increases without bound in a neighborhood of any point of the boundary \( \partial B(0, R) \). It is obvious from the following assertion that such boundary singularities cannot be isolated. Namely, we have
THEOREM 3. Suppose that $\Omega \subset \mathbb{R}^n$ is a domain and $q \in \Omega$ is a point. Let $f(x)$ be a solution of (2) in $\Omega \setminus \{q\}$. Then the function $H(f(x))$ is bounded in a neighborhood of $q$, and

$$\lim_{x \to q} |H(f(x))| \leq \left( \text{dist}(q, \partial \Omega) \right)^{-1}.$$

3. It is interesting to determine the exact value of the functional on the left-hand side of (5) in the case of an arbitrary domain $\Omega$ different from a ball. Below we give examples of domains for which this problem can be completely solved.

For any arbitrary integer $1 \leq p \leq n$ let $\Pi_{n,p}$ be a layer in space that, to within a motion and a homothety, is the coordinate product

$$\Pi_{n,p} = D^p(a) \times \gamma^{n-p},$$

where $\gamma^{n-p}$ is an $(n-p)$-dimensional plane, and $D^p(a)$ is a $p$-dimensional disk of radius $a > 0$.

THEOREM 4. Let $\Omega$ coincide with one of the domains of the form (6). Then

$$\sup_{x \in \Omega} \{ |H(f(x))| \cdot \text{dist}(x, \partial \Omega) \} \leq \frac{p}{n}$$

for any solution $f(x)$ of equation (2) in $\Omega$. Equality is attained in cases of special surfaces of constant mean curvature over $\Omega$.

We sketch the proof. Let $\Omega = P_{n,p}^1$, $1 \leq p \leq n - 1$. Using the properties of the mean curvature of a hypersurface in $\mathbb{R}^{n+1}$, we can assume without loss of generality that $\Pi_{n,p}$ has the form

$$\pi_{n,p} = \left\{ x \in \mathbb{R}^{n+1}; \sum_{i=1}^{p} x_i^2 < 1; \quad x_{n+1} = 0 \right\},$$

and it suffices to establish (7) for $x = 0$.

Let $v$ and $w$ denote the projections of a vector $x \in \mathbb{R}^{n+1}$ on the following mutually orthogonal subspaces of $\mathbb{R}^{n+1}$: $\gamma = \{ x \in \mathbb{R}^{n+1}; x_i = 0; \quad 1 \leq i \leq p \}$ and $\gamma^\perp$, respectively. Consider the $\lambda$-parametric family of tori

$$T_x(Q) = \{ x = (w, v) \in \mathbb{R}^{n+1}; \|v - \lambda e_{n+1} - R\|^2 + \|w\|^2 = r^2 \},$$

where $0 < r < 1 < R$ are fixed numbers, and $e_{n+1}$ is a coordinate vector. It is clear that $T_x(Q)$ does not intersect the surface $F$ for sufficiently large numbers $\lambda > 0$. We find the infimum $\lambda_0$ of such numbers $\lambda$. Then $T_{x_0}(Q)$ is everywhere not below the $F$, and touches it at the same point $x_0 \in \Pi_{n,p}$. Comparing the mean curvature of the torus $H_x$ at the point $x_0$, we arrive at the inequality

$$H(f(x_0)) \leq H_x(x_0) = \frac{1}{r} \left( \frac{p}{n} + \frac{n-p}{n} \frac{\|v_0\| - R}{\|v_0\|^2} \right)$$

for the mean curvature of the surface, where $x_0 = (w_0, v_0)$. But the point $(0; f(0))$ of $F$ lies no higher than the point $(x_0; f(x_0))$, and hence, since $H(t)$ is monotone,

$$H(f(x_0)) \leq \frac{1}{r} \left( \frac{p}{n} + \frac{n-p}{n} \frac{1}{R-1} \right).$$

The required estimate is obtained by passing to the limit as $r \to 1$ and $R \to \infty$. A lower estimate is established similarly. The case of equality holds, for example, when

$$f(x) = \sqrt{1 - x_1^2 - \cdots - x_p^2},$$

with $H \equiv p/n$. 

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