A NOTE ON JÖRGENS-CALABI-POGORELOV THEOREM

Tkachev Vladimir G.

1. Let $S_k(A)$ denote the $k$th principal symmetric function of the eigenfunctions of an $n \times n$ matrix $A$, i.e.

$$\det(A + tI) = \sum_{k=0}^{n} S_k(A) t^{n-k}.$$ 

The following classical result is well known.

**Theorem A** (Jörgens-Calabi-Pogorelov, [4], [2], [6]). Let $f(x)$ be a convex entire solution of

$$S_n(\operatorname{Hess} f) \equiv \det(\operatorname{Hess} f) = 1, \quad x \in \mathbb{R}^n,$$

where $\operatorname{Hess} f$ is the Hessian matrix of $f(x) = f(x_1, \ldots, x_n)$. Then $f(x)$ is a quadratic polynomial, i.e.

$$f(x) = a + \langle b, x \rangle + \langle x, Ax \rangle, \quad (1)$$

where $A$ is an $n \times n$ matrix with constant real coefficients and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$.

Let us consider the operator

$$L[f] \equiv \sum_{i=1}^{n} a_i(x) S_i(\operatorname{Hess} f) = 0. \quad (2)$$

In [1], A.A. Borisenko has established that affine functions $f(x) = a + \langle b, x \rangle$ are the only entire convex solutions of (2) in the following special cases, namely, when

$$L[f] = S_n(\operatorname{Hess} f) - S_1(\operatorname{Hess} f) = \det \operatorname{Hess} f - \Delta f = 0 \quad (3)$$

and

$$L[f] = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k S_{2k+1}(\operatorname{Hess} f) = 0. \quad (4)$$

Notice that solutions to (3) and (4) describe special Lagrangian submanifolds given a non-parametric form.

Let us consider the following condition.

(Q) either $a_k(x) \equiv 0$ on $\mathbb{R}^n$, or there exist two positive constants $\mu_1 \leq \mu_2$ such that $\mu_1 \leq |a_k(x)| \leq \mu_2$.

Let us denote by $J = J(L)$ the set of indices $i$, $1 \leq i \leq n$ such that $a_i(x) \neq 0$. The main purpose of this note is to establish the following generalization of [1].
THEOREM. Let \( f(x) \) be an entire convex \( C^2 \)-solution of (2) and that the structural condition (Q) is satisfied. If
\[
\lim_{\|x\| \to \infty} \sup_{i \in J} \frac{|f(x)|}{\|x\|^2} = 0
\]
then \( S_i(A(x)) \equiv 0 \) for any \( i \in J \), in particular, \( \det \text{Hess} f(x) = 0 \). If additionally \( a_1(x) \neq 0 \) then \( f(x) \) is an affine function.

REMARK 1. We construct an example in paragraph 4 below which shows that (5) is optimal in the sense that there exist operators \( L \) satisfying the condition (Q) and possessing solutions growing quadratically \( f(x) \sim \|x\|^2 \) as \( x \to \infty \) and such that \( \text{Hess} f(x) \neq 0 \).

2. We use the standard convention to write \( A \geq B \) if \( A - B \) is a positive semi-definite matrix.

**LEMMA 1.** Let \( A(x) \geq 0 \) be a continuous \( n \times n \) matrix solution of
\[
L(A(x)) = \sum_{i=1}^{n} a_i(x) S_i(A(x)) = 0, \quad x \in \mathbb{R}^n,
\]
where \( L \) is subject to the condition (Q). Then either \( S_i(A(x)) \equiv 0 \) for any \( i \in J \), or there exists \( k \in J \) and a constant \( \sigma_0 \) depending on \( \mu_1 \) and \( \mu_2 \) such that for all \( x \in \mathbb{R}^n \) the inequality holds
\[
S_k(A(x)) \geq \sigma_0 > 0.
\]

**PROOF OF LEMMA 1.** Note that \( S_k(A(x)) \geq 0 \) in virtue of the positive semi-definiteness of \( A(x) \). Then, if all (non-identically zero) coefficients \( a_i \) have the same sign then \( S_k(A(x)) \equiv 0 \) holds for any \( i \in J \). Now suppose that there exists \( x_0 \in \mathbb{R}^n \) and a number \( k \in J \) such that \( S_k(A(x_0)) > 0 \). In that case, there exist two coefficients \( a_i \) having different signs. Observe that by the condition (Q) this also holds true in the whole \( \mathbb{R}^n \). Let us rewrite (6) as
\[
|a_{i_1}(x_0)| S_{i_1}(A(x_0)) + \ldots + |a_{i_m}(x_0)| S_{i_m}(A(x_0)) = |a_{j_1}(x_0)| S_{j_1}(A(x_0)) + \ldots + |a_{j_p}(x_0)| S_{j_p}(A(x_0)),
\]
where \( i_1 < \ldots < i_m, j_1 < \ldots < j_p \), and also \( i_1 < j_1 \). We claim that \( k = i_1 \) satisfies the conclusion of the lemma. Indeed, we have
\[
S_{i_1}(A(x_0)) \leq b_1 S_{j_1}(A(x_0)) + \ldots + b_p S_{j_p}(A(x_0)),
\]
where \( b_k = |a_{j_k}(x_0)|/|a_{i_1}(x_0)| \leq \mu_2/\mu_1 \). Now, using Proposition 3.2.2 in [5, p. 106], we have
\[
\left( \frac{S_k(A(x_0))}{\binom{n}{k}} \right)^m \leq \left( \frac{S_m(A(x_0))}{\binom{n}{m}} \right)^k,
\]
for any \( 1 \leq m \leq k \leq n \), therefore by (7)
\[
S_{i_1}(A(x_0)) \leq \frac{\mu_2}{\mu_1} \sum_{k=1}^{p} \alpha_k \cdot (S_{i_1}(A(x_0)))^{\nu_k},
\]
where \( \nu_k = j_k/i_1 > 1 \) and \( \alpha_k = \binom{n}{j_k} \cdot \binom{n}{i_1}^{-\nu_k} \). Observe that the left hand side of the equation
\[
\frac{\mu_2}{\mu_1} \sum_{k=1}^{p} \alpha_k \cdot \sigma^{\nu_k - 1} = 1
\]
is an increasing function of $\sigma \geq 0$, and let $\sigma = \sigma_0$ denote its (unique) positive root. Then in virtue of the positiveness of $S_i(A(x_0))$ we conclude that $S_i(A(x_0)) \geq \sigma_0$. By the continuity assumption on $A(x)$, the latter inequality also holds in the whole $\mathbb{R}^n$ which proves the lemma.

\[ \square \]

**Corollary 1.** Let $f(x) \in C^2(\mathbb{R}^n)$ be a convex solution of (2) under the condition (Q). Then either $\det Hess f = 0$ in $\mathbb{R}^n$ or there exists $k \in J$ such that the inequality

$$S_k(Hess (f(x))) \geq \sigma_0 > 0$$

(8)

holds for all $x \in \mathbb{R}^n$ with $k, \sigma_0$ chosen as in Lemma 3.

3. **Proof of the Theorem.** We claim that under the hypotheses of the theorem there holds $S_i(Hess (f(x))) \equiv 0$ for any $i \in J$. Indeed, arguing by contradiction we have by Lemma 1 that $\Box$ holds in the whole $\mathbb{R}^n$ for some $k \in J$. One can assume without loss of generality, replacing if needed $f(x)$ by $f(x) + c + \langle a, x \rangle$, that $f(x) \geq 0$ in $\mathbb{R}^n$. Given an arbitrary $\epsilon > 0$, the condition (a) yields the existence of a constant $p \in \mathbb{R}$ such that $f(x) \leq \frac{\epsilon}{2}\|x\|^2 + p$ for any $x \in \mathbb{R}^n$. But $g(x) = \frac{\epsilon}{2}\|x\|^2 - f(x) \rightarrow \infty$ uniformly as $x \rightarrow \infty$, hence it attains its minimum value at some point, say $x_0 \in \mathbb{R}^n$, and there holds

$$\text{Hess} g(x_0) = \text{Hess}(\frac{\epsilon}{2}\|x\|^2 - f(x))|_{x_0} \geq 0,$$

which yields $\text{Hess} f(x_0) \leq \epsilon I$ with $I$ being the unit matrix.

Since $\text{Hess} f(x_0) \geq 0$ we obtain applying the majorization principle (see, for instance Corollary 4.3.3 in [4]) that

$$S_k(\text{Hess} f(x_0)) \leq S_k(\epsilon I) = \epsilon^k \binom{n}{k}.$$

But the assumption $S_k(\text{Hess} f(x)) \geq \sigma_0$ yields easily a contradiction with the arbitrariness of the $\epsilon$. This proves our claim. In particular, in virtue of the convexity of $f$ we also have $\text{Hess} f(x) \geq 0$, hence $\text{Hess} f(x)$ has zero eigenvalues for any $x \in \mathbb{R}^n$ implying $\det \text{Hess} f(x) \equiv 0$ in $\mathbb{R}^n$ (see also Corollary 3). If, additionally, $a_1(x) \neq 0$ then $1 \in J$ and the claim implies $S_1(\text{Hess} f(x)) = \Delta f(x) \equiv 0$. Applying again the convexity of $f(x)$ easily yields that $\text{Hess} f(x) \equiv 0$ in $\mathbb{R}^n$, hence $f(x)$ is an affine function and finishes the proof of the theorem.

4. **Example.** Let $\alpha(t)$ be a positive function, non-identically constant and such that $0 < q \leq \alpha(t) \leq q^{-1}$ for some fixed $0 < q < 1$. Let us consider the function

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \int_{0}^{x_i} (x_i - t)\alpha(t)dt.$$  

Then $\text{Hess} f(x) = (\alpha(x_i)\delta_{ij})_{1 \leq i, j \leq n}$, hence $f(x)$ is convex and satisfies

$$S_i(\text{Hess} f(x)) - \omega(x)S_i(\text{Hess} f) = 0$$

with $\omega(x) = \alpha(x_1) \ldots \alpha(x_n)$, $S_i(x) = \alpha(x_i)$. We have $\frac{1}{q}q^{n+1} \leq a_1(x) \leq \frac{1}{q}q^{-n-1}$, which establishes that $L$ satisfies the condition (Q). On the other hand,

$$\frac{q}{2}\|x\|^2 \leq f(x) \leq \frac{1}{2q}\|x\|^2,$$

thus $f(x)$ has the quadratic growth at infinity.
References