GENERALIZATIONS OF COMPLEX ANALYSIS
AND THEIR APPLICATIONS IN PHYSICS II

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LIFE-TIME OF MINIMAL TUBES AND COEFFICIENTS OF UNIVALENT FUNCTIONS IN A CIRCULAR RING

Summary

Estimate of life-time of two-dimensional minimal tubes in \( \mathbb{R}^3 \) have been obtained via potential theory method. The connection between this problem and coefficients of univalent functions in an annulus have been established.

1. Introduction

Let \( x = (x_1, x_2, \ldots, x_n, x_{n+1}) \) be a point in Euclidean space \( \mathbb{R}^{n+1} \) with the time axis \( Ox_{n+1} \) and \( M \) be a \( p \)-dimensional Riemannian manifold, \( 2 \leq p \leq n \).

Definition 1. We say that a surface \( \mathcal{M} = (M, u) \) given by \( C^2 \)-immersion \( u : M \to \mathbb{R}^{n+1} \) is a tube with the projection interval \( \tau(\mathcal{M}) \subset Ox_{n+1} \), if (i) for any \( \tau \in \tau(\mathcal{M}) \) the sections \( \Sigma_\tau = f(\mathcal{M}) \cap \Pi_\tau \) by hyperplanes \( \Pi_\tau = \{ x \in \mathbb{R}^{n+1} : x_{n+1} = \tau \} \) are not empty compact sets; (ii) for \( \tau, \tau' \in \tau(\mathcal{M}) \) any part of \( \mathcal{M} \) situated between two different \( \Pi_\tau \) and \( \Pi_{\tau'} \) is a compact set.

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Definition 2. A surface \( M \) is called minimal if the mean curvature of \( M \) vanishes everywhere.

It is the well-known fact (see [5], p.331) that the minimality condition of \( M \) is equivalent to that all co-ordinate functions of the immersion \( u \) are harmonic. For this reason, the two-dimensional minimal tubes can be considered as direct analog of the closed relative string conception in the modern nuclear physics (cf. [2]). This approach was proposed by V.M. Miklyukov and the author in [7] for an arbitrary dimension \( p \).

From this point of view many intrinsic geometric invariants of \( M \) have the natural physical meaning. Namely, the length of the projection interval \( |\tau(M)| \) can be interpreted as a life-time of the tube \( M \).

To introduce the following important characteristic we denote by \( \nu \) the unit normal to \( \Sigma \), with respect to \( M \) which is co-directed with the time-axis \( OX_{n+1} \). Then by virtue of the harmonicity of the coordinate functions \( u_k(m) = x_k \circ u(m) \), \( 1 \leq k \leq n+1 \), the flow integrals

\[
J_k = \int_{\Sigma_t} (\nabla u_k, \nu) \, d\Sigma
\]

are independent of \( \tau \in \tau(M) \). Here \( d\Sigma \) is the 1-Hausdorff measure along \( \Sigma_t \).

Definition 3. We call \( Q(M) = (J_1, J_2, \ldots, J_{n+1}) \in \mathbb{R}^{n+1} \) the full flow-vector of \( M \).

We notice the positiveness of \( J_{n+1} \) as a consequence of the choice of \( \nu \) direction. Moreover, \( Q(M) \) is an 1-homogeneous functional of \( M \) under the homotheties group action in \( \mathbb{R}^{n+1} \). Let us denote by \( s(M) \) the angle between \( Q(M) \) and the time-axis \( OX_{n+1} \).

In this paper we are interested in following question: What sufficient conditions yield the finiteness of the time-life of a two-dimensional minimal tube? As it was shown in the series of papers [6]–[8], in the case \( p \geq 3 \) this quantity is always finite and the following estimation holds

\[ |\tau(M)| \leq q(M) \varphi, \]

where \( \varphi \) depends only on \( p \), and \( q(M) \) is the smallest diameter of sections \( \Sigma_t \). The last relationship is sharp and the equality occurs if and only if \( M \) is a minimal surface of revolution.

A special feature of the two-dimensional case is that there exist tubes with finite as well as infinite values of the life-time. A crucial observation for that is existence of an additional family of the slanting minimal tubes having circular section \( \Sigma_t \) as against the many-dimensional case. This class of surfaces were discovered by Riemann [10]. Some recent examples can be found in [4].

In this paper we prove

Theorem 1. Let \( M, \dim M = 2 \), be a minimal two-connected tube with univalent Gaussian mapping. If the angle \( \alpha(M) \) is different from zero, then the life-time \( |\tau(M)| \) of \( M \) is finite and

\[ \tau(M) \leq \pi |Q| \frac{\cos \alpha(M)}{\ln \tan \left( \frac{\pi}{4} + \frac{\alpha}{2} \right)}. \]

Let us denote by \( a_0(f) \) the central coefficient of the Laurent decomposition of an holomorphic function \( f(z) \) in an annulus \( K_R = \{ z : 1/R < |z| < R \} \), i.e.

\[ a_0(f) \equiv \int_{C_1} \frac{d(z) \, dz}{\zeta}, \]

where \( C_1 \) is the unite circle \( \{ z \in \mathbb{C} : |z| = 1 \} \). The following auxiliary assertion is a key ingredient in the proof of Theorem 1.

Theorem 2. Let \( g(z) \) be a univalent holomorphic function defined in the annulus \( K_R \) and omitting zero. Assume that

\[ a_0(g) = \lambda, \quad a_0(1/g) = -\lambda, \]

for some real positive \( \lambda \). Then

\[ \ln R \leq \ln R_0(\lambda) := \frac{\pi^2}{\ln(\lambda + \sqrt{1 + \lambda^4})}. \]

Remark 1. We note that estimate (2) has well asymptotic behaviour for \( R \to \infty \) as shown Riemannian example mentioned above. But we can’t now present the sharp estimate for \( \ln R \). Nevertheless, it seemed us very probably that the following conjecture is true.

Conjecture. The best upper bound of the left side of (2) is achieved for the Weierstrass-type holomorphic function \( g_\alpha(z) \) which maps the annulus onto the plain \( \mathbb{C} \) with two slits: \( (-1/\alpha; 0) \) and \( (\alpha; +\infty) \), for the suitable choice of parameter \( \alpha \).

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2. Proof of Theorem 2

Let \( \Gamma = \{ C_\rho : 1/R < \rho < R \} \) be a family of all concentric circles \( C_\rho = \{ z : |z| = \rho \} \) in the annulus \( K_R \). It follows easily from the non-vanishing property of \( g(z) \) that the loop \( C_1 \) in the integrals (1) may be replaced by an arbitrary circle \( C_\rho \in \Gamma \). It follows from the mean value theorem and (1) that for every \( \rho \in (1/R, R) \) there exist \( t_1 \) and \( t_2 \) such that

\[ \text{Re} \, g(\rho e^{it_1}) = \lambda \quad \text{and} \quad \text{Re} \, \frac{1}{g(\rho e^{it_2})} = -\lambda. \]
Let \( \gamma_{\rho} = g(C_{\rho}) \). Then by virtue of the univalence of \( g(z) \), the curve \( \gamma_{\rho} \) is the simple Jordan one. Let \( g(\rho e^{i\theta}) = x(t) + iy(t) \) be the representation of \( \gamma_{\rho} \). Then we obtain from (3)

\[ x(t_1) = \lambda; \quad x''(t_2) + y''(t_2) + \frac{1}{\lambda} x(t_2) = 0. \]

The last relations have the helpful geometric interpretation:

(*) The curve \( \gamma_{\rho} \) intersects the vertical rightline \( L_1 = \{ z : \text{Re} z = \lambda \} \) and the circle \( L_2 = \{ z : |z + \frac{1}{2\lambda}| = \frac{1}{2\lambda} \} \).

Now we recall the following definition from the potential theory.

**Definition 4.** Let \( E \) be a family of locally rectifiable curves \( \gamma \) and \( \varphi(x) \geq 0 \) be a Baire function with the property

\[ \int_{\gamma} \varphi(x) \, dx \geq 1, \]

for every \( \gamma \in E \). The infimum

\[ \text{mod } E = \inf \int \varphi^2(x) \, dx \, dy \]

over all such \( \varphi(x) \) is called a conformal module of the family \( E \).

Then it is known (see [1]) that \( \text{mod } E \) is the conformal invariant. As a consequence we obtain in our situation

\[ \text{mod } \Gamma = \text{mod } \Gamma_1, \]

where \( \Gamma_1 = \{ \gamma_{\rho} : 1/R < \rho < R \} \).

Let us denote by \( D \) the two-dimensional domain

\[ D = \left\{ z : \text{Re} z < \lambda ; \left| z + \frac{1}{2\lambda} \right| > \frac{1}{2\lambda} \right\}. \]

Using (*)-property, we can find for every \( \rho \in (1/R, R) \) the continuum \( \gamma'_{\rho} \subset \gamma_{\rho} \) joining the boundary components of \( D \). Then a family \( \Gamma_2 \) consisting of all continua \( \gamma'_{\rho} \) is "shorter" than \( \Gamma_1 \) and it follows from Theorem 1.2, [1] that

\[ \text{mod } \Gamma_1 \leq \text{mod } \Gamma_2. \]

On the other hand, \( \Gamma_2 \) is the subfamily of \( \Gamma(D) \), where the last term means the family of all curves joining the boundary components of a domain \( D \). The monotonicity property of infimum and Definition 4 lead to the following inequality

\[ \text{mod } \Gamma_2 \leq \text{mod } \Gamma(D). \]

Now, combining the standard fact

\[ \text{mod } \Gamma = \frac{\ln R}{\pi} \]

with relations (4), (5) and (6) we arrive at the following inequality

\[ \frac{\ln R}{\pi} \leq \text{mod } \Gamma(D). \]

To compute the last module we note that the linear-fractional function

\[ f(z) = \frac{z + \lambda^*}{\lambda^* - 1 - z^2} \]

maps \( D \) onto an annulus \( K_1 = \{ w : 1 < |w| < 1/\lambda^* \} \), where \( \lambda^* = \sqrt{\lambda^2 + 1 - \lambda} \).

Thus, using the invariance property of conformal module we obtain

\[ \frac{\ln R}{\pi} \leq \text{mod } \Gamma(D) \leq \frac{2\pi}{\ln(1/\lambda^2)} = \frac{\pi}{\ln(\lambda + \sqrt{1 + \lambda})}, \]

and Theorem 2 is proved.

3. Gaussian map two-dimensional minimal tubes and the full-flow vector

In this section we express the full flow-vector of an arbitrary two-dimensional tube \( M \subset \mathbb{R}^n \) via Chern-Weyl equivalence representation for minimal surfaces. Namely, if \( M \) is a two-connected surface then we can arrange that \( M \) is conformally equivalent to an annulus \( K_R \) for the appropriate \( R > 1 \). Then there exists the corresponding parametrization of \( M \) (see [9]):

\[ u(z) = \text{Re} \int_{z_0}^z F(\zeta) \, d\zeta : K_R \rightarrow \mathbb{R}^n, \]

where

\[ F(z) = (\varphi_1(z), ..., \varphi_n(z)) \]

and \( \varphi_i(z) \) are holomorphic functions satisfying the following conditions

\[ \sum_{i=1}^n \varphi_i(\zeta)^2 = 0 \]

and

\[ \text{Re} \int_{|z|=1} F(\zeta) \, d\zeta = 0. \]
Lemma 1. Under the above hypotheses we have

\[ Q(\mathcal{M}) = \text{Im} \int_{|z|=1} F(\zeta) \, d\zeta. \]

Proof. It suffices to show that

\[ J_k \equiv \int_{\Sigma_k} (\nabla_{u_k} v) \, d\Sigma = \text{Im} \int_{|z|=1} \varphi_k(\zeta) \, d\zeta, \]

for every \( k = 1, 2, \ldots, n + 1. \)

To prove (11) we introduce the conjugate to \( u_k(z) \) function \( v_k(z) \) by

\[ v_k^*(z) = \text{Im} \int_{|z|=1} \varphi_k(\zeta) \, d\zeta. \]

We notice that \( v_k(z) \) in general is a multi-valued function. On the other hand, the covariant derivative \( \nabla_{u_k} \) is well defined and using the properties of Hodge * operator we have

\[ \int_{\Sigma_k} (\nabla_{u_k} v) \, d\Sigma = \int_{\Sigma_k} (\nabla_{v_k} u) \, d\Sigma = \int_{\Sigma_k} (\nabla_{u_k} v_k) \, d\Sigma = \int_{\Sigma_k} \partial v_k = \text{Im} \int_{|z|=1} \varphi_k(\zeta) \, d\zeta, \]

and (11) is proved.

In our case \( n = 2 \), Chern-Weilstrass representation can be simplified in the following classical way. Namely, there exist a holomorphic function \( f(z) \) and a meromorphic function \( g(z) \) which are well defined in the annulus \( K_R \) and such that

\[ F(z) = (1 - g^2) f(z) + (1 + g^2) f(z) + 2g f(z). \]

Moreover, poles of \( g(z) \) coincide with zeros of \( f(z) \) and the order of a pole of \( g(z) \) is precisely the order of the corresponding zero of \( f(z) \). We emphasize that \( g(z) \) is a composition of the stereographic projection and Gaussian map of \( M \).

Lemma 2. In our assumptions

\[ 2fg \equiv \frac{Q(\mathcal{M}), e_3}{2\pi z}, \]

and \( g(z) \) omits the zero and infinity values.

Proof. We use the method proposed by M. Schiffman in [11]. We recall that the coordinate function \( u_3(z) \) is harmonic in the annulus \( K_R \) and by virtue of Definition 1,

\[ \lim_{z \to 1/R} u_3(z) = \tau_1, \quad \lim_{z \to -R} u_3(z) = \tau_2. \]

where \( r(M) = (\tau_1; \tau_2) \) is the projection of the tube \( M \) onto \( z_p \)-axis.

We consider an auxiliary harmonic function

\[ h(z) = \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R} \ln |z|. \]

It is easily seen that \( h(z) \) satisfies (14). Thus \( h_1(z) = u_3(z) - h(z) \) is harmonic in the annulus and

\[ \lim_{z \to K_R} h_1(z) = 0. \]

Then the maximum principle implies that \( h_1(z) \equiv 0 \) everywhere in \( K_R \) and hence

\[ u_3(z) \equiv \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R} \ln |z|. \]

In particular, it follows from (15) that

\[ 2u_3(z) \equiv \frac{\tau_2 - \tau_1}{2\ln R} \frac{z}{|z|^2}. \]

doesn’t vanish in \( K_R \). We have, as a consequence, the normal \( n(z) \) to \( M \) isn’t parallel to \( e_3 \) at any point. Taking into account the above remark about the geometrical sense of \( g(z) \) we obtain that \( g(z) : K_R \to C - \{0; \infty\} \).

By comparing of (15) and (12) we deduce that

\[ 2g(z) f(z) = \frac{\tau_2 - \tau_1}{2\ln R} \frac{dz}{z}. \]

To exclude \( \ln R \) from the last equality we substitute (16) into (12), and after using (10) we obtain

\[ \ln R = \frac{\pi(\tau_2 - \tau_1)}{J_0}. \]

By substituting of the found relationship into (16) we arrive at the conclusion of Lemma 2.
4. Proof of Theorem 1

Let us denote \( w = (J_1 + iJ_2)/J_3 \). Combining Lemma 2, (12) and (9) we obtain
\[
\int_{\Omega} \frac{1-g_2(z)}{2g(z)} \frac{d\zeta}{\zeta} = 2\pi w_j, \quad \int_{\Omega} \frac{1+g_2(z)}{2g(z)} \frac{d\zeta}{\zeta} = 2\pi w_j.
\]
Simplifying the last expressions and denoting \( w = |w| e^{i\theta}, g_j(z) = -e^{-i\theta}g(z) \) we obtain the following system
\[
\frac{1}{2\pi} \int_{\Omega} \frac{g_1(z) d\zeta}{\zeta} = |w|, \quad \frac{1}{2\pi} \int_{\Omega} \frac{d\zeta}{g_1(z) \zeta} = -|w|.
\]
Applying Theorem 2 we arrive at the inequality
\[
\ln R \leq \frac{\pi^2}{|w| + \sqrt{1 + |w|^2}}
\]
where \( |w| \equiv |J_1 + iJ_2|/J_3 = \tan \alpha(M) \). Using (17) we obtain the required estimate and the theorem is proved.

References


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CZAS ŻYCIA MINIMALNYCH TUB I WSPÓŁCZYNNIKI FUNKCJI JEDNOŁISTNYCH W PIERŚCIENIU KOŁOWYM

Streszczenie

Ustalono oszacowanie czasu życia dwuwymiarowych tub minimalnych w \( \mathbb{R}^3 \) używając metod \( \varepsilon \)-regulacją teorii potencjału oraz zastosowano szereg zagadnień ze współczynnikami funkcji jednośmiernych w pierścieniu.