An analysis of non-conforming grid techniques for high order summation-by-parts methods

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Abstract

We derive a bound on the order of accuracy of interpolation operators for energy stable summation-by-parts discretizations on non-conforming multi-block meshes. The new theoretical result, which corroborate with experience from previous work, implies a local reduction in the formal accuracy of summation-by-parts discretizations based on diagonal norms. Numerical results confirm a corresponding reduction in convergence rate in both the maximum norm and the discrete $L_2$ norm for a hyperbolic model problem.

1. Introduction

Summation-by-parts (SBP) operators \cite{1, 2} together with weak boundary and interface conditions of simultaneous-approximation-term (SAT) type \cite{3} provide a natural framework for stable and high order accurate finite difference formulations on curvilinear and multi-block meshes \cite{4, 5, 6, 7, 8}. The SBP-SAT theoretical framework has more recently been shown to include other important classes of high order methods as well, including spectral elements methods \cite{9, 6}. For these, the SAT approach to stable element couplings can be seen as equivalent to the nodal discontinuous Galerkin (dG) method \cite{10}. See \cite{11, 12} and the references therein for comprehensive reviews of the development of SBP-SAT formulations including applications.

In \cite{13}, multi-block couplings of non-conforming finite difference grids could be treated within the SBP-SAT framework for the first time. The new technique was based on so-called SBP preserving interpolation schemes, and
it was extended in [14] to more general couplings between several finite difference blocks. A modification to the SBP preserving interpolation approach was proposed in [15], where the projection of grid solutions to an intermediate polynomial function space was used. This also allowed for stable hybrid couplings of finite difference and discontinuous Galerkin formulations. In [16], these SBP preserving coupling techniques were extended and investigated for second order hyperbolic problems.

The accuracy of the SBP preserving schemes implemented in the above cited literature is suboptimal in the sense that they lead to a reduction in the order of truncation error as compared to the errors introduced by numerical differentiation. At first this was not expected to have an effect on the global order of convergence, and it was seemingly confirmed by the initial numerical results in [13]. It has later been found that the suboptimal interpolation accuracy does impact convergence rates negatively [14, 16]. However, the numerical results presented in the literature have so far been insufficient for drawing general conclusions about when such a drop actually occurs and, if so, to what extent.

The main contribution of this present work consists of a new proof explaining the limited accuracy associated with SBP preserving interpolation. The theory is formulated in full generality such that it holds for any numerical scheme satisfying the SBP-SAT framework. In doing this, we also extend the theoretical notion of SBP preserving interpolation beyond finite difference methods, as exemplified by the important case of spectral element methods. For these, the accuracy orders are shown explicitly to corroborate with the general theoretical bound.

We also perform a detailed convergence study of a first order hyperbolic model problem, where we quantify the effects of the reduced accuracy in terms of asymptotic convergence rates. Even though we focus mostly on the convergence results for a class of finite difference methods, we formulate the theoretical results in full generality such that it holds for any numerical scheme satisfying the SBP-SAT theoretical framework.

2. SBP operators in 1D

In order to set the stage for the analysis of nonconforming interfaces in 2D domains, we begin with an introduction to the concept of SBP operators in one dimension. We let \( \mathbf{x} = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1} \) denote a grid vector.
for the line segment $[\alpha, \beta]$, such that $x_0 = \alpha$ and $x_N = \beta$, and let the discretization parameter be $\Delta x = (\beta - \alpha)/N$. An SBP first derivative operator, $D$, for this grid is defined by the decomposition

$$D = P^{-1}Q,$$  

(1)

where $P = \Delta x H$ is a (positive definite) discrete $L_2$ integration operator on $[\alpha, \beta]$. The associated norm $\|\Phi\|^2_P = \Phi^T P \Phi$ approximates the continuous $L_2$ norm $\|\phi\|^2 = \int_{\alpha}^{\beta} \phi^2 dx$, if $\Phi$ is the projection of a real function $\phi$ onto the grid. For brevity, we will sometimes refer to the matrix $P$ itself as the norm associated with the SBP operator.

Moreover, the matrix $Q$ in (1) satisfies the so-called SBP property

$$Q + Q^T = e_{\beta}^T e_{\beta} - e_{\alpha}^T e_{\alpha},$$  

(2)

where $e_{\alpha} = (1, 0, \ldots, 0)$ and $e_{\beta} = (0, \ldots, 0, 1)$ restricts any discrete function $\Phi$ to the boundary points of the continuous domain. Thus, we have

$$e_{\alpha} \Phi = \Phi_0, \quad e_{\beta} \Phi = \Phi_N,$$

and the following exact integration-by-parts rule for the discrete operator now follows as a direct result of (2),

$$(\Phi, D \Psi)_P = \Phi^T P (D \Psi) = \Phi_N \Psi_N - \Phi_0 \Psi_0 - (D \Phi, \Psi)_P.$$

In this paper we will only consider SBP operators with diagonal norms $P$, since this is needed in order to guarantee stability on curvilinear grids as well as for variable coefficient problems [17, 18].

2.1. Accuracy

The formal accuracy of an SBP operator $D$ can be expressed in terms of exact differentiation of grid polynomials. In particular, the conditions leading to an accuracy of order $s$ can be written as

$$D x^j = j x^{j-1}, \quad j = 0, 1, \ldots, s,$$  

(3)

where $x^j$ is the $j$:th order monomial projected on the grid $x$, with the convention $0^0 = 1$. For $j = 0$, the right-hand-side of (3) is zero, which eliminates the need to define the vector $x^{-1}$. Note that the left-hand-side of (3) is scaled by $1/\Delta x$ due to the scaling with $\Delta x$ of $P$ in (1). By Taylor’s formula, it can
easily be shown that the accuracy conditions (3) yield truncation errors of order $s$ when differentiating smooth grid functions.

It should also be noted that the parameter $s$ in (3) in general denotes the minimum order of accuracy which holds uniformly across the whole discrete domain. For example, consider the diagonal norm finite difference SBP operator given by $P = \Delta x \text{Diag}(1/2, 1, \ldots, 1, 1/2)$ (the standard second order trapezoidal rule), and

$$Q = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \ddots \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \ddots \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \ddots \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \ddots \end{pmatrix}. \quad (4)$$

Even though this operator is based on a second order accurate central stencil in the interior, the formal accuracy is limited by the first order one-sided approximations at the boundaries of the operator, and hence $s = 1$ in (3). In general, it can be shown that a diagonal norm finite difference SBP operator of order $s$ at the boundary closures, where $s = 1, 2, \ldots$, must have an interior stencil of at least order $2s$. We shall refer to discretizations using this class of finite difference operators with diagonal norms as SBP($2s,s$), where $2s$ denotes the order of accuracy in the interior of the SBP operator, and $s$ the order at the boundaries [1].

Another important class of diagonal norm SBP operators can be derived with a spectral element approach, by using the nodes of a Gauss-Lobatto (GL) quadrature rule as collocation points, see [10]. To this end, let $x$ contain a set of $N + 1$ GL nodes for $[\alpha, \beta]$, and let $L = (l_0(x), l_1(x), \ldots, l_N(x))^T$ denote the vector of $N$-th order Lagrange basis polynomials on $x$, satisfying the interpolation property,

$$L(x)^T x^j = x^j, \quad j = 0, 1, \ldots, N. \quad (5)$$

Moreover, we define $P$ as a diagonal matrix with the weights of the GL quadrature rule inserted on the diagonal. With the same type of explicit notation as was used in [18, 19], a diagonal norm SBP operator based on $P$ and $L$ can be expressed by the relations,

$$P = \int_P L L^T dx, \quad Q = \int L L_x^T dx = \int P L_x^T dx. \quad (6)$$
where $\int_P$ denotes inexact integration with the GL rule. The identity for $P$ in (6) follows directly from the defining property $l_j(x_i) = \delta_{ij}$ of the Lagrange basis, where $\delta_{ij}$ is the Kronecker delta function. The SBP rule (2) also follows from the same Lagrange property, after integrating $Q$ in (6) by parts. Note that $\int_P$ and $\int$ can be used interchangeably in the definition of $Q$, due to the fact that GL quadrature is exact for all polynomials up to order $2N - 1$. The same is not true for $P$, since the matrix $LL^T$ contains polynomials of order up to $2N$.

Using (5), we can finally confirm that the accuracy (3) is given uniformly by $s = N$ for the spectral SBP operator defined in (6). Indeed, we find

$$P^{-1}Q\mathbf{x}^j = P^{-1}(\int_P LL^T(x)dx)\mathbf{x}^j = P^{-1}\int_P L(\mathbf{L^T x}^j)dx$$
$$= P^{-1}\int_P L_jx^{j-1}dx$$
$$= P^{-1}\int_P (LL^T dx)jx^{j-1}$$
$$= jx^{j-1}, \quad j = 0, 1, \ldots, N. \quad (7)$$

2.2. Quadrature accuracy

In both examples discussed above, the diagonal norm $P$ is constructed based on a high order accurate quadrature rule. In fact, for any SBP operator (1), it can be shown that $P$ must satisfy a corresponding set of accuracy conditions to (3) in terms of exact integration of polynomials, see [20, 21]. For completeness, we include a full derivation of these general conditions. Consider (3) for two different parameter values $i$ and $j$ between 0 and $s$. We multiply the first of these relations with $(\mathbf{x}^j)^T P$, the second one with $(\mathbf{x}^i)^T P$, and add the results together. These operations yield the set of conditions

$$(\mathbf{x}^j)^T Q\mathbf{x}^i + (\mathbf{x}^i)^T Q\mathbf{x}^j = i(\mathbf{x}^j)^T Px^{i-1} + j(\mathbf{x}^i)^T Px^{j-1}, \quad i, j = 0, 1, \ldots, s. \quad (8)$$

Using the SBP property (2), the left-hand-side to (8) above can now be simplified into

$$(\mathbf{x}^j)^T Q\mathbf{x}^i + (\mathbf{x}^i)^T Q\mathbf{x}^j = (\mathbf{x}^j)^T (Q + Q^T)\mathbf{x}^i = \beta^{i+j} - \alpha^{i+j}.$$  

Moreover, since we assume $P$ to be a diagonal matrix, the right-hand-side to (8) satisfies

$$i(\mathbf{x}^j)^T Px^{i-1} + j(\mathbf{x}^i)^T Px^{j-1} = (i + j)1^T Px^{i+j-1}.$$  

5
After making the substitution \( \tau = i + j \), we can finally write (8) as

\[ 1^T P \mathbf{x}^{\tau-1} = \frac{\beta^\tau - \alpha^\tau}{\tau}, \quad \tau = 1, 2, \ldots, q, \]  

(9)

where \( q = 2s \). The diagonal norm \( P \) thus integrates polynomials up to order \( 2s - 1 \) exactly, where \( s \) is the order of the SBP operator itself (3).

As we shall find later, the quadrature conditions in (9) limits the accuracy for SBP preserving interpolation. Hence it is interesting to consider whether they can be improved upon, or in other words, if (9) can hold for \( q \) larger than \( 2s \). We note for instance that in the case of spectral SBP operators based on GL collocation, \( q = 2s = 2N \) in (9) corresponds strictly to the accuracy of the \( N + 1 \) point GL quadrature rule which defines \( P \). For finite difference SBP(\( 2s, s \)) operators, a similar strict result follows implicitly from the original theoretical analysis of SBP operators in [1], see Appendix Appendix B for a proof of this result. We conclude that the quadrature conditions (9) are indeed strict for two important classes of high order accurate SBP operators.

**Remark 1.** It is possible that new types of SBP operators can be constructed that satisfy (9) for \( q > 2s \), for example by reducing the accuracy to \( s < N \) in the element approach while keeping the same quadrature rule. Another solution for finite difference operators recently proposed [22] is to adapt the boundary closure to a specific block size \( N \) (which also requires increasing the size of the boundary closure). However, for classical definitions of finite difference operators with a variable number of interior rows, the result presented in Appendix Appendix B is strict.

### 3. SBP preserving interpolation

As our model problem we consider a scalar, constant coefficient advection problem with positive wave speeds \( a, b > 0 \), expressed by

\[
\begin{align*}
  u_t + au_x + bu_y &= 0, & (x, y) \in \Omega, & t \geq 0 \\
  u(t, x, -1) &= g_1(t, x), & t \geq 0 \\
  u(t, -1, y) &= g_2(t, y), & t \geq 0 \\
  u(0, x, y) &= f(x, y), & (x, y) \in \Omega,
\end{align*}
\]

(10)

and defined on the square reference domain \( \Omega = [-1, 1] \times [-1, 1] \). Without loss of generality, we consider the division of \( \Omega \) into two subdomains \( \Omega_L \)
and $\Omega_R$ by inserting a numerical interface along the line $x = 0$. We further introduce the four one-dimensional grid vectors $x_L$, $x_R$, $y_L$ and $y_R$ to define cartesian grids on the two subdomains. To each of the one-dimensional grid vectors we associate SBP operators $D_{x_L}$, $D_{x_R}$, $D_{y_L}$ and $D_{y_R}$, respectively.

We define a conforming numerical interface between $\Omega_L$ and $\Omega_R$ by the condition

$$P_{y_R} = P_{y_L}, \quad (11)$$

i.e. numerical integration is carried out using the same nodes and the same quadrature rule on both sides. If (11) is assumed to hold, a two-domain SBP-SAT discretization of (10) can be formulated in a straightforward way without losing orders of accuracy. In the nonconforming case, we introduce interpolation operators $P_L$ and $P_R$ acting between the grids $y_L$ and $y_R$ to facilitate the coupling.

In the same way as for discrete first derivative operators in (3), and for discrete $L_2$ operators in (9), we consider interpolation operators that satisfy accuracy conditions in terms of exact evaluation of grid polynomials. Thus, let

$$P_L y^j_L = y^j_R, \quad j = 0, \ldots, p$$
$$P_R y^j_R = y^j_L, \quad j = 0, \ldots, p. \quad (12)$$

Note that, since $P_L$ and $P_R$ are not scaled by $1/\Delta x$ (as opposed to the discrete differential operators), the accuracy conditions in (12) imply by Taylor’s formula that smooth grid functions on $y_L$ and $y_R$ are interpolated with an accuracy order of $p + 1$ rather than $p$. It should also be noted that $p$ in (12) denotes the largest exponent which yields exact interpolation uniformly across the whole interface. For example, the original interpolation operators introduced in [13] satisfy $p = s - 1$ at both ends of the numerical interface, while polynomials up to order $2s - 1$ are interpolated exactly in the interior.

In order to make energy stable couplings of nonconforming interfaces possible with the SBP-SAT technique, the following additional condition was identified in [13],

$$P_{y_L} P_R = P_L^T P_{y_R}. \quad (13)$$

A pair of interpolation operators satisfying (13) is referred to in [13] as SBP preserving.

3.1. Dual consistency and energy stability

We discretize (10) on the two subdomains $\Omega_L$ and $\Omega_R$, employing weak SAT conditions to couple the solution across the nonconforming interface.
Let \( U \) and \( V \) denote the discrete solution vectors on the two subdomains. We write the two-domain SBP-SAT discretization of (10) with a nonconforming interface as

\[
U_t + a(D_{xL} \otimes I_{yL}) U + b(I_{xL} \otimes D_{yL}) U = BC_L + IC_L \\
V_t + a(D_{xR} \otimes I_{yR}) V + b(I_{xR} \otimes D_{yR}) V = BC_R + IC_R,
\]

where \( I_{xL}, I_{xR}, I_{yL} \) and \( I_{yR} \) are identity matrices of appropriate dimensions, and \( \otimes \) denotes the Kronecker product. Moreover, the weak boundary and interface conditions in (14) are given by

\[
BC_L = - (u_{B1} - g_1(t, x_L)) \otimes P_{yL}^{-1} e_{xL,0}^T e_{yL,-1} \otimes (u_{B2} - g_2(t, y_L)) \\
BC_R = - (v_{B1} - g_1(t, x_R)) \otimes P_{yR}^{-1} e_{yR,-1}^T e_{xR,0} \otimes (v_{B2} - g_2(t, y_R)) \\
IC_L = \sigma_L P_{xL}^{-1} e_{xL,0}^T (u_I - P_L v_I) \\
IC_R = \sigma_R P_{xR}^{-1} e_{xR,0}^T (v_I - P_R u_I),
\]

where \( \sigma_L \) and \( \sigma_R \) are (yet unknown) penalty coefficients, and

\[
u_{B1} = (I_{xL} \otimes e_{yL,-1}) U, \quad u_{B2} = (e_{xL,-1} \otimes I_{yL}) U, \\
v_{B1} = (I_{xR} \otimes e_{yR,-1}) V, \\
u_I = (e_{xL,0} \otimes I_{yL}) U, \\
v_I = (e_{xR,0} \otimes I_{yR}) V.
\]

Remark 2. Recall that the accuracy conditions (12) imply that the interpolation operators are accurate to at least order \( p+1 \) uniformly. However, due to the scaling of the weak interface conditions with \( P_{xL}^{-1} \) and \( P_{xR}^{-1} \) in (14), the order of the resulting truncation errors is only \( p \). The order of truncation errors originating from numerical differentiation in (14) on the other hand coincides with the accuracy of the SBP operators themselves, i.e. they are at least order \( s \) uniformly (see (3)).
Before analyzing the semi-discrete problem (14) further, we write it in a more compact form similar to the so-called Q-formulation employed in [4] for the analysis of multi-dimensional SBP-SAT discretizations. We thus rewrite (14) into the system

\[
W_t + P^{-1}QW = -P^{-1}E_B^TP_B(E_BW - g(t)) \tag{17}
\]

where we have defined the full solution vector \(W = (U^T \, V^T)^T\), together with a corresponding discrete differential operator with the weak interface treatment built in, given by

\[
P = \begin{pmatrix}
P_{xL} \otimes P_{yL} & 0 \\
0 & P_{xR} \otimes P_{yR}
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
a(Q_{xL} \otimes P_{yL}) + b(P_{xL} \otimes Q_{yL}) & 0 \\
0 & a(Q_{xR} \otimes P_{yR}) + b(P_{xR} \otimes Q_{yR})
\end{pmatrix}
\tag{18}
\]

\[-E_I^T \Sigma_I E_I.
\]

Moreover, the boundary treatment in (17) is defined using

\[
E_B = \begin{pmatrix}
I_{xL} \otimes e_{yL,-1} \\
e_{xL,-1} \otimes I_{yL} \\
I_{xR} \otimes e_{yR,-1}
\end{pmatrix},
\]

\[
P_B = \begin{pmatrix}
P_{xL} \\
P_{yL} \\
P_{xR}
\end{pmatrix}
\tag{19}
\]

\[g(t) = (g_1(t, x_L)^T \, g_2(t, y_L)^T \, g_1(t, x_R)^T)^T,
\]

and the interface treatment in (18) by

\[
E_I = \begin{pmatrix}
e_{xL,0} \otimes I_{yL} & 0 \\
0 & e_{xR,0} \otimes I_{yR}
\end{pmatrix},
\]

\[
\Sigma_I = \begin{pmatrix}
\sigma_L P_{yL} & \sigma_R P_{yR} \\
\sigma_L P_{yR} & -\mathcal{P}_L & -\mathcal{P}_R
\end{pmatrix}.
\tag{20}
\]

In order to prove dual consistency and energy stability at the interface, we aim for an SBP rule analogous to (2) of the one-dimensional operators. We will for clarity of presentation only consider contributions from the nonconforming interface treatment. Boundary conditions have already been studied
extensively elsewhere with the SBP-SAT technique, and can be analyzed independently of the numerical interface. We thus get, from the definition in (18),
\[ Q + Q^T = E_I^T M_I E_I, \] (21)
where we have ignored all contributions from the boundaries of the domain Ω, and where \( M_I \) is given by
\[ M_I = -\Sigma_I - \Sigma_I^T + a \begin{pmatrix} P_{yL} & -P_{yR} \\ \sigma_L P_{yL} P_{yR} + \sigma_R P_L^T P_{yR} & (-a - 2\sigma_R) P_{yR} \end{pmatrix} . \]
The SBP preserving property (13) together with the choice \( \sigma_L = a/2 \) and \( \sigma_R = -a/2 \) leads to \( M_I = 0 \) in (21), thus leaving zero contribution from the interface to the two-dimensional SBP property. It follows immediately that such an interface treatment is both dual consistent and energy stable. Indeed, the discrete dual operator defined by \( -P_{y}^{-1} Q \) (approximating the dual continuous operator \( -a \partial_x - b \partial_y \)) can now be obtained from the primal one using
\[ P_{y}^{-1}(P_{y}^{-1} Q)^T P = P_{y}^{-1} Q^T = -P_{y}^{-1} Q, \]
again ignoring contributions from the outer boundaries. Compare this e.g. with the condition of dual consistency derived from equations (59)-(61) in [26]. Finally, applying the discrete energy method to (17) by multiplying with \( W^T P \) and adding the transpose, we get the energy rate
\[ \frac{d}{dt} \| W \|_P^2 = b \| g_1(t, x_L) \|_{P_{xL}}^2 + a \| g_2(t, y_L) \|_{P_{yL}}^2 + b \| g_1(t, x_R) \|_{P_{xR}}^2 \]
\[ - b \| u_{B_1} - g_1(t, x_L) \|_{P_{xL}}^2 - a \| u_{B_2} - g_2(t, y_L) \|_{P_{yL}}^2 - b \| v_{B_1} - g_1(t, x_R) \|_{P_{xR}}^2 \]
\[ - b \| (I_{xL} \otimes e_{yL,1}) U \|_{P_{xL}}^2 - a \| (I_{xR} \otimes e_{yR,1}) V \|_{P_{xR}}^2 - b \| (e_{xR,1} \otimes I_{yR}) V \|_{P_{yR}}^2 , \]
which is bounded by data, thus proving energy stability.

3.2. Example: spectral element methods

The SBP preserving interpolation schemes, which makes dual consistency and energy stability possible in the SBP-SAT formulation, unfortunately comes at a price in terms of accuracy. Before deriving a general result connecting the SBP preserving condition (13) to formal accuracy (12), we will
first study the special case of collocated spectral element methods. This partly serves to illustrate the usefulness of SBP preserving interpolation as a theoretical notion, in that it naturally arises in a broader range of settings than originally considered. It is also an illuminating example to study since the formal accuracy in this case can be determined by explicit derivation.

Non-conforming interfaces for the spectral element dG method has previously been treated with a so-called mortar approach [27, 28, 29]. For simplicity, we restrict the discussion here to the $p-$refinement case, and thus consider the coupling of just two neighboring elements as was done in the previous section. Recall that the accuracy is given by $s_L = N_L$ and $s_R = N_R$ for collocated spectral SBP operators (7), and we assume without loss of generality that $N_R > N_L$. A pair of interpolation operators between the grids $y_L$ and $y_R$ can now be defined through projection matrices involving the Lagrange bases; specifically in the form of equations (18) and (20) in [28]. Following our previous notation in (5)-(6), we write these interpolation operators as

\[
\mathcal{P}_L = P_{y_R}^{-1}\mathcal{R}_L, \\
\mathcal{P}_R = P_{y_L}^{-1}\mathcal{R}_R,
\]

where

\[
\mathcal{R}_L = \int_{P_{y_R}} \mathcal{L}_R \mathcal{L}_L^T dy, \quad \mathcal{R}_R = \int_{P_{y_R}} \mathcal{L}_L \mathcal{L}_R^T dy.
\]

Note that the most accurate of the two available GL quadrature rules, namely $P_{y_R}$, is used to define both $\mathcal{R}_L$ and $\mathcal{R}_R$. The SBP preserving property (13) follows automatically, since $\mathcal{R}_R = \mathcal{R}_L^T$.

Next, we consider the formal accuracy of $\mathcal{P}_L$ and $\mathcal{P}_R$ according to the definition in (12). In the case of $\mathcal{P}_L$, it suffices to consider the interpolation property (5) of Lagrange polynomials. Using this, we get

\[
\mathcal{P}_L y^i_L = P_{y_R}^{-1}\mathcal{R}_L y^i_L = P_{y_R}^{-1} \int_{P_{y_R}} \mathcal{L}_R (\mathcal{L}_L^T y^i_L) dy = P_{y_R}^{-1} \int_{P_{y_R}} \mathcal{L}_R (\mathcal{L}_R^T y^i_R) dy = P_{y_R}^{-1} \int_{P_{y_R}} \mathcal{L}_R \mathcal{L}_R^T dy y^i_R = y^i_R, \quad i \leq N_L = s_L,
\]
The third equality above is due to the fact that both $\mathcal{L}_L^T y^i_L$ and $\mathcal{L}_R^T y^i_R$ are given exactly by $y^i$, due to (12).

For the second interpolation operator $\mathcal{P}_R$, one must in addition to (5) also take into account the fact that $\int_{P^y_L}$ and $\int_{P^y_R}$ only produce the same result for polynomials of order $2N_L - 1$ or lower, due to (9). This leads to

$$\mathcal{P}_R y^i_R = P_{yL}^{-1} \mathcal{R}_R y^i_R$$

$$= P_{yL}^{-1} \left( \int_{P^y_R} \mathcal{L}_L (\mathcal{L}_R^T y^i_R) dy \right)$$

$$= P_{yL}^{-1} \left( \int_{P^y_L} \mathcal{L}_L (\mathcal{L}_L^T y^i_L) dy \right)$$

$$= P_{yL}^{-1} \left( \int_{P^y_L} \mathcal{L}_L \mathcal{L}_L^T dy y^i_L \right)$$

$$= y^i_L, \quad i \leq N_L - 1 = s_L - 1,$$

since $\mathcal{L}_L (\mathcal{L}_L^T y^i_L) = \mathcal{L}_L y^i$ then contains polynomials of at most order $2N_L - 1$. In other words, the formal accuracy of $\mathcal{P}_R$ is limited by aliasing errors introduced when changing the quadrature rule between $P^y_R$ and $P^y_L$.

We conclude that the operators defined in (22) satisfy the accuracy conditions (12) with $p = s_L - 1$. Note that the different results obtained for $\mathcal{P}_L$ and $\mathcal{P}_R$ do not contradict the formulation of (12), since $p$ is defined as the largest integer such that (12) holds for all rows in both $\mathcal{P}_L$ and $\mathcal{P}_R$.

3.3. A general result on the accuracy of SBP preserving schemes

In Lemma 1 below we derive a general condition which for all known classes of high order SBP operators implies an order reduction to $s - 1$ of truncation errors resulting from interpolation in (14).

**Lemma 1.** Let $\mathcal{P}_L$ and $\mathcal{P}_R$ be a pair of SBP preserving (13) interpolation operators with respect to the diagonal norms $P^y_L$ and $P^y_R$, satisfying the accuracy conditions (12). Then $P^y_L$ and $P^y_R$ must satisfy the additional set of conditions

$$1^T_R P^y_R y^i_R^{-1} = 1^T_L P^y_L y^i_L^{-1}, \quad \tau = 0, 1, \ldots, q,$$

where $q = 2p + 1$. 

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Proof. Let \( i \) and \( j \) be two integers such that \( 0 \leq i, j \leq p \). We begin by multiplying the second condition in (12) with \((y^i_L)^T P_{yl}\) from the left. This yields

\[
(y^i_L)^T P_{yl} P_R y^j_R = (y^i_L)^T P_{yl} y^j_L.
\]

Next, we apply the SBP preserving property (13) to rewrite this into

\[
(P_L y^i_L)^T P_{yl} y^j_R = (y^i_L)^T P_{yl} y^j_L.
\]

Finally, the first condition in (12) leads to \((y^j_R)^T P_{yr} y^i_R = (y^i_L)^T P_{yl} y^j_L\) which, since the norms are diagonal, is equivalent to

\[
1^T_{R, P_{yl}} y^{i+j}_R = 1^T_{L, P_{yl}} y^{i+j}_L.
\]

By assumption, \( i \) and \( j \) satisfy \( 0 \leq i, j \leq p \), and hence (23) follows.

Remark 3. Note the similarity between the additional conditions (23) and the quadrature conditions (9). In fact, (9) implies that (23) is automatically satisfied up to \( q = 2s \), since both of the discrete integrals in (23) are then exact. For \( q = 2s + 1 \) on the contrary, (23) can not in general be expected to hold, unless the two norms also satisfy (9) for \( q = 2s + 1 \). However, we recall that the quadrature conditions (9) are strict for the two main classes of diagonal norm SBP operators discussed in this paper, and can thus not be improved.

Given this observation, we are now ready to formulate the main theoretical result of this paper.

Proposition 1. Assume that the diagonal norms used in (14) do not satisfy the additional condition (23) for \( q = 2s + 1 \). Then Lemma 1 implies that \( p \) in (12) is bounded by \( p < s \), and hence can be at most \( s - 1 \).

Note that the lower bound \( p = s - 1 \) given by Proposition 1 implies truncation errors from interpolation of order \( s - 1 \) in the scheme (14), while numerical differentiation yields truncation errors of at most order \( s \), see also Remark 2. For SBP(2s,s) finite difference discretizations in two dimensions, the increased truncation errors of order \( s - 1 \) can be restricted to a finite number of points at both ends of the numerical interface, as was done in [13].
4. Numerical calculations

In this section we investigate the implications of the limited interpolation accuracy due to Proposition 1.

4.1. Finite difference discretizations

A convergence theory for difference approximations of initial boundary value problems was formulated in [30, 31] and further developed in [32, 33]. According to this, energy stable discretizations of first order hyperbolic problems can in general be predicted to converge with an order of \( s + 1 \), if \( s \) is the order of accuracy at the boundaries of the domain, and where the interior order is some value larger than \( s \). Hence, for a standard SBP(2s,s) discretization without a nonconforming interface, the order of convergence is given by \( s + 1 \). Dual consistent approximations has the further advantage of linear functionals superconverging with the order \( 2s \), as was originally shown in [23]. Unfortunately however, the existing convergence theory is insufficient for predicting the consequences of a reduction in formal accuracy to \( s - 1 \) at a finite number of points in the two-dimensional domain. Also, due to the local nature of this reduction, we should not be surprised to see larger errors in the maximum norm of solutions than in the \( L_2 \) norm, which was also observed in [14].

Since the existing convergence theory can not be directly applied to SBP-SAT discretizations involving a nonconforming interface, it remains to investigate the resulting convergence rates numerically. For this, we consider (10) with wave speeds \( a = b = 1 \), and employ the exact solution \( u = \sin(x+y-2t) \). The error in the solution to (14) is measured at \( t = 1 \), and the domain is discretized using four computational blocks, where the upper right one is more resolved than the others. The refined block has \( CN + 1 \) grid points in each coordinate direction, where \( C \) is a constant grid size ratio, and the remaining three blocks are each discretized using \( N + 1 \) grid points in each direction. This setting is illustrated in Figure 1 below for the case \( N = 8, C = 2 \). The classical fourth order Runge-Kutta scheme is employed for time integration, and the time step size is chosen to make temporal discretization errors negligible.

We employ three different classes of interpolation operators, which are described in more detail in Appendix Appendix A. In short, OPT and MBW denote two sets of operators based on repeated interior stencils that are optimized for accuracy and minimal bandwidth, respectively, while AUTO
Figure 1: Example of grid structure for $N = 8$ and $C = 2$. The bold lines denote numerical block boundaries.

Figure 2: Rates of convergence in the $L_2$ norm. Results are shown for SBP(2,1), SBP(4,2) and SBP(6,3).
Figure 3: Rates of convergence in the maximum norm. Results are shown for SBP(2,1), SBP(4,2) and SBP(6,3).

Figure 4: Rates of convergence for a linear functional. Results are shown for SBP(2,1), SBP(4,2) and SBP(6,3).

denotes a class of operators automatically generated on each grid. The re-
Figure 5: Spatial error distribution for one discretizations of SBP(4,2) with $N = 64$, using optimized SBP preserving interpolation operators.

Figure 6: Convergence in the $L_2$ norm as a function of total number of grid points, for a rapidly varying manufactured solution.
finement ratio $C$ is given by 2 for the $OPT$ and the $MBW$ operators, and 13/7 for the $AUTO$ operators. The numerical results in terms of convergence rates are shown in Figures 2-4 for the cases SBP(2,1), SBP(4,2) and SBP(6,3). In all cases we also compare with the conforming case, i.e. with $C = 1$.

As can be seen in Figure 2, the order is asymptotically reduced to $s + 1/2$ for all three sets of interpolation operators if the error is measured in the $L_2$ norm. The drop from $s + 1$ of the conforming case is only clearly visible at rather small error levels, a fact which probably explains why no drop in convergence was reported in some of the previous works. The drop is more severe in the maximum norm, for which the asymptotic rate is only $s$, see Figure 3. The observed difference between the $L_2$ and the maximum norm can perhaps be better understood by a closer study of the spatial error distribution. An example is given in Figure 5, showing the distribution of absolute error for a single realization of SBP(4,2). As the figure suggests, the shape of the region containing enhanced error levels approaches a one-dimensional set with grid refinement. This observation is consistent with a drop by only half an order in the $L_2$ norm, even though it is reduced by one full order in the maximum norm.

Finally, the convergence rate for a linear functional is shown in Figure 4, where the functional is chosen as simply the solution integrated over the domain. Again, the convergence rate is clearly reduced compared to the conforming case. We conjecture that the observed convergence rates in all three measures considered in this investigation can be explained by the local reduction in accuracy order to $s - 1$.

It is interesting to note that the three different types of interpolation operators show approximately the same performance in most of the cases considered. An exception is SBP(4,2) for which the optimized operators clearly outperform the minimum bandwidth ones, see figures 2 and 3. Also, the automatically generated operators remains competitive in all cases. We thus expect that a continued development of automatically generated SBP preserving interpolation schemes can enable both efficient and stable interface couplings of completely general grid interfaces.

As a final illustration of how Proposition 1 can affect the viability of using interpolation for gaining efficiency, we consider a manufactured solution including a rapidly decaying Gaussian pulse centered in the more finely resolved region of the domain. Specifically, the manufactured solution is given
by

\[ u = \sin(x + y - 2t) + e^{-(x-0.5)^2+(y-0.5)^2/2\sigma^2}, \]

where \( \sigma = 1/8 \), and is enforced in (10) by inserting suitable forcing functions to the right-hand-side of the equation. In Figure 6 we show the results for SBP(4,2), where the \( L_2 \) error is compared to the total number of grid points in the domain. This gives a good indication of the break-even points in terms of efficiency, since all methods require approximately the same number of time steps for the same accuracy. As one could expect, interpolation is more efficient for coarse grids. Asymptotically however, the reduced convergence rate by one half order means that the conforming approximation outperform any interpolation scheme, no matter how optimized.

4.2. Spectral element discretizations

![Figure 7: Rates of h-convergence in the \( L_2 \) norm for non-conforming spectral element discretizations with \( N_R = N_L + 1 \). Results are shown for \( N_L = 1 \) and \( N_L = 2 \) respectively.](image)

For a single uniform mesh, the spectral element discontinuous Galerkin method can be shown to yield convergence rates of order \( N + 1 = s + 1 \) [34], where \( N = s \) is the order of polynomials used in the approximation (which we assume here to be constant). In figures 7-8 below, we confirm that this optimal rate is not retained for the collocated Gauss-Lobatto SBP-SAT implementation in the presence of non-conforming interfaces. As in the
Figure 8: Rates of $h$-convergence in the maximum norm for non-conforming spectral element discretizations with $N_R = N_L + 1$. Results are shown for $N_L = 1$ and $N_L = 2$ respectively.

Figure 9: Rates of $h$-convergence for a linear functional for non-conforming spectral element discretizations with $N_R = N_L + 1$. Results are shown for $N_L = 1$ and $N_L = 2$ respectively.

previous section, we use a more accurate discretization in the top right corner
of the domain. We consider the $p$–refinement case, and thus let $N_R > N_L$, while the number of blocks $n$ in each coordinate direction is the same in all four subdomains. The results are compared to the conforming case where $N = N_L$ is used in all subdomains. In this setting, we can clearly see that the convergence rate drops from $s + 1$ to $s$ in the maximum norm, and to $s + 1/2$ in the $L_2$ norm, just as in the finite difference case. Contrary (and quite surprisingly) to this however, there is no reduction for the linear functional, see Figure 9. A plot of the spatial error is shown in Figure 10. The error distribution is quite different from the finite difference case in Figure 5.

5. Conclusions

We have derived a theoretical bound on the accuracy of SBP preserving interpolation operators based on diagonal norms. The bound is related to the quadrature rule associated with the norm, and it leads to suboptimal accuracy compared to numerical differentiation for both finite difference and spectral element methods on SBP-SAT form.

We find by numerical calculations that the order of convergence is reduced by half an order in the $L_2$ norm, and by one order in the maximum norm. The convergence rate of linear functionals is also considered, and the results indicate a reduction for the finite difference method, but surprisingly enough
not for the spectral element method.

By studying the efficiency, we confirm that interpolation schemes are more efficient on coarse grids while conforming schemes are more efficient on fine grids.

Appendix A. Description of interpolation schemes employed in the finite difference calculations

We briefly describe the different types of SBP preserving interpolation operators used in section 4.1.

Appendix A.1. Optimized operators for fixed grid size ratios

A general methodology for constructing SBP preserving operators based on the application of repeated, central interior stencils was proposed in [13] for finite difference applications, assuming a fixed grid size ratio between the blocks. The formal accuracy was set to $p = s - 1$ (B.1) in a small number of rows near to both ends of the boundaries, while polynomials up to order $2s - 1$ were interpolated exactly along the interface interior. This formulation allowed for a number of free parameters to be tuned in order to minimize leading order error terms. A class of such operators for the 2:1 ratio was presented and tested for convergence. We refer to this set of operators as “$OPT$” in the numerical section of this paper.

Appendix A.2. Minimum bandwidth operators for fixed grid size ratios

Mainly for the purpose of making practical comparisons with the existing optimized operators from [13], we have constructed a similar set for the 2:1 grid size ratio with the same formal accuracy as $OPT$, but without any free parameters and hence not optimized for accuracy. These new operators (which have stencils of minimal bandwidth and include a minimal number of non-zero coefficients in the boundary closures) are disclosed below for the cases SBP(2,1) and SBP(4,2). We denote these with ”$MBW$”.

The diagonal norm of a standard one-dimensional SBP(2,1) operator $P^{-1}Q$ is given by $P = \Delta x \text{Diag}(\frac{1}{2}, 1, \ldots)$. The corresponding new minimum bandwidth operator $\mathcal{P}_L$ for the 2:1 grid size ratio is given by

$$
\mathcal{P}_L = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.
$$

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Recall that the operator $\mathcal{P}_R$ can be obtained from the same data by applying the SBP preserving property (13).

In the SBP(4,2) case, the norm is given by $P = \Delta x \text{Diag}(\frac{17}{48}, \frac{59}{48}, \frac{43}{48}, \frac{49}{48}, 1, \ldots)$, and the interpolation operator $\mathcal{P}_L$ by

$$
\mathcal{P}_L = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{429}{544} & 0 & -\frac{103}{534} & \frac{27}{32} & 1 & \frac{23}{544} & 0 & -\frac{1}{272} \\
0 & \frac{1}{2} & \frac{544}{944} & \frac{43}{118} & \frac{1888}{118} & \frac{59}{118} & \frac{1888}{118} & \frac{1888}{118} & 0 & -\frac{1}{272} \\
-\frac{1}{32} & 0 & \frac{9}{32} & \frac{1}{32} & \frac{2}{32} & 0 & -\frac{1}{272} & 0 & -\frac{1}{272} & \frac{1}{32} \\
& & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} & 0 & 0 & 0 & 0 & \frac{1}{32} \\
& & & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} & 0 & 0 & 0 & \frac{1}{32} \\
& & & & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} & 0 & 0 & \frac{1}{32} \\
& & & & & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} & 0 & \frac{1}{32} \\
& & & & & & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} & \frac{1}{32} \\
& & & & & & & \frac{32}{32} & \frac{1}{32} & \frac{2}{32} \\
& & & & & & & & \frac{32}{32} & \frac{1}{32} \\
& & & & & & & & & \frac{32}{32} \\
\end{pmatrix},
$$

and $\mathcal{P}_R$ is again obtained from (13).

**Appendix A.3. Automatically generated operators for arbitrary grids**

As a first step toward fully automatic generation of interpolation schemes between arbitrary grid sets, we have designed a simple algorithm for constructing SBP preserving operators not based on repeated interior stencils as the previous ones. Instead, they are constructed on each grid realization by directly solving for all accuracy conditions along the whole interface, as a preprocessing step to calculations. First, the positions of a sufficient number of non-zero coefficients are specified, and in order to ensure that a feasible solution to the accuracy relations always exists, we set this number to be slightly larger than would be necessary with an optimal placement. Hence, the resulting linear system is slightly underspecified (as well as rank-deficient, due to the linear dependencies derived in the proof of Lemma 1), and can be solved for example by application of the pseudoinverse. This approach yields operators with relatively small leading order error terms, without notably increasing the stiffness of the problem.

We have chosen to restrict the order of exact polynomial evaluation in the interior of these new automatic operators to $s$ instead of $2s - 1$ as in the previous ones based on repeated interior stencils. We expect this lower interior accuracy to be sufficient given the fact that the resulting order of truncation errors in (14) then coincides with those from numerical differentiation along the interface, i.e. order $s$ (see (3)). For our numerical calculations we employ a set of operators generated in this way for blocks with a grid size ratio of 13:7, denoted with "AUTO".
Appendix B. A result on the quadrature order of SBP(2s,s) operators

To prove that the quadrature conditions given in (9) cannot be improved for finite difference SBP operators with diagonal norms, we draw upon the theory for such operators originally developed in [1], as well as to the connections to high order quadrature rules demonstrated in [20].

Consider a finite difference SBP(2s,s) operator \( D \) and the associated diagonal norm \( P \). By definition, such an operator is based on a repeated interior stencil with an accuracy of order \( 2s \), augmented with boundary closures of order \( s \). A central difference approximation with \( t \) unique coefficients \( a_1, a_2, \ldots, a_t \) is defined by
\[
(D \Phi)_i = \frac{a_t \Phi_{i+t} + \cdots + a_1 \Phi_{i+1} - a_1 \Phi_{i-1} - \cdots - a_t \Phi_{i-t}}{\Delta x},
\]
and the stencil coefficients must satisfy the following conditions for an accuracy order of \( 2s \),
\[
\sum_{\nu=1}^{t} a_{\nu} \nu = \frac{1}{2},
\]
\[
\sum_{\nu=1}^{t} a_{\nu} \nu^j = 0, \quad j = 3, 5, \ldots, 2s - 1.
\]

The structure of \( P \), assuming \( r \) points in each boundary closure, is given by
\[
P = \Delta x \text{Diag}(p_0, \ldots, p_{r-1}, 1, \ldots, 1, p_{r-1}, \ldots, p_0).
\]

As was demonstrated in the proof of Theorem 1 in [20], the accuracy conditions (9) for \( \tau \leq q \) are equivalent to the following set of conditions for the coefficients in the boundary closure of \( P \),
\[
\tau \sum_{j=0}^{r-1} p_j (r-j)^{\tau-1} = r^\tau - (-1)^\tau B_\tau, \quad \tau \leq q - 1 \text{ even}
\]
\[
\tau \sum_{j=0}^{r-1} p_j (r-j)^{\tau-1} = r^\tau \quad \tau \leq q - 1 \text{ odd},
\]
where \( B_\tau \) is the \( \tau \):th Bernoulli number. With this, we are now ready to prove the following result.
Lemma 2. The accuracy conditions (9) for a diagonal norm SBP(2s, s) operator cannot hold for q larger than 2s.

Proof. Since (9) is equivalent to (B.2) for diagonal norm SBP(2s, s) operators, it suffices to show that q in (B.2) cannot exceed 2s. We will show this result by contradiction.

In [1], it was shown that SBP(2s, s) operators must satisfy the following set of so-called compatibility conditions,

$$\tau \sum_{j=0}^{r-1} p_j (r-j)^{\tau-1} = r^\tau - 2(-1)^{\tau} \sum_{\nu=1}^{t} a_i N_\tau, \quad \tau = 1, 2, \ldots, 2s, \quad (B.3)$$

where

$$N_\tau = (\tau + 1)^{-1} v^{\tau+1} + \frac{1}{2} \binom{\tau}{1} B_2 v^{\tau-1} + \frac{1}{4} B_4 \binom{\tau}{3} v^{\tau-3} + \ldots + B_\tau \nu, \quad \text{for } \tau \text{ even}$$

$$N_\tau = (\tau + 1)^{-1} v^{\tau+1} + \frac{1}{2} \binom{\tau}{1} B_2 v^{\tau-1} + \frac{1}{4} B_4 \binom{\tau}{3} v^{\tau-3} + \ldots + \frac{\tau}{2} B_\tau \nu^2, \quad \text{for } \tau \text{ odd.}$$

Due to the accuracy conditions (B.1) of the interior stencil, one can easily verify that (B.3) automatically simplifies into (B.2) for $\tau \leq 2s - 1$. For the single remaining case $\tau = 2s$, (B.3) can instead be written as

$$\tau \sum_{j=0}^{r-1} p_j (r-j)^{2s-1} = r^{2s} - B_{2s} - 2(2s + 1)^{-1} \sum_{\nu=1}^{t} a_i \nu^{2s+1}. \quad (B.4)$$

Now assume that $q = 2s + 1$. Combining (B.3) for $\tau = q - 1 = 2s$ with (B.4) then leads to the condition

$$\sum_{\nu=1}^{t} a_i \nu^{2s+1} = 0,$$

which would imply that the interior stencil is of order $2s + 2$, see (B.1). This contradicts the premise that $D$ is an SBP(2s, s) operator, and it follows that $q$ in (B.2) (and equivalently $q$ in (9)) cannot exceed 2s. \qed
References


