Indirect System Identification for Unknown Input Problems

With Applications to Ships

Jonas Linder

Cover illustration: A ship performing a turn on the vastness of LiU blue.

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**Indirect System Identification for Unknown Input Problems
With Applications to Ships**

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To partial clarity...

Abstract

System identification is used in engineering sciences to build mathematical models from data. A common issue in system identification problems is that the true inputs to the system are not fully known. In this thesis, existing approaches to unknown input problems are classified and some of their properties are analyzed.

A new *indirect framework* is proposed to treat system identification problems with unknown inputs. The effects of the unknown inputs are assumed to be measured through possibly unknown dynamics. Furthermore, the measurements may also be dependent on other known or measured inputs and can in these cases be called *indirect input measurements*. Typically, these indirect input measurements can arise when a subsystem of a larger system is of interest and only a limited set of sensors is available. Two examples are when it is desired to estimate parts of a mechanical system or parts of a dynamic network without full knowledge of the signals in the system. The input measurements can be used to eliminate the unknown inputs from a mathematical model of the system through algebraic manipulations. The resulting *indirect model* structure only depends on known and measured signals and can be used to estimate the desired dynamics or properties. The effects of using the input measurements are analyzed in terms of identifiability, consistency and variance properties. It is shown that cancelation of shared dynamics can occur and that the resulting estimation problem is similar to errors-in-variables and closed-loop estimation problems because of the noisy inputs used in the model. In fact, the indirect framework unifies a number of already existing system identification problems that are contained as special cases.

For completeness, an instrumental variable method is proposed as one possibility for estimating the indirect model. It is shown that multiple datasets can be used to overcome certain identifiability issues and two approaches, the multi-stage and the joint identification approach, are suggested to utilize multiple datasets for estimation of models. Furthermore, the benefits of using the indirect model in filtering and for control synthesis are briefly discussed.

To show the applicability, the framework is applied to the roll dynamics of a ship for tracking of the loading conditions. The roll dynamics is very sensitive to changes in these conditions and a worst-case scenario is that the ship will capsize. It is assumed that only motion measurements from an inertial measurement unit (IMU) together with measurements of the rudder angle are available. The true inputs are thus not available, but the measurements from the IMU can be used to form an indirect model from a well-established ship model. It is shown that only a subset of the unknown parameters can be estimated simultaneously. Data was collected in experiments with a scale ship model in a basin and the joint identification approach was selected for this application due to the properties of the model. The approach was applied to the collected data and gave promising results.

Populärvetenskaplig sammanfattning

Till skillnad från många andra industrier där avancerade styrsystem har haft en omfattande utveckling under de senaste decennierna så har styrsystem för skepps- och marinindustrin inte alls utvecklats i samma utsträckning. Det är framförallt under de senaste 10 åren som lagkrav och stigande driftskostnader har ökat intresset för effektivitet och säkerhet genom användning av styrsystem. Rederier och den marina industrin är nu intresserade av hur de avancerade styrsystem som används inom andra områden kan tillämpas för marina ändamål. Huvudmålet är typiskt att minska den totala energianvändningen, och därmed också bränsleförbrukningen, genom att hela tiden planera om hur skeppet skall användas med hjälp av ny information samt styra skeppet och dess ingående system på ett sätt som maximerar effektiviteten.

För många av dessa avancerade styrsystem är det grundläggande att ha en god förståelse för beteendet hos det systemet som skall styras. Ofta används matematiska modeller av systemet för detta ändamål. Sådana modeller kan skapas genom att observera hur systemet reagerar på yttre påverkan och använda dessa observationer för att finna eller skatta den modell som bäst beskriver observationerna. Observationerna är mätningar som görs med så kallade sensorer och tekniken att skapa modeller från mätningarna kallas för systemidentifiering. Detta är i grunden ett utmanande problem och det kan försvåras ytterligare om de sensorer som behövs inte finns tillgängliga eller är för dyra att installera.

I denna avhandling föreslås en ny teknik där de mätningar som finns tillgängliga används på ett nytt och annorlunda sätt. Detta kan minska mängden nödvändiga sensorer eller möjliggöra användandet av alternativa sensorer i modellframtagningen. Med hjälp av denna nya teknik kan enkla sensorer användas för att skatta en matematisk modell för en del av skeppet på ett sätt som inte är möjligt med traditionella metoder. Genom att skatta denna modell kan fysikaliska egenskaper hos skeppet, så som dess massa och hur massan är fördelad över skeppet, övervakas för att upptäcka förändringar. Just dessa två egenskaper har stor inverkan på hur skeppet beter sig och om skeppet är fellastat kan det i värsta fall kapsejsa. Vetskapen om dessa fysikaliska egenskaper kan alltså utöver effektivisering användas för att varna besättningen eller påverka styrsystemen så att farliga manövrar undviks.

För att visa att tekniken fungerar i verkligheten har den använts på mätningar som har samlats in från ett skalenligt modellskepp. Experimenten utfördes i bassäng och resultaten visar att tekniken fungerar.

Denna nya teknik är inte specifik för marint bruk utan kan också vara användbar i andra typer av tillämpningar. Även i dessa tillämpningar möjliggörs användandet av färre eller alternativa sensorer för att skatta modeller. Tekniken kan vara speciellt användbar när en modell av ett system eller process som verkar i ett nätverk av många system är av intresse, något som också diskuteras i avhandlingen.

A smooth sea never made a skilled sailor.
– English proverb

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I have been very lucky in my life and always had people that believed in me. This has led to both opportunities but also challenges. Looking back at the past few years I only have one regret, writing (in my mind) good acknowledgements for the Licentiate's thesis since it made these acknowledgements much more difficult to write! Naturally, there is bound to be some overlap between them, after all, facts are facts! Hence, it still a fact that, one day on the *best* coast of Sweden in the sunny city of Göteborg, Simon Lindkvist suggested me for a summer internship at ABB. This eventually led to our Master's thesis which in turn led me to the position in Linköping. I am very grateful for your recommendation, thank you! To be frank, I never saw myself as a PhD student before I started working at ABB and I would like to extend my gratitude to my former colleagues for making me understand my potential and nurturing me towards academia. I would especially like to mention Shiva Sander Tavallaey for forcefully convincing me when I was hesitant and then, together with Johan Sjöberg, recommending me for a position at the university!

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Figure 1: A picture captured during a sunny day in Dalsland (a part of Sweden) during 2013. Note that the dynamics of the canoe is changed due to the center of mass shifting upwards, i.e. Isak standing up. If the controller in the front of the canoe, here called girlfriend, wants to safely maneuver the canoe, it is probably a good idea to have knowledge about this change, see Chapters 7 to 9.

been a great supervisor and have always found the time for me. To quote myself: *Your eye for detail and care put into every single thing are nothing but amazing!* Thank you very much, it has been fun and inspiring to work with you!

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Linköping, February 2017

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Notation

Systems and Models

Notation	Meaning
S	A system
S_O	The (sub)system of interest
G_X	The transfer function from u to y_X , $X = \{O, I, D\}$
H_X	The transfer function from τ to y_X , $X = \{O, I, D\}$
G_{XY}	The transfer function from u_Y , $Y = \{K, I, D, A\}$ to y_X , $X = \{O, I, D\}$
G_{jk}	The transfer function from w_k to w_j
$G_{j\mathcal{X}}$	The transfer function from $w_{\mathcal{X}}$ to w_j
$G_{\mathcal{X}Y}$	The transfer function from w_Y to $w_{\mathcal{X}}$
$f_{y\mathcal{X}}$	A filter such that $f_{y\mathcal{X}}G_{\mathcal{X}Y} = I$ if $G_{\mathcal{X}Y}$ has full-column rank

Signals

Notation	Meaning
$\delta \in \mathbb{R}^{n_\delta}$	External user-controllable signals
$\tau \in \mathbb{R}^{n_\tau}$	External uncontrollable disturbances including measurement noises
$\tau_X \in \mathbb{R}^{n_X}$	External uncontrollable disturbance acting on y_X , $X = \{O, I, D\}$
$w \in \mathbb{R}^{n_w}$	Internal signals (also called nodes in a dynamic network setting)
$y \in \mathbb{R}^{n_y}$	All measurements, not necessarily noisy; static functions of τ and w
$y_O \in \mathbb{R}^{n_O}$	The output measurements of S_O which are elements of y
$y_I \in \mathbb{R}^{n_I^y}$	The indirect input measurements which are elements of y
$y_D \in \mathbb{R}^{n_D^y}$	The direct input measurements which are elements of y
$u \in \mathbb{R}^{n_u}$	The inputs to S_O which are elements of w
$u_K \in \mathbb{R}^{n_K}$	The known inputs to S_O which are elements of u
$u_U \in \mathbb{R}^{n_U}$	The unknown inputs to S_O which are elements of u
$u_I \in \mathbb{R}^{n_I}$	The indirectly measured inputs to S_O which are elements of u_U
$u_D \in \mathbb{R}^{n_D}$	The directly measured inputs to S_O which are elements of u_U
$u_A \in \mathbb{R}^{n_A}$	Additional known inputs which are elements of w
$w_{\mathcal{X}} \in \mathbb{R}^{n_{\mathcal{X}}}$	The vector containing all w_k , $k \in \mathcal{X}_j$
$\delta_{\mathcal{X}} \in \mathbb{R}^{n_{\mathcal{X}}}$	The vector containing all δ_k , $k \in \mathcal{X}_j$
$\tau_{\mathcal{X}} \in \mathbb{R}^{n_{\mathcal{X}}}$	The vector containing all τ_k , $k \in \mathcal{X}_j$

Dynamic Systems and System Identification

Object	Description
$\mathbf{q}$	Shift operator, i.e. $\mathbf{q}y_t = y_{t+1}$
$\mathbf{p}$	Differentiation operator, i.e. $\mathbf{p}y_t = \dot{y}_t$
ϑ	Parameters in estimation problems
ϑ_p	Physical parameters in estimation problems
$\hat{y}_t(\vartheta)$	The one-step-ahead predictor of y_t
ϑ^*	The limit value of $\hat{\vartheta}_N$, i.e. $\hat{\vartheta}_N \rightarrow \vartheta^*$ w.p. 1 as $N \rightarrow \infty$
P_ϑ	The covariance of the asymptotic distribution, i.e. $\text{Cov}(\hat{\vartheta}) = P_\vartheta/N$
$\text{AMSE}(\vartheta)$	The asymptotic mean squared error given by $P_\vartheta + (\vartheta^* - \vartheta)(\vartheta^* - \vartheta)^T$

Sets

Notation	Meaning
$\mathcal{N}$	The set of all nodes $\{1, \dots, N\}$
$\mathcal{N}_j$	The set of nodes with direct paths to w_j
$\mathcal{Z}_j$	The set of nodes with <i>no</i> direct paths to w_j , i.e. $\mathcal{Z}_j = \mathcal{N} \setminus \mathcal{N}_j$
$\mathcal{S}_j$	The set of <i>observable</i> (known) nodes
$\mathcal{U}_j$	The set of <i>unobservable</i> nodes, i.e. $\mathcal{U}_j = \mathcal{N} \setminus \mathcal{S}_j$
$\mathcal{K}_j$	The set of <i>observable</i> nodes with direct paths to w_j , i.e. $\mathcal{K}_j = \mathcal{N}_j \cap \mathcal{S}_j$
$\bar{\mathcal{U}}_j$	The set of <i>unobservable predictor inputs</i> , i.e. $\bar{\mathcal{U}}_j = \mathcal{U}_j \cap \mathcal{N}_j$
$\tilde{\mathcal{U}}_j$	The set of <i>remaining unobservable nodes</i> , i.e. $\tilde{\mathcal{U}}_j = \mathcal{U}_j \setminus \bar{\mathcal{U}}_j$
$\mathcal{A}_j$	The set of additional nodes that are observable, i.e. $\mathcal{A}_j = \mathcal{S}_j \setminus (\mathcal{K}_j \cup \{j\})$
$\mathcal{D}_j$	The indices $i \in \mathcal{A}_j$ such that G_{ik} for $k \in \bar{\mathcal{U}}_j$ are <i>known</i> and at least one G_{ik} is non-zero, and $G_{in} = 0$ for all $n \notin \bar{\mathcal{U}}_j$, see Definition 4.2
$\mathcal{I}_j$	The indices $\{i \in \mathcal{A}_j \setminus \mathcal{D}_j \mid \exists k \in \bar{\mathcal{U}}_j, G_{ik} \neq 0\}$, see Definition 4.3
$\mathcal{O}_j$	The set of remaining observable nodes, i.e. $\mathcal{O}_j = \mathcal{A}_j \setminus (\mathcal{I}_j \cup \mathcal{D}_j)$

Abbreviations

Abbreviation	Meaning
CB	Center of Buoyancy, see Definition A.4 in Appendix A.1.1
CF	Center of Flotation, see Definition A.5 in Appendix A.1.1
CG	Center of Gravity, see Definition A.6 in Appendix A.1.1
CM	Center of Mass
CR	Center of Rotation, see Definition A.7 in Appendix A.1.1
CO	Center of Origin, see Definition A.8 in Appendix A.1.1
CS	Center of Sensor, see Definition A.9 in Appendix A.1.1
EIV	Errors-in-variables, see Chapter 3
IMU	Inertial Measurement Unit, see Appendix A.6.1
LCG	Lateral Center of Gravity, see Appendix A.3.1
NED	North East Down, a convention for coordinate systems
NWU	North West Up, a convention for coordinate systems

Rigid-body - Reference Frames

Frame	Description
b -frame	Body-fixed reference frame, see Definition A.1
n -frame	World-fixed reference frame, see Definition A.2
s -frame	Body-fixed sensor reference frame, see Definition A.3

Rigid-body - Generalized Positions

Variable	Description
x	Position relative the n -frame
y	Position relative the n -frame
z	Position relative the n -frame
ϕ	Roll angle relative the n -frame
θ	Pitch angle relative the n -frame
ψ	Yaw angle relative the n -frame

Rigid-body - Generalized Velocities

Variable	Description
u	Surge speed, linear velocity in the x_b -axis of the b -frame
v	Sway speed, linear velocity in the y_b -axis of the b -frame
w	Heave speed, linear velocity in the z_b -axis of the b -frame
p	Roll-rate, angular velocity about the x_b -axis of the b -frame
q	Pitch-rate, angular velocity about the y_b -axis of the b -frame
r	Yaw-rate, angular velocity about the z_b -axis of the b -frame

1

Introduction

Science may be described as the art of systematic oversimplification.

– Karl Popper

In this thesis, we study the problem of estimating models of systems with unknown inputs. An indirect framework is developed to deal with the unknown inputs in a systematic way. The framework is used to estimate physical properties of ships. This can be a difficult problem, partly due to the complex dynamics governing the interaction between the water and the ship and partly due to the limited set of sensors that is assumed to be available for the task. A complete model of a ship would be very complex and we have instead chosen to work with approximate models that are simple but still accurate enough to capture some important properties of the system that is studied.

This chapter will give a background and motivation, together with a description of contributions and papers, and finally, the structure of the thesis will be explained.

1.1 Research Motivation

In many engineering applications, there is an interest to study or to control a system's behavior under some unknown excitation. In robotics, for example, forces acting on a manipulator might be unknown due to a changing payload. In the automotive industry, there are unknown gravitational forces acting on the vehicles due to variations in the slope of the road, and in marine applications, the forces induced by wind and waves are typically unknown. Other examples include propagation of vibrations in structures, such as bridges or airplanes, where the forces inducing the vibrations are unknown, and telecommunication, where unknown data is sent through a channel with unknown properties.

Similar situations can also occur when a subsystem, i.e. a part of a larger system, is of interest and only a limited set of sensors is available. Two examples are when it is desired to estimate or to control parts of a mechanical system or parts of a network with interacting subsystems without full knowledge of the signals in the whole system. A few benefits of only considering a subsystem are that fewer sensors are needed, a higher accuracy can be achieved since the available information is used to estimate a less complex model or that a lower complexity controller potentially can be used to control the subsystem. Furthermore, more of the unknown excitation entering the system might be captured by using internal signals which are closer to the desired subsystem. One drawback is that the signals influencing the subsystem typically are correlated which means that their properties have to be considered carefully during modelling, estimation and control. This disadvantage might be compounded if some of the signals influencing the subsystem are unknown.

The treatment of the unknown inputs varies depending on the application and the characteristics of the signals. Sometimes, the unknown signals are considered to be disturbances and either neglected or partially mitigated, for example, by using feedback and feedforward control or noise modelling in estimation. Other situations might require a different approach, for instance, due to genuine interest in the unknown excitation, or a large impact of the unknown excitation which might lead to bad performance if it is ignored or only partially mitigated. The goal might thus be to retrieve the unknown signals. However, the goal might also be to construct a representation of the system, to predict its response or to control its output under the influence of the unknown signals. For many of these mentioned goals, a model of the system is desired or can be used to improve the performance in the intended application.

Models can be constructed in many ways ranging from first-principle modelling to data-based approaches. One benefit of the data-based approaches is that the model can be refined when new data is available which means that the model can be adapted to changes in the system (Åström and Wittenmark, 2008). System identification is one estimation approach for obtaining models using data collected from the system. The signals used for this purpose are typically assumed to be either measurable or known. Under these assumptions, one important issue is to mitigate the effects of disturbances, present in the signals, so that an accurate model of the input-output behavior can be estimated. A classical example is the open-loop system identification problem, where the output is corrupted by an unknown disturbance while the input is assumed to be known and uncorrelated with the disturbance (Ljung, 1999). This might be a challenging problem and it will be even more difficult if not all the desired signals are available.

In some of the already mentioned applications, all the relevant signals affecting the system might not be available. Problems with unknown inputs have been studied extensively within the field of system identification and the remedies can be divided into five classes. Firstly, *assumptions* can be made about the unknown inputs and this information is used as a substitute to either only estimate a model of the system (Shumway and Stoffer, 2010) or to estimate a model of the system and the

unknown inputs (Abed-Meraim et al., 1997). Secondly, the unknown inputs can simply be *neglected* and called disturbances (Ljung, 1999). Thirdly, the unknown inputs can be *approximated* using other known signals. Fourthly, *measurements* of the unknown inputs can be used as substitutes leading to an errors-in-variables problem (Söderström, 2007). Finally, several output signals affected by the inputs can be assumed to be available and the extra measurements can be used to *eliminate* the unknown inputs from the model. However, the resulting model does not describe the original system (D’Amato et al., 2009). Although these five remedies lead to satisfactory results in many applications, they are not always tractable. Neglecting inputs will typically lead to issues if the inputs are correlated, measuring the unknown signal might be impossible or impractical and the necessary assumptions might be too restrictive in many practical situations.

In this thesis, an *indirect modelling framework* for systems with unknown inputs will be proposed. The goal is to estimate a model of a subsystem from partially unknown inputs to some of the available output measurements. It is assumed that all the inputs might be correlated and hence, neglecting some inputs can cause a bias in the remaining parts of the model estimator. In the proposed indirect framework, the effects of the unknown inputs are assumed to be visible in several additional measurements that are divided into *direct* and *indirect input measurements*. The direct input measurements are similar to the input measurements in errors-in-variables problems and directly measure the unknown inputs with noise through known dynamics. In contrast, the remaining unknown inputs are only measured through possibly unknown dynamics and the measurements may also depend on all other known or measured inputs. These measurements thus indirectly contain information about the inputs and are called indirect input measurements. Similar to the aforementioned fifth approach to unknown inputs, the unknown inputs are eliminated and the resulting *indirect model* can be used in conjunction with the output, the input measurements and the known inputs to estimate the dynamical model of interest.

1.1.1 Motivational Example - Marine Vessels

Models are nowadays commonly used to increase performance or accuracy. This is essential, for instance, in model-based control where a poor model will affect the performance negatively (Skogestad and Postlethwaite, 2005), or in decision support systems where an error in the model can result in bad advice which might lead to the user eventually not trusting the system (Funk et al., 2006; Sanchez et al., 2011). Many mechanical systems, and certainly ships, have time-dependent physical properties. By estimating these properties online, a higher model accuracy can be achieved. Some examples are the change in the center of mass due to passengers entering or leaving a car, the change of the mass in an oil tanker due to oil being pumped into its tanks and the mass of a container ship being changed due to loading and unloading of containers as can be seen in Figure 1.1.

These changing properties affect the dynamic behavior of the ships and the variation of certain properties might be safety critical. Fujiwara and Haraguchi (2005) investigate the influence of water flooding the car deck of roll-on-roll-off passenger

ships, such as the disaster of the ship *Estonia* in 1994. The authors show that the restoring properties, both the roll damping and the roll stiffness, change with the amount of water on deck. This is a situation where online estimation could be used to improve the accuracy of the model for later use in, for instance, a decision support system that could aid the crew to operate the vessel safely.

A ship's roll dynamics is very sensitive to changes in the loading conditions and a worst-case scenario is that the ship will capsize (Fossen, 2011; Tannuri et al., 2003). Moreover, a ship's response to waves is sensitive to the direction of the

Figure 1.1: An aerial view of the Cape Town harbor showing ships being tended to by gantry cranes. A container ship can carry thousands of containers, each weighing several tons, and a significant part of the ship's weight is the cargo.

waves relative to the ship's heading (Faltinsen, 2005). For example, the ship's roll motion is more likely to be affected by waves approaching from the side than if the ship would face them with the bow first. Iseki and Terada (2001), Tannuri et al. (2003), Nielsen (2007, 2008), and Nielsen and Iseki (2010, 2011) study the problem of estimating the directional wave spectrum from the ship motion. The estimated wave spectrum can be used to simulate a ship's response over time, to apply control or to suggest appropriate actions. This is basically an inverse problem where the spectrum of the input to the system is estimated and the result depends on the ship's mass. Inaccurate knowledge of the ship's mass, and thus of the model, can lead to an inaccurate wave spectrum, which in the end might lead to poor performance if the wave spectrum is used in the earlier mentioned control or decision support systems.

Perez (2005) investigates methods for ship roll stabilization or ship roll reduction systems using model-based control. The model used in the approach is a linear maneuvering model where the parameters are assumed to be known. The effects of model errors are not investigated, but due to the big change in the ship roll dynamics shown in, for instance, Fossen (2011) and Tannuri et al. (2003), there are reasons to believe that, for example, significant changes in mass will affect the ship dynamics and that online estimation of the mass would make it possible to increase the performance of the controller.

Actually, the mass, or inertia, is one of the most influential parameters in most mechanical systems. However, it is difficult to uniquely determine the mass in many cases due to inadequate measurements. For example, the acceleration a of a particle with mass m , affected by the force F and moving in a straight path is directly proportional to F/m . If the acceleration a is measured, there are still multiple combinations of F and m that will solve the equation. Special experiments, where the forces acting on the system are known or can be measured, can be used to overcome this ambiguity, but doing special experiments is in many cases intractable. This is certainly true for the ship application that is studied here since it is difficult to measure all forces acting on the ship's hull.

In the last part of this thesis, a model of the ship's roll dynamics is estimated using only a limited set of sensors and the approach proposed in this thesis. More specifically, it is assumed that only motion measurements from an inertial measurement unit (IMU) together with the rudder angle are available. Hence, direct measurements of all the true inputs to the subsystem are not available, but the measurements supplied by the IMU indirectly contain information about the inputs and these measurements can be used to eliminate the unknown inputs. The main benefit of using the approach in the marine application is that it can be applied easily without installing any expensive additional sensor systems on the ship.

1.2 Contributions and Publications

There are six main contributions in this thesis. The first contribution is the indirect framework based on the indirect input measurements presented in Chapter 4. The second contribution is the dynamic network formulation of the indirect model that more clearly shows the correlation structure and properties of the indirect model in relation to the whole system also described in Chapter 4. The third contribution is the derivation of and connection between the indirect models for the pendulum and ship models presented in Chapters 7 and 8. The fourth contribution is the analysis of the identifiability and disturbance properties of the indirect model presented in Chapter 4 and the more concrete analyses for the applications presented in Chapters 7 and 8. The fifth contribution is the multiple dataset identification approaches to overcome some identifiability issues introduced in Chapter 5, including the two concrete implementations in Sections 7.4 and 8.5. The sixth contribution is the experimental work and the initial estimation results based on the collected ship data, see Chapter 9.

The notion of indirect input measurements, i.e. measurements that contain indirect information about the unknown input signal, was first introduced in

J. Linder. *Graybox Modelling of Ships Using Indirect Input Measurements*. Linköping Studies in Science and Technology. Thesis 1681. Linköping University, Sweden, 2014

The material of Chapters 3, 4 and 6 are partially based on the following three papers. The idea of the indirect framework was first introduced in the paper

J. Linder and M. Enqvist. On Indirect Input Measurements. In *Proceedings of the 17th IFAC Symposium on System Identification*, Beijing, China, October 2015

where the concept was treated from a general system point of view. Moreover, the applications of Chapters 8 and 9 were expressed in the indirect modelling framework and the results were briefly discussed for the general case. In

J. Linder and M. Enqvist. Identification of systems with unknown inputs using indirect input measurements. *International Journal of Control*, 2016a. doi: 10.1080/00207179.2016.1222557

the framework was extended and a deeper analysis was given. The experimental results of this paper are included in Chapter 7. The use and benefits of the framework in the dynamic networks setting were investigated in

J. Linder and M. Enqvist. Identification and Prediction in Dynamic Networks with Unobservable Nodes. Technical Report LiTH-ISY-R-3095, Department of Electrical Engineering, Linköping University, Linköping, 2016b

where more emphasis was given to correlation between variables and the implication of this. In addition, the use of indirect input measurements in black-box models for estimation and prediction was briefly discussed.

The results in Chapter 7 have also been published in

J. Linder, M. Enqvist, F. Gustafsson, and J. Sjöberg. Identifiability of physical parameters in systems with limited sensors. In *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, August 2014b

J. Linder, M. Enqvist, and F. Gustafsson. A closed-loop instrumental variable approach to mass and center of mass estimation using IMU data. In *Proceedings of the 53rd IEEE Conference on Decision & Control*, Los Angeles, CA, USA, December 2014a

The identifiability analysis is given in the first paper and the second paper describes the impact of disturbances and the use of the closed-loop instrumental variable approach to decrease the variance of the estimates.

The material of Chapter 8 (and parts from Appendix A) has been published in

J. Linder, M. Enqvist, T. I. Fossen, T. A. Johansen, and F. Gustafsson. Modeling for IMU-based Online Estimation of Ship's Mass and Center of Mass. In *Proceedings of the 10th IFAC Conference on Manoeuvring and Control of Marine Craft (MCMC)*, Copenhagen, Denmark, August 2015b

with the focus on using indirect input measurements to retain the same predictive power as a well-established four degrees-of-freedom model but for a lower order model. The material in Chapter 9, i.e. the experimental verification of Chapter 8, together with a summary of the model and the estimator have been published in

J. Linder, M. Enqvist, T. I. Fossen, T. A. Johansen, and F. Gustafsson. Online Estimation of Ship's Mass and Center of Mass Using Inertial Measurements. In *Proceedings of the 10th IFAC Conference on Manoeuvring and Control of Marine Craft (MCMC)*, Copenhagen, Denmark, August 2015a

Not included but published material are the patent application and the two papers

J. Linder, M. Enqvist, F. Gustafsson, and J. Sjöberg. Righting arm model determination, February 18 2016. WO Patent App. PCT/EP2014/067,460

J. Sjöberg, S. Lindkvist, J. Linder, and A. Daneryd. Interactive multiobjective optimization for the hot rolling process. In *Proceedings of the 51st IEEE Conference on Decision & Control*, Maui, Hawaii, USA, December 2012

J. Sjöberg, S. Lindkvist, J. Linder, and J. Öhr. Interactive multiobjective optimization for a grab-shift unloader crane. In *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, August 2014

The patent application describes how techniques similar to those described in Chapter 8 can be used to estimate the righting arm of a ship. The papers are based on Linder and Lindkvist (2011) and describe how optimization techniques can be used in decision support applications. Although not directly related, optimization is central in system identification and one of the intended applications for the work in this thesis is decision support systems. The first paper describes how the interactive multiobjective optimization framework developed in Linder and Lindkvist (2011) can be used to explore the solutions and learn about the hot-rolling process. The second paper describes how the interactive multiobjective optimization framework can be used to solve an optimal control problem to find a good trade-off between, for instance, minimum time and minimum energy.

1.3 Thesis Outline

The theoretical foundation of this thesis relies on two theories, system identification and ship modelling. Chapter 2 and Appendix A give a brief theoretical introduction to selected parts of these subjects. Although Chapter 2 can be considered background, Section 2.2 defines some of the notation used in the thesis.

The thesis is divided in two parts. The first part consists of Chapters 3 to 6 and presents the problem of identification of systems with unknown inputs, introduces the framework and describes the usage of the proposed indirect model. In the second part, comprised of Chapters 7 to 9, the framework is applied to estimation of a ship's physical properties.

Chapter 3 gives an introduction to systems with unknown inputs. A brief overview and a classification of the approaches to unknown input problems are given. A general model is given for systems with unknown inputs and the effects of using the earlier classified approaches are investigated.

In Chapter 4, the direct and indirect input measurements are defined and the indirect framework is derived. The effects of using the input measurements and some properties of the indirect model are investigated. In addition, the division of the measurements into output and input measurements is briefly discussed, and the connections to existing system identification problems are given.

In Chapter 5, an approach to estimate the parameters of the indirect model using an instrumental variable method is presented. In addition, two approaches, the multi-stage and the joint identification approaches, where multiple datasets are used to gain identifiability, are proposed.

Chapter 6 briefly presents how the indirect input measurements can be used in black-box models for filtering and control.

Chapter 7 presents an introduction to the approximate model of a ship's roll dynamics and an indirect model for this system together with an analysis of the identifiability properties of the model and the signals' dependencies on the disturbances. Furthermore, the sequential multi-stage instrumental variable method is applied to a specific indirect pendulum model. Finally, an investigation of the suggested estimator's properties is presented in an experimental study.

In Chapter 8, it is shown that a well-established ship model can be used to derive a generalization of the approximate model of Chapter 7, with more indirect input measurements and a few extra parameters. Furthermore, the analyses of Chapter 7 are extended to the generalized model. Finally, because of this extra complexity, the joint identification approach is applied to the suggested indirect model.

Chapter 9 introduces the experimental setup used to collect data from a scale ship model for verification purposes. It also describes the estimation results obtained by applying the algorithm in Chapter 8 to the collected data.

Finally, in Chapter 10 the thesis is concluded and ideas for future work are given.

2

Selected Topics in System Identification

... causes and effects are discoverable, not by reason but by experience ...

– David Hume

System identification is the science of building dynamical models based on measurements of signals in a system. The essence of system identification is the problem of induction, i.e. generalizing from a finite number of observations. In the nomenclature of system identification, the objective is to *estimate*, or *infer*, a connection between two signals, commonly called *input* (the cause) and the resulting *output* (the effect). This inference results in a *model* (generalization) that is gained from *measurements* and knowledge of the system such as the *structure* of the dynamical model or values of some *parameters* (experience/previous knowledge).

A model is in this context never claimed to be the true description of the system, but rather a simplified mathematical description that is *good enough* for the intended application. The notion of a system is quite broad and it might, for instance, be a ship, the world's economy or even the human body.

The validity of the model is typically verified by some independent test. The inferred model might, for instance, be validated with a second batch of measurements and be rejected if the outcome is too poor. In reality, the outcome from the estimation and validation is used in an iterative process to deduce new theories and then the loop starts from the beginning.

This chapter is not an exhaustive description of system identification, but is rather supposed to give a brief introduction to certain topics needed to understand the work in this thesis. The focus is on estimation of parametric time-domain models. There are many approaches to system identification, and for a thorough treatment, see, for instance, Söderström and Stoica (1989), Schoukens and Pintelon (1991), Ljung (1999) or Pintelon and Schoukens (2012).

2.1 Systems and Signals

A process or object that describes a cause and effect relationship is called a *system*. Both the cause and the effect are described by variables. These variables are called *signals* and they are sources of information which are functions of some independent variables, such as time and space (Phillips et al., 2008). Commonly, signals are physical quantities, for example, force or temperature. In this thesis, systems are modeled by mathematical equations and signals are modeled by mathematical functions. In most cases, these mathematical representations are approximations of reality. In classical systems theory, the signals affecting the system are called input signals and the resulting signals are called output signals, or simply inputs and outputs, respectively. A system transforms the input into the output. In this thesis, we are mainly interested in time-dependent signals. The output of a *static* system is only dependent on the input at the current time. In contrast to a static system, a *dynamic* system has an internal memory and the output is not only dependent on the input at the current time but also on the input at previous times. In this thesis, the system is assumed to be causal, i.e. the input at time t only affects the output at times $s \geq t$.

2.2 Representations of Systems and Signals

In engineering, the objects of interest are commonly collections of interconnected systems (components). The output of one of these systems might thus be the input to another system. In this setting, labelling signals in the underlying system as inputs or outputs is perhaps not the most natural approach. Instead, it is assumed that the system of interest has internal (dependent) signals $w \in \mathbb{R}^{n_w}$ that are affected (determined) by external (independent) signals. The external signals are classified as user-controllable signals $\delta \in \mathbb{R}^{n_\delta}$ and uncontrollable disturbances $\tau \in \mathbb{R}^{n_\tau}$ (including measurement noise). It is assumed that there are measurements denoted by $y \in \mathbb{R}^{n_y}$ that are static functions of the internal variables w and the external disturbances τ . The behavior of the internal variables w and hence, also of the measurements y , is thus fully determined by the dynamics and the external variables δ and τ , see Figure 2.1. Note that all the measurements not necessarily have to depend on τ , i.e. there might be noise-free measurements of w .

We are interested in a subset of the dynamics that is denoted by S_O . Here, we will actually return to the notation of input and output signals to coincide with the nomenclature used in the system identification literature. The output of S_O is denoted by $y_O \in \mathbb{R}^{n_O}$ and is a function of the input $u \in \mathbb{R}^{n_u}$ and the external

Figure 2.1: A schematic view of the system S with measurements y , external controllable signals δ and uncontrollable disturbances τ .

Figure 2.2: A common assumption is that the desired (sub-)system has an input u that might be correlated with the disturbance.

disturbances τ , see Figure 2.2. The output y_O and input u are elements of y and w , respectively. The input u is possibly correlated with τ and captures all contributions from the external user-controllable signals δ that affect the output y_O .

In the classical system identification discussed in this chapter, the input u is known. In our framework, this means that the input u consists of (noise-free) elements of the measurement y . The case with u partially unknown is the subject of this thesis and it will be discussed and treated in Chapter 3 and Chapter 4. Even if the input u is known, it can still be correlated with the disturbance τ . A well-known case in automatic control is a system in closed loop, which can be seen in Figure 2.3. The correlation between the input u and the disturbance τ has implications, for instance, which estimator and model structure that might be suitable choices.

Note that the input not necessarily is a physical signal in the system. For example, in automatic control it is common to consider the computer-generated output of the controller to be the input to the system while the physical input, for instance, is a force generated by an actuator. The discrepancy between the input and the physical input thus has to be captured by the description of the system. In the end, it is the relation between the signals defined as input and output that is interesting.

This description of the system is called a *model*. In a broad sense, a model is a concept that relates the variables of the estimation problem to each other and it is with this point of view a fairly abstract concept. In this thesis, three different types of mathematical models will be used to describe the systems. These representations all have different benefits and different levels of detail.

1. If the system S_O is linear and time-invariant, it can be represented by a transfer-function model given by

$$y_{O,t} = G_O(\mathbf{q})u_t + H_O(\mathbf{q})\tau_t \quad (2.1a)$$

$$u_t = F_\delta(\mathbf{q})\delta_t + F_\tau(\mathbf{q})\tau_t \quad (2.1b)$$

where $\mathbf{q}$ is the forward shift operator, i.e. $\mathbf{q}x_t = x_{t+1}$, and $G_O(\mathbf{q})$, $H_O(\mathbf{q})$, $F_\delta(\mathbf{q})$ and $F_\tau(\mathbf{q})$ are transfer-function matrices of suitable dimensions. The transfer-function matrices could have direct-terms which, for instance, means that $y_{O,t}$

Figure 2.3: The input u in a closed-loop system is correlated with τ due to the feedback from the output.

can depend directly on u_t and τ_t without delay. In this model, the internal signals of S_O have no direct representation, instead the output y_O at time t is described by all inputs u and disturbances τ up to time t . The model given by (2.1) is fairly general since both u and y_O are dependent on τ . For example, it can describe an open-loop experiment when $F_\tau(\mathbf{p})=0$. However, this representation also allows the system S_O to be part of a bigger more complicated system, such as a system operating in closed-loop (Forssell and Ljung, 1999) or a module in a dynamic network (Van den Hof et al., 2013) where $F_\tau(\mathbf{p})$ is dependent on the controller and the interconnection structure, respectively.

2. Sometimes it is convenient with a structured model of the whole system S . The dynamic network model is one example. Here, the dynamic network framework described in Van den Hof et al. (2013) and Dankers (2014) will be used as a basis. In this framework, the network is assumed to consist of L internal signals, or nodes, $w_k, k \in \mathcal{N}$ where $\mathcal{N} = \{1, \dots, L\}$. These nodes are assumed to be dynamically dependent on each other and, in addition, on the external user-controllable signals $\delta_k, k \in \mathcal{N}$, and disturbances $\tau_k, k \in \mathcal{N}$, that could enter at each node. Note that some τ_k or δ_k could be zero for all times. The j^{th} node can thus be described by

$$w_{j,t} = \sum_{k \in \mathcal{N}_j} G_{jk}(\mathbf{q})w_{k,t} + \delta_{j,t} + \tau_{j,t} \quad (2.2)$$

where $G_{jk}(\mathbf{q}), k \in \mathcal{N}_j$, are transfer functions and the set $\mathcal{N}_j$ is the indices $k \in \mathcal{N} \setminus \{j\}$ (i.e. no self-loops) for which $G_{jk}(\mathbf{q}) \neq 0$. The descriptions (2.2) of each node can be combined in vector notation and the entire network can be described by

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \\ \vdots \\ w_{L,t} \end{bmatrix} = \begin{bmatrix} 0 & G_{12}(\mathbf{q}) & \cdots & G_{1L}(\mathbf{q}) \\ G_{21}(\mathbf{q}) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1}(\mathbf{q}) & \cdots & G_{LL-1}(\mathbf{q}) & 0 \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \\ \vdots \\ w_{L,t} \end{bmatrix} + \begin{bmatrix} \delta_1 + \tau_1 \\ \delta_2 + \tau_2 \\ \vdots \\ \delta_L + \tau_L \end{bmatrix} \quad (2.3)$$

where the non-zero entries of the j^{th} row are defined by $\mathcal{N}_j$. Equation (2.3) can also be written on the compact form

$$w_t = G(\mathbf{q})w_t + \delta_t + \tau_t \quad (2.4)$$

and the output is given by $y_{O,t} = h_O(w_t)$ where $h_O(\cdot)$ is a static function. There is a *path* between the nodes w_i and w_j if there exists a sequence of integers $n_1, \dots, n_k$, such that $G_{jn_1}G_{n_1n_2} \cdots G_{n_k i} \neq 0$. There is a *direct path* between w_i and w_j if $G_{ji} \neq 0$. A path from w_j to w_j is called a *loop*. All $\tau_k, k \in \mathcal{N}$ are assumed to be independent and in addition, the network is assumed to satisfy Assumption 2.1.

Assumption 2.1 (Assumption 1 of Dankers et al. (2016)).

- (a) The network is well-posed in the sense that all principal minors of $\lim_{z \rightarrow \infty} (I - G(z))$ are non-zero.
- (b) $(I - G(z))^{-1}$ is stable.
- (c) All $\delta_m, m \in \mathcal{N}$ are uncorrelated with all $\tau_k, k \in \mathcal{N}$.

Here, the nodes will be grouped and reordered into sets denoted by large calligraphic letters and the dependencies of t will be dropped for brevity. The variable $w_{\mathcal{X}}$ is the vector containing all $w_k, k \in \mathcal{X}_j$. The ordering is not important as long as all vectors and matrices are ordered consistently. With this notation, the network can, for instance, be written

$$\begin{bmatrix} w_j \\ w_{\mathcal{N}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G_{j\mathcal{N}}(\mathbf{q}) & 0 \\ G_{\mathcal{N}j}(\mathbf{q}) & G_{\mathcal{N}\mathcal{N}}(\mathbf{q}) & G_{\mathcal{N}\mathcal{Z}}(\mathbf{q}) \\ G_{\mathcal{Z}j}(\mathbf{q}) & G_{\mathcal{Z}\mathcal{N}}(\mathbf{q}) & G_{\mathcal{Z}\mathcal{Z}}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{N}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} \delta_j + \tau_j \\ \delta_{\mathcal{N}} + \tau_{\mathcal{N}} \\ \delta_{\mathcal{Z}} + \tau_{\mathcal{Z}} \end{bmatrix} \quad (2.5)$$

where $\mathcal{Z}_j$ is the set of indices for which $G_{jk} = 0, k \in \mathcal{Z}_j$. This network description can be transformed into an input-output transfer function representation, for instance, by letting $y_{O,t} = w_{j,t}$ and $u_t = [w_{\mathcal{N},t}^T, \delta_{j,t}^T]^T$.

3. A state-space model is an even more detailed model of the system S . It stores the memory of the system S in a latent finite-dimensional vector x where each element is called a state. The states contain the information about the effects of all previous excitation to the system and are thus a representation of the internal signals w . A general nonlinear state-space model of S can be written

$$\dot{x}_t = f(x_t, \delta_t, \tau_t) \quad (2.6a)$$

$$y_t = h(x_t, \delta_t, \tau_t) \quad (2.6b)$$

where $f(\cdot)$ and $h(\cdot)$ are static functions. A state-space model is particularly useful in mechanical systems where the states usually are physical quantities, for instance, angular velocity or position. This type of model can also be used to describe the subsystem S_O where the states would represent the internal variables of S_O and δ_t would be replaced by u_t .

An example of a dynamic system in a control application is given in Example 2.1, where the concepts are related to a real world application, an autopilot for a ship.

Example 2.1: A dynamic system in a control application - ship autopilot

In an autopilot application, the goal is to steer the ship to a desired position and heading, using the available inputs to the system while the effects from disturbances are minimized. Figure 2.4 shows a ship controlled using a rudder and the speed of the motor. The dynamic system is the ship itself and its behavior defines the dynamics, for instance, the ship's response to a change in rudder angle. The input to the system is the rudder angle and the motor speed. The output $y_{O,t}$ is the position and possibly heading of the ship in relation to some global coordinate system, for instance, longitude, latitude and heading in relation to north. Furthermore, the signal τ_t represents the environmental disturbances affecting the ship. In this case, only the waves are considered. Since the input is the desired rudder angle and speed of the motor, the dynamics will also contain the rudder and motor dynamics. A complete model (if it were available) would be rather complex due to the ship's interaction with the environment. In a real application, simplified models are commonly used and are often sufficient for the application, see for instance, Källström et al. (1979) and Tzeng et al. (1999).

Figure 2.4: A ship can be seen as a dynamic system with user-controllable inputs $u_t = \delta_t$ from the rudder angle and the motor speed and it is affected by the environmental disturbances τ_t , here only the waves. The output $y_{O,t}$ is the position, and possibly heading, in a global coordinate system.

2.3 The Concepts of System Identification

The basis for all system identification is the observations in the form of a dataset. A dataset is a collection of N data points denoted by

$$Z_N = (y_{O,t}, u_t, o_t)_{t=1}^N \quad (2.7)$$

where $y_{O,t}$ is the output, u_t is the input and o_t contains any other measured signals of interest, for example, measured disturbances, all at time t . In this thesis, parametric models are considered. A parametric model is parameterized by the vector ϑ and changing the values of the parameters will alter the properties of the model. In general, the model can be represented by the one-step-ahead predictor

$$\hat{y}_{O,t}(\vartheta) = g(t, Z_{t-1}, u_t | \vartheta) \quad (2.8)$$

which is here given as a general nonlinear filter based on previous data and the parameter ϑ . The model is adapted to the data by estimating the parameters as

$$\hat{\vartheta} = \underset{\vartheta \in \mathcal{D}}{\operatorname{argmin}} V_N(\vartheta, Z_N) \quad (2.9)$$

where $V_N(\vartheta, Z_N)$ is a nonlinear criterion that depends on the data and the predictor. The criterion is typically a scalar measure of, for example, the prediction error

$$\varepsilon_{O,t}(\vartheta) = y_{O,t} - \hat{y}_{O,t}(\vartheta) \quad (2.10)$$

A model is usually considered to be good if the criterion function is small when the model is applied to a set of observed data that was not used in the estimation.

2.4 Parametric Models

A parametric model structure uses a finite number of parameters to specify the behavior of the model. For instance, a candidate set of models for (2.1a) is defined by

$$y_{O,t} = G_O(\mathbf{q}, \vartheta)u_t + H_O(\mathbf{q}, \vartheta)\tau_t = G_O(\mathbf{q}, \vartheta)u_t + H(\mathbf{q}, \vartheta)e_t \quad (2.11)$$

where $H(\mathbf{q}, \vartheta)$ is a monic transfer function, e_t is a white zero-mean Gaussian process, $\vartheta \in D_{\mathcal{M}}$ and the challenge is to find which of these models that is most suitable.

Comparison between different models in this set is usually done by looking at their ability to reproduce the output data, for instance, using prediction or simulation. An one-step-ahead predictor of $y_{O,t}$ is

$$\hat{y}_{O,t}(\vartheta) = H^{-1}(\mathbf{q}, \vartheta) G_O(\mathbf{q}, \vartheta) u_t + [1 - H^{-1}(\mathbf{q}, \vartheta)] y_{O,t} \quad (2.12)$$

where outputs up to time $t - 1$ and inputs up to time t are used in the predictor.

Proposing a model structure is by no means an easy task. At one end of the scale, there are first-principle (whitebox) model structures where everything is derived from physical laws and all needed parameters are known. On the other side of the scale are the blackbox model structures. The parameters in blackbox model structures have no physical meaning and they are used to tune the properties of the model to fit the data. In between these are the graybox model structures in which the parameters have physical interpretations, for example, mass, density or length.

2.4.1 Transfer-Function Model Structures

A typical example of a blackbox model structure is the general transfer-function model structure (Ljung, 1999)

$$A(\mathbf{q}, \vartheta) y_{O,t} = \frac{B(\mathbf{q}, \vartheta)}{F(\mathbf{q}, \vartheta)} u_t + \frac{C(\mathbf{q}, \vartheta)}{D(\mathbf{q}, \vartheta)} e_t \quad (2.13)$$

where

$$B(\mathbf{q}, \vartheta) = \mathbf{q}^{-n_k} (b_0 + b_1 \mathbf{q}^{-1} + b_2 \mathbf{q}^{-2} + \dots + b_{n_b-1} \mathbf{q}^{-n_b+1}), \quad (2.14)$$

$$A(\mathbf{q}, \vartheta) = 1 + a_1 \mathbf{q}^{-1} + a_2 \mathbf{q}^{-2} + \dots + a_{n_a} \mathbf{q}^{-n_a} \quad (2.15)$$

and where the structures for $C(\mathbf{q}, \vartheta)$, $D(\mathbf{q}, \vartheta)$ and $F(\mathbf{q}, \vartheta)$ are similar to the one for $A(\mathbf{q}, \vartheta)$ but not necessarily with the same number of parameters. However, this general structure is typically too flexible and special cases are used instead. These cases are obtained if some of the polynomials are set to unity and here we will list the ones that will be used later in this thesis in order of increasing complexity.

One of the most commonly used model structures, the autoregressive with exogenous input (ARX) model structure

$$A(\mathbf{q}, \vartheta) y_{O,t} = B(\mathbf{q}, \vartheta) u_t + e_t \quad (2.16)$$

is obtained by setting $C(\mathbf{q}, \vartheta) = D(\mathbf{q}, \vartheta) = F(\mathbf{q}, \vartheta) = 1$ in (2.13). The ARX model structure is especially attractive due to its simple structure and that the parameters can be found using efficient estimation methods, see the next section. However, for a low model order, the noise model can be too restrictive and to increase flexibility in the noise model we can introduce a moving average (MA) part. The autoregressive moving average with exogenous input (ARMAX) model structure

$$A(\mathbf{q}, \vartheta) y_{O,t} = B(\mathbf{q}, \vartheta) u_t + C(\mathbf{q}, \vartheta) e_t \quad (2.17)$$

is obtained by setting $D(\mathbf{q}, \vartheta) = F(\mathbf{q}, \vartheta) = 1$ in (2.13) but has the drawback of a more computationally heavy estimation procedure. Finally, if $G_O(\mathbf{q}, \vartheta)$ and $H(\mathbf{q}, \vartheta)$ are

parameterized individually, the Box-Jenkins (BJ) model structure

$$y_{O,t} = \frac{B(\mathbf{q}, \vartheta)}{F(\mathbf{q}, \vartheta)} u_t + \frac{C(\mathbf{q}, \vartheta)}{D(\mathbf{q}, \vartheta)} e_t \quad (2.18)$$

is obtained by setting $A(\mathbf{q}, \vartheta) = 1$ in (2.13). A special case of the BJ model structure is when $C(\mathbf{q}, \vartheta) = D(\mathbf{q}, \vartheta) = 1$ which gives the output error (OE) model structure.

An one-step-ahead predictor of (2.13) follows directly from (2.12) and is given by

$$\hat{y}_{O,t}(\vartheta) = \frac{D(\mathbf{q}, \vartheta)B(\mathbf{q}, \vartheta)}{C(\mathbf{q}, \vartheta)F(\mathbf{q}, \vartheta)} u_t + \left[1 - \frac{D(\mathbf{q}, \vartheta)A(\mathbf{q}, \vartheta)}{C(\mathbf{q}, \vartheta)} \right] y_{O,t} \quad (2.19)$$

and naturally, the one-step-ahead predictors of the ARX, ARMAX, BJ and OE model structures are all special cases of (2.19).

2.4.2 Linear Regression Model Structures

Let us consider the linear regression model structure that can be written as

$$\hat{y}_{O,t}(\vartheta) = \varphi_t^T \vartheta \quad (2.20)$$

where the elements of the regression vector φ_t are known which means that the parameters can be estimated using a least-squares (LS) method. The ARX model structure is one example that can be written on this form. An one-step-ahead predictor of the ARX model is given by

$$\hat{y}_{O,t}(\vartheta) = B(\mathbf{q}, \vartheta) u_t + [1 - A(\mathbf{q}, \vartheta)] y_{O,t} \quad (2.21)$$

and can be written on the linear regression model structure form by introducing

$$\varphi_t = [-y_{O,t-1}, \dots, -y_{O,t-n_a}, u_{t-n_k}, \dots, u_{t-n_b+1-n_k}]^T \quad (2.22)$$

and

$$\vartheta = [a_1, \dots, a_{n_a}, b_0, \dots, b_{n_b-1}]^T \quad (2.23)$$

2.4.3 State-space Model Structures

A general nonlinear state-space model structure can be written

$$\dot{x}_t = f(x_t, u_t, \tau_t, \vartheta) \quad (2.24a)$$

$$y_{O,t} = h(x_t, u_t, \tau_t, \vartheta) \quad (2.24b)$$

where x_t is the state vector. For a linear time-invariant system, the linear state-space model structure

$$\dot{x}_t = A(\vartheta)x_t + B(\vartheta)u_t + G(\vartheta)\tau_t \quad (2.25a)$$

$$y_{O,t} = C(\vartheta)x_t + D(\vartheta)u_t + E(\vartheta)\tau_t \quad (2.25b)$$

is commonly used and it can be transformed to a transfer-function model structure

$$y_{O,t} = \left(C(\vartheta) [sI - A(\vartheta)]^{-1} B(\vartheta) + D(\vartheta) \right) u_t + \tilde{\tau}_t \quad (2.26)$$

where $\tilde{\tau}_t$ represents the disturbance terms.

Similarly, a discrete-time system can be represented by

$$x_{t+1} = A(\vartheta)x_t + B(\vartheta)u_t + G(\vartheta)\tau_t \quad (2.27a)$$

$$y_{O,t} = C(\vartheta)x_t + D(\vartheta)u_t + E(\vartheta)\tau_t \quad (2.27b)$$

The one-step-ahead predictor of a linear state-space model is given by the Kalman filter (Ljung, 1999). Assuming that $G(\vartheta)\tau_t$ and $E(\vartheta)\tau_t$ are white zero-mean Gaussian processes with covariance given by

$$E \left[\begin{bmatrix} G(\vartheta) \\ E(\vartheta) \end{bmatrix} \tau_t \tau_t^T \begin{bmatrix} G(\vartheta) \\ E(\vartheta) \end{bmatrix}^T \right] = \begin{bmatrix} Q(\vartheta) & S(\vartheta) \\ S^T(\vartheta) & R(\vartheta) \end{bmatrix} \quad (2.28)$$

and that all data from minus infinity up to time $t-1$ is available, the predictor is

$$\hat{x}_{t+1}(\vartheta) = A(\vartheta)\hat{x}_t(\vartheta) + B(\vartheta)u_t + K(\vartheta)[y_{O,t} - C(\vartheta)\hat{x}_t(\vartheta) - D(\vartheta)u_t] \quad (2.29a)$$

$$\hat{y}_{O,t}(\vartheta) = C(\vartheta)\hat{x}_t(\vartheta) + D(\vartheta)u_t \quad (2.29b)$$

where

$$K(\vartheta) = [A(\vartheta)\bar{P}(\vartheta)C^T(\vartheta) + S(\vartheta)][C(\vartheta)\bar{P}(\vartheta)C^T(\vartheta) + R(\vartheta)]^{-1} \quad (2.30)$$

and $\bar{P}(\vartheta)$ is the steady-state solution to the algebraic Riccati equation

$$\begin{aligned} \bar{P}(\vartheta) = & A(\vartheta)\bar{P}(\vartheta)A^T(\vartheta) + Q(\vartheta) - [A(\vartheta)\bar{P}(\vartheta)C^T(\vartheta) + S(\vartheta)] \\ & \times [C(\vartheta)\bar{P}(\vartheta)C^T(\vartheta) + R(\vartheta)]^{-1} [A(\vartheta)\bar{P}(\vartheta)C^T(\vartheta) + S(\vartheta)]^T \end{aligned} \quad (2.31)$$

The prediction error $e_t = y_{O,t} - C(\vartheta)\hat{x}_t(\vartheta) - D(\vartheta)u_t$ is called the innovation, and using e_t , (2.29) can be written as

$$\hat{x}_{t+1}(\vartheta) = A(\vartheta)\hat{x}_t(\vartheta) + B(\vartheta)u_t + K(\vartheta)e_t \quad (2.32a)$$

$$y_{O,t} = C(\vartheta)\hat{x}_t(\vartheta) + D(\vartheta)u_t + e_t \quad (2.32b)$$

which is called the innovations form. It can be transformed into the transfer-function model structure (2.12) where direct calculations give

$$G_O(\mathbf{q}, \vartheta) = C(\vartheta)[\mathbf{q}I - A(\vartheta)]^{-1}B(\vartheta) + D(\vartheta) \quad (2.33)$$

$$H(\mathbf{q}, \vartheta) = C(\vartheta)[\mathbf{q}I - A(\vartheta)]^{-1}K(\vartheta) + I \quad (2.34)$$

If not all data is available, (2.29) is typically suboptimal. If the penalty of this suboptimality is too large, a time-varying Kalman-filter can be used instead (Ljung, 1999). The uncertainty regarding past data is represented by an initial state-estimate $x_0(\vartheta)$ and its uncertainty $\Pi_0(\vartheta)$. The time-varying predictor is given by

$$\hat{x}_{t+1}(\vartheta) = A(\vartheta)\hat{x}_t(\vartheta) + B(\vartheta)u_t + K_t(\vartheta)[y_{O,t} - C(\vartheta)\hat{x}_t(\vartheta) - D(\vartheta)u_t] \quad (2.35a)$$

$$\hat{y}_{O,t}(\vartheta) = C(\vartheta)\hat{x}_t(\vartheta) + D(\vartheta)u_t \quad (2.35b)$$

$$K_t(\vartheta) = [A(\vartheta)P_t(\vartheta)C^T(\vartheta) + S(\vartheta)][C(\vartheta)P_t(\vartheta)C^T(\vartheta) + R(\vartheta)]^{-1} \quad (2.35c)$$

$$\begin{aligned} P_{t+1}(\vartheta) = & A(\vartheta)P_t(\vartheta)A^T(\vartheta) + Q(\vartheta) \\ & - K_t(\vartheta)[C(\vartheta)P_t(\vartheta)C^T(\vartheta) + R(\vartheta)]K_t^T(\vartheta) \end{aligned} \quad (2.35d)$$

where $\hat{x}_0(\vartheta) = x_0(\vartheta)$ and $P_0(\vartheta) = \Pi_0(\vartheta)$.

2.5 Parameter Estimation Methods

Once the model structure has been selected and the data have been collected, the model should be fitted to the data. Here we will describe two methods, the prediction-error method (PEM) and the instrumental variable (IV) method.

2.5.1 Prediction-Error Method

In the PEM the parameters are computed by minimizing the criterion function

$$V_N(\vartheta, Z_N) = \frac{1}{N} \sum_{t=1}^N \ell(\varepsilon_{O,t}(\vartheta)) \quad (2.36)$$

with respect to ϑ (Ljung, 1999). Here

$$\varepsilon_{O,t}(\vartheta) = y_{O,t} - \hat{y}_{O,t}(\vartheta) \quad (2.37)$$

is the prediction error. The scalar-valued function $\ell(\cdot)$ will here be assumed to be quadratic $\ell(x) = \frac{1}{2}x^T x$. The parameters are thus found by computing

$$\hat{\vartheta} = \underset{\vartheta}{\operatorname{argmin}} \frac{1}{N} \sum_{t=1}^N \frac{1}{2} (y_{O,t} - \hat{y}_{O,t}(\vartheta))^T (y_{O,t} - \hat{y}_{O,t}(\vartheta)) \quad (2.38)$$

Note that this might be a nonconvex optimization problem for a general nonlinear one-step-ahead predictor. However, for certain model structures, (2.38) will simplify. In case that the one-step-ahead predictor can be written as a linear regression, the estimate of the parameters ϑ is given by solving the LS problem

$$\hat{\vartheta}^{\text{LS}} = \underset{\vartheta}{\operatorname{argmin}} \|Y_N - \Phi_N \vartheta\|_2^2 \quad (2.39)$$

where $\|x\|_2^2 = x^T x$,

$$\Phi_N = [\varphi_1, \dots, \varphi_N]^T \quad \text{and} \quad Y_N = [y_{O,1}^T, \dots, y_{O,N}^T]^T \quad (2.40)$$

For results on consistency, convergence and a deeper treatment of the PEM and the properties of its estimates, see Ljung (1999, Ch. 7–9).

2.5.2 Correlation Method - Instrumental Variable Method

The IV method is an alternative to the PEM. As the model (2.1a) suggests, there are two terms contributing to $y_{O,t}$, one containing information about the interesting input-output relation and the second containing a contribution from the noise. An IV method is a correlation-based approach where the instrument vector ζ_t is used to extract the interesting information from the data. In principle, the interesting information, i.e. the parameters ϑ , is found by requiring that

$$f_N(\vartheta, Z_N) = \frac{1}{N} \sum_{t=1}^N \zeta_t^T \alpha [L(\mathbf{q})(y_{O,t} - \varphi_t^T \vartheta)] = 0 \quad (2.41)$$

where $\alpha(\cdot)$ is a transformation, typically given by $\alpha(\varepsilon_{O,t}) = \varepsilon_{O,t}$ and $L(\mathbf{q})$ is a prefilter. In other words, the idea is that the sample covariance between ζ_t and the (prefiltered and transformed) prediction error should be zero. A good in-

strument should be correlated with the output $y_{O,t}$ but be uncorrelated with the disturbance τ_t . A generalization is given by the extended IV method, where the parameters are found by

$$\hat{\vartheta}^{\text{IV}} = \underset{\vartheta}{\operatorname{argmin}} \|Y_N - \Phi_N \vartheta\|_Q^2 \quad (2.42)$$

where $\|x\|_Q^2 = x^T Q x$, $Q \geq 0$ is a weighting matrix,

$$\Phi_N = \frac{1}{N} [\zeta_1 \cdots \zeta_N] \begin{bmatrix} \bar{\varphi}_1^T \\ \vdots \\ \bar{\varphi}_N^T \end{bmatrix}, \quad Y_N = \frac{1}{N} [\zeta_1 \cdots \zeta_N] \begin{bmatrix} \bar{y}_{O,1} \\ \vdots \\ \bar{y}_{O,N} \end{bmatrix}, \quad (2.43)$$

$\bar{y}_{O,t} = L(\mathbf{q})y_{O,t}$, $\bar{\varphi}_t^T = L(\mathbf{q})\varphi_t^T$ and $L(\mathbf{q})$ is a stable prefilter (Söderström and Stoica, 1989). The IV estimator will typically be consistent if the matrix

$$\lim_{N \rightarrow \infty} \Phi_N \quad (2.44a)$$

has full rank and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \zeta_t L(\mathbf{q}) \tau_t = 0 \quad (2.44b)$$

with probability 1. See, for instance, Söderström and Stoica (1983) or Ljung (1999) for more details.

Any instruments that satisfy the conditions in (2.44) will provide consistency, but the variance properties of the parameter estimator are greatly affected by the choice of instruments and prefilter, for more details see, for instance, Söderström and Stoica (1989), Ljung (1999) or Forssell (1999).

Instrumental Variable Method for Closed-loop Identification

Figure 2.3 shows a system operating in closed loop and certain care has to be taken when the IV method is applied to such systems. The closed-loop IV method and its properties have been investigated in Forssell and Chou (1998), Gilson and Van den Hof (2005) and Gilson et al. (2011) among others. Further insights about the choice of prefilter, the optimal variance instruments and approximately optimal instruments for a closed-loop system are given in, for instance, Gilson et al. (2011). These results are all based on using a noise model together with the external reference signal δ instead of the input u to create the optimal instruments.

Assume that the model (2.12) is given on the form

$$y_{O,t} = G_O(\mathbf{q}, \vartheta_0)u_t + H_O(\mathbf{q}, \vartheta_0)\tau_t = \varphi_t^T \vartheta_0 + \tau_{O,t} \quad (2.45)$$

where $u_t = \delta_t + F_y(\mathbf{q})y_{O,t}$, that the regression vector is given by

$$\varphi_t = [-y_{O,t-1}, \dots, -y_{O,t-n_a}, u_{t-n_k}, \dots, u_{t-n_b-n_k+1}]^T$$

defined in (2.22) and that the disturbance is

$$\tau_{O,t} = A(\mathbf{q}, \vartheta_0)H_O(\mathbf{q}, \vartheta_0)\tau_t = A(\mathbf{q}, \vartheta_0)H(\mathbf{q}, \vartheta_0)e_t \quad (2.46)$$

where $A(\mathbf{q}, \vartheta_0)$ is the denominator of $G_O(\mathbf{q}, \vartheta_0)$ and it is assumed that e_t is white dis-

crete-time zero-mean noise. The optimal prefilter is given by (Gilson et al., 2011)

$$L(\mathbf{q}, \vartheta_0) = \frac{1}{A(\mathbf{q}, \vartheta_0)H(\mathbf{q}, \vartheta_0)} \quad (2.47)$$

and the optimal instruments are given by the filtered noise-free regression vector

$$\zeta_t = L(\mathbf{q}, \vartheta_0) [-\hat{y}_{O,t-1}, \dots, -\hat{y}_{O,t-n_a}, \hat{u}_{t-n_k}, \dots, \hat{u}_{t-n_b-n_k+1}]^T \quad (2.48)$$

where $\hat{z}$ is the noise-free part of the signal z . However, these prefilters and instrument vectors are not implementable in reality since they require knowledge of the true system and perfect measurements. Instead, a variant of the iterative approach presented in Gilson et al. (2011) can be used. The method, described in Algorithm 2.1, is used to estimate $L(\mathbf{q})$ and create the instruments ζ_t . In this method, the instrument vector is

$$\zeta_t = L(\mathbf{q}) [\hat{y}_{O,t-1}, \dots, \hat{y}_{O,t-\hat{n}_y}, \hat{u}_{t-n_k}, \dots, \hat{u}_{t-n_k-\hat{n}_u+1}]^T \quad (2.49)$$

where $L(\mathbf{q})$ is a prefilter,

$$\hat{y}_{O,t} = \frac{G_O(\mathbf{q}, \hat{\vartheta})}{1 - G_O(\mathbf{q}, \hat{\vartheta})F_y(\mathbf{q})} \delta_t \quad \text{and} \quad \hat{u}_t = \frac{1}{1 - G_O(\mathbf{q}, \hat{\vartheta})F_y(\mathbf{q})} \delta_t \quad (2.50)$$

are the simulated output and input, respectively, and $\hat{\vartheta}$ is an estimate of the parameters. The constants $\hat{n}_y$ and $\hat{n}_u$ are the number of time lags included in ζ_t for the output and the input, respectively, i.e. $\hat{n}_y = 0$ means that $\hat{y}_{O,t}$ is not included in ζ_t . The prefilter is given by

$$L(\mathbf{q}, \eta) = \frac{1}{\hat{H}(\mathbf{q}, \eta)} \quad (2.51)$$

where $\hat{H}(\mathbf{q}, \hat{\eta})$ is an ARMA model estimated from the residual $\varepsilon_{O,t} = y_{O,t} - \varphi_t^T \hat{\vartheta}$.

Algorithm 2.1 (The refined IV method (to decrease the variance of $\hat{\vartheta}$)).

(A) Initialize:

(a) Set $L^{(0)}(\mathbf{q}) = 1$

(b) Set $\zeta_t^{(0)} = [\delta_t, \dots, \delta_{t-\hat{n}_{r,0}+1}]^T$ where $\hat{n}_{r,0}$ is the number of time lags included

(c) Set $j = 1$

(B) Estimate parameters:

(a) Estimate $\hat{\vartheta}^{(j)}$ using (2.42) with $L^{(j-1)}(\mathbf{q})$ and $\zeta_t^{(j-1)}$

(b) Estimate the parameters $\hat{\eta}^{(j)}$ of the model $\hat{H}(\mathbf{q}, \eta)$ from $\varepsilon_{O,t} = \hat{H}(\mathbf{q}, \eta) \bar{e}_t$ where $\varepsilon_{O,t} = y_{O,t} - \varphi_t^T \hat{\vartheta}^{(j)}$

(c) Create the prefilter $L^{(j)}(\mathbf{q}) = \frac{1}{\hat{H}(\mathbf{q}, \hat{\eta}^{(j)})}$

(d) Simulate $\hat{y}_{O,t}^{(j)}$ and $\hat{u}_t^{(j)}$ according to (2.50) using $\hat{\vartheta}^{(j)}$ and create the instrument $\zeta_t^{(j)}$ according to (2.49)

(C) Terminate: Increase j and go to Step B while

$\|\hat{\vartheta}^{(j)} - \hat{\vartheta}^{(j-1)}\|_2 + \|\hat{\eta}^{(j)} - \hat{\eta}^{(j-1)}\|_2$ is above a threshold or as long as a maximum number of iterations is not reached, return $\hat{\vartheta} = \hat{\vartheta}^{(j)}$ and terminate otherwise.

2.6 Statistical Properties of Parameter Estimates

To understand what happens with the estimates when the amount of data tends to infinity, the asymptotic analysis in Ljung (1999) can be used. In the asymptotic analysis, the data is assumed to have certain statistical properties which are inherited by the parameter estimates. This section is only a short summary and technical details are not given, see Chapters 8 and 9 in Ljung (1999) for all details.

The notation ϑ^* will be used to represent the limit value, i.e. $\hat{\vartheta} \rightarrow \vartheta^*$ with probability 1 (w.p. 1) as the number of data points tends towards infinity. For the prediction-error approach, the limit value is given by

$$\vartheta^* \rightarrow \underset{\vartheta \in D_{\mathcal{M}}}{\operatorname{argmin}} \bar{E}[\ell(\varepsilon_{O,t}(\vartheta), \vartheta)], \quad \text{w.p. 1 as } N \rightarrow \infty \quad (2.52)$$

where $\bar{E}[\cdot]$ is the ensemble average. For the correlation approach, the limit value is

$$\vartheta^* \rightarrow \underset{\vartheta \in D_{\mathcal{M}}}{\operatorname{sol}} \bar{E}[\zeta_t(\vartheta)\alpha(\bar{\varepsilon}_{O,t}(\vartheta))] = 0, \quad \text{w.p. 1 as } N \rightarrow \infty \quad (2.53)$$

where $\zeta_t(\vartheta)$ is the correlation vector and $\operatorname{sol} f(x)=0$ is the solution(s) to $f(x)=0$.

The asymptotic estimates can thus be seen as the best approximation for the given statistical properties of the system. If the true system is part of the model set and under certain assumptions, such that the data is informative enough, the estimated model will converge to the true description of the system (Ljung, 1999, Ch. 8).

The covariance of the parameter estimator $\hat{\vartheta}$ is given by $\operatorname{Cov}(\hat{\vartheta}) = P_{\vartheta}/N$ where P_{ϑ} is the covariance of the asymptotic distribution. The asymptotic distribution for the prediction-error method with quadratic criterion is given by

$$P_{\vartheta} = \bar{V}''(\vartheta^*)^{-1} Q \bar{V}''(\vartheta^*)^{-1} \quad (2.54)$$

where

$$Q = \lim_{N \rightarrow \infty} N \cdot E[V_N'(\vartheta^*, Z_N) V_N'(\vartheta^*, Z_N)^T], \quad (2.55)$$

$$\bar{V}''(\vartheta) = \bar{E}[\Psi_t(\vartheta)\Psi_t^T(\vartheta)] - \bar{E}[\Psi_t'(\vartheta)\varepsilon_{O,t}(\vartheta)], \quad (2.56)$$

$\Psi_t(\vartheta^*) = -\frac{d}{d\vartheta} \varepsilon_{O,t}(\vartheta)|_{\vartheta=\vartheta^*} = \frac{d}{d\vartheta} y_{O,t}(\vartheta)|_{\vartheta=\vartheta^*}$ and $E[\cdot]$ is the expected value. Similarly, the asymptotic distribution for the correlation approach when $\alpha(\varepsilon_{O,t}) = \varepsilon_{O,t}$ is

$$P_{\vartheta} = \bar{f}'(\vartheta^*)^{-1} Q (\bar{f}'(\vartheta^*)^{-1})^T \quad (2.57)$$

where

$$\bar{f}'(\vartheta) = \bar{E}[\zeta_t'(\vartheta)\bar{\varepsilon}_{O,t}(\vartheta)] - \bar{E}[\zeta_t(\vartheta)(L(\mathbf{q})\Psi_t(\vartheta))^T] \quad (2.58)$$

and

$$Q = \lim_{N \rightarrow \infty} N \cdot E[f_N(\vartheta^*, Z_N) f_N(\vartheta^*, Z_N)^T] \quad (2.59)$$

To take both variance and bias into account, we have used

$$\operatorname{AMSE}(\hat{\vartheta}) = P_{\vartheta} + (\vartheta^* - \vartheta)(\vartheta^* - \vartheta)^T \quad (2.60)$$

which we call the asymptotic mean squared error.

2.7 Structural Identifiability of Linear Systems

Unique estimation of a model for a chosen model structure is both connected to identifiability of the selected model structure and the informativity of the data set used for identification (Bellman and Åström, 1970; Ljung and Glad, 1994; Bazanella et al., 2010). This section is focused on the structural identifiability of linear systems in state-space form. For a discussion of identifiability and informativity in dynamic networks, see Gevers and Bazanella (2015). The outcome of the analysis is a qualitative measure, i.e. if it is possible (simply yes or no) to uniquely determine the parameters of a model structure, without giving a measure of the quality of the estimates. In the general case, a linear model parameterized with parameter vector $\vartheta \in \mathbb{R}^{n_\vartheta}$ can be written on the state-space form

$$\begin{aligned}\dot{x} &= A(\vartheta)x + B(\vartheta)u \\ y_O &= C(\vartheta)x\end{aligned}\tag{2.61}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input, $y_O \in \mathbb{R}^{n_O}$ is the output and the matrices are of suitable sizes. Note that the parameterization might be nonlinear even though the system is linear in its state and input. Identifiability can be seen as observability of the extended nonlinear system

$$\begin{aligned}\begin{bmatrix} \dot{x} \\ \dot{\vartheta} \end{bmatrix} &= \begin{bmatrix} f(x, u, \vartheta) \\ 0 \end{bmatrix} = g^0(x, \vartheta) + g(x, \vartheta)u \\ y_O &= h(x, \vartheta)\end{aligned}\tag{2.62}$$

where $g^0(x, \vartheta)$, $g(x, \vartheta)$ and $h(x, \vartheta)$ are assumed to be rational in the extended states (Walter, 1982). To show observability for this nonlinear system (identifiability for the linear system), the system of equations

$$\begin{aligned}y_{O,1}^{(0)} &= h_1^{(0)}(x, \vartheta) \\ y_{O,1}^{(1)} &= h_1^{(1)}(x, \vartheta, u_1, \dots, u_{n_u}) \\ &\vdots \\ y_{O,1}^{(K)} &= h_1^{(K)}(x, \vartheta, u_1, \dots, u_1^{(K-1)}, \dots, u_{n_u}, \dots, u_{n_u}^{(K-1)}) \\ &\vdots \\ y_{O,n_O}^{(0)} &= h_{n_O}^{(0)}(x, \vartheta) \\ y_{O,n_O}^{(1)} &= h_{n_O}^{(1)}(x, \vartheta, u_1, \dots, u_{n_u}) \\ &\vdots \\ y_{O,n_O}^{(K)} &= h_{n_O}^{(K)}(x, \vartheta, u_1, \dots, u_1^{(K-1)}, \dots, u_{n_u}, \dots, u_{n_u}^{(K-1)})\end{aligned}\tag{2.63}$$

has to be solved for x and ϑ , where y , u and their time derivatives are assumed to be known (Diop and Fliess, 1991). Here, the notation $(\cdot)^{(k)}$ means the k^{th} derivative of $(\cdot)$ with respect to time. For a general nonlinear system, there is no upper limit on K and it might be infeasible to solve this system of equations or difficult to analyze the result. Instead of solving the system of equations, it can be linearized around the extended states and an analysis can be performed on the

Jacobian (Hermann and Krener, 1977)

$$\mathcal{O}(x, \vartheta, u_1, \dots, u_{n_u}^{(K-1)}) = \begin{bmatrix} \frac{\partial h_1^{(0)}(x, \vartheta)}{\partial(x, \vartheta)} \\ \frac{\partial h_1^{(1)}(x, \vartheta, u_1, \dots, u_{n_u})}{\partial(x, \vartheta)} \\ \vdots \\ \frac{\partial h_1^{(K)}(x, \vartheta, u_1, \dots, u_1^{(K-1)}, \dots, u_{n_u}, \dots, u_{n_u}^{(K-1)})}{\partial(x, \vartheta)} \\ \vdots \\ \frac{\partial h_{n_O}^{(0)}(x, \vartheta)}{\partial(x, \vartheta)} \\ \frac{\partial h_{n_O}^{(1)}(x, \vartheta, u_1, \dots, u_{n_u})}{\partial(x, \vartheta)} \\ \vdots \\ \frac{\partial h_{n_O}^{(K)}(x, \vartheta, u_1, \dots, u_1^{(K-1)}, \dots, u_{n_u}, \dots, u_{n_u}^{(K-1)})}{\partial(x, \vartheta)} \end{bmatrix} \quad (2.64)$$

which hereafter will be called the extended observability matrix (for simplicity denoted $\mathcal{O}$). The system is locally weakly identifiable if $\mathcal{O}$ satisfies an observability rank condition (Hermann and Krener, 1977).

In Sections 4 and 5 of Anguelova (2007), it is shown that under the assumption that the system equations, i.e. $h(x, \vartheta)$, $g^0(x, \vartheta)$ and $g(x, \vartheta)$ in (2.62), are rational functions in the states and parameters, it is enough to include the first $K = n + n_\vartheta - 1$ derivatives of $h_j(x, \vartheta)$, $j = 1, \dots, n_O$, with respect to time. It is also shown that if $\text{rank}(\mathcal{O}) = n_q < n + n_\vartheta$, it is enough to form $\mathcal{O}$ using the first $n_q - 1$ derivatives with respect to time. This is useful if the dimension of the extended system is large since the complexity of the rows of $\mathcal{O}$ increases for high values of K . If the rank is less than $n + n_\vartheta$, one can exclude columns from the observability matrix one-by-one to identify the unobservable parameters or states since excluding a parameter in the extended system is equivalent to dropping the corresponding column in $\mathcal{O}$. See Anguelova (2007) for a thorough discussion about identifiability of systems that are rational in the extended state vector.

Part I

Identification of Systems with Unknown Inputs

3

Systems with Unknown Inputs

In this chapter, a model for systems that are excited by unknown signals will be presented with the goal of unifying system identification for these types of problems. The model and the discussion surrounding it are the foundation of the framework that will be presented in Chapter 4. A brief overview of the approaches for unknown input problems in the field of system identification will be given. Then, the possible effects of using these approaches are shown by applying them to the proposed unknown input model.

3.1 Coping with Unknown Signals in System Identification

In this thesis, the approaches to handle system identification problems with unknown inputs are divided into five classes.

The first class of remedies is when assumptions are made on the signal acting on the system. One example is time-series analysis where the input is assumed to be a stochastic process, for example, filtered white noise, and the goal is to estimate a model that can predict future outputs based on current and previous outputs only (Shumway and Stoffer, 2010), see Figure 3.1(a). Another example of problems with assumptions on the input is the related problem of blind identification. In blind identification, the input is not necessarily assumed to be stochastic and it is estimated in conjunction with a model of the system (Abed-Meraim et al., 1997). We will not consider this class of approaches in this thesis.

The second class, simply neglecting an unknown input u_U , is not really a method but is commonly used in modelling of systems. This is often done due to the properties of the signal. For instance, the signal might not contribute significantly to the

Figure 3.1: Different approaches to estimation of systems with unknown inputs.

output or the signal might be independent and is seen as a disturbance, and there are thus implicit assumptions made on the signal being neglected.

The third class of remedies is to approximate the unknown signal u_U by using other signals or approximate the contribution of u_U by including other signals in the predictor, see Figure 3.1(b).

Another class of problems is where the input is unknown but is assumed to be measured with noise. A typical example is the errors-in-variables (EIV) problem where the emphasis lies in compensating for the effects of noise on the input measurement y_U and output measurement y_O (Söderström, 2007), see Figure 3.1(c). Note that this is a fundamentally different problem to the closed-loop problem discussed in Section 2.2 where the input disturbance actually enters the system through the input and excites the system. However, closed-loop EIV problems have also been considered (Söderström et al., 2013) and in the general dynamic network identification setup, all signals are typically assumed to be measured with noise (Dankers, 2014).

The fifth and final class is sensor-to-sensor system identification problems (S2SID). In the S2SID framework, several measurements are assumed to be dependent on the unknown input, see Figure 3.1(d). Some measurements y_O are used as output while other measurements y_S are used as input to what is called a pseudo transfer function. Unfortunately, dynamics that is shared in the transfer functions from the unknown input to the measurements will cancel, but more importantly, so does the unknown input (D’Amato et al., 2009; Aljanaideh et al., 2012).

3.2 Modelling Systems with Unknown Inputs

Recall Figure 2.1 on page 10 which shows the system S , here assumed to be linear, with the measurements $y \in \mathbb{R}^{n_y}$ that are influenced by two types of external signals, user-controllable signals $\delta \in \mathbb{R}^{n_\delta}$ and uncontrollable signals $\tau \in \mathbb{R}^{n_\tau}$ called disturbances. The system contains some unknown dynamics or properties that are desired to be estimated from data. Estimating a complete model of the system S is assumed to be intractable, for instance, due to the complexity of the system or due to essential measurements not being available. Furthermore, estimating a model from δ to y_O will not explain the excitation induced by the disturbances τ and

the model will typically be of high complexity since a large part of the system S needs to be modeled. Instead, it is desired to estimate only a subsystem S_O of S with the input u and the output y_O , which contains the interesting dynamics or properties, by using the model

$$y_O = G_O(\mathbf{p})u + \underbrace{H_O(\mathbf{p})\tau}_{\tau_O} \quad (3.1)$$

where the output $y_O \in \mathbb{R}^{n_O}$ contains elements of the available measurements y , $u \in \mathbb{R}^{n_u}$ is the input, $\mathbf{p}$ is the differential operator and $G_O(\mathbf{p})$ and $H_O(\mathbf{p})$ are proper transfer-function matrices of suitable size, see Figure 2.2.

The input u is assumed to be related to the external signals δ and τ through

$$u = F_\delta(\mathbf{p})\delta + F_\tau(\mathbf{p})\tau \quad (3.2)$$

where $F_\delta(\mathbf{p})$ and $F_\tau(\mathbf{p})$ are possibly unknown transfer-function matrices of suitable dimensions, see Figure 3.2(a).

The input u is assumed to capture all contributions from the external signal δ that exists in the output y_O , i.e. there are no paths from the signal δ to the output y_O not passing through the input u . This implies that the input u will describe *at least* as much of the output y_O as the external user-controllable signal δ since the input u also can contain excitation from the disturbance τ . Furthermore, the input u is assumed to be *partially unknown* and is divided into

$$u = \begin{bmatrix} u_K \\ u_U \end{bmatrix} \begin{matrix} \text{Known} \\ \text{Unknown} \end{matrix} \quad (3.3)$$

where $u_K \in \mathbb{R}^{n_K}$ and $u_U \in \mathbb{R}^{n_U}$ are known and unknown inputs, respectively. The ordering is not important as long as all vectors and matrices are ordered consistently.

To be consistent with the input partitioning, $G_O(\mathbf{p})$ is partitioned into

$$G_O(\mathbf{p}) = \begin{bmatrix} G_{OK}(\mathbf{p}) & G_{OU}(\mathbf{p}) \end{bmatrix} \quad (3.4)$$

(a) Equations (3.1) and (3.2) represented as a block diagram showing the dependencies on the external signals δ and τ .

(b) Equations (3.2) and (3.5) represented as a block diagram showing the dependencies on the external signals δ and τ .

Figure 3.2: The model (3.1) of the subsystem S_O with the input u and the output y_O and its dependencies on signals viewed as a block diagram. It is assumed that estimating a model of the system S is intractable. Instead, a subsystem S_O , which contains the interesting dynamics or properties, with the output y_O and the input u is estimated. The input u is divided according to (3.3) which results in (3.5).

such that (3.1) can be written

$$y_O = G_{OK}(\mathbf{p})u_K + G_{OU}(\mathbf{p})u_U + \underbrace{H_O(\mathbf{p})\tau}_{\tau_O} \quad (3.5)$$

which can be seen in Figure 3.2(b). There are no assumptions made on the partitioning of $G_O(\mathbf{p})$, for instance, the individual transfer functions could share dynamics. This division is due to convenience and is only a means to an end. This could be considered in the modelling process, for instance, by choosing common parameters, or it could be neglected which would result in a model of higher order.

Remark 3.1 (Continuous- or discrete-time modelling). In Section 2.2, the transfer-function model (2.1) was given in discrete time. Here, the models will instead be given in continuous time. The choice between continuous- and discrete-time models is dependent on what best suits the application and the chosen estimation method. Using a continuous-time model might be beneficial, for instance, if the system is stiff or the data has varying sample time, see for instance Garnier and Young (2014) for details.

A model of the entire system S will also be considered in order to understand the behavior of the estimated model when certain signals are unknown. Here, the dynamic network model introduced in Section 2.2 with the j^{th} node as the desired output y_O will be used for this purpose. In this model, the unknown inputs will be represented by the unobservable nodes introduced in Definition 3.1 below.

Definition 3.1 (Observable and unobservable nodes).

Given a sensor setup, the sets of indices of all *observable* (known) and *unobservable* (unknown) nodes are denoted $\mathcal{S}_j$ and $\mathcal{U}_j = \mathcal{N} \setminus \mathcal{S}_j$, respectively, where the j^{th} node is assumed to be observable. Furthermore, the observable nodes are divided into the set of observable nodes that have a direct path to w_j , which is defined as $\mathcal{K}_j = \mathcal{N}_j \cap \mathcal{S}_j$, and the set $\mathcal{A}_j = \mathcal{S}_j \setminus (\mathcal{K}_j \cup \{j\})$, i.e. the set of indices of all additional nodes that are observable except j .

With this set notation, the dynamic network model (2.3) can be written

$$\begin{bmatrix} w_j \\ w_K \\ w_A \\ w_U \end{bmatrix} = \begin{bmatrix} 0 & G_{jK} & 0 & G_{jU} \\ G_{Kj} & G_{KK} & G_{KA} & G_{KU} \\ G_{Aj} & G_{AK} & G_{AA} & G_{AU} \\ G_{Uj} & G_{UK} & G_{UA} & G_{UU} \end{bmatrix} \begin{bmatrix} w_j \\ w_K \\ w_A \\ w_U \end{bmatrix} + \begin{bmatrix} \delta_j + \tau_j \\ \delta_K + \tau_K \\ \delta_A + \tau_A \\ \delta_U + \tau_U \end{bmatrix} \quad (3.6)$$

where G_{KK} , G_{AA} and G_{UU} are zero on the diagonals. The network model (3.6) is a special case of (3.1) and (3.2) since only one output is considered. The model (3.1) is given by the first row of (3.6) if the known input u_K is given by $[\delta_j, w_K^T]^T$ and the elements of the unknown input u_U are given by $w_k, k \in \mathcal{U}_j$ such that $G_{jk} \neq 0$. The benefit of this model is that the correlation structure is more clearly defined.

The necessity or benefit of including an input (or compensating for an unknown input) depends on the significance of the contribution to the output and the correlation to the other input signals. In the following sections, the simple system with one known and one unknown input shown in Figure 3.3 and the dynamic network shown in Figure 3.4 will be used to illustrate some important aspects in a number

of examples. To get familiar with the systems, set notation and to get a baseline, the first two examples will be used to see what is possible if all inputs are known.

In the first example, the static system of Figure 3.3, the goal is to estimate b_K and we will look at the consistency and variance properties of the estimator.

Example 3.1: Static system with an unknown input

Consider the static system shown in Figure 3.3 where the goal is to estimate the parameter b_K accurately. The system is described by

$$y_O = b_K u_K + b_U u_U + \tau_O \quad (3.7)$$

where b_K and b_U are constant parameters and $\tau_O \sim \mathcal{N}(0, \lambda_O)$. Here, $\mathcal{N}(0, \lambda)$ describes Gaussian noise with zero mean and variance λ . The input is given by

$$\begin{bmatrix} u_K \\ u_U \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} \beta \\ 1 \end{bmatrix} \tau_u \quad (3.8)$$

where α and β are constants controlling the correlation of the inputs, $\delta_1 \sim \mathcal{N}(0, \lambda_1)$, $\delta_2 \sim \mathcal{N}(0, \lambda_2)$ and $\tau_u \sim \mathcal{N}(0, \lambda_u)$. The input u_U is unknown, but to get a baseline, let us assume that both inputs are known and consider the predictor

$$\hat{y}_O(\vartheta) = \hat{b}_K u_K + \hat{b}_U u_U \quad (3.9)$$

where the parameters are found using the standard non-weighted least-squares method described in Section 2.5.1. Using the asymptotic results of Section 2.6, this results in the estimates $b_K^* = b_K$ and $b_U^* = b_U$ with the asymptotic covariance

$$P_{\vartheta}^{3.1} = \begin{bmatrix} P_{b_K}^{3.1} & \times \\ \times & \times \end{bmatrix} = \frac{\lambda_O}{\lambda_1 \lambda_2 + (\alpha\beta - 1)^2 \lambda_1 \lambda_u + \beta^2 \lambda_2 \lambda_u} \begin{bmatrix} \alpha^2 \lambda_1 + \lambda_2 + \lambda_u & -\alpha \lambda_1 - \beta \lambda_u \\ -\alpha \lambda_1 - \beta \lambda_u & \lambda_1 + \beta^2 \lambda_u \end{bmatrix} \quad (3.10)$$

Hence, all external signals will contribute to $P_{b_K}^{3.1}$ in the general case. If the input u_K is uncorrelated with τ_u , the asymptotic covariance becomes

$$P_{\vartheta}^{3.1} \big|_{\beta=0} = \lambda_O \begin{bmatrix} \frac{\alpha^2}{\lambda_2 + \lambda_u} + \frac{1}{\lambda_1} & -\frac{\alpha}{\lambda_2 + \lambda_u} \\ -\frac{\alpha}{\lambda_2 + \lambda_u} & \frac{1}{\lambda_2 + \lambda_u} \end{bmatrix} \quad (3.11)$$

Figure 3.3: A static system that is dependent on one known and one unknown input. The inputs are dependent on the external signals δ_1 , δ_2 and τ_u where α and β control the correlation and the output y_O is measured with noise.

In the second example shown in Figure 3.4, the goal is to understand the properties of the dynamics related to the second node. The main focus will be changes in dynamics among nodes and induced correlation from external signals.

Example 3.2: Dynamic network with unobservable nodes

Consider the dynamic network shown in Figure 3.4 which is described by

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & 0 & 0 & 0 & 0 & G_{17} & 0 \\ G_{21} & 0 & G_{23} & 0 & G_{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{36} & 0 & 0 \\ G_{41} & 0 & 0 & 0 & G_{45} & 0 & 0 & 0 \\ 0 & G_{52} & 0 & 0 & 0 & 0 & 0 & G_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{76} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{86} & G_{87} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \delta_6 + \tau_6 \\ \tau_7 \\ \tau_8 \end{bmatrix} \quad (3.12)$$

where the goal is to estimate the dynamics with $y_O = w_2$ as output, i.e. $j = 2$ and $\mathcal{N}_2 = \{1, 3, 5\}$, see (2.2). Hence, if $w_k, k \in \{j\} \cup \mathcal{N}_2$ were available, the model

$$y_O = G_O(\mathbf{q}, \vartheta)u + \tau_2 \quad (3.13)$$

where $G_O(\mathbf{q}, \vartheta) = [G_{21}(\mathbf{q}, \vartheta) \ G_{23}(\mathbf{q}, \vartheta) \ G_{25}(\mathbf{q}, \vartheta)]$ and $u^T = [w_1 \ w_3 \ w_5]$, could be used to get consistent estimates of the desired transfer functions (Van den Hof et al., 2013). However, there is only a limited set of sensors in the network and in accordance with Definition 3.1, the set of observable nodes is given by $\mathcal{S}_2 = \{1, 2, 3, 4, 7\}$ which means that $\mathcal{U}_2 = \{5, 6, 8\}$ is unobservable. Hence, the set of observable nodes with direct path to w_2 is $\mathcal{K}_2 = \{1, 3\}$ which gives $\mathcal{A}_2 = \{4, 7\}$. The unknown input model is thus given by

$$y_O = G_{OK}(\mathbf{q}, \vartheta)u_K + G_{OU}(\mathbf{q}, \vartheta)u_U + \tau_2 \quad (3.14)$$

where $u_K^T = [w_1 \ w_3]$, $u_U = w_5$ and $G_O(\mathbf{q}, \vartheta)$ is partitioned accordingly.

Figure 3.4: A dynamic network where the circles, the boxes and the rounded boxes correspond to the nodes, the dynamic subsystems and the measurements, respectively. The desired dynamics is marked with orange.

3.3 Neglecting Inputs

The simplest approach to the unknown input u_U in (3.5) is simply to neglect it, i.e.

$$y_O = G_{OK}u_K + G_{OU}u_U + \tau_O = \check{G}_{OK}u_K + \check{\tau}_O \quad (3.15)$$

There are multiple $\check{G}_{OK}$ and $\check{\tau}_O$ that satisfy this equation. The *effective disturbance* $\check{\tau}_O$ could, for instance, be given by $G_{OU}u_U + \tau_O$ but other choices of $\check{G}_{OK}$ could lead to a smaller effective disturbance. This choice will potentially have two effects. Firstly, there could be changes to the dynamics between u_K and y_O . Secondly, neglecting inputs will have an impact on the effective disturbance $\check{\tau}_O$ acting on the output since less of the output potentially can be described by the remaining inputs. Due to correlation that might exist between the effective disturbance $\check{\tau}_O$ and the remaining inputs u_K , neglecting u_U might have unintended consequences. This is caused by something called *confounding variables*, which are external signals that affect both the input and output, but not through the direct causal relation (system) between the input and the output (Dankers et al., 2016).

Example 3.3: Consequences of neglecting input

Let us again consider the problem of Example 3.1 but now assume that u_U is unknown and determine what happens if it is neglected. Since the goal is to estimate b_K with good accuracy, let us instead try the simple predictor model

$$\hat{y}_O(\vartheta) = \hat{b}_K u_K \quad (3.16)$$

which is identifiable with respect to the desired parameter and where the parameter is found by minimizing the least-squares criterion described in Section 2.5.1. Using the asymptotic results of Section 2.6, the resulting estimate is given by

$$b_K^* = b_K + \frac{\alpha\lambda_1 + \beta\lambda_u}{\lambda_1 + \beta^2\lambda_u} b_U \quad (3.17)$$

which is biased when α or β and b_U are nonzero. Here, both δ_1 and τ_u act as confounding variables, see Figure 3.3. The asymptotic variance is given by

$$P_{b_K}^{3.3} = \frac{b_U^2\lambda_2 + \lambda_O}{\lambda_1 + \beta^2\lambda_u} + \frac{(\alpha\beta - 1)^2 b_U^2 \lambda_1 \lambda_u}{(\lambda_1 + \beta^2\lambda_u)^2} \quad (3.18)$$

Note that in contrast to Example 3.1, increasing λ_2 will *decrease* the accuracy of $\hat{b}_K$ regardless of the values of α and β . This is not surprising since the predictor cannot describe the contribution from δ_2 . Furthermore, note that there is a cost to include an input if it is correlated, for instance, for $\beta = 0$ and $\lambda_1(\lambda_2 + \lambda_u) > 0$, $P_{b_K}^{3.3} > P_{b_K}^{3.1} \Rightarrow b_U^2(\lambda_2 + \lambda_u)^2 > \alpha^2\lambda_O\lambda_1$ which means that if the contribution from u_U is small enough, then it is better to neglect the input.

The least-squares predictor is not the sole cause of the bias. If instead the instrumental variable method described in Section 2.5.2 is used with the instrument given by $\zeta = \delta_1$ and $L(\mathbf{q}) = 1$, then the limit value becomes

$$b_{K,IV}^* = b_K + \alpha b_U \quad (3.19)$$

which is biased unless $\alpha = 0$ and with the corresponding asymptotic variance

$$P_{b_{K,IV}}^{3.3} = \frac{\lambda_O + b_U^2(\lambda_2 + (\alpha\beta - 1)^2\lambda_u)}{\lambda_1} \quad (3.20)$$

In the network setting, the idea of predictor input selection was introduced in Dankers et al. (2013a) and Dankers et al. (2013b). This idea can be used to investigate what will happen to a model of a system when inputs are neglected, for example, the model that best describes the contribution of the remaining signals in the output. The changes to the dynamics among the remaining nodes $w_k, k \in \{j\} \cup \mathcal{K}_j$ when the variables $w_k, k \in \mathcal{Z}_j = \mathcal{U}_j \cup \mathcal{A}_j$, are neglected can be evaluated by looking at the *immersed network* for $\{j\} \cup \mathcal{K}_j$. This is the equivalent network, from all external signals δ and τ to the remaining nodes $w_k, k \in \{j\} \cup \mathcal{K}_j$ when $w_{\mathcal{Z}}$ are eliminated (Dankers, 2014; Dankers et al., 2016). The immersed network for $\{j\} \cup \mathcal{K}_j$ of (3.6) can be formed by solving row three and four of (3.6) for $w_{\mathcal{Z}}$ and inserting into the other rows, which after normalization gives

$$\begin{bmatrix} w_j \\ w_{\mathcal{K}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \check{G}_{j\mathcal{K}} \\ \check{G}_{\mathcal{K}j} & \check{G}_{\mathcal{K}\mathcal{K}} \end{bmatrix}}_{\check{G}} \begin{bmatrix} w_j \\ w_{\mathcal{K}} \end{bmatrix} + \underbrace{\begin{bmatrix} \check{F}_{jj} & 0 & \check{F}_{j\mathcal{A}} & \check{F}_{j\mathcal{U}} \\ 0 & \check{F}_{\mathcal{K}\mathcal{K}} & \check{F}_{\mathcal{K}\mathcal{A}} & \check{F}_{\mathcal{K}\mathcal{U}} \end{bmatrix}}_{\check{F}} \begin{bmatrix} \delta_j + \tau_j \\ \delta_{\mathcal{K}} + \tau_{\mathcal{K}} \\ \delta_{\mathcal{A}} + \tau_{\mathcal{A}} \\ \delta_{\mathcal{U}} + \tau_{\mathcal{U}} \end{bmatrix} \quad (3.21)$$

where $\check{G}_{\mathcal{K}\mathcal{K}}$ is zero on the diagonal. Note that the external signals $\delta_n, n \in \mathcal{P}_j$, where $\mathcal{P}_j$ is the set of indices such that $\check{F}_{jn}\delta_n \neq 0$ and $\tau_n, n \in \mathcal{R}_j$, where $\mathcal{R}_j$ is the set of indices such that $\check{F}_{jn}\tau_n \neq 0$, are needed in addition to δ_j and the nodes $w_k, k \in \mathcal{K}_j$ to fully describe w_j after $w_{\mathcal{Z}}$ has been eliminated. The transfer-function matrix $\check{F}$ will typically depend on the dynamics of the eliminated variables. Furthermore, a structural change might have occurred, which means that the dynamics $\check{G}_{j\mathcal{K}}$ is not necessarily equal to $G_{j\mathcal{K}}$ if crucial nodes are unobserved and thus eliminated. In other words, the dynamics describing the (noise-free) relationship between w_j and $w_{\mathcal{K}}$ might have changed. Conditions for guaranteeing equality of $\check{G}_{ji}$ and G_{ji} were given in Dankers et al. (2016) and are restated below.

Proposition 3.1 (Proposition 4 of Dankers et al. (2016)).

The transfer function $\check{G}_{ji}$ in the immersed network is equal to G_{ji} if $\mathcal{K}_j$ satisfies the following conditions:

- (a) $i \in \mathcal{K}_j, j \notin \mathcal{K}_j$
- (b) every path w_i to w_j , excluding the path G_{ji} , goes through a node $w_k, k \in \mathcal{K}_j$
- (c) every loop w_j to w_j goes through a node $w_k, k \in \mathcal{K}_j$. $\square$

Note that Proposition 3.1 is fulfilled if all predictor inputs to w_j are observable.

In addition to the structural changes, there might also be issues due to the disturbance properties of the immersed network. Assume that we use the model

$$y_O = \check{G}_{OK}u_K + \check{\tau}_O = \check{G}_{j\mathcal{K}}w_{\mathcal{K}} + \check{F}_{jj}\delta_j + \check{\tau}_O \quad (3.22)$$

i.e. only $w_k, k \in \mathcal{K}_j$ and δ_j are used as inputs, then the effective noise is given by

$$\check{\tau}_j = \check{F}_{jj}\tau_j + \check{F}_{j\mathcal{A}}(\delta_{\mathcal{A}} + \tau_{\mathcal{A}}) + \check{F}_{j\mathcal{U}}(\delta_{\mathcal{U}} + \tau_{\mathcal{U}}) \quad (3.23)$$

This is potentially an issue if there are external variables $\tau_n, n \in \mathcal{Z}_j$ or $\delta_n, n \in \mathcal{Z}_j$ that have paths to both w_j and $w_{\mathcal{K}}$, i.e. both the output y_O and the inputs u_K , and thus act as confounding variables.

Example 3.4: Using only known predictor inputs

The immersed network for $\{2\} \cup \mathcal{K}_2$ of the network described in Example 3.2 is

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & 0 \\ \frac{G_{21}}{1-G_{25}G_{52}} & 0 & \frac{G_{23}}{1-G_{25}G_{52}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} \tau_1 + G_{17}(G_{76}(\delta_6 + \tau_6) + \tau_7) \\ \frac{\tau_2 + G_{25}(\tau_5 + G_{58}((G_{86} + G_{87}G_{76})(\delta_6 + \tau_6) + G_{87}\tau_7 + \tau_8))}{1-G_{25}G_{52}} \\ \tau_3 + G_{36}(\delta_6 + \tau_6) \end{bmatrix} \quad (3.24)$$

which means that $\mathcal{P}_2 = \{6\}$ and $\mathcal{R}_2 = \{2, 5, 6, 7, 8\}$. The transfer functions $\check{G}_{21} \neq G_{21}$ and $\check{G}_{23} \neq G_{23}$ due to the feedback between w_5 and w_2 which violates condition (c) of Proposition 3.1. In addition, a significant part of the eliminated dynamics related to the unobserved nodes is contained in $\check{F}$. Note that if $G_{52} = 0$, then Proposition 3.1 is fulfilled. If the predictor model

$$\hat{w}_2 = \check{G}_{21}w_1 + \check{G}_{23}w_3 \quad (3.25)$$

is used, then the external variable τ_6 will act as a confounding variable since $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 = \{6\}$. This can be mitigated by using δ_6 as an instrumental variable. However, δ_6 is not included as an input to the predictor model and will also be a confounding variable. A natural solution is to include δ_6 as an input, something that will be investigated in the next section.

In conclusions, when inputs are neglected, less of the output can typically be described which will affect the variance properties.

Bias in the model might occur due to two reasons. Firstly, the model structure might change, i.e. the noise-free relationship between the output and input has changed, since the remaining variables have to explain more of the dynamics. Secondly, correlation might be induced through confounding variables which might cause bias regardless if the model structure is unchanged.

3.4 Approximating Inputs Using Other Known Signals

Due to properties of the system, neglecting inputs can lead to unwanted effects as discussed in Section 3.3. In this section, it will be shown that it is possible to mitigate these effects by utilizing other available signals. Two different approaches will be discussed, firstly, substituting the unknown input u_U with an approximation $\hat{u}_U$, and secondly, approximating the contribution from the unknown input $G_{OU}u_U$ in the output y_O using other available signals u_A .

The first approach is to replace u_U in (3.5) with $\hat{u}_U$ and this can be described by

$$y_O = G_{OK}u_K + G_{OU}u_U + \tau_O = \dot{G}_{OK}u_K + \dot{G}_{OU}\hat{u}_U + \dot{\tau}_O \quad (3.26)$$

where $\hat{u}_U$ is an estimate of the unknown input u_U that is created using a model. This method will typically affect the effective disturbance $\dot{\tau}_O$ since the disturbances acting on the input normally are unknown and not included in the approximation. Although some issues are avoided by this approach, confounding variables might still appear and there might still be structural changes in the resulting model.

Example 3.5: Using an estimate of the unknown input

Let us again consider the problem of Example 3.1 but now assume that u_U is unknown and $\beta = 0$. Assume that an estimate of the unknown input is given by

$$\hat{u}_U = \hat{\alpha}_1\delta_1 + \hat{\alpha}_2\delta_2 \quad (3.27)$$

which gives $u_U - \hat{u}_U = (\alpha - \hat{\alpha}_1)\delta_1 + (1 - \hat{\alpha}_2)\delta_2 + \tau_u$. Consider the predictor

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_U \hat{u}_U \quad (3.28)$$

The parameters b_K and b_U are found by minimizing the least-squares criterion described in Section 2.5.1 and the results of Section 2.6 give the limit value

$$\begin{bmatrix} b_K^* \\ b_U^* \end{bmatrix} = \begin{bmatrix} b_K \\ b_U \end{bmatrix} + \begin{bmatrix} \alpha - \frac{\hat{\alpha}_1}{\hat{\alpha}_2} \\ \frac{1}{\hat{\alpha}_2} - 1 \end{bmatrix} b_U \quad (3.29)$$

which is biased unless $\hat{\alpha}_1 = \alpha$ and $\hat{\alpha}_2 = 1$. The asymptotic covariance is given by

$$P_{\vartheta}^{3.5} = \begin{bmatrix} P_{b_K}^{3.5} & \times \\ \times & \times \end{bmatrix} = (\lambda_O + b_U^2 \lambda_u) \begin{bmatrix} \frac{\hat{\alpha}_1^2}{\hat{\alpha}_2^2 \lambda_2} + \frac{1}{\lambda_1} & -\frac{\hat{\alpha}_1}{\hat{\alpha}_2^2 \lambda_2} \\ -\frac{\hat{\alpha}_1}{\hat{\alpha}_2^2 \lambda_2} & \frac{1}{\hat{\alpha}_2^2 \lambda_2} \end{bmatrix} \quad (3.30)$$

and thus will be larger than $P_{\vartheta}^{3.1}$ unless $\lambda_u = 0$. This is an expected result since $\hat{u}_U$ captures less of the exciting signal than u_U , i.e. $\varepsilon_O = y_O - \hat{y}_O = b_U \tau_u + \tau_O$.

The results of Example 3.5 are based on the fact that $\beta = 0$. If $\beta \neq 0$, $P_{b_K}^{3.5}$ might actually be smaller than $P_{b_K}^{3.1}$. However, Example 3.5 shows the basic issues with using an estimated signal as input to the estimator. Firstly, there will typically be a bias unless the model used to simulate $\hat{u}_U$ is correct. Estimating the model might be difficult unless a measurement of the unknown input u_U is available, this will be discussed in Section 3.5. Secondly, less of the input can be described compared to when the input is known since the disturbance is not included in $\hat{u}_U$ and only a simulation of u_U can be performed. This could lead to changes in the disturbance properties.

The second approach is to replace $G_{OU}u_U$ of (3.5) with an approximation of the unknown input using other signals. This approach can be described by

$$y_O = G_{OK}u_K + G_{OU}u_U + \tau_O = \check{G}_{OK}u_K + \check{G}_{OA}u_A + \check{\tau}_O \quad (3.31)$$

where u_A are additional known internal and external signals. Similar to the approach of approximating the unknown input u_U , some issues are avoided by approximating the contribution from the unknown input. However, confounding variables might still appear and there might still be structural changes in the model.

Example 3.6: Including additional known signals

Consider the problem of Example 3.1 but with the predictor given by

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_2 \delta_2 \quad (3.32)$$

where the least-squares method described in Section 2.5.1 is used to estimate the parameters. Using the asymptotic results of Section 2.6, the limit values are

$$b_K^* = b_K + \frac{\alpha\lambda_1 + \beta\lambda_u}{\lambda_1 + \beta^2\lambda_u} b_U, \quad b_2^* = b_U, \quad (3.33)$$

and the asymptotic covariance is given by

$$P_{\vartheta,1}^{3.6} = \begin{bmatrix} P_{b_K,1}^{3.6} & \times \\ \times & \times \end{bmatrix} = \begin{bmatrix} \frac{\lambda_O}{\lambda_1 + \beta^2\lambda_u} + \frac{(\alpha\beta-1)^2 b_U^2 \lambda_1 \lambda_u}{(\lambda_1 + \beta^2\lambda_u)^2} & 0 \\ 0 & \frac{\lambda_O}{\lambda_2} + \frac{(\alpha\beta-1)^2 b_U^2 \lambda_1 \lambda_u}{\lambda_2(\lambda_1 + \beta^2\lambda_u)} \end{bmatrix} \quad (3.34)$$

which is the same limit value as when neglecting the input but with lower variance, compare with Example 3.3. Note that there is no cost of estimating b_2 since u_K and δ_2 are independent. It might be tempting to include additional signals, for instance, consider the predictor given by

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_1 \delta_1 + \hat{b}_2 \delta_2 \quad (3.35)$$

Again, the parameters are found using the least-squares method of Section 2.5.1. This results in the limit values given by

$$b_K^* = b_K + \frac{b_U}{\beta}, \quad b_1^* = \left(\alpha - \frac{1}{\beta} \right) b_U, \quad b_2^* = b_U, \quad (3.36)$$

and the asymptotic covariance is given by

$$P_{\vartheta,2}^{3.6} = \begin{bmatrix} P_{b_K,2}^{3.6} & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} = \lambda_O \begin{bmatrix} \frac{1}{\beta^2\lambda_u} & -\frac{1}{\beta^2\lambda_u} & 0 \\ -\frac{1}{\beta^2\lambda_u} & \frac{1}{\lambda_1} + \frac{1}{\beta^2\lambda_u} & 0 \\ 0 & 0 & \frac{1}{\lambda_2} \end{bmatrix} \quad (3.37)$$

Perhaps surprisingly, increasing λ_1 will *not* increase the accuracy of $\hat{b}_K$. It is totally dependent on the power of the unknown disturbance τ_u . If λ_u tends towards zero, there will be identifiability issues and $P_{b_K,2}^{3.6}$ will tend towards infinity.

This approach requires a model of a larger part of the system S since the model has to describe the dynamics from δ to y_O . It is not obvious in Example 3.6 due to the simplicity of the system, but this can make a big difference in the complexity of the model if S_O only is a small part of the system S . Furthermore, since a model of S_O is estimated in conjunction with F it might be difficult to separate between them.

The approach of including additional known signals as inputs to the predictor can be done systematically in the network setting as it was hinted in Example 3.4. A

first step could be to use all the known external signals $\delta_k, k \in \mathcal{P}_j$ in addition to w_κ as inputs to the predictor. A further improvement is to instead eliminate only the unobservable nodes $w_\mathcal{U}$ and thus utilize all observable nodes, i.e. forming the immersed network for $\mathcal{S}_j = \{j\} \cup \mathcal{K}_j \cup \mathcal{A}_j$. The immersed network for $\mathcal{S}_j$ is given by

$$\begin{bmatrix} w_j \\ w_\kappa \\ w_\mathcal{A} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \check{G}_{j\kappa} & \check{G}_{j\mathcal{A}} \\ \check{G}_{\kappa j} & \check{G}_{\kappa\kappa} & \check{G}_{\kappa\mathcal{A}} \\ \check{G}_{\mathcal{A}j} & \check{G}_{\mathcal{A}\kappa} & \check{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix}}_{\check{G}} \begin{bmatrix} w_j \\ w_\kappa \\ w_\mathcal{A} \end{bmatrix} + \underbrace{\begin{bmatrix} \check{F}_{jj} & 0 & 0 & \check{F}_{j\mathcal{U}} \\ 0 & \check{F}_{\kappa\kappa} & 0 & \check{F}_{\kappa\mathcal{U}} \\ 0 & 0 & \check{F}_{\mathcal{A}\mathcal{A}} & \check{F}_{\mathcal{A}\mathcal{U}} \end{bmatrix}}_{\check{F}} \begin{bmatrix} \delta_j + \tau_j \\ \delta_\kappa + \tau_\kappa \\ \delta_\mathcal{A} + \tau_\mathcal{A} \\ \delta_\mathcal{U} + \tau_\mathcal{U} \end{bmatrix} \quad (3.38)$$

where $\check{G}_{\kappa\kappa}$ and $\check{G}_{\mathcal{A}\mathcal{A}}$ are zero on the diagonal. Again, $\check{G}_{j\kappa}$ is not necessarily equal to $G_{j\kappa}$ if crucial nodes are unobserved and the conditions of Proposition 3.1 can still be used to check if $\check{G}_{ji}$ and G_{ji} is equal. Note that compared with (3.21), i.e. the immersed network for $\{j\} \cup \mathcal{K}_j$, more of the output is potentially described by the known signals since $w_\mathcal{A}$ contains information about all external signals that have paths to $w_\mathcal{A}$, including the external disturbances $\tau_\mathcal{A}$. This can be understood by looking at the effective disturbance $\check{\tau}_O$. Assume that the model

$$y_O = \check{G}_{OK}u_K + \check{G}_{OA}u_\mathcal{A} + \check{\tau}_O = \check{G}_{j\kappa}w_\kappa + \check{G}_{j\mathcal{A}}w_\mathcal{A} + \check{F}_{jj}\delta_j + \check{F}_{j\mathcal{U}}\delta_\mathcal{U} + \check{\tau}_O \quad (3.39)$$

is used, then the effective noise is given by

$$\check{\tau}_O = \check{F}_{jj}\tau_j + \check{F}_{j\mathcal{U}}\tau_\mathcal{U} \quad (3.40)$$

This could still potentially be an issue if the external variables $\tau_\mathcal{U}$ have paths to the output y_O and the inputs u_K and $u_\mathcal{A}$, and thus act as confounding variables.

Example 3.7: Immersed network for all observable nodes

Consider the immersed network for $\mathcal{S}_2$ of the network in Example 3.2 given by

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_7 \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & 0 & 0 & G_{17} \\ \frac{G_{21}}{1-G_{25}G_{52}} & 0 & \frac{G_{23}}{1-G_{25}G_{52}} & 0 & \frac{G_{25}G_{58}G_{87}}{1-G_{25}G_{52}} \\ 0 & 0 & 0 & 0 & 0 \\ G_{41} & G_{45}G_{52} & 0 & 0 & G_{45}G_{58}G_{87} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_7 \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \frac{\tau_2 + G_{25}(\tau_5 + G_{58}(\tau_8 + G_{86}(\delta_6 + \tau_6)))}{1-G_{25}G_{52}} \\ \frac{\tau_3 + G_{36}(\delta_6 + \tau_6)}{\tau_4 + G_{45}(\tau_5 + G_{58}(\tau_8 + G_{86}(\delta_6 + \tau_6)))} \\ \tau_7 + G_{76}(\delta_6 + \tau_6) \end{bmatrix} \quad (3.41)$$

The observable node w_7 thus supplies additional information and a larger part of the excitation that enters in the network is described compared with (3.24). The external disturbance τ_6 still act as a confounding variable since it has paths to both w_2 and w_3 but it can be mitigated, for instance, by using an instrumental variable method with δ_6 as instrument.

To conclude, using additional signals, either to create an estimate of the unknown inputs or to use them directly as signals in the estimator, might thus be beneficial since potentially more of the output can be described. However, the issues seen when neglecting inputs still exist since the input disturbances are not explained by the approximation and might thus still be confounding variables.

3.5 Using Measured Inputs

In this section, the unknown inputs are assumed to be measured with noise. The input measurement of u_U in (3.5) is assumed to be given by

$$y_U = u_U + e_U \quad (3.42)$$

where e_U is measurement noise. Here, we will consider two approaches, using the measurement to create an approximation of the unknown input, and substituting the unknown input with the measurement y_U , i.e. errors-in-variables estimation.

The first approach, to use the measurement to estimate a model from δ and u_A to u_U in order to form an approximation $\hat{u}_U$ was technically already considered in Section 3.4 and can be represented by

$$y_O = G_{OK}u_K + G_{OU}u_U + \tau_O = \dot{G}_{OK}u_K + \dot{G}_{OU}\hat{u}_U + \dot{\tau}_O$$

as defined in (3.26). However, the measurement y_U gives two major benefits. Firstly, the model from u_A and δ to u_U can be estimated, whereas it in Section 3.4 was assumed to be given or guessed. Secondly, the measurement can be used to predict u_U rather than just to simulate it like in Section 3.4 which potentially is more accurate. In the network setting, an exact model of the unobservable nodes is given by solving row four of (3.6) for w_U . This gives

$$w_U = (I - G_{UU})^{-1}(G_{Uj}w_j + G_{UK}w_K + G_{UA}w_A + \delta_U + \tau_U) \quad (3.43)$$

and the prediction error of the best approximation that can be created is $w_U - \hat{w}_U = \tau_U$ since τ_U typically is unknown. As discussed in the previous section, another possibility is to approximate w_U using only the known external signals δ . However, this will describe less of w_U and typically require a more complex model. Using the input measurement y_U to first estimate a model and creating the approximation $\hat{u}_U$ is similar to the two-stage method introduced in Van Den Hof and Schrama (1993). In the two-stage method, a known input is projected onto known external signals and then used as input to the estimation. Here, a noisy measurement is instead projected onto known external signals and then used as input to the estimation.

Example 3.8: Two-stage approach

Let us again consider the problem of Example 3.1 but now assume that u_U is unknown. However, there is a noisy measurement of the unknown input given by

$$y_U = u_U + e_U \quad (3.44)$$

where $e_U \sim \mathcal{N}(0, \lambda_e)$. A two-stage approach can be used as a remedy for the measurement noise. In a two-stage approach, a model is first estimated from δ to y_U and then u_U is simulated from δ . The model is found by using the predictor

$$\hat{y}_U = \hat{\gamma}_1 \delta_1 + \hat{\gamma}_2 \delta_2 \quad (3.45)$$

where the parameters are found by using a standard unweighted least-squares estimator resulting in $\gamma_1^* = \alpha$, $\gamma_2^* = 1$ and

$$P_\gamma = (\lambda_e + \lambda_u) \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} \quad (3.46)$$

The resulting signal $\hat{u}_U$ is used in the predictor

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_U \hat{u}_U \quad (3.47)$$

The result was given in Example 3.5 for $\beta = 0$ with $\alpha_1 = \gamma_1^*$ and $\alpha_2 = \gamma_2^*$. The estimator is thus consistent with the covariance given by

$$P_{\vartheta}^{3.5}|_{\hat{\alpha}_1=\alpha, \hat{\alpha}_2=1} = (\lambda_O + b_U^2 \lambda_u) \begin{bmatrix} \frac{\alpha^2}{\lambda_2} + \frac{1}{\lambda_1} & -\frac{\alpha}{\lambda_2} \\ -\frac{\alpha}{\lambda_2} & \frac{1}{\lambda_2} \end{bmatrix} \quad (3.48)$$

However, if both inputs are correlated with τ_u , i.e. if $\beta \neq 0$, then the limit value is

$$b_K^* = b_K + \frac{\beta \lambda_u (\alpha^2 \lambda_1 + \lambda_2)}{\lambda_1 \lambda_2 + \beta^2 \lambda_u (\alpha^2 \lambda_1 + \lambda_2)} b_U \quad (3.49)$$

and the asymptotic variance is

$$P_{b_K}^{3.8} = \frac{\lambda_1 \lambda_2 (\lambda_O + b_U^2 \lambda_u) (\alpha^2 \lambda_1 + \lambda_2) + \beta^2 \lambda_O \lambda_u (\alpha^2 \lambda_1 + \lambda_2)^2}{(\beta^2 \lambda_u (\alpha^2 \lambda_1 + \lambda_2) + \lambda_1 \lambda_2)^2} \quad (3.50)$$

The input disturbance τ_u thus act as a confounding variable.

The second approach can be derived by substituting u_U with $y_U - e_U$ which gives

$$y_O = G_{OK} u_K + G_{OU} u_U + \tau_O = G_{OK} u_K + G_{OU} y_U + \bar{\tau}_O \quad (3.51)$$

The effective disturbance is given by

$$\bar{\tau}_O = \tau_O - G_{OU} e_U \quad (3.52)$$

and hence, is correlated with y_U . The difficulty is thus that the measurement noise e_U acts as a confounding variable. This section will not be a comprehensive survey of the errors-in-variables framework or methods, the interested reader can, for instance, see Söderström (2007) or Söderström (2011), but will show the basic problem and how one specific estimator, an instrumental variable estimator, can be used to mitigate the measurement noise.

Example 3.9: Using input measurement – Errors-in-variables (EIV)

Again, consider the problem in Example 3.1 for $\beta=0$ where the goal is to estimate b_K . In this example, the unknown input u_U is assumed to be measured as

$$y_U = u_U + e_U \quad (3.53)$$

where $e_U \sim \mathcal{N}(0, \lambda_e)$ is measurement noise. Consider the EIV predictor

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_U y_U \quad (3.54)$$

which is identifiable with respect to the parameter b_K . The parameters are found by using the predictor (3.54) together with the least-squares method described in Section 2.5.1. Using the asymptotic theory, described in Section 2.6, results in

$$\vartheta^* = \begin{bmatrix} b_K \\ b_U \end{bmatrix} + \begin{bmatrix} \frac{\alpha \lambda_e}{\lambda_2 + \lambda_u + \lambda_e} \\ -\frac{\lambda_e}{\lambda_2 + \lambda_u + \lambda_e} \end{bmatrix} b_U \quad (3.55)$$

Note that a bias is introduced due to using a noisy input in the least-squares estimator. The bias is dependent on the signal to noise ratio of the input measurement, i.e. it will decrease if λ_2 or λ_u are increased or λ_e is decreased. Furthermore, if the input is neglected, the bias αb_U of Example 3.3 (for $\beta = 0$)

will always be larger than or equal to the bias $\alpha b_U \lambda_e / (\lambda_2 + \lambda_e)$ of the least-squares EIV estimator since $\lambda_e / (\lambda_2 + \lambda_e) \leq 1$. The asymptotic covariance is given by

$$P_{\vartheta}^{3.9} = \begin{bmatrix} P_{b_K}^{3.9} & \times \\ \times & \times \end{bmatrix} = \begin{bmatrix} \frac{b_U^2 \lambda_2}{\lambda_1} \cdot \frac{\lambda_e (\alpha^2 \lambda_1 + \lambda_2 + \lambda_e)}{(\lambda_2 + \lambda_e)^2} + \lambda_O \left(\frac{\alpha^2}{\lambda_2 + \lambda_e} + \frac{1}{\lambda_1} \right) & - \frac{\alpha (\lambda_2 \lambda_e b_U^2 + \lambda_2 \lambda_O + \lambda_e \lambda_O)}{(\lambda_2 + \lambda_e)^2} \\ - \frac{\alpha (\lambda_2 \lambda_e b_U^2 + \lambda_2 \lambda_O + \lambda_e \lambda_O)}{(\lambda_2 + \lambda_e)^2} & \frac{\lambda_2 \lambda_e b_U^2 + \lambda_2 \lambda_O + \lambda_e \lambda_O}{(\lambda_2 + \lambda_e)^2} \end{bmatrix} \quad (3.56)$$

In a variance sense, it is beneficial to use the input measurement rather than neglecting the input if $P_{b_K}^{3.9}$ is smaller than $P_{b_K}^{3.3}$ (of Example 3.3), i.e.

$$\frac{b_U^2 \lambda_2}{\lambda_1} \cdot \frac{\lambda_e (\alpha^2 \lambda_1 + \lambda_2 + \lambda_e)}{(\lambda_2 + \lambda_e)^2} + \lambda_O \left(\frac{\alpha^2}{\lambda_2 + \lambda_e} + \frac{1}{\lambda_1} \right) < \frac{b_U^2 \lambda_2 + \lambda_O}{\lambda_1} \quad (3.57)$$

which is equivalent to

$$\underbrace{(\alpha^2 \lambda_O \lambda_1) \lambda_2}_{C_1} - \underbrace{(b_U^2) \lambda_2^3}_{C_2} + \lambda_e \left[\underbrace{(\alpha^2 \lambda_O \lambda_1)}_{C_1} + \underbrace{(\alpha^2 b_U^2 \lambda_1)}_{C_3} \right] \lambda_2 - \underbrace{(b_U^2) \lambda_2^2}_{C_2} < 0 \quad (3.58)$$

if $\lambda_1 (\lambda_2 + \lambda_e)^2 > 0$. If $b_U = 0$, then it is always better to neglect the unknown input since it does not affect the output. On the other hand, if $b_U \neq 0$ and the inputs are independent, i.e. $\alpha = 0$ and $\beta = 0$, then the input measurement is always useful since (3.58) is fulfilled and $b_K^* = b_K$. Hence, if $\lambda_e > 0$, then using the input measurement in the least-squares estimator will always give an unbiased estimator of b_K with variance less than or equal to the case when u_U was neglected.

In general, when all parameters are non-zero, then all the coefficients of (3.58) are positive. It is thus sufficient to consider the special case $\alpha = \lambda_1 = \lambda_O = b_U = 1$ that captures the essence of the inequality. With these values, the inequality is

$$\lambda_2 - \lambda_2^3 + \lambda_e (1 + \lambda_2 - \lambda_2^2) < 0 \quad (3.59)$$

Note that if $1 + \lambda_2 - \lambda_2^2$ is negative, which is true for $\lambda_2 > 1/2 + \sqrt{5}/4 \approx 1.618$, then (3.59) is fulfilled for all values of λ_e . Figure 3.5(a) shows the contour of $P_{b_K}^{3.9} - P_{b_K}^{3.3}$, i.e. the differences in asymptotic variance of $\hat{b}_K$ between the estimators as a function of λ_2 and λ_e . The curve corresponding to $\lambda_2 - \lambda_2^3 + \lambda_e (1 + \lambda_2 - \lambda_2^2) = 0$ (black line) and the line $\lambda_2 = 1.618$ (dashed black line) are also drawn. Here, (3.59) is fulfilled for all values to the right of the black line. For instance, it can be seen that if $\lambda_e = 0$ and $\lambda_2 > 1$, then it is better to use the input measurement rather than neglecting the unknown input. Furthermore, if λ_2 is small, it is better to neglect the unknown input.

Note that the asymptotic mean squared error (AMSE), described in Section 2.6, will be even more favorable when using the input measurement since the bias is always smaller if the input measurement is used instead of neglecting the input. This is confirmed in Figure 3.5(b) which shows the difference in AMSE. In this case, the AMSE is smaller for the EIV model if $\lambda_2 > 0.62$ regardless of λ_e .

Figure 3.5: The differences between neglecting u_U (Example 3.3) and using an input measurement (Example 3.9) as a function of the input variance λ_2 and measurement noise of the input measurement λ_e . The figures show the asymptotic variance/AMSE of $\hat{b}_K$ in Example 3.9 minus the asymptotic variance/AMSE of $\hat{b}_K$ in Example 3.3. The values to the right of the black line correspond to when it is beneficial to use the input measurement. For values $\lambda_2 > 1/2 + \sqrt{5}/4 \approx 1.618$ (dashed line) it is always beneficial to use the input measurement regardless of the value of λ_e in a variance sense.

Note that, in contrast to Example 3.3, the bias of Example 3.9 is a result of using the least-squares estimator. The bias of (3.55) decreases with decreasing measurement noise variance λ_e of the input measurement and the estimator is consistent if $\lambda_e = 0$ (which corresponds to a known input and the results then coincide with Example 3.1). The influence of the noise can be mitigated with a proper choice of estimator, for instance, an instrumental variable estimator, see for example, Söderström (2011) for a thorough treatment.

Example 3.10: EIV with IV estimator

Let us again consider the problem of Example 3.1 where there is an input measurement of the unknown input u_U given by

$$y_U = u_U + e_U \quad (3.60)$$

and $e_U \sim \mathcal{N}(0, \lambda_e)$. Consider the predictor

$$\hat{y}_O = \hat{b}_K u_K + \hat{b}_U y_U \quad (3.61)$$

where the parameters are found using an instrumental variable method with the instrument vector given by

$$\zeta = \begin{bmatrix} \delta_1 \\ \delta_1 + \delta_2 \end{bmatrix} \quad (3.62)$$

This results in the limit values $b_K^* = b_K$ and $b_U^* = b_U$ and the asymptotic covariance

$$P_{b_K}^{3.10} = (\lambda_O + b_U^2 \lambda_e) \begin{bmatrix} \frac{\alpha^2}{\lambda_2} + \frac{1}{\lambda_1} & -\frac{\alpha}{\lambda_2} \\ -\frac{\alpha}{\lambda_2} & \frac{1}{\lambda_2} \end{bmatrix} \quad (3.63)$$

Hence, increasing λ_2 will improve the accuracy also of $\hat{b}_K$ unless $\alpha = 0$. However, note that the excitation induced by τ_u is not contributing to lowering the variance. This is because τ_u is uncorrelated with the vector of instruments ζ . This result is valid also for $\beta \neq 0$.

Working with noisy measurements of the unknown inputs introduces challenges and methods to deal with these issues exist, for example, see the previously mentioned Van Den Hof and Schrama (1993), Söderström (2007) or Söderström (2011). The benefit of using the input measurements directly in the predictor is that they contain information about all the excitation from the external signals, including disturbances, that affect the measured inputs. However, it might be difficult to use this information and still get an unbiased estimator due to the measurement noise. As we saw in Example 3.9, the benefit of using the input measurements depends on the signal-to-noise ratio. However, there might also be other benefits of using an EIV approach. For example, the model complexity might be decreased since a smaller part of the system is modeled.

3.6 Eliminating Unknown Inputs

In Sections 3.3 and 3.4, different approximations for the unknown inputs u_U in (3.5) were considered. Furthermore, input measurements were used as a substitute for the unknown inputs u_U in Section 3.5. In health monitoring of mechanical structures, it is common that the input is fully unknown, i.e. (3.5) with $n_K = 0$, given by

$$y_O = G_{OU}u_U + \tau_O \quad (3.64)$$

and the unknown input is typically environmental disturbances, for example, wind causing vibrations in a bridge, and is thus not user-controllable, see for instance, Maia et al. (2001), Devriendt and Guillaume (2008) and D'Amato et al. (2009). Hence, the solutions discussed in previous sections or assumptions on the input are not always tractable. In this section, methods rely on the extra measurements

$$y_S = G_{SU}u_U + H_S\tau = G_{SU}u_U + e_S \quad (3.65)$$

where G_{SU} is an invertible and unknown transfer-function matrix and e_S is measurement noise. These extra measurements are used to *eliminate* the unknown input u_U from the model (3.64). In principle, the unknown input u_U is eliminated by solving (3.65) for u_U and inserting it in (3.64). This gives

$$y_O = G_{OU}G_{SU}^{-1}(y_S - e_S) + \tau_O = G_{OS}y_S + \bar{\tau}_O \quad (3.66)$$

where G_{OS} is a transfer function called a *transmissibility operator* or *pseudo transfer function*. Here, the disturbance acting on y_S will act as a confounding variable. Moreover, common factors in G_{OU} and G_{SU} will typically cancel and since both transfer functions are unknown, it is in the general case not possible to recover the dynamics of G_{OU} unless additional knowledge of the system is introduced.

Example 3.11: Cancellation in the S2SID framework

Consider the models

$$y_1 = G_1(p)u_U = \bar{G}_1(p)G(p)u_U \quad \text{and} \quad y_2 = G_2(p)u_U = \bar{G}_2(p)G(p)u_U \quad (3.67)$$

If both the transfer functions and the input u_U are unknown, the pseudo-transfer function is given by solving for u_U in either of the equations in (3.67) and inserting into the other. The shared dynamics $G(p)$ can never be estimated since

$$y_1 = \bar{G}_1(p)\bar{G}_2^{-1}(p)y_2 \quad \text{or} \quad y_2 = \bar{G}_2(p)\bar{G}_1^{-1}(p)y_1 \quad (3.68)$$

Besides the system identification problems above, the idea of elimination of unknown inputs or disturbances has also been approached in other fields. A closely related approach is the *Unknown input observers* (UIO), also called *disturbance decoupled observers*, which have been developed since the 1970's and have mainly been used in the field of fault detection. The UIO is an observer where the difference between the state estimate and the true state is independent of the unknown input. Many observer designs have been proposed to solve this problem. Instead of directly designing an observer, Hou and Müller (1994) introduced, and Hou and Patton (1998) later extended, the idea of a disturbance-decoupled *equivalent model* where the unknown input has been eliminated. This equivalent description thus separates the step of eliminating the disturbance from the actual observer design, which means that an arbitrary design scheme can be used to obtain the observer.

The derivation of the equivalent model starts with the *known* state-space model

$$\dot{x} = Ax + Bu_K + Eu_U \quad (3.69a)$$

$$y = Cx + Du_K + Fu_U \quad (3.69b)$$

where $x \in \mathbb{R}^n$ is the state. Let us introduce $\tilde{C} = \bar{C}E(I_{n_U} - F^\dagger F)$,

$$\bar{A} = (I_n - \bar{B}\bar{C})(A - EF^\dagger C), \quad \bar{B} = E(I_{n_U} - F^\dagger F)\tilde{C}^\dagger, \quad (3.70a)$$

$$\bar{C} = (I_{n_y} - FF^\dagger)C, \quad \bar{D} = (I_{n_y} - FF^\dagger)D, \quad (3.70b)$$

$$\bar{B}_1 = (I - \bar{B}\bar{C})(B - EF^\dagger D), \quad \bar{B}_2 = -\bar{B}\bar{D}, \quad (3.70c)$$

$$\bar{G}_1 = (I - \bar{B}\bar{C})EF^\dagger, \quad \bar{G}_2 = \bar{B}(I_{n_y} - FF^\dagger) \quad (3.70d)$$

and $\bar{y} = \bar{C}y$ where $(\cdot)^\dagger$ is the Moore-Penrose pseduo-inverse (Penrose, 1955).

Assumption 3.1. The pair $(\bar{A}, \bar{C})$ is detectable and $E(I_{n_U} - F^\dagger F)[I_{n_U} - \tilde{C}^\dagger \tilde{C}] = 0$.

The second part of Assumption 3.1 in principle implies that the remaining part of the unknown input does not affect the state vector. Moreover, the first part of Assumption 3.1 can equivalently be stated as (Hou and Müller, 1994)

$$\text{rank} \begin{bmatrix} -sI_{n \times n} + A & E \\ C & F \end{bmatrix} = n + \text{rank} \begin{bmatrix} E \\ F \end{bmatrix} \quad \forall s \in \mathbb{C}, \quad \text{Re}(s) \geq 0 \quad (3.71)$$

which means that no transmission zeros are allowed to be in the right half-plane (Hautus, 1983) and hence, that the system from u_U to y has to be minimum phase or equivalently, be inversely stable.

If Assumption 3.1 is fulfilled, then the equivalent model is given by

$$\dot{x} = \bar{A}x + \bar{B}_1 u_K + \bar{B}_2 \dot{u}_K + \bar{G}_1 y + \bar{G}_2 \dot{y} \quad (3.72a)$$

$$\bar{y} = \bar{C}x + \bar{D}u_K \quad (3.72b)$$

For more details, see Hou and Müller (1994) and Hou and Patton (1998). Note that the output $\bar{y}$ of the disturbance decoupled system can be a linear combination of y if several components of the measurements have direct-terms from u_U . This also implies that $\bar{G}_1 = (I - \bar{B}\bar{C})EF^\dagger y$ is a linear combination of all components of y and thus, that the input to the system is directly correlated with the output.

Example 3.12: UIO with direct terms

Consider the model given by

$$\dot{x} = Ax + Eu_U \quad (3.73a)$$

$$\underbrace{\begin{bmatrix} y_O \\ y_I \end{bmatrix}}_y = \underbrace{\begin{bmatrix} C_O \\ C_I \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_F u_U \quad (3.73b)$$

In this case, $F^\dagger = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$ and the equivalent disturbance-decoupled model is

$$\dot{x} = (A - EF^\dagger C)x + EF^\dagger y \quad (3.74a)$$

$$\bar{y} = \bar{C}x \quad (3.74b)$$

where $\bar{y} = y_1 - y_2$ is the output and $F^\dagger y = 0.5y_1 + 0.5y_2$ is the effective input.

3.7 Conclusions

In this chapter, the model (3.5) for systems with unknown inputs u_U

$$y_O = G_{OK}u_K + G_{OU}u_U + \tau_O$$

has been defined. Previous work has briefly been surveyed and the necessities or benefits of using some remedy for an unknown input have been discussed in the context of identifiability, consistency and variance properties. Due to the complexity of the model and the fact that consistency and variance properties are dependent on the estimation method, it is difficult to draw conclusions in the general case. Instead, the discussion was centered around simple and illustrative examples that highlighted important aspects.

The first approach was to neglect the unknown inputs, and resulted in the model

$$y_O = \check{G}_{OK}u_K + \check{\tau}_O$$

defined in (3.15) and investigated in Section 3.3. To increase the performance, other signals can be used to either approximate the unknown input or to approximate the contribution of the unknown input. These approaches resulted in the models

$$y_O = \dot{G}_{OK}u_K + \dot{G}_{OU}\hat{u}_U + \dot{\tau}_O$$

defined in (3.26) and

$$y_O = \check{G}_{OK}u_K + \check{G}_{OA}u_A + \check{\tau}_O$$

defined in (3.31), which were investigated in Section 3.4. In Section 3.5, the use of measurements of the unknown inputs was investigated. Firstly, it was discussed how input measurements could be used to improve the approximation of the unknown inputs. Secondly, it was discussed how the measurements could be used as substitutes for the unknown inputs leading to the errors-in-variables model

$$y_O = G_{OK}u_K + G_{OU}y_U + \bar{\tau}_O$$

defined in (3.51). Finally, in Section 3.6, it was discussed how additional measurements could be used to eliminate the unknown inputs which lead to the sensor-to-sensor system identification problem

$$y_O = G_{OS}y_S + \bar{\tau}_O$$

defined in (3.66). One common issue with all these approaches was that confounding variables might be introduced into the estimation problem, either through signals that were not described by the approximations or noise introduced when measurements were used. However, these confounding variables might not always be an issue and might in some cases be mitigated by a suitable choice of estimator.

4

The Indirect Framework

In this chapter, the indirect modelling framework will be derived and its properties will be discussed. The goal of the framework is to estimate a model of a subsystem S_O from the *partially unknown inputs* u to some of the available measurements y_O . The framework provides a systematic approach for dealing with unknown inputs u_U using additional measurements where the effects of the unknown inputs u_U are visible. The unknown inputs u_U are eliminated through algebraic manipulations and the resulting *indirect model* can be used in conjunction with the output, the input measurements and the known inputs, to estimate the dynamical model of interest.

Some of the approaches described in Chapter 3 can be described by the concept of downstream and upstream signals, see Figure 4.1. In the third class, upstream internal and external signals are used to approximate the unknown inputs or their effects on the output. The fourth class can be seen as using a type of downstream signals as substitutes for the unknown inputs. In the fifth class, downstream signals are used to eliminate the unknown inputs. The framework proposed in this section is mainly related to the approaches of class four and five in Chapter 3. Here, additional downstream signals, called the *direct* and *indirect input measurements*, are used to eliminate the unknown inputs. The indirect model can be seen as a generalization of class four and five and they will be shown to be special cases of the indirect model in Section 4.5.

Figure 4.1: An illustration of down- and upstream signals where a dashed line represents a path and a circle represents a node, i.e. signal or variable. An upstream signal has a path to (affect) the node. A downstream signal has path from (is affected by) the node. For example, w_3 is an upstream node to w_1 and w_2 while w_1 is a downstream node to w_2 and w_3 .

4.1 Direct and Indirect Input Measurements

In Chapter 3, the necessity and benefits of properly treating unknown inputs were discussed. The unknown input model

$$y_O = G_{OK}(\mathbf{p})u_K + G_{OU}(\mathbf{p})u_U + \underbrace{H_O(\mathbf{p})\tau}_{\tau_O}$$

where $u_K \in \mathbb{R}^{n_K}$ are known inputs and $u_U \in \mathbb{R}^{n_U}$ are unknown inputs, was defined in (3.5). In this chapter, additional measurements will be used to eliminate elements of the unknown input u_U . To this end, the inputs u are divided into

$$u = \begin{bmatrix} u_K \\ u_U \end{bmatrix} = \begin{bmatrix} u_K \\ u_I \\ u_D \end{bmatrix} \left\{ \begin{array}{l} \text{known} \\ \text{unknown} \end{array} \right. \quad (4.1)$$

where $u_I \in \mathbb{R}^{n_I}$ and $u_D \in \mathbb{R}^{n_D}$ will be referred to as *indirectly* and *directly measured inputs*, respectively. To be consistent with the input division, $G_O(\mathbf{p})$ is defined as

$$G_O(\mathbf{p}) = \begin{bmatrix} G_{OK}(\mathbf{p}) & G_{OI}(\mathbf{p}) & G_{OD}(\mathbf{p}) \end{bmatrix} \quad (4.2)$$

such that (3.5) can be written

$$y_O = G_{OK}(\mathbf{p})u_K + G_{OI}(\mathbf{p})u_I + G_{OD}(\mathbf{p})u_D + \underbrace{H_O(\mathbf{p})\tau}_{\tau_O} \quad (4.3)$$

It is assumed that the directly measured inputs u_D can be measured through some *known* dynamics with an additive disturbance. These measurements are called *direct input measurements* and are given by

$$y_D = G_D(\mathbf{p})u + H_D(\mathbf{p})\tau = G_{DD}(\mathbf{p})u_D + \underbrace{H_D(\mathbf{p})\tau}_{\tau_D} \quad (4.4)$$

where $G_{DD}(\mathbf{p})$ is a known transfer-function matrix of suitable dimension, $H_D(\mathbf{p})$ is a transfer-function matrix and $y_D \in \mathbb{R}^{n_D^y}$ consists of elements of the measurements y . The direct input measurements are thus a generalization of the input measurements (3.42) in the errors-in-variables framework.

Furthermore, the remaining unknown inputs are indirectly measured through

$$\begin{aligned} y_I &= G_I(\mathbf{p})u + H_I(\mathbf{p})\tau \\ &= G_{IK}(\mathbf{p})u_K + G_{II}(\mathbf{p})u_I + G_{ID}(\mathbf{p})u_D + G_{IA}(\mathbf{p})u_A + \underbrace{H_I(\mathbf{p})\tau}_{\tau_I} \end{aligned} \quad (4.5)$$

where u_A are additional known signals, the transfer-function matrices are of suitable dimensions and possibly unknown, and $y_I \in \mathbb{R}^{n_I^y}$ consists of elements of the measurements y . The measurements y_I are called *indirect input measurements* since they indirectly contain information about the unknown inputs u_I .

Assumption 4.1. The transfer-function matrices $G_{DD}(\mathbf{p})$ and $G_{II}(\mathbf{p})$ are assumed to have full column rank. In addition, the transfer-function matrix $G_{DD}(\mathbf{p})$ is assumed to be known. Note that the rank assumptions imply that: Firstly, $\dim(y_D) = n_D^y \geq n_D$ and $\dim(y_I) = n_I^y \geq n_I$, respectively. Secondly, that the input measurements contain information about all the unknown inputs.

The categorization of the unknown inputs u_U is based on the properties of the input measurements and there is a fundamental difference between the direct and indirect input measurements. To understand this difference, consider the case where the input measurements are noise-free. Then a known filter and the direct input measurement y_D can be used directly to represent the unknown input u_D . In contrast, the indirect input measurements y_I cannot be used without additional information, such as knowledge of the unknown dynamics and/or additional signals, to represent the unknown input u_I even if the input measurements are noise-free.

Similarly to Chapter 3, a dynamic network model will be used to investigate properties of the resulting indirect model when the framework is applied to a model of the system S . Recall from Section 3.2, and more specifically Definition 3.1, that the unknown inputs u_U are elements of the unobservable nodes w_U . Naturally, not all of the unobservable nodes w_U need to be directly or indirectly measured if only the subsystem S_O is of interest. For this reason, the following sets are introduced.

Definition 4.1 (Unobservable predictor inputs). The unobservable nodes are divided into the set of *unobservable predictor inputs* $\bar{U}_j = \mathcal{U}_j \cap \mathcal{N}_j$ and the set of *remaining unobservable nodes*, i.e. $\tilde{U}_j = \mathcal{U}_j \setminus \bar{U}_j$.

With this definition, the unknown inputs u_U are equivalent to $w_{\bar{U}}$. Moreover, Definition 3.1 implies that the direct and indirect input measurements must be elements of the additional observable nodes w_A . To more clearly show the connection between the transfer-function model and the dynamic network model, the following definitions are introduced.

Definition 4.2 (Direct node observations). The set $\mathcal{D}_j$ is the indices $i \in \mathcal{A}_j$ such that G_{ik} for $k \in \bar{U}_j$ are *known* and at least one G_{ik} is non-zero, and $G_{in} = 0$ for all $n \notin \bar{U}_j$.

The direct node observations are thus the elements of w_A such that

$$w_D = G_{D\bar{U}}(\mathbf{p})w_{\bar{U}} + \delta_D + \tau_D \quad (4.6)$$

where the transfer function $G_{D\bar{U}}(\mathbf{p})$ is known. In correspondence with the direct input measurements, the direct node observations w_D (together with δ_D) can be used directly to represent a part of $w_{\bar{U}}$ corresponding to u_D .

Definition 4.3 (Indirect node observations). The set $\mathcal{I}_j$ is the indices $i \in \mathcal{A}_j \setminus \mathcal{D}_j$ such that $G_{ik} \neq 0$ for some $k \in \tilde{U}_j$. The nodes w_I will be called *indirect node observations* since they contain indirect information about the unobservable predictor inputs $w_{\bar{U}}$. The set $\mathcal{O}_j = \mathcal{A}_j \setminus (\mathcal{I}_j \cup \mathcal{D}_j)$ is the indices of the remaining observable nodes.

The indirect node observations are thus the elements of w_A such that

$$w_I = G_{Ij}w_j + G_{IK}w_K + G_{IO}w_O + G_{II}w_I + G_{ID}w_D + G_{IU}w_U + \delta_I + \tau_I \quad (4.7)$$

where all transfer functions possibly are unknown and the dependencies on $\mathbf{p}$ have been dropped for brevity. Note that (4.7) potentially is dependent on all nodes w_U . Like the direct node observations, the indirect node observations are defined in correspondence with the indirect input measurements and it is thus *not* sufficient with knowledge of w_I (and δ_I) to represent the elements of $w_{\bar{U}}$ corresponding to u_I even in the noise-free case. The following assumptions are made on the network model.

Table 4.1: Connection between signals in the transfer function and dynamic network models. The notation z^* means elements of z .

Model	Signals						
Transfer-function	y_O	y_I	y_D	u_K	u_I	u_D	u_A
Dynamic network	w_j	w_I	w_D and δ_D	w_K and δ_j	$w_{\mathcal{U}}^*$	$w_{\mathcal{U}}^*$	$w_{\mathcal{O}}^*, \delta_{\mathcal{U}}^*$ and δ_I

Assumption 4.2. The transfer-function matrix $[G_{D\mathcal{U}}^T(\mathbf{p}), G_{\mathcal{I}\mathcal{U}}^T(\mathbf{p})]^T$ has full column rank. In addition, the transfer-function matrix $G_{D\mathcal{U}}(\mathbf{p})$ is assumed to be known and has n non-zero columns where $n \leq \dim(w_D)$ and $\text{rank}(G_{D\mathcal{U}}(\mathbf{p})) = n$.

The indirect node observations (4.7) are dependent on the remaining unobservable signals $w_{\mathcal{U}}$. Furthermore, elements of w_I can be dependent on w_D and other elements of w_I . In (4.5), this is represented by u_A and a more complex noise model. The known signals, i.e. δ_I and elements of w_O and $\delta_{\mathcal{U}}$, are part of u_A while the contribution from elements of $\tau_{\mathcal{U}}$ is captured by the more complex noise model of τ_I . A mapping of the signals between the two models is given in Table 4.1.

4.2 Indirect Model – Elimination of Unknown Inputs

The main idea of the framework is that the input measurements (4.4) and (4.5) can be used to eliminate the unknown inputs of (4.3) by algebraic manipulation.

Assumption 4.3. There exist linear filters $f_{DD}(\mathbf{p})$ and $f_{II}(\mathbf{p})$ such that

$$\begin{bmatrix} f_{DD}(\mathbf{p}) & 0 \\ 0 & f_{II}(\mathbf{p}) \end{bmatrix} \begin{bmatrix} G_{DD}(\mathbf{p}) & 0 \\ 0 & G_{II}(\mathbf{p}) \end{bmatrix} = I_{n_U \times n_U} \quad (4.8)$$

where $I_{k \times k}$ is a k by k identity matrix.

By using the filter f_{DD} and (4.4), the directly measured inputs can be written

$$u_D = f_{DD}(\mathbf{p})[y_D - H_D(\mathbf{p})\tau] \quad (4.9)$$

Similarly, (4.5) and $f_{II}(\mathbf{p})$ can be used to recover the indirectly measured inputs

$$u_I = f_{II}(\mathbf{p})[y_I - G_{IK}(\mathbf{p})u_K - G_{ID}(\mathbf{p})u_D - G_{IA}(\mathbf{p})u_A - H_I(\mathbf{p})\tau] \quad (4.10)$$

Finally, by inserting (4.9) and (4.10) into (4.3) we obtain the *indirect model*

$$\begin{aligned} y_O &= \tilde{G}_{OK}(\mathbf{p})u_K + \tilde{G}_{OI}(\mathbf{p})y_I + \tilde{G}_{OD}(\mathbf{p})y_D + \tilde{G}_{OA}(\mathbf{p})u_A + \tilde{H}_O(\mathbf{p})\tau \\ &= \tilde{G}_O(\mathbf{p})\tilde{u} + \tilde{\tau}_O \end{aligned} \quad (4.11)$$

where

$$\tilde{u} = \begin{bmatrix} u_K^T & y_I^T & y_D^T & u_A^T \end{bmatrix}^T, \quad (4.12)$$

$$\tilde{G}_O(\mathbf{p}) = \begin{bmatrix} \tilde{G}_{OK}(\mathbf{p}) & \tilde{G}_{OI}(\mathbf{p}) & \tilde{G}_{OD}(\mathbf{p}) & \tilde{G}_{OA}(\mathbf{p}) \end{bmatrix}, \quad (4.13)$$

$$\tilde{G}_{OI}(\mathbf{p}) = G_{OI}(\mathbf{p})f_{II}(\mathbf{p}), \quad (4.14)$$

$$\tilde{G}_{OK}(\mathbf{p}) = G_{OK}(\mathbf{p}) - \tilde{G}_{OI}(\mathbf{p})G_{IK}(\mathbf{p}), \quad (4.15)$$

$$\tilde{G}_{OD}(\mathbf{p}) = (G_{OD}(\mathbf{p}) - \tilde{G}_{OI}(\mathbf{p})G_{ID}(\mathbf{p}))f_{DD}(\mathbf{p}), \quad (4.16)$$

$$\tilde{G}_{OA}(\mathbf{p}) = -\tilde{G}_{OI}(\mathbf{p})G_{IA}(\mathbf{p}) \quad (4.17)$$

and $\tilde{\tau}_O = \tilde{H}_O(\mathbf{p})\tau$ with

$$\tilde{H}_O(\mathbf{p}) = H_O(\mathbf{p}) - \tilde{G}_{OI}(\mathbf{p})H_I(\mathbf{p}) - \tilde{G}_{OD}(\mathbf{p})H_D(\mathbf{p}) \quad (4.18)$$

In Linder and Enqvist (2015), $G_{DD}(\mathbf{p})$ and $G_{II}(\mathbf{p})$ were both assumed to be invertible. This implies that the dimension of the input measurements is equal to the number of directly and indirectly measurable unknown inputs. If there are additional input measurements, this invertibility assumption might be limiting since the extra measurements possibly can be used to improve the accuracy. Here, this assumption is relaxed and by Assumption 4.1, $G_{DD}(\mathbf{p})$ and $G_{II}(\mathbf{p})$ are both assumed to have full column rank which is equivalent to the invertibility assumption if they are square matrices (Lancaster and Tismenetsky, 1985). Hence, if $\dim(y_D) = n_D$ and $\dim(y_I) = n_I$, then the filters can be chosen as $f_{DD}(\mathbf{p}) = G_{DD}^{-1}(\mathbf{p})$ and $f_{II}(\mathbf{p}) = G_{II}^{-1}(\mathbf{p})$ which is equivalent to Linder and Enqvist (2015).

In the network model, if the additional observable nodes w_A contain information about all the unobservable nodes w_U , then they can potentially be used to form an indirect model. Similar to Assumptions 4.1 and 4.3, assume that $G_{AU}(\mathbf{p})$ has full column-rank and that there exists a filter $f_{UA}(\mathbf{p})$ such that

$$f_{UA}(\mathbf{p})G_{AU}(\mathbf{p}) = I_{n_U \times n_U} \quad (4.19)$$

Solving row three of (3.6) for $G_{AU}(\mathbf{p})w_U$ and filtering with $f_{UA}(\mathbf{p})$ gives

$$w_U = f_{UA}(-G_{Aj}w_j - G_{AK}w_K + (I - G_{AA})w_A - \delta_A - \tau_A) \quad (4.20)$$

Inserting (4.20) into (3.6) gives the *fully-eliminated indirect model*

$$\begin{bmatrix} w_j \\ w_K \\ w_A \end{bmatrix} = \begin{bmatrix} 0 & \dot{G}_{jK} & \dot{G}_{jA} \\ \dot{G}_{Kj} & \dot{G}_{KK} & \dot{G}_{KA} \\ \dot{G}_{Aj} & \dot{G}_{AK} & \dot{G}_{AA} \end{bmatrix} \begin{bmatrix} w_j \\ w_K \\ w_A \end{bmatrix} + \begin{bmatrix} \dot{F}_{jj} & 0 & \dot{F}_{jA} & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & \times & \times \end{bmatrix} \begin{bmatrix} \delta_j + \tau_j \\ \delta_K + \tau_K \\ \delta_A + \tau_A \\ \delta_U + \tau_U \end{bmatrix} \quad (4.21)$$

after normalization, where $\times$ denotes a possibly non-zero entry and $\dot{G}_{KK}(\mathbf{p})$ and $\dot{G}_{AA}(\mathbf{p})$ have zeros on the diagonal. Note that the indirect model of w_j only depends on the observable nodes w_S and the external variables δ_j , δ_A , τ_j and τ_A . Furthermore, the observable nodes w_A contain information about all excitation that enters into the network and has paths to w_A , also from the external unmeasured disturbances τ_U . By using this extra excitation, the variance can potentially be reduced. However, note that τ_A potentially will act as a confounding variable.

The rank assumption on $G_{AU}(\mathbf{p})$ can be restrictive since all unobservable nodes w_U have to be directly or indirectly observed, even nodes far away from the node of interest. Let us instead assume that *all* the unobservable predictor inputs $w_{\bar{U}}$ are directly or indirectly observable, i.e. that Assumption 4.2 is fulfilled.

Assumption 4.4. There exist filters $f_{\bar{U}D}(\mathbf{p})$ and $f_{\bar{U}I}(\mathbf{p})$ such that

$$f_{\bar{U}D}(\mathbf{p})G_{D\bar{U}}(\mathbf{p}) + f_{\bar{U}I}(\mathbf{p})G_{I\bar{U}}(\mathbf{p}) = I_{n_{\bar{U}} \times n_{\bar{U}}} \quad (4.22)$$

By using the filters, $w_{\mathcal{D}}$ and $w_{\mathcal{I}}$, $w_{\bar{\mathcal{U}}}$ can be eliminated in a first step, which gives

$$\begin{bmatrix} w_j \\ w_{\mathcal{K}} \\ w_{\mathcal{O}} \\ w_{\mathcal{I}} \\ w_{\mathcal{D}} \\ w_{\bar{\mathcal{U}}} \end{bmatrix} = \begin{bmatrix} 0 & \bar{G}_{j\mathcal{K}} & \bar{G}_{j\mathcal{O}} & \bar{G}_{j\mathcal{I}} & \bar{G}_{j\mathcal{D}} & \bar{G}_{j\bar{\mathcal{U}}} \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{K}} \\ w_{\mathcal{O}} \\ w_{\mathcal{I}} \\ w_{\mathcal{D}} \\ w_{\bar{\mathcal{U}}} \end{bmatrix} + \begin{bmatrix} \bar{F}_{jj} & 0 & 0 & \bar{F}_{j\mathcal{I}} & \bar{F}_{j\mathcal{D}} & 0 & 0 \\ 0 & \times & 0 & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & 0 & \times \end{bmatrix} \begin{bmatrix} \delta_j + \tau_j \\ \delta_{\mathcal{K}} + \tau_{\mathcal{K}} \\ \delta_{\mathcal{O}} + \tau_{\mathcal{O}} \\ \delta_{\mathcal{I}} + \tau_{\mathcal{I}} \\ \delta_{\mathcal{D}} + \tau_{\mathcal{D}} \\ \delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}} \end{bmatrix} \quad (4.23)$$

after normalization. The resulting *partially-eliminated indirect model* is given by

$$\begin{bmatrix} w_j \\ w_{\mathcal{K}} \\ w_{\mathcal{O}} \\ w_{\mathcal{I}} \\ w_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{G}_{j\mathcal{K}} & \tilde{G}_{j\mathcal{O}} & \tilde{G}_{j\mathcal{I}} & \tilde{G}_{j\mathcal{D}} \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{K}} \\ w_{\mathcal{O}} \\ w_{\mathcal{I}} \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} \tilde{F}_{jj} & 0 & 0 & \tilde{F}_{j\mathcal{I}} & \tilde{F}_{j\mathcal{D}} & 0 & \tilde{F}_{j\bar{\mathcal{U}}} \\ 0 & \times & 0 & \times & \times & 0 & \times \\ 0 & 0 & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \delta_j + \tau_j \\ \delta_{\mathcal{K}} + \tau_{\mathcal{K}} \\ \delta_{\mathcal{O}} + \tau_{\mathcal{O}} \\ \delta_{\mathcal{I}} + \tau_{\mathcal{I}} \\ \delta_{\mathcal{D}} + \tau_{\mathcal{D}} \\ \delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}} \end{bmatrix} \quad (4.24)$$

where $w_{\bar{\mathcal{U}}}$ has been eliminated using the last equation of (4.23), i.e. the immersed network for $\mathcal{S}_j$ of (4.23). Note that the same result is obtained if the elimination is reversed, i.e. by first forming the immersed network by eliminating $w_{\bar{\mathcal{U}}}$ and then eliminating $w_{\bar{\mathcal{U}}}$ using $w_{\mathcal{D}}$ and $w_{\mathcal{I}}$. In the case where the remaining unobservable nodes $w_{\bar{\mathcal{U}}}$ have no paths to the indirect node observations $w_{\mathcal{I}}$, i.e. $G_{\bar{\mathcal{U}}\mathcal{I}}(\mathbf{p}) = 0$ and hence, $\tilde{F}_{j\bar{\mathcal{U}}} = 0$, then the indirect model of the j^{th} node will only depend on locally observable nodes and external signals with at most two modules between the predictor inputs and w_j . This is shown in Figure 4.2 where the distances between the nodes are illustrated.

Here, no assumptions, such as statistical properties, have been made on the unknown inputs since they have been eliminated through algebraic manipulations. In fact, except for Assumptions 4.1/4.2 and 4.3/4.4 there are no additional assumptions, such as stability, on the indirect model (4.11)/(4.24). This is an intentional choice since such requirements could be too restrictive and thus lead to a conservative set of models. However, in practice, additional requirements might be necessary depending on the method chosen for estimation of the indirect model. With this in mind, it is important to realize that:

1. Given a certain system/model of the system, only some estimation methods might be suitable due to properties of the resulting indirect model.
2. Given a specific estimation method, additional requirements might have to be imposed on the indirect model.

4.3 Properties of the Indirect Model

The proposed indirect model

$$y_O = \tilde{G}_{OK}(\mathbf{p})u_K + \tilde{G}_{OI}(\mathbf{p})y_I + \tilde{G}_{OD}(\mathbf{p})y_D + \tilde{G}_{OA}(\mathbf{p})u_A + \underbrace{\tilde{H}_O(\mathbf{p})\tau}_{\tilde{\tau}_O}$$

defined in (4.11), only has known or measured inputs besides the disturbance term. The elimination of the unknown inputs results in new challenges due to the disturbances on the input measurements and in this section, the effects of using the input measurements will be investigated in the context of identifiability, consistency and variance properties.

4.3.1 Effects of Using Direct and Indirect Input Measurements

To understand the changes from G_O to $\tilde{G}_O$ and the properties of $\tilde{\tau}_O$, it is convenient to look at the details of the indirect model (4.24) of a dynamic network. To gain insight and separate the effects of the direct and indirect input measurements as well as the remaining unobservable nodes, two special cases will be considered. The first case is when there are direct node observations of all unobservable predictor inputs. The unknown input can then be written

$$w_{\bar{U}} = f_{\bar{U}D} [w_D - \delta_D - \tau_D] \quad (4.25)$$

where $f_{\bar{U}D}$ is known and the indirect model is given by

$$w_j = G_{jK}w_K + G_{j\bar{U}}f_{\bar{U}D}w_D + \delta_j + \tau_j - G_{j\bar{U}}f_{\bar{U}D}(\delta_D + \tau_D) \quad (4.26)$$

The direct input measurements can thus be seen as a generalization of the additive input measurements in the errors-in-variables framework. This implies that when all the unknown predictor inputs are directly observable, the direct node observations contain (filtered) information about all the excitation that enters into the unobservable predictor inputs $w_{\bar{U}}$. However, similar to the errors-in-variables problem, the noise τ_D acting on the measurements will act as a confounding variable.

If instead $w_{\bar{U}}$ is fully indirectly measured, then the unknown input can be written

$$w_{\bar{U}} = f_{\bar{U}\mathcal{I}} [(I - G_{\mathcal{I}\mathcal{I}})w_{\mathcal{I}} - G_{\mathcal{I}j}w_j - G_{\mathcal{I}K}w_K - G_{\mathcal{I}O}w_O - \delta_{\mathcal{I}} - \tau_{\mathcal{I}}] \quad (4.27)$$

where it has been assumed that $w_{\bar{U}}$ has no direct path to $w_{\mathcal{I}}$, i.e. $G_{\bar{U}\mathcal{I}} = 0$.

Inserting (4.27) into the first row of (3.6) gives

$$w_j = (I + G_{j\bar{U}}f_{\bar{U}\mathcal{I}}G_{\mathcal{I}j})^{-1} [(G_{jK} - G_{j\bar{U}}f_{\bar{U}\mathcal{I}}G_{\mathcal{I}K})w_K - G_{j\bar{U}}f_{\bar{U}\mathcal{I}}G_{\mathcal{I}O}w_O + G_{j\bar{U}}f_{\bar{U}\mathcal{I}}(I - G_{\mathcal{I}\mathcal{I}})w_{\mathcal{I}} + \delta_j + \tau_j - G_{j\bar{U}}f_{\bar{U}\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}})] \quad (4.28)$$

Like the direct input measurements, the indirect input measurements contain (filtered) information about all excitation that enters the network and has paths to $w_{\mathcal{I}}$, also from the external disturbances τ . However, the indirect measurements will in general affect all the dynamics and similar to the sensor-to-sensor system identification, it is in general not possible to recover the dynamics of G_O unless additional knowledge of the system is introduced.

Example 4.1: Partial elimination with indirect node observations

For the dynamic network of Example 3.2, $\mathcal{I}_2 = \{4\}$ and the indirect observation is

$$w_4 = G_{41}w_1 + G_{45}w_5 + \tau_4 \quad (4.29)$$

A partial elimination of w_5 using $f_{\bar{u}\mathcal{I}} = G_{45}^{-1}$ gives

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} &= \begin{bmatrix} 0 & G_{12} & 0 & 0 & G_{17} \\ G_{21} - G_{25}G_{45}^{-1}G_{41} & 0 & G_{23} & G_{25}G_{45}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ G_{41} & G_{45}G_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{87} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ G_{36} & 0 \\ 0 & G_{45}G_{58} \\ 0 & 0 \\ G_{76} & 0 \\ G_{86} & 0 \end{bmatrix} \begin{bmatrix} w_6 \\ w_8 \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 - G_{25}G_{45}^{-1}\tau_4 \\ \tau_3 \\ \tau_4 + G_{45}^{-1}\tau_5 \\ \delta_6 + \tau_6 \\ \tau_7 \\ \tau_8 \end{bmatrix} \quad (4.30) \end{aligned}$$

The indirect model, i.e. the immersed network for $\mathcal{S}_2$ of (4.30), is given by

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_7 \end{bmatrix} &= \begin{bmatrix} 0 & G_{12} & 0 & 0 & G_{17} \\ G_{21} - G_{25}G_{45}^{-1}G_{41} & 0 & G_{23} & G_{25}G_{45}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ G_{41} & G_{45}G_{52} & 0 & 0 & G_{45}G_{58}G_{87} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_7 \end{bmatrix} \\ &+ \begin{bmatrix} \tau_1 \\ \tau_2 - G_{25}G_{45}^{-1}\tau_4 \\ \tau_3 + G_{36}(\delta_6 + \tau_6) \\ \tau_4 + G_{45}(\tau_5 + G_{58}(\tau_8 + G_{86}(\delta_6 + \tau_6))) \\ \tau_7 + G_{76}(\delta_6 + \tau_6) \end{bmatrix} \quad (4.31) \end{aligned}$$

Note that the second row is unaffected by the immersion since $w_{\mathcal{I}}$ is independent of $w_{\bar{\mathcal{U}}}$ and that the other four rows are equivalent to (3.41). Furthermore, w_2 only depends on $w_{\mathcal{S}}$, τ_2 and τ_4 . The variables w_1 , w_3 and w_4 capture all excitation from δ_6 and more importantly $\tau_k, k = \{1, 3, 5, 6, 7, 8\}$. Finally, note that τ_4 is correlated with both $\tilde{\tau}_2$ and $\tilde{\tau}_4$ and thus act as a confounding variable.

In the general case, assuming that there exist filters $f_{\bar{u}\mathcal{D}}$ and $f_{\bar{u}\mathcal{I}}$ such that (4.22) holds, then the unknown input can be described by

$$\begin{aligned} w_{\bar{\mathcal{U}}} &= f_{\bar{u}\mathcal{D}}(w_{\mathcal{D}} - \delta_{\mathcal{D}} - \tau_{\mathcal{D}}) \\ &+ f_{\bar{u}\mathcal{I}}(-G_{\mathcal{I}\mathcal{J}}w_{\mathcal{J}} - G_{\mathcal{I}\mathcal{K}}w_{\mathcal{K}} - G_{\mathcal{I}\mathcal{O}}w_{\mathcal{O}} + (I - G_{\mathcal{I}\mathcal{I}})w_{\mathcal{I}} - G_{\mathcal{I}\mathcal{D}}w_{\mathcal{D}} - G_{\mathcal{I}\bar{\mathcal{U}}}w_{\bar{\mathcal{U}}} - \delta_{\mathcal{I}} - \tau_{\mathcal{I}}) \quad (4.32) \end{aligned}$$

Inserting (4.32) into the first row of (3.6) and normalizing gives

$$\begin{aligned} w_{\mathcal{J}} &= \bar{G}_{j\mathcal{K}}w_{\mathcal{K}} + \bar{G}_{j\mathcal{O}}w_{\mathcal{O}} + \bar{G}_{j\mathcal{I}}w_{\mathcal{I}} + \bar{G}_{j\mathcal{D}}w_{\mathcal{D}} + \bar{G}_{j\bar{\mathcal{U}}}w_{\bar{\mathcal{U}}} + \bar{\delta}_j + \bar{\tau}_j \\ &= (I + G_{j\bar{\mathcal{U}}}f_{\bar{u}\mathcal{I}}G_{\mathcal{I}\mathcal{J}})^{-1} [(G_{j\mathcal{K}} - G_{j\bar{\mathcal{U}}}f_{\bar{u}\mathcal{I}}G_{\mathcal{I}\mathcal{K}})w_{\mathcal{K}} - G_{j\bar{\mathcal{U}}}f_{\bar{u}\mathcal{I}}G_{\mathcal{I}\mathcal{O}}w_{\mathcal{O}} \\ &\quad + G_{j\bar{\mathcal{U}}}f_{\bar{u}\mathcal{I}}(I - G_{\mathcal{I}\mathcal{I}})w_{\mathcal{I}} + G_{j\bar{\mathcal{U}}}(f_{\bar{u}\mathcal{D}} - f_{\bar{u}\mathcal{I}}G_{\mathcal{I}\mathcal{D}})w_{\mathcal{D}} - G_{j\bar{\mathcal{U}}}f_{\bar{u}\mathcal{I}}G_{\mathcal{I}\bar{\mathcal{U}}}w_{\bar{\mathcal{U}}}] + \bar{\delta}_j + \bar{\tau}_j \quad (4.33) \end{aligned}$$

which is the first row of (4.23) and where

$$\begin{aligned}\bar{\delta}_j + \bar{\tau}_j &= \bar{F}_{jj}(\delta_j + \tau_j) + \bar{F}_{j\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) + \bar{F}_{j\mathcal{D}}(\delta_{\mathcal{D}} + \tau_{\mathcal{D}}) \\ &= (I + G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}j})^{-1} [\delta_j + \tau_j - G_{j\bar{u}} f_{\bar{u}\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) - G_{j\bar{u}} f_{\bar{u}\mathcal{D}}(\delta_{\mathcal{D}} + \tau_{\mathcal{D}})]\end{aligned}\quad (4.34)$$

As previously discussed in connection to the special cases, using the direct and indirect input measurements creates a correlation between the model inputs and the disturbance $\bar{\tau}_j$ due to the unknown disturbances $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{D}}$. In some sense, the use of the input measurements and the transformation to the indirect model create artificial paths that make it possible to utilize these measurements. If the input $\tilde{u}$ and the disturbance $\tau_{\mathcal{O}}$ on the output are uncorrelated, the remaining disturbances $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{I}}$ of the input measurements will lead to an errors-in-variables formulation and $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{I}}$ will act as confounding variables.

Example 4.2: Using an indirect input measurement

Let us once again consider the problem in Example 3.1 with the system

$$y_{\mathcal{O}} = b_K u_K + b_U u_U + \tau_{\mathcal{O}}$$

where the goal is to estimate b_K . Here, it is assumed that the unknown input $u_U = u_I$ is indirectly measured and that the measurement is given by

$$y_I = u_K + u_I + \tau_I \quad (4.35)$$

where $\tau_I \sim \mathcal{N}(0, \lambda_I)$ is measurement noise. Solving (4.35) for u_I , i.e. choosing $f_{II} = 1$, and inserting into (3.7) results in the indirect predictor model given by

$$\hat{y}_{\mathcal{O}}(\vartheta) = \hat{b}_K u_K + \hat{b}_I (y_I - u_K) \quad (4.36)$$

which is identifiable with respect to the parameter b_K . Note that $y_I - u_K = u_I + \tau_I$ which means this is equivalent to the errors-in-variables problem discussed in Examples 3.9 and 3.10 for $u_U = u_I$ and $\tau_e = \tau_I$. If on the other hand, the indirect input measurement is given by

$$y_I = a_K u_K + a_I u_I + \tau_I \quad (4.37)$$

where a_K and a_I are unknown, then the indirect model is given by

$$y_{\mathcal{O}} = b_K u_K + \frac{b_I}{a_I} (y_I - a_K u_K - \tau_I) + \tau_{\mathcal{O}} = \left(b_K - \frac{b_I}{a_I} a_K \right) u_K + \frac{b_I}{a_I} y_I + \tilde{\tau}_{\mathcal{O}} \quad (4.38)$$

and b_K is unidentifiable unless more information is known.

If the disturbances $\tau_{\mathcal{O}}$ on the other hand are correlated with the inputs $w_{\mathcal{I}}$ and $w_{\mathcal{D}}$ of the indirect model, then regardless whether the original system is open loop, the indirect estimation problem shares the challenges of a closed-loop estimation problem. The indirect model can in some sense be viewed as an artificial closed-loop system with loops induced by using the measurements as inputs, or a dynamic network with artificial paths. The model is thus not representing a part of the actual physical system. Unlike a system with a real controller in the loop, the artificial loops of the indirect model might contain direct terms, something that will be seen in Chapter 7. These direct terms are potentially a problem since there is an algebraic relation between the input $\tilde{u}$ and the output $y_{\mathcal{O}}$ caused by *both* the subsystem of interest and the feedback structure (Van den Hof et al., 1992). Some closed-loop identification methods, such as the direct prediction-error method, might fail if this is not considered (Van den Hof et al., 1992; Dankers, 2014).

The impact of the remaining unobservable signals $w_{\bar{u}}$ depends on properties of the network as well as the approach of dealing with them as discussed in Chapter 3. If there are no direct paths to the indirect node observations $w_{\mathcal{I}}$ from $w_{\bar{u}}$, then $w_{\bar{u}}$ will not have an impact on the indirect model of w_j , for instance, see Example 4.1. The impact of eliminating the remaining unobservable nodes can be investigated by looking at the immersed network for $\mathcal{S}_j$ of (4.33). The immersed network is formed by solving the last row of (4.23) for $w_{\bar{u}}$ which gives

$$w_{\bar{u}} = (I - \bar{G}_{\bar{u}\bar{u}})^{-1} [\bar{G}_{\bar{u}j} w_j + \bar{G}_{\bar{u}\mathcal{K}} w_{\mathcal{K}} + \bar{G}_{\bar{u}\mathcal{O}} w_{\mathcal{O}} + \bar{G}_{\bar{u}\mathcal{I}} w_{\mathcal{I}} + \bar{G}_{\bar{u}\mathcal{D}} w_{\mathcal{D}}] + \bar{\delta}_{\bar{u}} + \bar{\tau}_{\bar{u}} \quad (4.39)$$

where

$$\begin{aligned} \bar{\delta}_{\bar{u}} + \bar{\tau}_{\bar{u}} &= \bar{F}_{\bar{u}\bar{u}}(\delta_{\bar{u}} + \tau_{\bar{u}}) + \bar{F}_{\bar{u}\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) + \bar{F}_{\bar{u}\mathcal{D}}(\delta_{\mathcal{D}} + \tau_{\mathcal{D}}) \\ &= (I - \bar{G}_{\bar{u}\bar{u}})^{-1} [\delta_{\bar{u}} + \tau_{\bar{u}} - \bar{G}_{\bar{u}\mathcal{I}} f_{\bar{u}\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) - \bar{G}_{\bar{u}\mathcal{D}} f_{\bar{u}\mathcal{D}}(\delta_{\mathcal{D}} + \tau_{\mathcal{D}})] \end{aligned} \quad (4.40)$$

Inserting (4.39) into (4.33) and normalizing gives the indirect model

$$w_j = \tilde{G}_{j\mathcal{K}} w_{\mathcal{K}} + \tilde{G}_{j\mathcal{O}} w_{\mathcal{O}} + \tilde{G}_{j\mathcal{I}} w_{\mathcal{I}} + \tilde{G}_{j\mathcal{D}} w_{\mathcal{D}} + \tilde{\delta}_j + \tilde{\tau}_j \quad (4.41)$$

where

$$\tilde{G}_{j\mathcal{X}} = (I - \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}\bar{G}_{\bar{u}j})^{-1}(\bar{G}_{j\mathcal{X}} + \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}\bar{G}_{\bar{u}\mathcal{X}}) \quad (4.42)$$

for $\mathcal{X} = \mathcal{K}, \mathcal{O}, \mathcal{I}, \mathcal{D}$, and

$$\begin{aligned} \tilde{\delta}_j + \tilde{\tau}_j &= (I - \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}\bar{G}_{\bar{u}j})^{-1} [\bar{F}_{jj}(\delta_j + \tau_j) \\ &\quad + (\bar{F}_{j\mathcal{I}} - \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}\bar{G}_{\bar{u}\mathcal{I}} f_{\bar{u}\mathcal{I}})(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) \\ &\quad + (\bar{F}_{j\mathcal{D}} - \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}\bar{G}_{\bar{u}\mathcal{D}} f_{\bar{u}\mathcal{D}})(\delta_{\mathcal{D}} + \tau_{\mathcal{D}}) \\ &\quad + \bar{G}_{j\bar{u}}(I - \bar{G}_{\bar{u}\bar{u}})^{-1}(\delta_{\bar{u}} + \tau_{\bar{u}})] \end{aligned} \quad (4.43)$$

Unsurprisingly, eliminating the remaining unobservable signals $w_{\bar{u}}$ can affect the dynamics between the output $y_{\mathcal{O}}$ and the remaining predictor inputs and have an impact on the effective disturbance, potentially introducing confounding variables, as shown in Section 3.3. Moreover, this elimination might increase the complexity of the model since each element of $\delta_{\bar{u}}$ that has a path to w_j has to be included in the model. However, neglecting the unobservable predictor inputs $w_{\bar{u}}$ directly might potentially mean that a larger part of the system S has to be modeled. For instance, consider the resulting model in Example 3.4 or Example 3.7 where the unknown inputs are neglected compared to the indirect model of Example 4.1. Furthermore, since all the unobservable predictor inputs $w_{\bar{u}}$ are directly or indirectly observed, this implies that all excitation that enters into $w_{\bar{u}}$ will be described by $w_{\mathcal{I}}$ and $w_{\mathcal{D}}$ and hence, cannot act as confounding variables.

In Chapter 3, it was shown that neglecting the unobservable inputs might not be the best choice. An alternative is to recursively apply the indirect framework until all unobservable nodes are eliminated. This can be achieved if there exist elements in $w_{\mathcal{O}}$ that can be used as direct and indirect measurements of the elements of $w_{\bar{u}}$ that have direct paths to $w_{\mathcal{I}}$, etc. Using additional measurements in this way will increase the model complexity with every step as is illustrated in Figure 4.2.

Figure 4.2: An illustration of which nodes that can be used when an unknown node is eliminated. When nodes further away are eliminated, more of the network has to be described and the complexity increases (red arrow). Each arc corresponds to nodes with the same “distance” to w_j . For example, it can be seen that there only is one module between w_j and its predictor inputs w_k and w_u while there are two modules between w_j and w_I and w_D .

4.3.2 Identifiability Properties of the Indirect Model

Once a model structure and parameterization of the indirect model have been chosen, an initial step should be to investigate the identifiability properties. In this section, the focus is on the structural identifiability (henceforth called identifiability) according to Section 2.7. Here, the models are assumed to have a specific parameterization. The set of parameters ϑ is assumed to describe the desired dynamics or properties of the model as described in Section 2.4. Regardless if the indirect model (4.11) is parameterized directly or if the desired model (4.3) and the input measurement models (4.4) and (4.5) are first parameterized to then form the indirect model (4.11), it is assumed that they are all dependent on the parameters ϑ . Note that the models defined in Sections 3.2 and 4.1 are assumed to be linear in the inputs but that there is no linearity assumption on the parameterization.

Firstly, consider the trivial case when $G_{OK}(\mathbf{p}, \vartheta)$ only depends on a subset of the parameters. The predictor where the unknown inputs u_U are neglected would thus not be dependent on all parameters and would then trivially not be identifiable.

It is difficult to give an explicit condition for the identifiability of the resulting indirect model (4.11) given an arbitrary nonlinear parameterization of (4.3). As discussed earlier in this chapter, using the indirect input measurements might change the model in two ways. Firstly, the dynamics $G_{OK}(\mathbf{p}, \vartheta)$ between the known inputs u_K and the output y_O might change. Secondly, factors in the dynamics $G_{OU}(\mathbf{p}, \vartheta)$ related to the unknown inputs might cancel when the indirect model is formed. Both these changes might introduce identifiability issues. However, the input measurements might also contribute with new information. This new information can actually solve identifiability issues or even simplify the original model. It is thus not necessary that the original model (4.3) is identifiable with respect to the parameters in order for the indirect model to be identifiable.

Example 4.3: Gaining identifiability from an input measurement

Consider the simple static model of Example 3.1 with $b_K = b_1$ and $b_U = b_2^2$ given by

$$y_O = b_1 u_K + b_2^2 u_U + \tau_O \quad (4.44)$$

where the goal is to estimate both b_1 and b_2 . Naturally, b_2 is not identifiable if the unknown input is neglected. Moreover, since b_2 only appears squared, it is impossible to distinguish between $b_2 = b_2^0$ and $b_2 = -b_2^0$ where b_2^0 is the true value. This implies that the model is not identifiable even if both inputs are known. However, if the input measurement $y_I = b_2 u_U$ is available, then the resulting model

$$y_O = b_1 u_K + b_2 y_I + \tau_O \quad (4.45)$$

is identifiable with respect to the parameters due to the cancelation.

Unfortunately, identifiability might also be lost if the input measurement shares dynamics or parameters with the desired system as discussed in Sections 3.6 and 4.3.1. Let us again look at an example.

Example 4.4: Loss of identifiability due to shared dynamics

Again, consider the model of Example 3.1 but with $b_K = b_1$ and $b_U = b_2$ given by

$$y_O = b_1 u_K + b_2 u_U + \tau_O \quad (4.46)$$

where the goal is to estimate the parameters b_1 and b_2 . Both parameters would be identifiable if the inputs were known. However, the input u_U is unknown and the input measurement $y_I = b_2 u_I$ is available. The resulting indirect model

$$y_O = b_1 u_K + y_I + \tau_O \quad (4.47)$$

is not identifiable with respect to b_2 due to cancelation when the model is formed. Note that the loss of identifiability is in relation to the original model (4.46) where all inputs are known. However, this does not imply that it is practically possible to estimate both parameters since u_I is unknown.

Hence, if the models have a known parameterization, it is the indirect model (4.11) that has to be identifiable with respect to the parameters and neglecting inputs might lead to trivial losses of identifiability as discussed above. If (4.3), (4.4) and (4.5) are first parameterized, identifiability of (4.3) might both be gained and lost when the indirect model (4.11) is formed due to shared dynamics or parameters with the input measurements.

In other applications, it is the dynamics, for instance, the transfer function between the output and one specific input, that is desired and the parameters are of less importance. This is typically the situation when physical models are not known or are not desirable to derive and black-box models are used. In these situations, it is thus certain dynamics that should be identifiable, rather than the parameters, and certain properties of the original system are required.

It was shown in Section 4.3.1 that the indirect input measurements' dependencies on other signals can change the dynamics between the known inputs and the out-

put in the indirect model. Hence, knowledge of the structural properties of the system is needed in order to guarantee that certain dynamics are unaltered when the indirect model is formed. For this reason, it is beneficial to work with a dynamic network description of the system. This topological information is perhaps naturally available in networked systems, such as power networks or water supply networks, where the interconnection structure typically is known. In more traditional systems, structural information might be available from a physical model of the system as we will see in Chapters 7 to 9. However, guarantees cannot be made in the general case when nothing is known about the system but an indirect model might still be useful in certain applications as we will see in Chapter 6.

Assuming that there are no direct paths from the remaining unobservable nodes $w_{\bar{\mathcal{U}}}$ to the indirect input measurement $w_{\mathcal{I}}$, then the indirect model (4.41) is unaffected by the elimination $w_{\bar{\mathcal{U}}}$. However, using the indirect node observations to eliminate $w_{\bar{\mathcal{U}}}$ will alter the dynamics according to (4.33). Sufficient but not necessary conditions for guaranteeing module equality $\tilde{G}_{ji} = G_{ji}$ for an observable predictor input $i \in \mathcal{K}_j$ are formalized below.

Proposition 4.1 (Unknown predictor inputs independent of remaining unobservable nodes). *For the dynamic network (2.3), the transfer function $\tilde{G}_{ji}, i \in \mathcal{K}_j$ of the indirect model (4.24) will be equal to G_{ji} of (2.3) if the following conditions are satisfied:*

- (a) There exist filters $f_{\bar{\mathcal{U}}\mathcal{D}}$ and $f_{\bar{\mathcal{U}}\mathcal{I}}$ such that $f_{\bar{\mathcal{U}}\mathcal{D}}G_{\mathcal{D}\bar{\mathcal{U}}} + f_{\bar{\mathcal{U}}\mathcal{I}}G_{\mathcal{I}\bar{\mathcal{U}}} = I_{n_{\bar{\mathcal{U}}} \times n_{\bar{\mathcal{U}}}}$
- (b) $G_{\mathcal{I}\bar{\mathcal{U}}} = 0$
- (c) $G_{\mathcal{I}j} = 0$
- (d) $G_{\mathcal{I}i} = 0$

□

Proof: Condition (a) implies that all predictor inputs $w_k, k \in \mathcal{N}_j$, are directly or indirectly observed ($w_{\bar{\mathcal{U}}}$ are not predictor inputs by Definition 4.1) while Condition (b) guarantees that the indirect model will be independent of $w_{\bar{\mathcal{U}}}$. Hence, no change of G_{ji} will occur when the remaining unobservable nodes $w_{\bar{\mathcal{U}}}$ are eliminated. Conditions (c) and (d) ensure that G_{ji} is unaltered when the indirect observations are used to eliminate $w_{\bar{\mathcal{U}}}$. Direct insertion of conditions (c) and (d) into (4.33) gives

$$w_j = \tilde{G}_{j\mathcal{K} \setminus i} w_{\mathcal{K} \setminus i} + G_{ji} w_i + \tilde{G}_{j\mathcal{O}} w_{\mathcal{O}} + \tilde{G}_{j\mathcal{I}} w_{\mathcal{I}} + \tilde{G}_{j\mathcal{D}} w_{\mathcal{D}} + \tilde{\delta}_j + \tilde{\tau}_j \quad (4.48)$$

which shows that $\tilde{G}_{ji} = G_{ji}$. □

Example 4.5: Proposition 4.1 applied to Example 3.2

In Example 3.2, the indirect observation is given by

$$w_4 = G_{41}w_1 + G_{45}w_5, \quad (4.49)$$

$\mathcal{K}_2 = \{1, 3\}$, $\mathcal{U}_2 = \{5, 6, 8\}$, $\bar{\mathcal{U}}_2 = \{5\}$ and $\mathcal{I}_2 = \{4\}$. The filter $f_{\bar{\mathcal{U}}\mathcal{I}}$ can be chosen as G_{45}^{-1} and the indirect node observation has no direct paths from w_2, w_6 or w_8 . This implies that Conditions (a) and (b) of Proposition 4.1 are satisfied. Since $G_{43} = 0$, Proposition 4.1 gives $\tilde{G}_{23} = G_{23}$ which was shown in Example 4.1.

Proposition 4.1 gives conditions on the coupling between internal signals in the system, i.e. that links between certain internal signals cannot exist, but it does not require knowledge about any of the modules. When the input w_i to the module of interest is unobservable, i.e. $i \in \mathcal{U}_j$, more information about the modules in the network is typically needed. Instead of listing a number of special cases, it is perhaps more natural to talk about identifiability of the indirect model as we did in the beginning of this section. Assume that (3.6) is parameterized with ϑ , then the module $G_{ji}(\vartheta)$ can be recovered if the resulting indirect model

$$w_j = \tilde{G}_{j\kappa}(\vartheta)w_\kappa + \tilde{G}_{j\mathcal{O}}(\vartheta)w_\mathcal{O} + \tilde{G}_{j\mathcal{I}}(\vartheta)w_\mathcal{I} + \tilde{G}_{j\mathcal{D}}(\vartheta)w_\mathcal{D} + \tilde{\delta}_j(\vartheta) + \tilde{\tau}_j(\vartheta) \quad (4.50)$$

is identifiable with respect to ϑ .

4.3.3 Data Informativity in the Indirect Model

Consider the dataset

$$Z_N = (y_{\mathcal{O}}, \tilde{u}_t, \delta_t)_{t=1+t_N}^{N+t_N} \quad (4.51)$$

consisting of N data points where t_N is the starting time. As mentioned in Section 2.7, to get an unique solution to the system identification problem, the model structure has to be structurally identifiable and the dataset (4.51) has to be informative, i.e. that the dynamics of the system is excited during the data collection experiment, such that the dataset contains sufficient information about the input-output behavior (Bellman and Åström, 1970). In an open-loop case, a sufficient condition for data informativity is that the input is persistently exciting (Ljung, 1999). However, in closed-loop a persistently exciting input is not a sufficient condition for informativity due to the feedback mechanism mentioned in Section 4.5 and the disturbances acting on the input measurements. A sufficient condition in the general case, i.e. when $H_I(\mathbf{p})$, $H_D(\mathbf{p})$ and $F_\tau(\mathbf{p})$ all are nonzero, is that the external signal $F_\delta(\mathbf{p})\delta$ is informative enough. However, note that the informativity requirement on $F_\delta(\mathbf{p})\delta$ might be relaxed depending on the complexity of the feedback structure $F_\tau(\mathbf{p})$, see Bazanella et al. (2010) for a discussion on the closed-loop case or Weerts et al. (2015) for the more general dynamic network case.

4.4 Predictor Input and Output Selection

So far in this chapter, it has been assumed that the division of the measurements y into the output y_O and the input measurements y_D and y_I is given. In practice, this division has to be done by the user. Given the desirable dynamics or properties, the measurements y and the external signals δ and τ , it is not always obvious which subsystem that should be estimated, i.e. how to divide the measurements. This ambiguity stems partially from the symmetry of (4.3) and (4.5) which is illustrated by the following example.

Example 4.6: Ambiguity in predictor input and output selection

The choice of output y_O and indirect input measurement y_I is not always simple and it is dependent on the application. Consider the model

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_1(p) \\ G_2(p) \end{bmatrix} \begin{bmatrix} u_K \\ u_I \end{bmatrix} = \begin{bmatrix} G_{1K}(p) \\ G_{2K}(p) \end{bmatrix} u_K + \begin{bmatrix} G_{1I}(p) \\ G_{2I}(p) \end{bmatrix} u_I \quad (4.52)$$

where y_1 , y_2 , u_K and u_I are all scalar. We are interested in the dynamics of $G_1(p)$ and both measurements can be used as input measurement. If the second measurement is used as input measurement, the indirect model is given by

$$y_1 = (G_{1K}(p) - G_{1I}(p)G_{2I}^{-1}(p)G_{2K}(p))u_K + G_{1I}(p)G_{2I}^{-1}(p)y_2 \quad (4.53)$$

Note that if (4.53) is multiplied by $(G_{1I}(p)G_{2I}^{-1}(p))^{-1} = G_{2I}(p)G_{1I}^{-1}(p)$, then

$$y_2 = (G_{2K}(p) - G_{2I}(p)G_{1I}^{-1}(p)G_{1K}(p))u_K + G_{2I}(p)G_{1I}^{-1}(p)y_1 \quad (4.54)$$

is obtained after some algebraic manipulations. Equation (4.54) is the model obtained if the first measurement is used as input measurement. Estimating the model (4.53) is thus in some sense the inverse problem to estimating the one in (4.54). Which formulation that is preferable depends on the application, for instance, in estimation of inverse models it might be beneficial to directly estimate the inverse rather than to first estimate a model and then inverting it, see Jung and Enqvist (2013).

The choice of predictor inputs and outputs can be based on properties of the indirect model, such as stability of the predictor or that the transfer-function matrices are proper. Furthermore, the choice might depend on the usage of the model as was briefly mentioned in Example 4.6 and one application, feedforward control synthesis, will be discussed in Section 6.2. Hence, the division of measurements into the output measurements y_O and the input measurements y_D and y_I depends on the application and ultimately, many choices can lead to satisfactory results.

4.5 Connection to Existing System Identification Problems

One strength of the indirect modelling framework is its flexibility. In addition to solving a new set of system identification problems, many special cases of the indirect model (4.11), such as when there are only indirectly measured inputs, are equivalent to already existing problems and (4.11) can in this sense be seen as a generalization of these classical problems.

4.5.1 Open-loop Configurations

The most elementary case is when $F_\tau = 0$ and hence, that the input u defined in (3.2) is uncorrelated with the disturbance τ , i.e. that (3.5) is in open-loop operation.

If the dimension of u is equal to the number of known inputs n_K , then the classical open-loop multi-input-multi-output (MIMO) case shown in Figure 4.3(a) is recovered. See, for instance, Ljung (1999) for the discrete-time case and Garnier and Wang (2008) for the related continuous-time counterpart.

If all the inputs are directly measured, i.e. $n_D = \dim(u)$, then the indirect model will result in an errors-in-variable (EIV) problem, see Figure 4.3(b). The classical EIV problem is given if $G_{DD} = 1$ and τ_D is measurement noise. Note that the framework of this paper allows τ_O and τ_D to be correlated and that extra consideration has to be taken in this case. Söderström (2007) and Söderström (2011) are surveys for EIV problems and methods, and Mahata and Garnier (2006) deals with one method of estimating the model in a continuous-time EIV problem.

Finally, if $n_I = \dim(u)$ and hence, that there are only indirectly measured inputs, then the resulting model is a sensor-to-sensor system identification problem (S2SID) and the resulting transfer function is called a pseudo transfer function (D'Amato et al., 2009). Note that the principal structure of the S2SID problem shown in Figure 4.3(c) is misleadingly similar to the EIV problem shown in Figure 4.3(b). However, there is a significant difference not shown in the superficial structures of the problems in Figures 4.3(b) and 4.3(c). In the S2SID framework, dynamics shared between $G_{II}(\mathbf{q})$ and $G_{OI}(\mathbf{q})$ will cancel as discussed in Section 3.6.

Figure 4.3: In some sense, the indirect framework is a generalization of existing system identification problems. The classical open-loop problem (a), the classical EIV problem (b) and the S2SID problem (c) are all special cases if the system is acting in open loop ($F_\tau = 0$).

Figure 4.4: If F_τ is nonzero, more complex systems can be modeled.

4.5.2 Closed-loop and Network Configurations

More flexible models could be obtained if $F_\tau \tau$ is correlated with τ_O . This means that the input u defined in (3.2) is correlated with the disturbance τ_O , i.e. that there is some feedback mechanism from the measurements y to the input u .

Figure 4.4(a) shows the classical closed-loop system identification problem given by

$$y_O = G_{OK}(\mathbf{p})u_K + H_O(\mathbf{p})\tau \quad (4.55)$$

and

$$u_K = F_y(\mathbf{p})y_O + F_r(\mathbf{p})\delta \quad (4.56)$$

where the input u is assumed to be known, i.e. $n_K = \dim(u)$. If (4.55) and (4.56) are combined, the input can be written as

$$u_K = \underbrace{(I - F_y(\mathbf{p})G_{OK}(\mathbf{p}))^{-1}F_y(\mathbf{p})H_O(\mathbf{p})}_{F_\tau(\mathbf{p})}\tau + \underbrace{(I - F_y(\mathbf{p})G_{OK}(\mathbf{p}))^{-1}F_r(\mathbf{p})}_{F_\delta(\mathbf{p})}\delta \quad (4.57)$$

which fits the indirect framework. For details on closed-loop system identification, see for instance, Forssell and Ljung (1999).

Similarly, an EIV closed-loop system identification problem is recovered if the input is directly measured ($u = u_D$). For more details on the EIV closed-loop problem, see for instance, Söderström et al. (2013).

The dynamic network model (2.3), which has been used in Chapter 3 and this chapter, is an even more general model and the previously mentioned closed-loop problems are special cases. It was shown in Chapter 3 that the input measurements w_D and w_τ cannot be used directly in the framework of Dankers (2014) to estimate the transfer function $G_O(\mathbf{p})$ if there are no paths from the indirect and direct node observations w_τ and w_D to the output w_j . In some sense, the use of the input measurements and the transformation into the indirect model (4.24) create artificial paths that make it possible to utilize these measurements. The input measurements w_D and w_τ can with this point of view be seen as a generalization of the additive measurement error in the framework of Dankers (2014).

5

Estimation of the Indirect Model

The choices made during modelling, such as using a continuous- or discrete-time model, the predictor input-output selection and the parameterization, will decide the properties of the resulting indirect model. Depending on these properties, several methods can be used to estimate the parameters of the indirect model.

An instrumental variable method has been used to estimate the parameters in the applications of this thesis. Here, this method will be described for the general indirect model and the emphasis will be on the creation of instruments. The benefits of this method are low requirements on prior knowledge about the disturbances, and that process and measurement disturbances on the predictor in- and output as well as direct terms in a potential feedback mechanism can be managed. This type of approach can in many situations give sufficient accuracy while giving a low computational cost in the estimation problem. Note that the indirect framework is not dependent on the approach suggested in this chapter, it is merely one example and other alternatives might be more suitable as discussed in Section 4.2.

In addition, this chapter will detail two multiple dataset approaches that can be used to overcome certain identifiability issues. By using these approaches, the amount of necessary *a priori* information is reduced which could simplify the identification process of real systems.

5.1 Extended Instrumental Variable Method

There are two terms contributing to the output of the indirect model (4.11), one that contains information about the interesting input-output relation and one that corresponds to the contribution from disturbances that cannot be described by the inputs used in the model. Recall the instrumental variable method described in Section 2.5.2, which in principle finds the parameters by solving the equations

$$\frac{1}{N_k} \sum_{t=1}^{N_k} \zeta_t^k \left[y_{O,t}^k - (\varphi_t^k)^T \theta_k(\vartheta) \right] = 0, \quad k = 1, \dots, n_O \quad (5.1)$$

simultaneously. In this thesis, the extended instrumental variable method is used and the parameters are estimated by computing

$$\hat{\vartheta} = \underset{\vartheta}{\operatorname{argmin}} \sum_{k=1}^{n_O} \|Y^k - \Phi^k \theta_k(\vartheta)\|_{Q_k}^2 \quad (5.2)$$

where $\|x\|_{Q_k}^2 = x^T Q_k x$ and $Q_k \geq 0$ is a weighting matrix. The matrices

$$Y^k = \frac{1}{N_k} \begin{bmatrix} \zeta_1^k & \dots & \zeta_{N_k}^k \end{bmatrix} \begin{bmatrix} \bar{y}_{O,1}^k \\ \vdots \\ \bar{y}_{O,N_k}^k \end{bmatrix}, \text{ and } \Phi^k = \frac{1}{N_k} \begin{bmatrix} \zeta_1^k & \dots & \zeta_{N_k}^k \end{bmatrix} \begin{bmatrix} (\bar{\varphi}_1^k)^T \\ \vdots \\ (\bar{\varphi}_{N_k}^k)^T \end{bmatrix} \quad (5.3)$$

are the data and regressions matrices, respectively, where $\bar{y}_{O,t}^k = L^k(\mathbf{q}) y_{O,t}^k$, $(\bar{\varphi}_t^k)^T = L^k(\mathbf{q}) (\varphi_t^k)^T$ and $L^k(\mathbf{q})$, $k = 1, \dots, n_O$, are stable prefilters. Note that this formulation allows for datasets of different lengths for each component.

The challenge is to design the instruments ζ_t^k and the prefilters $L^k(\mathbf{q})$, $k = 1, \dots, n_O$, such that the accuracy of the predictor is maximized. A good instrument should be correlated with the regressors but be uncorrelated with the process disturbance, sensor biases and measurement noises. The closed-loop IV method and its properties have been investigated in Forsell and Chou (1998), Gilson and Van den Hof (2005) and Gilson et al. (2011) among others. It is shown that the optimal instruments are given by the filtered noise-free regression vector and that the optimal prefilter is given by the true inverse noise model. Naturally, this choice is unrealistic since it requires knowledge of the true system and noise-free measurements to be implemented. The approach used in this thesis is the iterative approach introduced by Gilson et al. (2011) and described in Section 2.5.2. During each iteration, the current estimate is used to create the instruments ζ_t^k and the prefilters $L^k(\mathbf{q})$, $k = 1, \dots, n_O$, which are then used to get a new estimate, and the iterations are terminated when the parameters have converged, see Algorithm 2.1.

For the indirect model, the instruments for the k^{th} component is given by

$$\zeta_t^k = L^k(\mathbf{q}) \left[-\hat{y}_{O,t-1}^k \dots - \hat{y}_{O,t-\hat{n}_O}^k, \hat{u}_{K,t-1}^k \dots \hat{u}_{K,t-\hat{n}_K}^k, \right. \\ \left. \hat{y}_{I,t-1}^k \dots \hat{y}_{I,t-\hat{n}_I}^k, \hat{y}_{D,t-1}^k \dots \hat{y}_{D,t-\hat{n}_D}^k, \hat{u}_{A,t-1}^k \dots \hat{u}_{A,t-\hat{n}_A}^k \right] \quad (5.4)$$

where the constants $\hat{n}_O^k$, $\hat{n}_K^k$, $\hat{n}_I^k$, $\hat{n}_D^k$ and $\hat{n}_A^k$ are the number of time lags included

in ζ_t^k for the output and the inputs of the indirect model. Here, $\hat{n}_O^k = 0$ means that $\hat{y}_{O,t}^k$ is not included in ζ_t^k . The in- and outputs are simulated with the current parameter estimate $\hat{\vartheta}$, for instance, by

$$\begin{bmatrix} \hat{y}_{O,t}^k \\ \hat{u}_t \end{bmatrix} = \begin{bmatrix} \hat{y}_{O,t}^k \\ \hat{u}_{K,t} \\ \hat{y}_{I,t} \\ \hat{y}_{D,t} \\ \hat{u}_{A,t} \end{bmatrix} = \begin{bmatrix} G_{OK}(\mathbf{q}, \hat{\vartheta}) & G_{OI}(\mathbf{q}, \hat{\vartheta}) & G_{OD}(\mathbf{q}, \hat{\vartheta}) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ G_{IK}(\mathbf{q}, \hat{\vartheta}) & G_{II}(\mathbf{q}, \hat{\vartheta}) & G_{ID}(\mathbf{q}, \hat{\vartheta}) & G_{IA}(\mathbf{q}, \hat{\vartheta}) & 0 \\ 0 & 0 & G_{DD}(\mathbf{q}, \hat{\vartheta}) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \hat{F}_\delta(\mathbf{q}) \delta_t \quad (5.5)$$

where $\hat{F}_\delta(\mathbf{q})$ is a discrete-time estimate of $F_\delta(\mathbf{p})$. The prefilters $L^k(\mathbf{q})$, $k = 1, \dots, n_O$, are estimated from the residuals and the best choice depends on the model structure, see Gilson et al. (2011). Note that (5.5) is only used to create the instruments. This means that if (2.44) is fulfilled, than an incorrect estimate of $F_\delta(\mathbf{p})$ only will affect the second-order properties of the estimator (Söderström and Stoica, 1989).

It is important to realize that, although the proposed instrumental variable method is simple to apply and requires little prior knowledge about the system of interest, it is typically not variance optimal. To understand this, consider the special case of the dynamic network model shown in Figure 5.1 and given by

$$\begin{bmatrix} w_j \\ w_{\mathcal{I}} \\ w_{\bar{\mathcal{U}}} \\ w_{\bar{\mathcal{U}}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{I}} \\ w_{\bar{\mathcal{U}}} \\ w_{\bar{\mathcal{U}}} \end{bmatrix} + \begin{bmatrix} \delta_j + \tau_j \\ \delta_{\mathcal{I}} + \tau_{\mathcal{I}} \\ \delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}} \\ \delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}} \end{bmatrix} \quad (5.6)$$

where $\times$ is a non-zero entry. The indirect model is given by

$$\begin{bmatrix} w_j \\ w_{\mathcal{I}} \end{bmatrix} = \begin{bmatrix} 0 & \times \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{I}} \end{bmatrix} + \begin{bmatrix} \delta_j + \tau_j - f_{\bar{\mathcal{U}}\mathcal{I}}(\delta_{\mathcal{I}} + \tau_{\mathcal{I}}) \\ \delta_{\mathcal{I}} + \tau_{\mathcal{I}} + G_{\bar{\mathcal{U}}\mathcal{I}}(\delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}}) + G_{\bar{\mathcal{U}}\bar{\mathcal{U}}}G_{\bar{\mathcal{U}}\mathcal{I}}(\delta_{\bar{\mathcal{U}}} + \tau_{\bar{\mathcal{U}}}) \end{bmatrix} \quad (5.7)$$

Note that $w_{\mathcal{I}}$ thus captures all the excitations that enters $w_{\bar{\mathcal{U}}}$ and $w_{\bar{\mathcal{U}}}$ and have paths to $w_{\mathcal{I}}$ as previously discussed in Section 4.3.1. Furthermore, $\tau_{\mathcal{I}}$ will act as a confounding variable since it is unknown and thus cannot be used as an input. The proposed instrumental variable method mitigates this by using instruments created by δ . This means that the excitation supplied by $\tau_{\bar{\mathcal{U}}}$ and $\tau_{\bar{\mathcal{U}}}$ will not be used to estimate the parameters since it is not correlated with the instruments. As suggested in Section 4.3.1, the estimation problem is basically an errors-in-variables problem due to the correlation between $w_{\mathcal{I}}$ and $\tau_{\mathcal{I}}$ and an alternative to the instrumental variable method is to use any of the methods suggested in, for instance, Söderström (2011). However, these methods typically require more knowledge about the disturbances which might not be available.

Figure 5.1: A dynamic network model where the dotted arrows correspond to coupling between the variables. Note that $w_{\mathcal{I}}$ captures all the excitation that enters into $w_{\bar{\mathcal{U}}}$ and this information is available if the indirect model is used.

5.2 Multiple Dataset Identification

In this section, two approaches for combining several sources of information and datasets will be proposed to overcome identifiability limitations. The first approach is to use a multi-stage identification procedure where the datasets are used consecutively to gain some information in each step. One benefit of this approach is that it can be computationally cheap under the right circumstances. A drawback is that the information in the previous step is locked down in the next step. The second approach is a joint estimation procedure where all of the datasets are used to simultaneously estimate all the information in one step. A major benefit of this approach is that the information in all the datasets is used in the estimator. However, it will typically lead to an estimator that is nonlinear in the parameters.

5.2.1 Gaining A Priori Information by Using Multiple Datasets

Identifiability issues can typically be avoided by introducing more information. This can be achieved by introducing new sensors or introducing additional *a priori* information, such as known parameters. If these solutions are intractable, an alternative is to change the experimental conditions, for instance, the mass of the system of interest, in a known way and collect new datasets. Since all the datasets are collected from the same system and the experimental conditions have been changed in a known way, this can be used to avoid identifiability issues. To introduce the concept of multiple datasets, let us firstly consider the following example.

Example 5.1: Overcoming identifiability issues with extra measurements

Consider the model

$$y_O^{(1)} = (a_1 + b_1)u_K \quad (5.8)$$

where both a_1 and b_1 are assumed to be unknown and hence, the model is unidentifiable. A model can be made identifiable by incorporating additional knowledge. For instance, if it was possible to introduce a second measurement

$$y_O^{(2)} = a_1 u_K \quad (5.9)$$

then, the extended model

$$y_O = \begin{bmatrix} y_O^{(1)} \\ y_O^{(2)} \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1 \end{bmatrix} u_K \quad (5.10)$$

would become identifiable. If it is not possible to install extra sensors in the experiment but it is possible to change the experimental setup in a known way between experiments, then measurements from several consecutive experiments can be used as an alternative. For example, if (5.8) and (5.9) were collected from different experiments, they can be combined into the artificial MIMO model

$$y_O = \begin{bmatrix} y_O^{(1)} \\ y_O^{(2)} \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} u_K^{(1)} \\ u_K^{(2)} \end{bmatrix} \quad (5.11)$$

where the submodels are related through the parameters and the complete model is structurally identifiable.

Hence, in some situations it is possible to eliminate identifiability issues by introducing additional information by collecting multiple datasets

$$Z_i = \left(y_O^{(i)}, \tilde{u}_t^{(i)}, \delta_t^{(i)} \right)_{t=1+t_{N_i}}^{N_i+t_{N_i}}, \quad i = 1, \dots, k \quad (5.12)$$

for different operational conditions which are related through the parameters. The identifiability properties can be investigated by looking at an extended model similar to (5.11). Consider the corresponding indirect models given on state-space form

$$\begin{aligned} \dot{x}^{(i)} &= A(\vartheta^{(i)})x^{(i)} + B(\vartheta^{(i)})\tilde{u}^{(i)} \\ y_O^{(i)} &= C(\vartheta^{(i)})x^{(i)} + D(\vartheta^{(i)})\tilde{u}^{(i)}, \quad i = 1, \dots, k \end{aligned} \quad (5.13)$$

which are each unidentifiable if the parameters $\vartheta^{(i)}$ are fully unknown. In some situations, such as (5.11) of Example 5.1, elements of $\vartheta^{(i)}$ are manipulable by the user and can be used to gain identifiability of the extended model given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A(\vartheta^{(1)}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A(\vartheta^{(k)}) \end{bmatrix} x + \begin{bmatrix} B(\vartheta^{(1)}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B(\vartheta^{(k)}) \end{bmatrix} \tilde{u} \\ y_O &= \begin{bmatrix} C(\vartheta^{(1)}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C(\vartheta^{(k)}) \end{bmatrix} x + \begin{bmatrix} D(\vartheta^{(1)}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D(\vartheta^{(k)}) \end{bmatrix} \tilde{u} \end{aligned} \quad (5.14)$$

which is identifiable with respect to the union of all the parameters denoted ϑ . This typically requires a large overlap of parameters between $\vartheta^{(i)}$, $i = 1, \dots, k$, such that the number of unknown parameters do not increase when the extra datasets are included. For examples in real applications, see for instance, Sections 7.2.1 and 8.3.

5.2.2 Relating Datasets Through Parameters

To utilize the approach in Section 5.2.1, a parameterization that can relate the datasets is desired. For example, if a physical model of the system is available, then the continuous-time physical parameters is one possible parameterization. If the indirect model (4.11) was derived in continuous time but a discrete-time version is desired, there are a number of discretization methods to choose from. Here, the bilinear transform will be used. The bilinear transform is given by

$$\mathbf{p} = \frac{2}{T} \cdot \frac{\mathbf{q} - 1}{\mathbf{q} + 1} \quad (5.15)$$

where T is the sample period and $\mathbf{q}$ is the shift operator, i.e. $\mathbf{q}z_t = z_{t+1}$. In addition to preserving the stability properties and being reasonably accurate for short sampling periods, it has an algebraic relation between $\mathbf{p}$ and $\mathbf{q}$ which implies that it is possible to have a discrete-time model with continuous-time parameters. A discrete-time model is given by the matrix fraction description

$$\begin{aligned} y_{O,t} &= \tilde{G}_O \left(\frac{2}{T} \frac{\mathbf{q}-1}{\mathbf{q}+1}, \vartheta \right) \tilde{u}_t + \tilde{\tau}_{O,t} = \tilde{G}_{O,d}(\mathbf{q}, \vartheta) \tilde{u}_t + \tilde{\tau}_{O,t} \\ &= \tilde{A}_{O,d}^{-1}(\mathbf{q}, \vartheta) \tilde{B}_{O,d}(\mathbf{q}, \vartheta) \tilde{u}_t + \tilde{\tau}_{O,t} \end{aligned} \quad (5.16)$$

assuming that the continuous-time model is finite-dimensional (Kailath, 1980).

Here, the matrices of (5.16) are normalized such that

$$\tilde{A}_{O,d}(\mathbf{q}, \vartheta) = I + A_1(\vartheta)\mathbf{q}^{-1} + \dots + A_{n_a}(\vartheta)\mathbf{q}^{-n_a} \quad (5.17a)$$

$$\tilde{B}_{O,d}(\mathbf{q}, \vartheta) = B_0(\vartheta)\mathbf{q}^{-n_k} + \dots + B_{n_b-1}(\vartheta)\mathbf{q}^{-n_b-n_k+1} \quad (5.17b)$$

The model (5.16) can alternatively be written as the difference equation

$$\begin{aligned} y_{O,t} = & -A_1(\vartheta)y_{O,t-1} - \dots - A_{n_a}(\vartheta)y_{O,t-n_a} \\ & + B_0(\vartheta)\tilde{u}_{t-n_k} + \dots + B_{n_b-1}(\vartheta)\tilde{u}_{t-n_b-n_k+1} + \tilde{A}_{O,d}(\mathbf{q}, \vartheta)\tilde{\tau}_{O,t} \end{aligned} \quad (5.18)$$

Note that $\tilde{A}_{O,d}(\mathbf{q}, \vartheta)\tilde{\tau}_{O,t}$ is a colored noise process even if $\tilde{\tau}_{O,t}$ is white.

In order to enable and simplify the possibility to extend the model according to the discussion in Section 5.2.1, it is assumed that the i^{th} component can be written as

$$y_{O,t}^{(i)} = \left(\varphi_t^{(i)} \right)^T \theta_i(\vartheta) + \tau_{O,t}^{(i)} \quad (5.19)$$

which allows datasets of different lengths that are related through ϑ .

5.2.3 Multi-Stage Identification Approach

The multi-stage identification approach uses the k datasets (5.12) in a sequential manner to obtain more information in each step. In each step, all the previously estimated information can be used together with a new dataset to gain identifiability in the current step. For generality, the approach allows for an initial step where *a priori* information can be introduced. This information, here denoted by $\hat{\vartheta}_{p,0}$ can either be measured or estimated by some special experiment. For examples, see Sections 7.4 and 8.5. The multi-stage approach is summarized below.

Approach 5.1 (Multi-stage identification approach).

1. Obtain the *a priori* information, here, represented by $\hat{\vartheta}_{p,0}$.
2. For each of the dataset $Z_i, i = 1, \dots, k$, use the estimates $\hat{\vartheta}_{p,j}, j = 0, \dots, i-1$ and Z_i to estimate $\vartheta_{p,i}$ by computing

$$\begin{aligned} \hat{\theta} = \underset{\vartheta}{\operatorname{argmin}} \quad & V_i(\hat{\vartheta}_{p,0}, \dots, \hat{\vartheta}_{p,i-1}, \theta, Z_i) \\ \text{s.t.} \quad & h_i(\hat{\vartheta}_{p,0}, \dots, \hat{\vartheta}_{p,i-1}, \theta) = 0 \end{aligned} \quad (5.20)$$

where $V_i(\hat{\vartheta}_{p,0}, \dots, \hat{\vartheta}_{p,i-1}, \vartheta, Z_i)$ is a nonlinear criterion function and $h_i(\cdot) = 0$ is a possibly nonlinear equality constraint, and then, depending on the parameterization, transform $\hat{\theta}$ to $\hat{\vartheta}_i$.

One downside of this approach is that the uncertainty information is not incorporated into the next step. The major benefit of this approach is when each step becomes computationally cheap to solve. An example is when the cost function of (5.20) can be written on the linear regression form

$$V_i(\hat{\vartheta}_{p,0}, \dots, \hat{\vartheta}_{p,i-1}, \theta, Z_i) = \|\Phi^i \theta - Y^i\|_2^2 \quad (5.21)$$

and the constraints of (5.20) are linear and thus can be written on the form

$$h_i(\hat{\vartheta}_{p,0}, \dots, \hat{\vartheta}_{p,i-1}, \theta) = B_i \theta - b_i = 0 \quad (5.22)$$

Then each step can be efficiently solved by using factorization, see Golub and Van Loan (1996, Ch. 12). Note that the constraints are needed to relate the datasets. Two commonly used estimators that can be written on this form are:

1. The least-squares estimator given by

$$\Phi^i = \frac{1}{N_i} \begin{bmatrix} \left(\varphi_1^{(i)} \right)^T \\ \vdots \\ \left(\varphi_{N_i}^{(i)} \right)^T \end{bmatrix} \text{ and } Y^i = \frac{1}{N_i} \begin{bmatrix} y_{O,1}^{(i)} \\ \vdots \\ y_{O,N_i}^{(i)} \end{bmatrix} \quad (5.23)$$

2. The (extended) instrumental variable estimator given by

$$\Phi^i = \frac{1}{N_i} \begin{bmatrix} \zeta_1^{(i)} & \dots & \zeta_{N_i}^{(i)} \end{bmatrix} \begin{bmatrix} \left(\bar{\varphi}_1^{(i)} \right)^T \\ \vdots \\ \left(\bar{\varphi}_{N_i}^{(i)} \right)^T \end{bmatrix} \text{ and } Y^i = \frac{1}{N_i} \begin{bmatrix} \zeta_1^{(i)} & \dots & \zeta_{N_i}^{(i)} \end{bmatrix} \begin{bmatrix} \bar{y}_{O,1}^{(i)} \\ \vdots \\ \bar{y}_{O,N_i}^{(i)} \end{bmatrix} \quad (5.24)$$

where ζ_t is the vector of instruments, $\bar{y}_{O,t} = L(\mathbf{q})y_{O,t}$, $\bar{\varphi}_t^T = L(\mathbf{q})\varphi_t^T$ and $L(\mathbf{q})$ is a stable prefilter. Here, the prefilter is either set a priori or estimated in an iterative scheme, and for a minimum variance estimator the prefilter should be equal to the inverse noise model, see Section 2.5.2.

Note that each step can be linear in some auxiliary variables that can be transformed into the wanted parameters after estimation, for example, see Section 7.4.

5.2.4 Joint Identification Approach

An alternative to the sequential Approach 5.1 in the previous section is to use all the k datasets simultaneously and jointly estimate the parameters in the extended model. One issue with Approach 5.1 is that the estimates from the earlier steps are used in current step without considering their uncertainty. The joint approach deals with this automatically at the expense of a possibly more complex estimator. Again, *a priori* information is allowed for generality like in the previous section. The joint identification approach is summarized below.

Approach 5.2 (Joint identification approach).

The *a priori* information $\hat{\vartheta}_{p,0}$ is used together with the datasets, $Z_i, i = 1, \dots, k$, to estimate $\hat{\vartheta}_p$ by computing

$$\begin{aligned} \hat{\vartheta}_p &= \underset{\tilde{\vartheta}_p}{\operatorname{argmin}} V(\hat{\vartheta}_{p,0}, \tilde{\vartheta}_p, Z_1, \dots, Z_k) \\ \text{s.t.} \quad & h(\hat{\vartheta}_{p,0}, \tilde{\vartheta}_p) = 0 \\ & \tilde{\vartheta}_p^{\text{lb}} \leq \tilde{\vartheta}_p \leq \tilde{\vartheta}_p^{\text{ub}} \end{aligned} \quad (5.25)$$

where $V(\cdot)$ is a nonlinear criterion function, $h(\cdot) = 0$ is a nonlinear equality constraint. Moreover, $\tilde{\vartheta}_p^{\text{lb}}$ and $\tilde{\vartheta}_p^{\text{ub}}$ represents the lower and the upper parameter bounds, respectively. Here, the inequalities should be interpreted element-wise.

A benefit of the joint approach is that the model can be parameterized using physical parameters. Here, lower and upper bounds on the parameters are introduced in the form of inequality constraints. These bounds can be used to ensure that the estimated parameters become physically reasonable, for example, that a mass is positive. As in the previous section, the datasets do not necessarily have to be of the same length.

In accordance with the previous section, Approach 5.2 can be further specified if the model can be expressed as a linear regression and the criterion function is quadratic in the linear parameters, see Section 8.5 for an example.

6

Filtering and Control Using the Indirect Model

There are many possible uses for a model of a dynamical system, for example, control, design, diagnosis, prediction or simulation (Ljung, 1999). For some of these applications, it is desired to have a model of the system, or part of the system, where the parameters have physical significance and where the model describes the reality as well as possible. This case will be discussed in Chapters 7 to 9. For other applications, it might be sufficient to work with a black-box model that describes the output well enough. In this chapter, the usefulness of a discrete-time indirect black-box model will briefly be discussed in relation to prediction and feedforward control synthesis.

6.1 Prediction Using Input Measurements

Prediction of signals in a system is an important topic in automatic control. In this section, the prediction of the j^{th} node in the network model (3.6) will be considered. Here, the models are assumed to be known while the disturbances τ and inputs $w_{\mathcal{U}}$ are assumed to be unknown. Moreover, the disturbances $\tau_k, k \in \mathcal{N}$ are assumed to be white zero-mean Gaussian noise. A one-step-ahead predictor for

$$w_{j,t} = G_{j\mathcal{K}}w_{\mathcal{K},t} + G_{j\bar{\mathcal{U}}}w_{\bar{\mathcal{U}},t} + \delta_{j,t} + \tau_{j,t}$$

defined in the first row of (3.6), is given by (Ljung, 1999)

$$\hat{w}_{j,t} = G_{j\mathcal{K}}w_{\mathcal{K},t} + G_{j\bar{\mathcal{U}}}w_{\bar{\mathcal{U}},t} + \delta_{j,t} \quad (6.1)$$

which would lead to the output error residual $\varepsilon_{j,t} = \tau_{j,t}$ if all inputs were known. However, it might not be useful in practice due to the unknown input $w_{\bar{\mathcal{U}},t}$. For the purpose of comparison, consider if the unknown input $w_{\bar{\mathcal{U}},t}$ is neglected, which will lead to the predictor

$$\hat{w}_{j,t} = G_{j\mathcal{K}}w_{\mathcal{K},t} + \delta_{j,t} \quad (6.2)$$

and the prediction error becomes

$$\varepsilon_{j,t} = w_{j,t} - \hat{w}_{j,t} = \tau_{j,t} + G_{j\bar{u}} w_{\bar{u},t} \quad (6.3)$$

As shown in Chapter 3, rather than simply neglecting the unknown inputs, a better approach is to work with the immersed model using all known signals. Assuming that the model is known, the one-step-ahead predictor of the model

$$w_{j,t} = \check{G}_{j\mathcal{K}} w_{\mathcal{K},t} + \check{G}_{j\mathcal{A}} w_{\mathcal{A},t} + \check{F}_{jj}(\delta_{j,t} + \tau_{j,t}) + \check{F}_{j\mathcal{U}}(\delta_{\mathcal{U},t} + \tau_{\mathcal{U},t}) \quad (6.4)$$

defined in (3.38), is given by

$$\hat{w}_{j,t} = \check{H}^{-1}[\check{G}_{j\mathcal{K}} w_{\mathcal{K},t} + \check{G}_{j\mathcal{A}} w_{\mathcal{A},t} + \check{F}_{jj} \delta_{j,t} + \check{F}_{j\mathcal{U}} \delta_{\mathcal{U},t}] + [1 - \check{H}^{-1}] w_{j,t} \quad (6.5)$$

where $\check{H}$ is a monic filter. The prediction error is

$$\check{\varepsilon}_{j,t} = w_{j,t} - \hat{w}_{j,t} = \check{H}^{-1}[\check{F}_{jj} \tau_{j,t} + \check{F}_{j\mathcal{U}} \tau_{\mathcal{U},t}] \quad (6.6)$$

and the covariance of the prediction error is thus dependent on the unknown excitation given by $\check{F}_{j\mathcal{U}} \tau_{\mathcal{U},t}$, i.e. all the external disturbances that excite the network and have paths to w_j not passing through any of the predictor inputs. The predictor based on the immersed model gives more accurate predictions since part of the unknown excitation is described by the upstream nodes $w_{\mathcal{A}}$ and part is described by the external user-controllable signals $\delta_{\mathcal{U}}$. However, some of the external disturbances will be unobserved unless all predictor inputs $\mathcal{N}_j$ are observable. For example, compare the effective noise

$$\check{\tau}_{2,t} = \frac{\tau_{2,t} + G_{25} \tau_{5,t} + G_{25} G_{58} ((G_{86} + G_{87} G_{76})(\delta_{6,t} + \tau_{6,t}) + G_{87} \tau_{7,t} + \tau_{8,t})}{1 - G_{25} G_{52}} \quad (6.7)$$

of (3.24) in Example 3.4 where the additional signals are neglected with the noise

$$\check{\tau}_{2,t} = \frac{\tau_{2,t} + G_{25} \tau_{5,t} + G_{25} G_{58} (G_{86} \tau_{6,t} + \tau_{8,t})}{1 - G_{25} G_{52}} \quad (6.8)$$

of (3.41) in Example 3.7 where w_7 supply additional information about the unknown excitation given by the disturbances τ_6 and τ_7 . Note that δ_6 is not a predictor input in Example 3.4 and thus part of the effective noise.

An alternative is to use a predictor based on j^{th} node of the indirect model

$$w_{j,t} = \tilde{G}_{j\mathcal{K}} w_{\mathcal{K},t} + \tilde{G}_{j\mathcal{O}} w_{\mathcal{O},t} + \tilde{G}_{j\mathcal{I}} w_{\mathcal{I},t} + \tilde{G}_{j\mathcal{D}} w_{\mathcal{D},t} \\ + \tilde{F}_{jj}(\delta_{j,t} + \tau_{j,t}) + \tilde{F}_{j\mathcal{I}}(\delta_{\mathcal{I},t} + \tau_{\mathcal{I},t}) + \tilde{F}_{j\mathcal{D}}(\delta_{\mathcal{D},t} + \tau_{\mathcal{D},t}) + \tilde{F}_{j\bar{u}}(\delta_{\bar{u},t} + \tau_{\bar{u},t})$$

defined in (4.24). To derive the prediction error, note that w_j can be expressed as

$$w_{j,t} = \tilde{G}_{j\mathcal{K}} w_{\mathcal{K},t} + \tilde{G}_{j\mathcal{O}} w_{\mathcal{O},t} \\ + (I + G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}j} - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\bar{u}\mathcal{I}} \bar{G}_{\bar{u}j})^{-1} \left[-G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{I}} w_{\mathcal{I},t} - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{D}} w_{\mathcal{D},t} \right. \\ \left. - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\bar{u}\mathcal{I}} (I - \bar{G}_{\bar{u}\bar{u}})^{-1} (G_{\bar{u}\mathcal{I}} - G_{\bar{u}\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{I}}) w_{\mathcal{I},t} \right. \\ \left. - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\bar{u}\mathcal{I}} (I - \bar{G}_{\bar{u}\bar{u}})^{-1} (G_{\bar{u}\mathcal{D}} - G_{\bar{u}\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{D}}) w_{\mathcal{D},t} \right] \\ + \tilde{F}_{j\mathcal{I}}(w_{\mathcal{I},t} - \tau_{\mathcal{I},t} - \delta_{\mathcal{I},t}) + \tilde{F}_{j\mathcal{D}}(w_{\mathcal{D},t} - \tau_{\mathcal{D},t} - \delta_{\mathcal{D},t}) \\ + \tilde{F}_{jj}(\delta_{j,t} + \tau_{j,t}) + \tilde{F}_{j\bar{u}}(\delta_{\bar{u},t} + \tau_{\bar{u},t}) \quad (6.9)$$

A predictor of (6.9) is given by

$$\begin{aligned}
 \hat{w}_{j,t} = \tilde{H}^{-1} & \left[\tilde{G}_{j\mathcal{K}} w_{\mathcal{K},t} + \tilde{G}_{j\mathcal{O}} w_{\mathcal{O},t} \right. \\
 & + (I + G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}j} - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\bar{u}} \bar{G}_{\bar{u}j})^{-1} \left[-G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{I}} w_{\mathcal{I},t} \right. \\
 & \quad - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{D}} w_{\mathcal{D},t} \\
 & \quad - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\bar{u}} (I - \bar{G}_{\bar{u}\bar{u}})^{-1} (G_{\bar{u}\mathcal{I}} - G_{\bar{u}\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{I}}) w_{\mathcal{I},t} \\
 & \quad \left. - G_{j\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\bar{u}} (I - \bar{G}_{\bar{u}\bar{u}})^{-1} (G_{\bar{u}\mathcal{D}} - G_{\bar{u}\bar{u}} f_{\bar{u}\mathcal{I}} G_{\mathcal{I}\mathcal{D}}) w_{\mathcal{D},t} \right] \\
 & + \tilde{F}_{j\mathcal{I}} (w_{\mathcal{I},t} - \delta_{\mathcal{I},t}) + \tilde{F}_{j\mathcal{D}} (w_{\mathcal{D},t} - \delta_{\mathcal{D},t}) \\
 & \left. + \tilde{F}_{jj} \delta_{j,t} + \tilde{F}_{j\bar{u}} \delta_{\bar{u},t} \right] + [1 - \tilde{H}^{-1}] w_{j,t}
 \end{aligned} \tag{6.10}$$

where $\tilde{H}$ is a monic filter. The prediction error is

$$\tilde{\varepsilon}_{j,t} = w_{j,t} - \hat{w}_{j,t} = \tilde{H}^{-1} [\tilde{F}_{jj} \tau_{j,t} + \tilde{F}_{j\bar{u}} \tau_{\bar{u},t} - \tilde{G}_{j\mathcal{I}} \tau_{\mathcal{I},t} - \tilde{G}_{j\mathcal{D}} \tau_{\mathcal{D},t}] \tag{6.11}$$

Note that the residual $\tilde{\varepsilon}_{j,t}$ is uncorrelated with the external disturbances $\tau_{\bar{u},t}$ since the direct and indirect node observations contain information about the unobservable predictor inputs $w_{\bar{u}}$ as noted and discussed in Section 4.3.1. However, this is an errors-in-variables filtering problem since the noisy $w_{\mathcal{I},t}$ and $w_{\mathcal{D},t}$ are used as predictor inputs.

From a prediction perspective, it is thus beneficial to use the indirect model if the covariance of the prediction error $\tilde{\varepsilon}_{j,t}$ of the indirect predictor (6.10) is smaller than the covariance of the prediction error $\tilde{\varepsilon}_{j,t}$ of the immersed predictor (6.5). It is thus the signal-to-noise ratio in the input measurements that determines if it is worthwhile to use the indirect predictor, i.e. if the contributions from the unknown inputs are larger than the uncertainty in the measurements.

Example 6.1: Output-error predictors comparison

Let us look at the performance of the discussed predictors for Example 3.2 with the noise models given by $\check{H} = \tilde{H} = 1$. The modules are described by

$$G_{nk} = \frac{\beta_{nk} \mathbf{q}^{-1}}{1 - \alpha_{nk} \mathbf{q}^{-1}} \tag{6.12}$$

with the parameters given in Table 6.1. The external disturbances τ_k , $k \in \mathcal{N}$ and the external user-controllable signal δ_6 were all white zero-mean Gaussian noise with unit variance while all other external user-controllable signals δ_n , $n \in \mathcal{N} \setminus \{6\}$ were zero for all times. 1000 samples were created by simulating $(I - G)^{-1}$ with $\delta + \tau$ as input and the resulting signals can be seen in Figure 6.1.

Four predictors were tested, one based on (6.2) ($\hat{w}_2 = G_{21} w_1 + G_{23} w_3$) where the node w_5 is simply neglected, two based on (6.5) with the immersed networks corresponding to (3.24) ($\hat{w}_2$) and (3.41) ($\hat{w}_2$), and one based on (6.10) with the indirect model corresponding to (4.31) ($\hat{w}_2$). The outputs of the predictors were simulated with the signals in Figure 6.1 as inputs and no filtering was performed.

The results can be seen in Figure 6.2 where the numbers in the legend describe the model fit. As expected, the predictor $\hat{w}_2$ gives the worst result. The predictor $\hat{w}_2$ partially compensates for the unobservable nodes but the excitation from

Table 6.1: The values of the transfer function parameters in Example 6.1.

Module	G_{12}	G_{18}	G_{21}	G_{23}	G_{25}	G_{36}	G_{41}	G_{45}	G_{52}	G_{58}	G_{76}	G_{86}	G_{87}
β_{nk}	0.07	0.20	0.19	0.18	0.20	0.17	0.16	0.17	0.11	0.17	0.18	0.21	0.19
α_{nk}	0.11	0.92	0.91	0.53	0.89	0.77	0.93	0.75	0.41	0.93	0.89	0.61	0.92

$\tau_n, n \in \{2, 5, 6, 7, 8\}$ is not fully captured. The predictor $\hat{w}_2$ does considerably better by including w_7 as an input. It partially compensates for the unobservable nodes but the excitation from $\tau_n, n \in \{2, 5, 6, 8\}$ is still not fully captured which can be seen around the peaks. The indirect predictor $\hat{w}_2$ follows the true signal well. The variation around the true signal is due to τ_4 entering the predictor and the unknown τ_2 . Note that the performance of the indirect model predictor is dependent on a sufficiently large signal-to-noise ratio. Finally, both $\hat{w}_2$ and $\hat{w}_2$ require knowledge about more modules than $\hat{w}_2$ as mentioned in Section 4.3.1.

Optimal filtering for errors-in-variables models has been considered in, for instance, Guidorzi et al. (2003), Diversi et al. (2003), Markovsky and De Moor (2005) and Diversi et al. (2005). In Markovsky and De Moor (2005) and Diversi et al. (2005), models on state-space form with inputs corrupted by white-noise disturbances are considered. These results can straightforwardly be applied to the indirect model (4.41) if G_{II} , G_{ID} and G_{II} are zero. Then (4.41) can be written as

$$w_{j,t} = \tilde{G}_{jK} w_{K,t} + \tilde{G}_{jO} w_{O,t} + \tilde{G}_{jI}(w_{I,t} - \delta_{I,t} - \tau_{I,t}) + \tilde{G}_{jD}(w_{D,t} - \delta_{D,t} - \tau_{D,t}) + \tilde{F}_{jj}(\delta_{j,t} + \tau_{j,t}) \quad (6.13)$$

and can be transformed into the state-space model

$$x_{t+1} = Ax_t + B_K w_{K,t} + B_O w_{O,t} + B_I(w_{I,t} - \delta_{I,t} - \tau_{I,t}) \quad (6.14a)$$

$$w_{j,t} = Cx_t + D_j(\delta_{j,t} + \tau_{j,t}) + D_K w_{K,t} + D_O w_{O,t} + D_I(w_{I,t} - \delta_{I,t} - \tau_{I,t}) + D_D(w_{D,t} - \delta_{D,t} - \tau_{D,t}) \quad (6.14b)$$

Following Diversi et al. (2005), the model (6.14) can be written as

$$x_{t+1} = Ax_t + B\tilde{u}_t + n_{x,t} \quad (6.15a)$$

$$z_{j,t} = Cx_t + n_{y,t} \quad (6.15b)$$

where $z_t = w_{j,t} - D\tilde{u}_t$, $\tilde{u}_t = [w_{K,t}^T \quad w_{O,t}^T \quad w_{I,t}^T - \delta_{I,t}^T \quad w_{D,t}^T - \delta_{D,t}^T \quad \delta_{j,t}^T]^T$,

$$B = [B_K \quad B_O \quad B_I \quad B_D \quad 0] \quad \text{and} \quad D = [D_K \quad D_O \quad D_I \quad D_D \quad D_j] \quad (6.16)$$

The aggregated noise terms are given by

$$n_{x,t} = -B_I\tau_{I,t} - B_D\tau_{D,t} \quad \text{and} \quad n_{y,t} = D_j\tau_{j,t} - D_I\tau_{I,t} - D_D\tau_{D,t} \quad (6.17)$$

which are zero-mean and with covariances given by

$$Q = E[n_{x,t} n_{x,t}^T] = B_I \lambda_I B_I^T + B_D \lambda_D B_D^T, \quad (6.18)$$

$$R = E[n_{y,t} n_{y,t}^T] = D_j \lambda_j D_j^T + D_I \lambda_I D_I^T + D_D \lambda_D D_D^T \quad (6.19)$$

Figure 6.1: A simulation of the network in Example 6.1.

Figure 6.2: A comparison of the predictors in Example 6.1.

and

$$S = E[n_{x,t}n_{y,t}^T] = B_I\lambda_I D_I^T + B_D\lambda_D D_D^T \quad (6.20)$$

The optimal one-step-ahead predictor is then given by the time-varying Kalman filter described in Section 2.4.3 with the output measurement given by z_t . The minimum variance estimates of the $w_{j,t} - \tau_{j,t}$, $w_{I,t} - \tau_{I,t}$ and $w_{D,t} - \tau_{D,t}$ are given by

$$\hat{w}_{j,t} = w_{j,t} - \lambda_j \lambda_{\varepsilon,t}^{-1} \varepsilon_t, \quad \hat{w}_{I,t} = w_{I,t} + \lambda_I D_I^T \lambda_{\varepsilon,t}^{-1} \varepsilon_t \quad \text{and} \quad \hat{w}_{D,t} = w_{D,t} + \lambda_D D_D^T \lambda_{\varepsilon,t}^{-1} \varepsilon_t \quad (6.21)$$

where $\varepsilon_t = z_t - C\hat{x}_t$ is the innovation and $\lambda_{\varepsilon,t} = CP_t C^T + R$ is the innovation's covariance (Diversi et al., 2005). Approaches for treating colored noise processes in Kalman filtering are described in, for instance, Bryson and Johansen (1965) or Bryson and Henrikson (1968).

6.2 Feedforward Control Synthesis

Assume that we are interested in controlling the output of the unknown input model

$$y_O = G_O u + \tau_O = G_{OK} u_K + G_{OU} u_U + \tau_O$$

defined in (3.5). Even if an accurate model of G_O is available, it cannot be fully utilized due to the unknown input u_U . An alternative is to use the indirect model

$$y_O = \tilde{G}_{OK} u_K + \tilde{G}_{OI} y_I + \tilde{G}_{OD} y_D + \tilde{G}_{OA} u_A + \tilde{\tau}_O$$

defined in (4.11). The issue with the indirect model is that we might not be able to control the additional signals nor the direct and indirect measurements accurately. However, the signals and the indirect model can be used in a feedforward scheme. One choice is the input given by

$$u_K = u_K^* - \tilde{G}_{OK}^{-1}(\tilde{G}_{OI} y_I + \tilde{G}_{OD} y_D + \tilde{G}_{OA} u_A) = u_K^* + \tilde{G}_{KI} y_I + \tilde{G}_{KD} y_D + \tilde{G}_{KA} u_A \quad (6.22)$$

Inserting (6.22) into (4.11) gives

$$y_O = \tilde{G}_{OK} u_K^* + \tilde{\tau}_O \quad (6.23)$$

where u_K^* can be used for controlling the output, for instance, using feedback control. In practice, the models of the system will not be perfect and the influence of the unknown inputs will only be approximately canceled.

In order for the control strategy (6.22) to be useful, the transfer-function matrices $\tilde{G}_{KI}$, $\tilde{G}_{KD}$ and $\tilde{G}_{KA}$ have to be realizable. Recall from Section 4.4 that it might be beneficial to directly estimate the inverse if a model of the inverse is desired in the application. This can be achieved by solving (4.11) for u_K , which gives

$$\begin{aligned} u_K &= \tilde{G}_{OK}^{-1} y_O - \tilde{G}_{OK}^{-1} \tilde{G}_{OI} y_I - \tilde{G}_{OK}^{-1} \tilde{G}_{OD} y_D - \tilde{G}_{OK}^{-1} \tilde{G}_{OA} u_A - \tilde{G}_{OK}^{-1} \tilde{\tau}_O \\ &= \tilde{G}_{KO} y_O + \tilde{G}_{KI} y_I + \tilde{G}_{KD} y_D + \tilde{G}_{KA} u_A + \tilde{\tau}_K \end{aligned} \quad (6.24)$$

and this model can be used to estimate the wanted transfer-function matrices directly. Here, the model structure can be chosen such that the transfer-function matrices are realizable. However, the disturbances on y_O , y_I and y_D will act as confounding variables and make it an errors-in-variables problem. For a discussion about estimation of inverse systems, see for instance, Jung and Enqvist (2013).

Part II

Applications

7

Graybox Modelling of a Second-order Subsystem

This chapter presents a study on the estimation of the physical parameters in a graybox model of an inverted pendulum. Estimation of physical parameters in a mechanical system is notoriously difficult if the input to the system is unknown. This is partly due to identifiability issues, i.e. several combinations of inputs and physical parameters will all satisfy the equations of motion. In this work, it is assumed that the true input to the system is unknown and that only measurements of the motion are available, together with a signal related to the true input. The unknown input issues are resolved by eliminating the true input from the model and instead using an acceleration measurement as the input to the identification procedure. The resulting indirect model is of second order and despite its simplicity, this approach can be used for estimation of physical parameters in a multitude of vehicles, such as, cars, trucks, trains and ships. The model derived in this chapter will be further discussed in Chapter 8 where it will be shown to be a subsystem of the complete dynamics of a ship.

7.1 The Force-Mass Ambiguity

Mass and its position are two of the most influential parameters in most mechanical systems. For example, consider the change in the dynamics due to the person standing up in the canoe shown in Figure 1 on page ix. However, if only motion data is known, it is difficult to uniquely determine the system's mass in many cases.

Figure 7.1: This is an example of the force-mass ambiguity. If only the acceleration is measured, the left and right systems are indistinguishable.

A central issue when only the motion is measured is the force–mass ambiguity.

For instance, consider the right figure of Figure 7.1, in which the acceleration a of a box with mass m , affected by the force F and moving in one dimension is directly proportional to F/m . If only a is measured, there are still multiple combinations of F and m that will satisfy the equation as, for instance, shown in Figure 7.1. This ambiguity can be overcome with special experiments where the force acting on the system is known, or by measuring the force acting on the system. However, doing special experiments is in many cases intractable or too expensive. Instead, the multiple dataset approach described in Chapter 5 will be considered. Here, the *change* in mass is estimated by using two datasets where the mass is known in the first dataset and unknown in the second dataset. The first dataset will be called the *nominal* or *calibration dataset* since it corresponds to a nominal loading case. The introduction of this extra information is sufficient to overcome the ambiguity. The advantage of this method is that once the nominal case is known, it can be used as a basis for comparison for data collected during normal operation. This makes it possible to estimate the current mass at any given time, assuming that enough data has been collected.

7.2 Approximate Model of a Stabilized Pendulum

Consider the system in Figure 7.2, which is an inverted pendulum hinged on a cart. The pendulum's mass is M with its center of gravity a distance z_g from the center of rotation (CR) and with the moment of inertia I_x about its center

Figure 7.2: A sketch relating the important quantities of the approximate inverted pendulum model. Rigidly attached to the pendulum is the NED body-fixed coordinate system (in green) located at the center of rotation. There is also a world-fixed coordinate system and the angle ϕ is the relative rotation of the x_b -axis about the x_n -axis. The orange arrows correspond to forces acting on the pendulum while the red straight arrow corresponds to the unknown acceleration a_y . Note that the CR coincides with the x_b -axis.

Figure 7.3: The acceleration a_y is unknown but is correlated with the known external signal δ . The signals a_y and δ are assumed to be related through the unknown subsystem $F_\delta(\mathbf{p})$.

of gravity. Furthermore, a mass m with moment of inertia $I_{x,m}$ is introduced a distance z_m from the CR. There are two torsional torques acting on the pendulum, one linearly dependent on the angle ϕ and on the angular velocity $\dot{\phi}$ corresponding to the restoring forces and damping/friction, respectively.

The cart is disturbed by an external force that results in the acceleration a_y which will be considered to be unknown. Here we will focus on the modelling of the pendulum. However, the pendulum is assumed to be a subsystem of a larger system and the acceleration a_y is assumed to be related with a known signal δ through some unknown system $F_\delta(\mathbf{p})$, see Figure 7.3. As an example, in Chapter 8 the pendulum will correspond to the roll dynamics, the larger system is the ship itself and δ is the rudder angle.

There is a torque disturbance $\bar{\tau}$ acting on the center of rotation. The torque disturbance $\bar{\tau}$ is not assumed to have any specific distribution. The disturbance is an aggregate of neglected couplings to other subsystems and other types of disturbances. For the ship in Chapter 8, examples of disturbances are vibrations from the engine and environmental disturbances such as waves or wind.

The dynamic model can be derived, for instance, by using Newton's second law. For a linear motion this relates the mass m , the acceleration a and the forces F_i acting on the system according to

$$\sum_i F_i = ma \quad (7.1)$$

Similarly, for a rotational motion the law relates the inertia I , the angular velocity ω and the torques τ_i acting on the system according to

$$\sum_i \tau_i = I\dot{\omega} \quad (7.2)$$

Consider Figure 7.4 where the cart–pendulum system in Figure 7.2 has been split into the cart and the pendulum, and the forces acting on each part have been marked with orange arrows and the accelerations with red straight arrows. The forces N and R_y are reaction forces and f_y is the force acting on the pendulum cart. The two masses are lumped into one point mass $\bar{M}$, at the position $\bar{z}_g$ with the inertia $\bar{I}_x$. These lumped parameters are given by basic mechanics and are defined as

$$\bar{M} = M + m, \quad \bar{z}_g = \frac{Mz_g + mz_m}{M + m} \quad (7.3a)$$

and

$$\bar{I}_x = I_x + I_{x,m} + M(\bar{z}_g - z_g)^2 + m(\bar{z}_g - z_m)^2 \quad (7.3b)$$

Figure 7.4: A sketch relating the important quantities of the approximate inverted pendulum model.

Firstly, note that the tangential acceleration at the lumped center of gravity is

$$\ddot{y}_b = \ddot{y} \cos(\phi) - \bar{z}_g \ddot{\phi} \quad (7.4)$$

where $\ddot{y}$ is the acceleration in the world-fixed y_n -axis and $\ddot{\phi}$ is the angular acceleration about the body-fixed x_b -axis. Hence, the force equilibrium in the lumped center of gravity in the (pendulum) body-fixed y_b -axis is given by

$$\bar{M}(\ddot{y} \cos(\phi) - \bar{z}_g \ddot{\phi}) = -N \sin(\phi) + R_y \cos(\phi) + \bar{M}g \sin(\phi) \quad (7.5)$$

and moment equilibrium about the lumped center of gravity gives

$$\bar{I}_x \ddot{\phi} = -\bar{z}_g (N \sin(\phi) - R_y \cos(\phi)) - d\dot{\phi} - k\phi + \bar{\tau} \quad (7.6)$$

By combining (7.5) with (7.6) and introducing $a_y = \ddot{y}$, the resulting dynamic model can be written as the differential equation

$$(\bar{I}_x + \bar{M}\bar{z}_g^2) \ddot{\phi} = -k\phi - \bar{M}\bar{z}_g g \sin(\phi) - d\dot{\phi} + \bar{M}\bar{z}_g \cos(\phi) a_y + \bar{\tau} \quad (7.7)$$

which is a mass–spring–damper model. Now note that

$$\bar{M}\bar{z}_g \stackrel{(7.3)}{=} Mz_g + mz_m \quad \text{and} \quad \bar{I}_x + \bar{M}\bar{z}_g^2 \stackrel{(7.3)}{=} I_x + Mz_g^2 + I_{x,m} + mz_m^2 \quad (7.8)$$

which gives

$$I_1 \ddot{\phi} = -k\phi - (Mz_g + mz_m)g \sin(\phi) - d\dot{\phi} + (Mz_g + mz_m) \cos(\phi) a_y + \bar{\tau} \quad (7.9)$$

where $I_1 = I_x + Mz_g^2 + I_{x,m} + mz_m^2$. Equation (7.9) can be simplified to

$$I_1 \ddot{\phi} = \underbrace{-(k + Mgz_g + mgz_m)\phi}_{\text{restoring torque}} \underbrace{-d\dot{\phi}}_{\text{damping}} \underbrace{+(Mz_g + mz_m)a_y + \bar{\tau}}_{\text{disturbances}} \quad (7.10)$$

for small angles ϕ , i.e. linearizing about zero. For brevity, the time-dependency of the states and disturbances has been dropped from the expression.

In the same way, force equilibrium in the world-fixed y_n -axis gives

$$\bar{M}\ddot{y} = f_y + \bar{M}\bar{z}_g\dot{\phi}^2 \sin(\phi) - \bar{M}\bar{z}_g\ddot{\phi} \cos(\phi) \quad (7.11)$$

which by inserting (7.3) is equivalent to

$$(M + m)\ddot{y} = f_y + (M\bar{z}_g + mz_m)\dot{\phi}^2 \sin(\phi) - (M\bar{z}_g + mz_m)\ddot{\phi} \cos(\phi) \quad (7.12)$$

but since the model has the acceleration as input, the force generating the acceleration is of less importance and hence, (7.12) will not be used.

The motion of the pendulum is assumed to be measured by an IMU. The details are described in Appendix A.6.1. The measurement equations are recapitulated here for readability. The IMU measures the angular velocity about the x_b -axis

$$y_{1,t} = \dot{\phi}_t + e_{1,t} \quad (7.13)$$

defined in (A.41) and the tangential acceleration

$$a_s = z_s\ddot{\phi} + g \sin(\phi) - a_y \cos(\phi) \stackrel{\text{small}}{\approx} z_s\ddot{\phi} + g\phi - a_y \quad (7.14)$$

defined in (A.43) is measured by

$$y_{2,t} = a_{s,t} + e_{2,t}, \quad (7.15)$$

defined in (A.44). Note that the IMU is assumed to have a NWU coordinate system while the body-fixed coordinate system is NED, see Figure 7.5.

Figure 7.5: A sketch showing the coordinate system of the IMU in relation to the (pendulum) body-fixed coordinate system. Note that the body-fixed system is NED while the sensor system is NWU.

7.2.1 Identifiability of the Approximate Model

Uniqueness is a key aspect for models containing a large number of parameters. For a model parameterized with parameters corresponding to physical quantities this is especially important since a non-unique description will imply that the parameters can assume several different values and still satisfy the dynamic model.

To investigate the uniqueness of the model (7.10), an identifiability analysis will be performed. The goal of the analysis is to understand what is possible to estimate given that the true input a_y is known. The structural identifiability method described in Section 2.7 will be used and the analysis is performed using symbolic software. It should be stressed that this method can be used to investigate local identifiability; for global methods, see for instance, Ljung and Glad (1994).

The model (7.10), with the output defined by (7.13) and (7.15), can be cast into a state-space form by choosing the states $x = [\phi, \dot{\phi}]^T$ and inserting (7.10) into (7.15). The effects of measurement and process noises are neglected in this section, i.e. $e_1 = e_2 = \bar{\tau} = 0$. The resulting state-space model is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{k+Mgz_g+mgz_m}{I_1} & -\frac{d}{I_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{Mz_g+mz_m}{I_1} \end{bmatrix} u = f(x, u, \vartheta) \\ y_O &= \begin{bmatrix} 0 & 1 \\ -\frac{k+Mgz_g+mgz_m}{I_1} + g & -\frac{d}{I_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{Mz_g+mz_m-\tilde{I}_1}{I_1} \end{bmatrix} u = h_O(x, u, \vartheta) \end{aligned} \quad (7.16)$$

where I_1 is defined in Section 7.2, $\tilde{I}_1 = I_1/z_s$ and $u = a_y$. Note that $f(x, u, \vartheta)$ and $h_O(x, u, \vartheta)$ symbolize the extended state-space model (2.62). The parameters considered unknown are

$$\vartheta_p = [M \quad z_g \quad k \quad I_x \quad d \quad I_{x,m} \quad m \quad z_m]^T \quad (7.17)$$

The remaining parameters are assumed to be known. There are $n_x = 2$ states and $n_\vartheta = 8$ parameters, i.e. the extended nonlinear system has dimension 10. All parameters are assumed to be non-zero and distinctly different from each other, unless otherwise stated, e.g. z_g and z_m are never treated as equal even if they might assume the same numerical value.

The complete extended observability matrix $\mathcal{O}$ for the general case is unfortunately not possible to recite due to its size and complexity. Instead, an illustrative special case will be presented in detail to show the typical structure and then, the extended observability matrix $\mathcal{O}$ for the general case is presented in row-reduced form together with conclusions.

Illustrative Example – A Special Case

This section presents an example that will show identifiability of the parameters I_x , k and d in the model (7.16) when there is no additional mass, i.e. $I_{x,m} = m = z_m = 0$. The intention is to get a feeling for the structure of the extended observability matrix $\mathcal{O}$. In this example, only the first measurement is used, the output is $y_O = \dot{\phi}$ and the input $u = a_y$, which for the analysis is assumed to be known. For notational simplicity, the parameters z_g and g are assumed to be equal to 1. Since

there are five states in the extended state-space model, i.e. two states and three parameters, at most four time derivatives of the output have to be considered. In this case, the extended observability matrix is

$$\mathcal{O} = \begin{bmatrix} \frac{\partial h(x,u,\vartheta)}{\partial x_1} & \frac{\partial h(x,u,\vartheta)}{\partial x_2} & \frac{\partial h(x,u,\vartheta)}{\partial I_x} & \frac{\partial h(x,u,\vartheta)}{\partial k} & \frac{\partial h(x,u,\vartheta)}{\partial d} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial x_1} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial x_2} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial I_x} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial k} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial d} \end{bmatrix} \quad (7.18)$$

$\underbrace{\hspace{1.5cm}}_{\mathcal{O}_{x_1}} \quad \underbrace{\hspace{1.5cm}}_{\mathcal{O}_{x_2}} \quad \underbrace{\hspace{1.5cm}}_{\mathcal{O}_{I_x}} \quad \underbrace{\hspace{1.5cm}}_{\mathcal{O}_k} \quad \underbrace{\hspace{1.5cm}}_{\mathcal{O}_d}$

and the corresponding columns are

$$\mathcal{O}_{x_1} = \begin{bmatrix} 0 \\ -\frac{M_k}{I_1^2} \\ \frac{dM_k}{I_1^3} \\ \frac{I_k - d^2}{I_1^2} \\ \frac{(I_k - d^2)M_k}{I_1^3} \\ \frac{d(d^2 - 2I_k)M_k}{I_1^4} \end{bmatrix}, \quad \mathcal{O}_{x_2} = \begin{bmatrix} 1 \\ -\frac{d}{I_1} \\ \frac{d^2 - I_k}{I_1^2} \\ \frac{d(2I_k - d^2)}{I_1^3} \\ \frac{d^4 - (3d^2 - I_k)I_k}{I_1^4} \end{bmatrix}, \quad (7.19a)$$

$$\mathcal{O}_k = \begin{bmatrix} 0 \\ -\frac{1}{I_1}x_1 \\ \frac{d}{I_1^2}x_1 - \frac{1}{I_1}x_2 \\ \frac{2I_k - d^2}{I_1^3}x_1 + \frac{2d}{I_1^2}x_2 + U_k(0,u) \\ \frac{d(d^2 - 4I_k)}{I_1^4}x_1 + \frac{2I_k - 3d^2}{I_1^3}x_2 + U_k(u,\dot{u}) \end{bmatrix}, \quad (7.19b)$$

$$\mathcal{O}_d = \begin{bmatrix} 0 \\ -\frac{1}{I_1}x_2 \\ \frac{M_k}{I_1^2}x_1 + \frac{2d}{I_1^2}x_2 + U_d(0,0,u) \\ -\frac{2dM_k}{I_1^3}x_1 + \frac{2I_k - 3d^2}{I_1^3}x_2 + U_d(0,u,\dot{u}) \\ \frac{(3d^2 - 2I_k)M_k}{I_1^4}x_1 + \frac{d(4d^2 - 6I_k)}{I_1^4}x_2 + U_d(u,\dot{u},\ddot{u}) \end{bmatrix}, \quad (7.19c)$$

$$\mathcal{O}_{I_x} = \begin{bmatrix} 0 \\ \frac{M_k}{I_1^2}x_1 + \frac{d}{I_1^2}x_2 + U_I(0,0,0,u) \\ -\frac{2dM_k}{I_1^3}x_1 + \frac{I_k - 2d^2}{I_1^3}x_2 + U_I(0,0,u,\dot{u}) \\ \frac{(3d^2 - 2I_k)M_k}{I_1^4}x_1 + \frac{d(3d^2 - 4I_k)}{I_1^4}x_2 + U_I(0,u,\dot{u},\ddot{u}) \\ \frac{2d(3I_k - 2d^2)M_k}{I_1^5}x_1 - \frac{(d^2 - 2I_k)(4d^2 - I_k)}{I_1^5}x_2 + U_I(u,\dot{u},\ddot{u},u^{(3)}) \end{bmatrix}, \quad (7.19d)$$

$$U_k(u_1, u_2) = \frac{2dM}{I_1^3}u_1 - \frac{M}{I_1^2}u_2 \quad (7.19e)$$

$$U_d(u_1, u_2, u_3) = \frac{M(2I_k - 3d^2)}{I_1^4}u_1 + \frac{2dM}{I_1^3}u_2 - \frac{M}{I_1^2}u_3, \quad (7.19f)$$

and

$$U_I(u_1, u_2, u_3, u_4) = \frac{2d(2d^2 - 3I_k)M}{I_1^5}u_1 + \frac{M(2I_k - 3d^2)}{I_1^4}u_2 + \frac{2dM}{I_1^3}u_3 - \frac{M}{I_1^2}u_4 \quad (7.19g)$$

where, in this case, $I_1 = I_x + M$, $M_k = k + M$ and $I_k = I_1 M_k$. Note the structure appearing in the columns due to the recursive properties of the matrix.

Non-zero informative input: Assuming that the input is informative enough, $\mathcal{O}$ is full rank for almost all values of x , i.e. given a persistently exciting input, the states will be informative enough to distinguish between the parameters, and hence, it is possible to identify all three parameters. However, note that this might not be true for all numerical values of the parameters. One, possibly unlikely case, is when all parameters except for I_x are zero, then no input will affect the output.

No input: Assuming that the input is zero but that the output still contains information, then the maximum rank of $\mathcal{O}$ drops to 4. This occurs, for instance, if the system has stored energy induced by perturbing it from its equilibrium and releasing it. The drop in rank is due to the linear dependency of the columns corresponding to the parameters. It can be seen in the row-reduced (normalized per column) form of the extended observability matrix

$$\bar{\mathcal{O}} = \begin{bmatrix} x_1 & x_2 & k & d & I_x \\ \text{orange } 1 & 0 & 0 & 0 & 0 \\ 0 & \text{orange } 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \text{orange } 1 \\ 0 & 0 & 0 & 1 & \text{yellow } \frac{M_k}{I_1} \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (7.20)$$

where the same color denotes that the elements assume the same value. Hence, it is only possible to identify two-combinations of parameters simultaneously. This result is reasonable since it is a second order system without input, i.e. a time-series model of second order.

Identifiability of the Original Model - General Case

For the general case of (7.16), the output is $y = [\dot{\phi}, a_s]^T$ and the input is $u = a_y$. For the analysis in this section, it is again assumed that a_y is known.

Non-zero informative input: For the general case, the row reduced extended observability matrix (normalized per column) has the following structure

$$\bar{\mathcal{O}} = \begin{bmatrix} x_1 & x_2 & k & d & I_x & M & z_g & I_{x,m} & m & z_m \\ \text{orange } 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{orange } 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & \text{gray } \times & \text{gray } \times & 0 & \text{gray } \times & \text{gray } \times \\ 0 & 0 & 0 & 0 & 1 & \text{yellow } \times & \text{green } \times & 1 & \text{red } \times & \text{blue } \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (7.21)$$

Note that the matrix is symbolic, i.e. $\times$ corresponds to a symbolic expression. This row-reduced form shows that the maximum rank of the extended observability matrix $\mathcal{O}$ is 5, assuming that the input is informative enough. This implies that at most 3 parameters can be uniquely identified. This is an expected result since (7.16) is a second order system with one external input.

Furthermore, it is possible to see that certain combinations of parameters will create issues with identifiability. In principle, this is equivalent to creating sub-matrices for each one-, two- and three-combination of parameters of the form

$$\mathcal{O}_s = \begin{bmatrix} \frac{\partial h(x,u,\vartheta)}{\partial x_1} & \frac{\partial h(x,u,\vartheta)}{\partial x_2} & \frac{\partial h(x,u,\vartheta)}{\partial \vartheta_1} & \frac{\partial h(x,u,\vartheta)}{\partial \vartheta_2} & \frac{\partial h(x,u,\vartheta)}{\partial \vartheta_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial x_1} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial x_2} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial \vartheta_1} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial \vartheta_2} & \frac{\partial h^{(4)}(x,u,\vartheta)}{\partial \vartheta_3} \end{bmatrix} \quad (7.22)$$

and checking the rank condition of these sub-matrices.

Firstly, note that each column corresponding to a parameter in (7.21) is linearly independent of the columns corresponding to the states and hence, if only one parameter is unknown, it is identifiable.

Secondly, any two columns in (7.21) corresponding to the parameters are linearly independent of each other, except for the columns corresponding to the parameters I_x and $I_{x,m}$. This implies that except for I_x and $I_{x,m}$, any two-combination of parameters is identifiable.

Finally, the analysis reveals that any three-combination of parameters is identifiable, except for any three-combination of $\{I_x, M, z_g, I_{x,m}, m, z_m\}$ or any combination of parameters where both I_x and $I_{x,m}$ are in the set. For instance, this means that if the entire system is known except for the additional load, it is not possible to estimate its inertia, mass and its center of mass at the same time.

These results are summarized in Table 7.1. It is worth noting that since the analysis is performed with a symbolic software, the parameters are assumed to have distinctly different values. Hence, identifiability might be lost for certain numerical values of the parameters.

No input: When the input $a_y = 0$, the row-reduced extended observability matrix (normalized per column) reduces to

$$\bar{\mathcal{O}} = \begin{bmatrix} x_1 & x_2 & k & d & I_x & M & z_g & I_{x,m} & m & z_m \\ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \times \\ 0 \\ \vdots \end{array} \end{bmatrix} \quad (7.23)$$

and hence, the maximum rank of $\mathcal{O}$ drops to 4 and dropping column by column shows that any two-combination of parameters is identifiable except for the combination of I_x and $I_{x,m}$, assuming the output still is informative enough. The results are summarized in Table 7.1.

Table 7.1: The results of the identifiability analysis for the general case of the approximate model. The exceptions are: a) Combinations including both the parameters I_x and $I_{x,m}$. b) Any three-combination of $\{I_x, M, z_g, I_{x,m}, m, z_m\}$.

Case	# Parameters	Identifiable	Exceptions
$a_y \neq 0$	1	Yes	–
	2	Yes	a
	3	Yes	a,b
$a_y = 0$	1	Yes	–
	2	Yes	a
	3	No	–

7.3 The Indirect Pendulum Model

One of the main issues with estimating the parameters of the system (7.16) is the unknown and highly influential acceleration a_y . Several approaches are possible as discussed in Chapters 3 and 4:

- (A) For sake of completeness, if a_y was known or measured, it could be used as an input: $x = [\phi, \dot{\phi}]^T$, $y = [y_1, y_2]^T$ and $u = a_y$. The estimation would then be straightforward using, for example, a standard prediction-error method (PEM) if a_y is known or an errors-in-variables (EIV) method if a_y is measured with noise (Söderström, 1981; Ljung, 1999).
- (B) The variable a_y could be introduced as a state, i.e. to model a larger part of the system. However, this requires a model of how a_y behaves. Possible alternatives are, for instance,
 - (a) an extended model with the states $x = [\phi, \dot{\phi}, a_y]^T$, the outputs $y = [y_1, y_2]^T$, defined by (7.13) and (7.15), and where the model of a_y is assumed to be a random walk.
 - (b) if the lateral velocity was measured, for instance using a GPS receiver, then the states could be chosen as $x = [\phi, \dot{\phi}, v_y, a_y]^T$ and the outputs $y = [y_1, y_2, v_y]^T$ defined by (7.13) and (7.15). The velocity v_y obtained with the GPS receiver would then implicitly supply information about a_y .
- (C) The variable a_y is eliminated using the indirect input measurement

$$a_s = z_s \ddot{\phi} + g\phi - a_y$$

defined in (7.14), according to Chapter 4. Note that in reality a_s is measured which leads to an EIV formulation.

Here, alternative (C) is used since both a_y and its model are unknown. Equa-

tion (7.14) is solved for a_y and inserted into (7.10) resulting in

$$I_2 \ddot{\phi} = -k\phi - d\dot{\phi} - (Mz_g + mz_m)a_s + \bar{\tau} \quad (7.24)$$

with

$$I_2 = I_x + Mz_g(z_g - z_s) + I_{x,m} + mz_m(z_m - z_s) \quad (7.25)$$

and a_s is now treated as the input.

7.3.1 Identifiability of the Indirect Model

The goal of the analysis in this section is to investigate if there is any change in the identifiability properties when a_s is used as an input instead of the true input a_y . As a basis for comparison, the results for the original model (7.10) in Section 7.2.1 show what is possible to estimate given that the true input of the system is known.

In this section the output is chosen as $y_O = \dot{\phi}$ and the input is $u = a_s$, i.e. the measured tangential acceleration. By defining the states $x = [\phi, \dot{\phi}]^T$, the indirect model (7.24) and the measurement equation (7.13) can similarly to Section 7.2.1 be cast into the state-space form (neglecting the disturbances)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{I_2} & -\frac{d}{I_2} \end{bmatrix} x - \begin{bmatrix} 0 \\ \frac{Mz_g + mz_m}{I_2} \end{bmatrix} a_s \\ y_O &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \quad (7.26)$$

where I_2 is defined by (7.25).

Non-zero informative a_y : As in Section 7.2.1, assuming that the true input a_y is informative enough, the maximum rank of $\mathcal{O}$ is 5. Also the identifiable one-, two- and three-combinations of the parameters are the same. Hence, the identifiability properties of the original model are retained when using the mathematical reformulation.

No input ($a_y = 0$): An apparent advantage with the reformulation (7.26) is that the unknown acceleration a_y is replaced by the measured acceleration a_s . As defined in (7.14), the tangential acceleration is

$$a_s = z_s \ddot{\phi} + g\phi - a_y$$

and if the system is perturbed from its equilibrium, then a_s will be non-zero for almost all t even if a_y is zero. At a first glance, this might seem beneficial since it suggests that the input a_s is informative even though a_y is zero. However, it is the input a_s that is used in the analysis and it is assumed that the input can be chosen arbitrarily. The input is a linear combination of a_y and the states and is thus not free to be chosen if a_y is zero. By inserting the known relation (7.15), the maximum rank of $\mathcal{O}$ is 4 as in Section 7.2.1, again retaining the same identifiability properties as for the original model.

Result 7.1. *In a local structural identifiability sense, nothing is gained or lost by using the indirect model (7.24) instead of the original model (7.10).*

7.3.2 Indirect Model Analysis – A Closed-loop Viewpoint

The system (7.24) with the output $y_O = \dot{\phi}$ can be represented by

$$y_O = G_O(p)a_s + H_O(p)\bar{\tau} = G_O(p)(a_s + \tau) \quad (7.27)$$

with

$$G_O(p) = \frac{\beta_1 p}{p^2 + \alpha_1 p + \alpha_2}, \quad H_O(p) = \frac{\gamma_1 p}{p^2 + \alpha_1 p + \alpha_2} = \frac{\gamma_1}{\beta_1} G_O(p), \quad (7.28)$$

$$\alpha_1 = \frac{d}{I_2}, \quad \alpha_2 = \frac{k}{I_2}, \quad \beta_1 = -\frac{Mz_g + mz_m}{I_2}, \quad \gamma_1 = \frac{1}{I_2} \quad (7.29)$$

and with I_2 according to (7.25). The transfer operator from a_s to y_O is $G_O(p)$, $H_O(p)$ is the transfer operator from $\bar{\tau}$ to y_O , $\tau = G_O^{-1}(p)H_O(p)\bar{\tau} = \frac{\gamma_1}{\beta_1}\bar{\tau}$ and p is the differentiation operator.

At first, it appears to be straightforward to use a second order ARMAX structure, see Figure 7.6. However, $\bar{\tau}$ is not white. Moreover, the indirect model (7.24) does not reveal the correlation between the input a_s and the disturbance $\bar{\tau}$, and that a_s is dependent on the output y_O .

To gain more insight, the original system (7.10) with the output y_O can be expressed in terms of $G_O(p)$ together with the tangential acceleration's dependency on the true input a_y and the output y_O .

Proposition 7.1. *The original system (7.10) with the output y_O is equivalent to $G_O(p)(a_s + \tau)$ in positive feedback with*

$$F_y(p) = \frac{z_s p^2 + g}{p} \quad (7.30)$$

and $-a_y$ as an external signal, see Figure 7.7.

Proof: The proof follows immediately from inserting the known input

$$a_s = \frac{z_s p^2 + g}{p} y - a_y = (z_s p^2 + g)\phi - a_y$$

into (7.27) which after algebraic manipulations results in the model (7.10). $\square$

Figure 7.6: The system (7.27) represented in a block diagram showing the apparent simple structure. However, note that a_s is dependent on y_O according to (7.14) which also means that it is correlated with τ .

Figure 7.7: The system (7.10) expressed in terms of $G_O(\mathbf{p})$ shown as a block diagram. This representation reveals the dependencies in the system more clearly than Figure 7.6.

The structure revealed by this point of view shows that in principle, this is identification in closed-loop. This means that in case a_s , the input to $G_O(\mathbf{p})$, is persistently exciting, the input might not be sufficiently informative since a_y is the true input to the system. It is well-known in system identification theory that the reference has to be persistently exciting and that a too simple linear feedback term does not contribute with new information, see for instance, Söderström and Stoica (1989, Sec. 10.3), Ljung (1999, Sec. 13.4) or Bazanella et al. (2010). Note that the linear measurement equation is due to the assumption that the IMU only measures the tangential acceleration. If the IMU is rotated, it will also measure the normal or centripetal acceleration, which depends nonlinearly on the angular velocity. Hence, if the IMU is deliberately rotated, the feedback term might contribute with information even if the true input is zero. However, this will lead to a nonlinear estimation problem.

Furthermore, $\lim_{s \rightarrow \infty} G_O(s)F_y(s) \neq 0$ which means that the loop contains a direct term. Hence, any accurate discrete approximation of this continuous-time system will also include a direct term. Assume that an accurate discretization method has been applied to (7.27), giving

$$y_{O,t} = G_{O,d}(\mathbf{q})u_t + H_{O,d}(\mathbf{q})\bar{\tau}_t = G_{O,d}(\mathbf{q})u_t + \bar{H}_{O,d}(\mathbf{q})e_t \quad (7.31)$$

where $\mathbf{q}$ is the shift operator. Also, assume that the noise can be described as $\bar{\tau}_t = H_1(\mathbf{q})e_t$ where e_t is white noise and that $H_1(\mathbf{q})$ is a stable discrete-time linear filter. Even if the correct noise model $H_{O,d}(\mathbf{q})H_1(\mathbf{q})$ is known, the residual

$$\varepsilon_t = H_1^{-1}(\mathbf{q})H_{O,d}^{-1}(\mathbf{q})[y_t - G_{O,d}(\mathbf{q})u_t] = H_1^{-1}(\mathbf{q})\bar{\tau}_t \quad (7.32)$$

and the input-dependent term $H_1^{-1}(\mathbf{q})H_{O,d}^{-1}(\mathbf{q})G_{O,d}(\mathbf{q})u_t = cH_1^{-1}(\mathbf{q})u_t$, where c is a constant, will be correlated and the consequence is a risk that the parameter estimator will be biased as discussed in Section 4.3.1.

Finally, when the true a_s and y_O are not known but measured according to Section 7.2, the input will be noisy which results in a different noise model, i.e. this is a closed-loop EIV problem, see Figure 7.8.

all identifiable combinations of parameters in Section 7.2.1 by solving selected parts of (7.29) for the wanted parameters. Here, two important cases are listed.

If I_x , k and d are unknown, then the numerator of β_1 is known and the solution is

$$\begin{aligned} d &= \frac{\alpha_1}{\beta_1}(-Mz_g - mz_m) \\ k &= \frac{\alpha_2}{\beta_1}(-Mz_g - mz_m) \\ I_x &= \frac{-Mz_g - mz_m - \beta_1[Mz_g(z_g - z_s) + I_{x,m} + mz_m(z_m - z_s)]}{\beta_1} \end{aligned} \quad (7.37)$$

If the additional mass m and the center of mass z_m are unknown then the transform is given by, for instance, solving α_1 and β_1 in (7.29) for m and z_m , which results in

$$\begin{aligned} m &= -\frac{(\beta_1 d + \alpha_1 M z_g)^2}{\alpha_1^2(I_x + M z_g^2) + \alpha_1 d(\beta_1 z_s - 1)} \\ z_m &= \frac{\alpha_1(I_x + M z_g^2) + d(\beta_1 z_s - 1)}{\alpha_1 M z_g + \beta_1 d} \end{aligned} \quad (7.38)$$

Direct Parameterization using Physical Parameters

An alternative to estimating the discrete-time linear parameters and then transforming them to the physical parameters is to directly parameterize the discrete-time model with the physical parameters. This is achieved by inserting (7.29) into (7.34) which results in the nonlinear transformations

$$\begin{aligned} \bar{\alpha}_1 &= \frac{2kT^2 - 8[I_x + Mz_g(z_g - z_s) + I_{x,m} + mz_m(z_m - z_s)]}{\mathcal{I}_2}, \\ \bar{\alpha}_2 &= \frac{4[I_x + Mz_g(z_g - z_s) + I_{x,m} + mz_m(z_m - z_s)] - 2dT + kT^2}{\mathcal{I}_2}, \\ \bar{\beta}_0 &= -\frac{2T(Mz_g + mz_m)}{\mathcal{I}_2}, \end{aligned} \quad (7.39)$$

where $\mathcal{I}_2 = 4[I_x + Mz_g(z_g - z_s) + I_{x,m} + mz_m(z_m - z_s)] + 2dT + kT^2$.

7.3.4 Obeying Nonlinear Parameterization Using Constraints

As discussed in Section 7.3.1, only a subset of all parameters can be estimated. The transfer function (7.33) has three free parameters, i.e. three degrees of freedom to fit the model to the data. When all three parameters are estimated, this gives a model that is more flexible than permitted with the consequence of increased variance of the parameter estimates (Ljung, 1999). The dependency between the linear parameters is complex since they are dependent through the original nonlinear parameterization. Fortunately, for certain parameter combinations, it is possible to use linear constraints together with (7.33) and still obey the original nonlinear parameterization. The flexibility is thus reduced which results in a reduced estimator variance. Arbitrary nonlinear constraints can also be used but since the model given by (7.33) can be written as a linear regression, using linear constraints makes

the estimation algorithms computationally cheaper as discussed in Section 5.2.3. Below, a few examples of these linear constraints are presented. These are all based on mathematical manipulations involving known numerators or known co-efficients. In all cases, the discrete linear parameter vector is $\vartheta = [\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0]^T$.

If only I_x and d are unknown, the numerators of α_2 and β_1 are known and

$$-\frac{Mz_g + mz_m}{k} = \frac{\beta_1}{\alpha_2} = \frac{2T\bar{\beta}_0}{1 + \bar{\alpha}_1 + \bar{\alpha}_2} \quad (7.40)$$

is thus known giving the constraint

$$\left[-\frac{Mz_g + mz_m}{k} \quad -\frac{Mz_g + mz_m}{k} \quad -2T \right] \vartheta = \frac{Mz_g + mz_m}{k} \quad (7.41)$$

Similarly, in the case where only I_x and k are unknown, the numerators of α_1 and β_1 are known and the quotient

$$-\frac{d}{Mz_g + mz_m} = \frac{\alpha_1}{\beta_1} = \frac{1 - \bar{\alpha}_2}{2\bar{\beta}_0} \quad (7.42)$$

between them is known giving the constraint

$$\begin{bmatrix} 0 & 1 & -2\frac{d}{Mz_g + mz_m} \end{bmatrix} \vartheta = 1 \quad (7.43)$$

If k and d are the only unknown parameters then

$$\beta_1 = \frac{8}{T} \frac{\bar{\beta}_0}{1 - \bar{\alpha}_1 + \bar{\alpha}_2} \quad (7.44)$$

is known giving the constraint

$$\begin{bmatrix} \beta_1 T & -\beta_1 T & 8 \end{bmatrix} \vartheta = \beta_1 T \quad (7.45)$$

If any two-combination of $\{I_x, M, z_g, I_{x,m}, m, z_m\}$ is unknown, the quotient

$$\frac{d}{k} = \frac{\alpha_1}{\alpha_2} = \frac{T(1 - \bar{\alpha}_2)}{1 + \bar{\alpha}_1 + \bar{\alpha}_2} \quad (7.46)$$

is known which gives the constraint

$$\begin{bmatrix} \frac{d}{k} & \frac{d}{k} + T & 0 \end{bmatrix} \vartheta = T - \frac{d}{k} \quad (7.47)$$

7.4 Parameter Estimation

In order to estimate the change in mass and the change in the center of mass on a real system, several sources of information and datasets have to be combined to overcome the identifiability limitations. This can be solved by using the multi-stage identification approach described in Section 5.2.3 or the joint estimation approach in Section 5.2.4. If no a priori information is known about the system parameters, this is indeed necessary since a maximum of three parameters can be identified simultaneously according to the results of Section 7.3.1. To counteract the force-mass ambiguity, a nominal and a loaded datasets are used, see Figure 7.9.

The datasets are defined by

$$Z_n = (y_t, u_t, \delta_t)_{t=1+t_n}^{N_n+t_n} \text{ and } Z_l = (y_t, u_t, \delta_t)_{t=1+t_l}^{N_l+t_l} \quad (7.48)$$

where N_n and N_l denote the lengths of the datasets and t_n and t_l denote the starting times. They are related through the physical parameters

$$\vartheta_p = \underbrace{\begin{bmatrix} M, & z_g \end{bmatrix}}_{\vartheta_{p,1}} \underbrace{\begin{bmatrix} k, & I_x, & d \end{bmatrix}}_{\vartheta_{p,2}} \underbrace{\begin{bmatrix} m, & z_m \end{bmatrix}}_{\vartheta_{p,3}}^T \quad (7.49)$$

Regardless of which approach that is used, two of the parameters M , z_g and I_x need to be known or estimated in a first step. This step involves some special kind of experiment. For instance, $\vartheta_{p,1}$, i.e. the mass and the center of mass, can be determined by specific procedures, such as the tilting procedure which will be described in Section 9.2.1 or $\vartheta_{\text{static}}$ can be determined by using the ballasting approach which will be described in Appendix B.

In the multi-stage approach described in Approach 5.1, depending on the parameters estimated in the first step, either the dynamic parameters $\vartheta_{\text{dynamic}}$ or $\vartheta_{p,2}$ are estimated using the nominal dataset in the second step. In the third step, all the parameters of the nominal case are known or estimated and the loaded dataset is used to estimate $\vartheta_{p,3}$ or to monitor changes in some other parameters. The real benefit is that the loaded dataset can be collected at any point in time and the change in mass and the change in center of mass can be estimated during normal operation.

As discussed in Section 7.3.3, the indirect pendulum model can be written as a linear regression and if the criterion functions are quadratic in the linear discrete-time parameters, Steps 2 and 3 in Approach 5.1 can be performed by solving a least-squares problem with linear constraints and then transforming the obtained linear parameters. In these cases, given that $\hat{\vartheta}_{p,1}$ and z_s are known, Approach 5.1 can be further specified to the linear Approach 7.1 which is computationally inexpensive as discussed in Section 5.2.3. A major benefit of this solution is that each step can

Figure 7.9: Data are collected from two different load cases, a nominal case giving the dataset Z_n and a case with additional load giving the dataset Z_l .

be efficiently solved using factorization, see Golub and Van Loan (1996, Ch. 12).

The second approach is to instead solve a joint problem by computing the estimate according to Approach 7.2. In this approach, both datasets are used simultaneously and the parameters $\vartheta_{p,2}$ and $\vartheta_{p,3}$ are estimated in one joint estimation problem assuming that $\vartheta_{p,1}$ is known or estimated. One issue with the sequential Approach 7.1 is that the estimate $\hat{\vartheta}_{p,2}$ is used in Step 3 without considering its uncertainty. The joint approach deals with this automatically at the expense of a nonlinear estimation problem. In Approach 7.2, the criterion function and constraints have deliberately been separated to emphasize the structure in the estimation problem, i.e. the parameters' relation to the different datasets. Note that the datasets do not need to be of equal length for this approach to work and that even several shorter datasets can be incorporated into the same methodology as described in Section 5.2.

For both Approach 7.1 and 7.2, the properties of the solutions in Steps 2 and 3 are dictated by the choices of Φ^i and Y^i , $i = \{n, l\}$ as discussed in Section 5.2.

Approach 7.1 (The linear multi-stage method).

1. The parameters $\vartheta_{p,1}$ of the nominal system are estimated or measured.
- 2a. The estimate $\hat{\vartheta}_{p,1}$ is used together with the nominal dataset Z_n to compute

$$\begin{aligned} \hat{\vartheta}_2 &= \underset{\vartheta}{\operatorname{argmin}} \|\Phi^n \vartheta - Y^n\|_2^2 \\ \text{s.t. } B_2 \vartheta &= b_2 \end{aligned} \tag{7.50}$$

where B_2 and b_2 are either

- (i) empty if I_x , k and d are unknown.
- (ii) given by (7.41) if k is known.
- (iii) given by (7.43) if d is known.

- 2b. $\hat{\vartheta}_{p,2}$ is given by (7.36) and (7.37).

- 3a. The loaded dataset Z_l and the estimates $\hat{\vartheta}_{p,1}$ and $\hat{\vartheta}_{p,2}$ are used to compute

$$\begin{aligned} \hat{\vartheta}_3 &= \underset{\vartheta}{\operatorname{argmin}} \|\Phi^l \vartheta - Y^l\|_2^2 \\ \text{s.t. } B_3 \vartheta &= b_3 \end{aligned} \tag{7.51}$$

where B_3 and b_3 are given by (7.47).

- 3b. Finally, the changes in mass m and the CM z_m are given by (7.36) and (7.38).

Note that the physical parameters are replaced with the estimates where necessary, for instance, in (7.41). _____

Approach 7.2 (Example of the joint identification approach).

1. The parameters $\vartheta_{p,1}$ of the nominal system are estimated or measured.
- 2-3. The estimate $\hat{\vartheta}_{p,1}$ is used together with the nominal and the loaded datasets to estimate I_x , d , k , m and z_m by computing

$$\begin{aligned} \{\hat{\vartheta}_{p,2}, \hat{\vartheta}_{p,3}\} = & \underset{\vartheta_2, \vartheta_3, \vartheta_{p,2}, \vartheta_{p,3}}{\operatorname{argmin}} \quad \|\Phi^n \vartheta_2 - Y^n\|_2^2 + \|\Phi^l \vartheta_3 - Y^l\|_2^2 \\ \text{s.t.} \quad & h_2(\vartheta_2, \hat{\vartheta}_{p,1}, \vartheta_{p,2}) = 0 \\ & h_3(\vartheta_3, \hat{\vartheta}_{p,1}, \vartheta_{p,2}, \vartheta_{p,3}) = 0 \end{aligned} \quad (7.52)$$

where ϑ_2 and ϑ_3 are auxiliary parameters, $h_2(\cdot) = 0$ is the nonlinear equality constraints given by (7.39) with $m = z_m = 0$ and $h_3(\cdot) = 0$ is given by (7.39). Note that an equivalent unconstrained estimation problem can be formulated by eliminating the auxiliary linear parameters.

Choice of Instruments for the Extended IV Estimator

As discussed in Section 2.5.2, any instruments that fulfill the conditions in (2.44) will guarantee consistency, but the variance properties of the parameter estimator are greatly affected by the choice of instruments and prefilter (Forsell, 1999; Ljung, 1999). Further insight about the choice of prefilter, the optimal instruments and approximate optimal instruments for a closed-loop system is discussed in Section 2.5.2. These results are all based on using the noise model and an external reference signal, instead of the input, to create the optimal instruments. Even though none of these sources consider optimal instruments for the case with noisy input and output data according to Figure 7.8, it seems likely that it should be possible to decrease the variance by using a similar strategy also in the present case. Preferably, the variable a_y should be used as the external reference, but it is unknown and instead an estimate $\hat{a}_y$ is used. As mentioned in Section 7.2 and shown in Figure 7.3 and Figure 7.8, a_y is correlated with the external reference δ through the relation $a_y = F_\delta(\mathbf{p})\delta$ but $F_\delta(\mathbf{p})$ is assumed to be unknown and hence, an estimate is used according to

$$\hat{a}_{y,t} = \hat{F}_{\delta,d}(\mathbf{q})\delta_t \quad (7.53)$$

Here, the instruments are chosen as

$$\zeta_t = L(\mathbf{q}) \begin{bmatrix} \hat{a}_{y,t} & \cdots & \hat{a}_{y,t-n_\delta-1} & \hat{y}_{O,t} & \cdots & \hat{y}_{O,t-n_y-1} & \hat{a}_{s,t} & \cdots & \hat{a}_{s,t-n_u-1} \end{bmatrix}^T \quad (7.54)$$

where $L(\mathbf{q})$ is a prefilter, $(\cdot)_t$ is the signal $(\cdot)$ at time t ,

$$\hat{y}_{O,t} = \frac{G_{O,d}(\mathbf{q}, \hat{\vartheta})}{1 - G_{O,d}(\mathbf{q}, \hat{\vartheta})F_y(\mathbf{q})} \hat{a}_{y,t} \quad \text{and} \quad \hat{a}_{s,t} = \frac{1}{1 - G_{O,d}(\mathbf{q}, \hat{\vartheta})F_y(\mathbf{q})} \hat{a}_{y,t} \quad (7.55)$$

are the simulated output and input, respectively, and $\hat{\vartheta}$ is an estimate of the parameters. The constants n_δ , n_y and n_u are the numbers of time lags (including the non-delay signal) included in ζ_t for the reference signal, the output and the input, respectively, i.e. $n_\delta = 0$ means that $\hat{a}_{y,t}$ is not included in ζ_t .

7.5 Experimental Verification

In this section, two experiments are presented. The first one is based on the simulation study presented in Linder et al. (2014b) and the second one is based on data from a modified inverted pendulum process presented in Linder et al. (2014a) and Linder and Enqvist (2016a). The goals of the first simulation study are to show the impact of measurement noise and to verify the applicability of the proposed method. The goals of the second experiment are to decrease the variance of the estimator, to show the applicability on real data and to show the difference between the multi-stage and joint identification approaches.

7.5.1 Simulation Study

To investigate the properties of the suggested method, it was applied to simulated data from the pendulum system without any process disturbance acting on it, i.e. $\bar{\tau}$ is zero. The dataset was synthesized with the approximate model (7.16). The signal a_y was formed by feeding the reference signal δ through the transfer function $F_\delta(s) = \frac{1}{0.2s+1}$. The true parameters are listed in Table 7.2.

Two datasets were created, one with nominal mass and one with additional mass. Both have 150 000 data points, corresponding to 25 min of data and a sampling period of 0.01 s. White zero-mean Gaussian measurement noise with a standard deviation of 0.01 was added to the simulated measurement signals, where the amplitude of the noise was chosen to be similar to the Xsens IMU in Appendix A.6.1. Note that this is actually a higher noise level on the gyro than in the real sensor. The noisy signals were subjected to a preprocessing stage where the signals were filtered through a zero-phase FIR equiripple low-pass filter of order 20 with a cut-off frequency of 0.5 Hz. This filtering is similar to what would be performed on data collected from a real IMU. Note that this preprocessing should not be confused with the prefilter in Section 7.4. Figure 7.10(a) shows the first 110 s of the nominal dataset (with no additional mass) and Figure 7.10(b) presents a more detailed view of a typical behavior of the system. The black boxes in Figure 7.10(a) correspond to the data shown in Figure 7.10(b).

The parameters were estimated using the IV method with the noise model

$$\bar{H}_{O,d}(q) = \frac{1}{A_O(q)} \quad (7.56)$$

where $A_O(q)$ is the denominator of $G_{O,d}(q)$. This gives the prefilter $L(q) = 1$,

Table 7.2: Parameters used in the simulation. Note that the additional mass is less than 1 % of the total mass of the pendulum. For an explanation of the parameters see Figure 7.2 on page 82.

M	z_g	I_x	d	k	m	z_m	z_s
22.04 kg	-0.018 m	0.1385 kgm ²	0.1067 Nms	10.39 Nm	0.2 kg	-0.274 m	-0.218 m

(a) The first 110 s of the nominal dataset.

(b) A detailed view of data showing a typical behavior. This data corresponds to the black boxes marked in Figure 7.10(a).

Figure 7.10: The nominal dataset that was used in the identification. The measured signal is shown in gray, the true signal in blue and the measured signal after the LP filter was applied in red. The left plot shows the angular velocity and the right plot shows the acceleration measurement.

see Section 2.5.2 for a discussion of the relation between the noise model and the prefilter. Furthermore, the transfer function $\hat{F}(s) = \frac{1}{0.1s+1}$ was used to form $\hat{a}_y$. Note that the time constant is different from the true one. For comparison, a constrained least-squares approach was also used to estimate the parameters.

In Step 2 of Approach 7.1, M , z_g and k were assumed to be known and I_x and d were estimated using the linear constraint (7.41). In Step 3, m and z_m were estimated using the linear constraint (7.47). A Monte Carlo experiment with 10 000 runs was performed.

The resulting parameters from Step 2a in Approach 7.1 can be seen in Figure 7.11(a) and the corresponding physical parameters from Step 2b, i.e. I_x and d , can be seen in Figure 7.11(b)–(c). For both the physical parameters, the LS estimator has less variance but is biased. However, note that the bias is in the order of 10^{-3} for the I_x estimator and in the order of 10^{-2} for the d estimator. These are both small in relation to the true values.

The results from the third and final stage, i.e. estimation of m and z_m , can be seen

in Figure 7.12. Figure 7.12(b) shows the result for the mass m and Figure 7.12(c) shows the result for the distance z_m from the center of rotation. In this stage, the estimates, i.e. $\hat{I}_x$ and $\hat{d}$, from the previous stage, were used to replace the true parameters. Again, the IV estimator gave a smaller bias than the LS estimator for both m and z_m but a larger variance. For the corresponding discrete-time parameters, see Figure 7.12(a). Note the increased variance compared to Figure 7.11(a). This was probably a result of that the estimator was less sensitive with respect to these parameters. However, this is something that needs further investigation.

According to theory, the estimated discrete-time linear parameters in the IV method will tend to the true ones as the number of data points goes to infinity (Söderström and Stoica, 1989). With the suggested IV estimator and the data length used, there was an obvious variation in the estimated physical parameters.

(a) The discrete-time linear parameters from Step 2a minus their true value. From the top, $\bar{\alpha}_1 - \bar{\alpha}_1^0$, $\bar{\alpha}_2 - \bar{\alpha}_2^0$ and $\bar{\beta}_0 - \bar{\beta}_0^0$.

(b) The results for I_x from Step 2b

(c) The result for d from Step 2b

Figure 7.11: The histograms of a Monte Carlo experiment with 10 000 runs for Step 2 of Approach 7.1. The red line corresponds to the true value, the blue bars correspond to the LS method, the dashed blue line corresponds to the mean of the LS estimator, the gray bars correspond to the IV method and the dashed black line corresponds to the mean of the IV estimator. Note that the LS estimator has smaller variance but larger bias.

This was probably due to the variance properties of the IV estimator and the sensitivity of the nonlinear transformation to the discrete-time linear parameters.

However, it should be emphasized that the increase in the mass was less than 1 % of the total mass. Hence, this simple version of the IV estimator was performing quite well since it actually estimated the mass above zero in all cases, within ± 0.1 kg in 85% of the cases and above 0.1 kg in 95% of the cases. Hence, the estimator captured the change in mass quite well. It was also interesting to note that even though the LS estimator in the previous stage was biased, the estimator for the mass and the center of mass had low bias and variance. In this limited experiment, only measurement noise was considered and the real benefit of the IV method is when both the input and output are correlated with the same noise source.

(a) The discrete-time linear parameters from Step 3a minus their true value. From the top, $\bar{\alpha}_1 - \bar{\alpha}_1^0$, $\bar{\alpha}_2 - \bar{\alpha}_2^0$ and $\bar{\beta}_0 - \bar{\beta}_0^0$. Note the increase in variance of the IV estimator compared to Figure 7.11(a).

(b) The results for m from Step 3b.

(c) The result for z_m from Step 3b.

Figure 7.12: The histograms of a Monte Carlo experiment with 10000 runs for Step 3 of Approach 7.1. The red line corresponds to the true value, the blue bars correspond to the LS method, the dashed blue line corresponds to the mean of the LS estimator, the gray bars correspond to the IV method and the dashed black line corresponds to the mean of the IV estimator. Note that the LS estimator has smaller variance but larger bias.

7.5.2 A Stabilized Inverted Pendulum Experimental Setup

This section describes an inverted pendulum experiment, together with results from the estimation procedure and an analysis of the results. The benefit of the experimental setup was that the system was very similar to the approximate model (7.10) and it could thus be used to validate the proposed method.

The experiment was based on real data collected from IMUs that were attached to an inverted pendulum modified with extra weights and springs to stabilize the pendulum. The system that was used in this experiment can be seen in Figure 7.13. The system consisted of a cart, six weights with adjustable positions, four springs and a mounting plate for the IMU. The known or measured masses, shapes and positions of each component were used to calculate the total inertia of the pendulum using theoretical results from basic mechanics. The weights and lengths were measured using a scale and slide caliper, respectively. The center of mass was calculated using the known positions and weights of each component. The physical quantities are listed in Table 7.3.

Figure 7.13: A picture of the pendulum. Uppermost in the picture is the IMU (orange) measuring the pendulum motion. In front of it is the additional load fastened with white tape. To the left on the cart is the reference IMU.

Table 7.3: The physical quantities of the system measured or calculated from known physical quantities.

Moment of Inertia (I_x)	Mass (M)	CM (z_g)	IMU position (z_s)
0.00707 kgm ²	1.324 kg	-0.0211 m	-0.1163 m

The damping coefficient d was unknown and it was unfortunately difficult to get an independent estimate using the tools that were available at the time of the experiment. The damping torque was due to the friction in the bearings of the pendulum rotation axis and the energy dissipated due to the interaction with the data cable attached to the IMU on the pendulum.

The spring coefficient k was also unknown. The spring torque was created with four linear springs and the elongation of the springs were roughly proportional to the angle of the pendulum. The largest deviation from a pure linear relation between the angle and the elongation was calculated to be 10^{-4} m up to 20° . This deviation was due the attachment on the pendulum moving in a slight arc. However, tests showed that the spring force from each individual spring was not proportional to the elongation and the restoring nonlinear restoring torque is assumed to be described by

$$k(\phi)\phi \quad (7.57)$$

where $k(\phi)$ is an unknown nonlinear function.

The additional mass was a cylindrical piece of metal that can be seen on the top in Figure 7.13. The known, measured or calculated physical quantities of the additional mass are listed in Table 7.4. The inertia was neglected due to its small size. The second row in Table 7.4 describes the relative difference to the nominal system, for instance, the second column corresponds to $100m/M\%$. It can be seen that the increase in the mass was relatively small, but that the inertia was increased with approximately 20% and the torque affecting the pendulum was increased roughly 0.2 times for the same input a_y since the coefficient β_1 becomes

$$\beta_1 = -\frac{Mz_g + mz_m}{I_x + Mz_g^2 + mz_m^2} = 4.37$$

which was a 19.9% increase from the nominal case.

Table 7.4: The physical quantities of the added mass measured or calculated from known physical quantities. The second row describes the relative difference to the original system.

Moment of Inertia (I_x)	Mass (m)	CM (z_m)	(mz_m)	(mz_m^2)
$5.1 \cdot 10^{-6}$ kgm ²	0.0975 kg	-0.1225 m	-0.0119 kgm	0.0015 kgm ²
—	7.37%	—	42.38%	20.68%

There were two IMUs mounted on the cart and the pendulum as can be seen in Figure 7.13. They were an Xsens MTi (cart) and an Xsens MTi-G (pendulum). Both were equipped with a three-axis accelerometer, a three-axis gyro and a three-axis magnetometer. Only a few of the axes were used to collect data. The Y -axis of the accelerometer on the pendulum-IMU corresponds to the tangential acceleration (7.15) while the X -axis of gyro corresponds to the angular velocity (7.13). The Y -axis of the cart-IMU measures the acceleration $y_{4,t} = a_{y,t} + e_{4,t}$, where $e_{4,t}$ was assumed to be a discrete-time white zero mean measurement noise, i.e. $F_\delta(\mathbf{q}) = 1$. Both IMUs were using a sampling rate of 100Hz. A hammer was used to perform a series of knocks on the rotation axis in the beginning of each measurement series. Synchronization was then achieved by carrying out a correlation analysis on the corresponding acceleration data.

The instruments were created using the constants $n_{\delta,0}=4$, $n_\delta=0$, $n_y=4$ and $n_u=4$. To investigate the necessity of a noise model, two different models were tested:

1. The noise model

$$\bar{H}_{O,d}(\mathbf{q}) = \frac{1 + \bar{c}_1\mathbf{q}^{-1} + \bar{c}_2\mathbf{q}^{-2} + \bar{c}_3\mathbf{q}^{-3} + \bar{c}_4\mathbf{q}^{-4}}{1 + \bar{\alpha}_1\mathbf{q}^{-1} + \bar{\alpha}_2\mathbf{q}^{-2}} \quad (7.58)$$

which corresponds to the prefilter

$$L(\mathbf{q}) = \frac{1}{1 + \bar{c}_1\mathbf{q}^{-1} + \bar{c}_2\mathbf{q}^{-2} + \bar{c}_3\mathbf{q}^{-3} + \bar{c}_4\mathbf{q}^{-4}} \quad (7.59)$$

Note that, this noise model was flexible enough to capture $H_{O,d}(\mathbf{q})$ and also gives some flexibility for coloring in $\bar{\tau}$.

2. The noise model

$$\bar{H}_{O,d}(\mathbf{q}) = \frac{1}{A_o(\mathbf{q})} \quad (7.60)$$

which corresponds to the prefilter $L(\mathbf{q}) = 1$, i.e. the same noise model that was used in the previous section. Note that this noise model was not flexible enough to capture even $H_{O,d}(\mathbf{q})$. It did however result in a less computationally expensive algorithm.

Two datasets were collected according to Section 7.4, one with the nominal weight and one with the additional mass. Both datasets were 10 minutes long, the signals were filtered through an FIR equiripple low-pass filter of order 20 with a cut-off frequency of 0.5 Hz and were down-sampled to 50Hz with a total of 30 000 samples in each dataset. The system was excited by dragging the cart back and forth. Figure 7.14(a) shows a typical part of the nominal dataset.

The result using Approach 7.1 with (5.24) for $\bar{H}_O(\mathbf{q})$ defined in (7.58) can be seen in the top rows of Table 7.5 and the gray bars of Figure 7.15(a). The figure is created by drawing 100 000 samples from $\mathcal{N}(\hat{\theta}, \hat{P}_\theta)$ and passing these through the nonlinear transformation in Step 2a and 3a to the physical parameters. The red lines correspond to the true value and the dashed black lines correspond to the sample mean.

The estimation of the noise model and filtering with $L(\mathbf{q})$ had a significant impact. As a comparison, the middle rows of Table 7.5 and the cyan bars of Figure 7.15(a)

(a) A closer look on the nominal dataset used for estimation.

(b) A typical part of the validation data with additional mass (blue). The simulated model output (dashed black) captures the periods of the data well but has troubles with both over- and underestimation of the peaks.

Figure 7.14: Examples of datasets used in Section 7.5.2.

show the result when the more restrictive noise model of Case 2 was used. Even though there was just a small difference in the second step, the estimates in the third step had large errors and the variance was significantly increased, compare for instance, the results shown in the top plot of Figure 7.15(a).

The parameters were also estimated with the joint identification approach described in Approach 7.2. The results can be seen in the lower rows of Table 7.5 and in Figure 7.15(b). Compared to the results of the multi-stage approach seen in Figure 7.15(a), the variance for the estimate of the mass has increased while the variance for the estimate of the center of mass is slightly smaller. Note that the true value of the additional mass is within the uncertainty interval (three standard deviations from the mean). The observed difference is probably due to the lack of some uncertainty information in the multi-stage approach that leads to overconfidence in the mass estimate. By simultaneously using all datasets in the joint approach, the uncertainty is incorporated and the result is an increased variance for the mass estimate.

Note that no apparent negative effects can be seen from the nonlinear restoring force (7.57) on the parameter estimates. This was another positive aspect since the restoring force of a ship is nonlinear due to its dependency on the hull form as described in Appendix A.2.1.

In addition to comparing the physical quantities, the model was validated by simulating (7.55) using the measured cart acceleration δ as input and with the parameters estimated with the joint approach given in Table 7.5. The fit to the nominal data was 75.4% for y_O and 75.8% for a_s . Similarly, the fit to the loaded data was 73.8% for y_O and 74.3% for a_s . Here, the fit is defined as the normalized root mean square error given by Tangirala (2014, p. 364). Figure 7.14(b) shows 24 typical seconds of the loaded validation dataset together with the simulated output from (7.55). The model captures the period of the signal well, but both under- and overestimates the peaks. This could be due to the disturbance in the measured input δ or the nonlinearities in the restoring and friction torques of the system.

Table 7.5: The results from multi-stage and joint identification of the physical parameters of the system shown in Figure 7.13. The first row in each case contains the estimated values and the second row contains the relative errors.

Case	I_x	d	k	m	z_m	mz_m	mz_m^2
Multi (7.58)	0.0071	0.0032	1.84	0.0918	-0.116	-0.0106	0.0012
	0.602%	–	–	5.81%	5.52%	10.5%	17.6%
Multi (7.60)	0.0073	0.0031	1.87	0.0719	-0.136	-0.0098	0.0006
	3.2%	–	–	26.2%	10.8%	17.8%	11.3%
Joint (7.58)	0.0071	0.0032	1.84	0.0922	0.115	0.0106	0.0012
	0.575%	–	–	5.40%	5.77%	10.9%	16.0%

Figure 7.15: (a): multistage approach and (b): joint approach. Red corresponds to the measured physical quantity in Table 7.3 and Table 7.4. Gray corresponds to the results from using the noise model $\bar{H}_{O,d}(\mathbf{q})$ defined in (7.58) and cyan corresponds to the results from using the noise model $\bar{H}_{O,d}(\mathbf{q})$ defined in (7.60). Blue corresponds to the sample mean for the estimates using the noise model in (7.60). Black corresponds to the sample mean for the estimates using the noise model in (7.58). Note that the IV estimator with the noise model (7.58) has a quite significant decrease in both bias and variance.

7.5.3 Summary of Experimental Studies

The two experimental studies presented in this section show the applicability of the method to both simulated and real data. The sensitivity shown in Section 7.5.1 could probably be reduced by the more flexible noise model used in Section 7.5.2. However, the intention of this section was not to do an exhaustive experimental study but rather to present a couple of examples with the goal of understanding the properties of the estimator. In Section 7.5.2, the closed-loop viewpoint of Section 7.3.2 enabled a better choice of instruments with inspiration from previous work in closed-loop identification. The suggested algorithms were applied to a set of data from an experimental setup with good results. The results could probably be improved further by choosing instruments for closed-loop identification with noisy input and output data (EIV). Both studies in this section had a quite low impact from the process disturbance and the real challenge lies in getting consistent estimators in situations with more severe disturbances.

8

Estimation of a Ship's Roll Dynamics

This chapter is devoted to graybox modelling of roll dynamics for ships. To avoid a physical model of the entire ship, here, the roll dynamics is seen as a subsystem of the ship's dynamics and the couplings from the other degrees of freedoms are seen as inputs to the roll subsystem, see Figure 8.1.

A linear model of the roll subsystem will be developed from the surge-sway-roll-yaw maneuvering model with rudder input (A.39) and an indirect model is derived for estimation of the parameters. Due to identifiability issues, the joint identification approach described in Chapter 5 will be used. Finally, simulation results showing the sensitivity of initial values will be presented. For results from this approach on real experimental data from a ship model basin, see Chapter 9.

Figure 8.1: The generalized position η and the generalized speed ν of a ship model are assumed to be affected by the rudder angle δ , the surge speed U and the disturbance τ . The roll dynamics is seen as a subsystem of the ship with inputs from the sway and yaw subsystems. Here, the surge speed U is assumed to be decoupled from the other subsystems and the double-pointed arrows symbolize two-way interaction between the subsystems.

8.1 Derivation of the Indirect 4DOF Ship Model

The surge-sway-roll-yaw model (A.39) can be written as

$$\bar{\mathbf{M}}\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \delta) + T\bar{\tau} \quad (8.1)$$

where $\bar{\tau}$ is the process disturbance, T is a matrix describing the process disturbance's couplings to the states, $\bar{\mathbf{M}}$ is the inertia matrix and $\mathbf{F}(\mathbf{x}, \delta)$ is the nonlinear state transition function. The states are given by

$$\mathbf{x} = [\phi, \psi, v, p, r]^T \quad (8.2)$$

where ϕ is the roll angle and ψ is the yaw angle, both expressed in the world-fixed coordinate system. The variables v , p and r are the linear surge speed, the roll angular velocity and the yaw angular velocity, respectively, all expressed in the body-fixed frame. For more details on the coordinate systems and notation, see Appendix A. Finally, δ is the rudder angle defined in Appendix A.3.1.

Assuming that the ship is port-starboard symmetric ($\bar{y}_g = 0$), that the ship is fore-aft symmetric ($\bar{x}_g = 0$) and that the surge component can be decoupled due to much slower dynamics, the linearization about $\bar{\mathbf{x}} = 0$ and $\bar{\delta} = 0$ is

$$\bar{\mathbf{M}}\dot{\mathbf{x}} = \left. \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}, \bar{\delta}=0} \mathbf{x} + \left. \frac{\partial \mathbf{F}}{\partial \delta} \right|_{\bar{\mathbf{x}}, \bar{\delta}=0} \delta + T\bar{\tau} \quad (8.3)$$

where

$$\bar{\mathbf{M}} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad (8.4)$$

$$\mathbf{M} = \begin{bmatrix} M + m - Y_{\dot{v}} & -Mz_g - mz_m - Y_{\dot{p}} & -Y_{\dot{r}} \\ -Mz_g - mz_m - K_{\dot{v}} & \bar{A}_x & -K_{\dot{r}} \\ -N_{\dot{v}} & -N_{\dot{p}} & \bar{A}_z \end{bmatrix}, \quad (8.5)$$

$$\left. \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}, \bar{\delta}=0} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -Y_{\phi uu}U^2 & Y_{|u|v}|U| & 0 & 0 & (Y_{ur} - M - m)U \\ K_{\phi uu}U^2 - \rho g \nabla \overline{GM}_T & K_{|u|v}|U| & K_p + K_{|u|p}|U| & (K_{ur} + Mz_g + mz_m)U \\ N_{\phi u|u}|U| & N_{|u|v}|U| & N_p + N_{|u|p}|U| & N_{|u|r}|U| \end{bmatrix} \quad (8.6)$$

and

$$\left. \frac{\partial \mathbf{F}}{\partial \delta} \right|_{\bar{\mathbf{x}}, \bar{\delta}=0} = \begin{bmatrix} 0 \\ 0 \\ C_2 \\ -z_r C_2 \\ x_r C_2 \end{bmatrix} \quad (8.7)$$

The fourth row of (8.3) given by

$$\begin{aligned} \bar{A}_x \dot{p} - (K_{\dot{v}} + Mz_g + mz_m)\dot{v} - K_{\dot{r}} \dot{r} &= (K_{\phi uu}U^2 - \rho g \nabla \overline{GM}_T)\phi + K_{|u|v}|U|v \\ &+ (K_p + K_{|u|p}|U|)p + (K_{ur} + Mz_g + mz_m)Ur - z_r C_2 \delta + \tau \end{aligned} \quad (8.8)$$

is a model of the roll subsystem in component form where τ is the component of

$T\bar{\tau}$ that affects the roll subsystem. Here, the influence from $K_{\dot{r}}\dot{r}$ and $K_{|u|v}|U|v$ is assumed to be negligible, i.e. these terms are assumed to be zero. To ease notation, three lumped parameters are introduced. These are

$$k = -K_{\phi uu}U^2 + \rho g \nabla \overline{GM}_T - Mg z_g - mg z_m, \quad (8.9a)$$

$$d = -K_p - K_{|u|p}|U| \quad (8.9b)$$

and

$$K_\delta = -z_r C_2 \quad (8.9c)$$

Noting that $\dot{\phi} = p$ due to the linearization, i.e. small angle assumption, and using the lumped parameters, the approximate roll dynamics can be written as

$$\begin{aligned} A_1 \ddot{\phi} = & -(k + Mg z_g + mg z_m) \phi - d \dot{\phi} \\ & + (K_{\dot{v}} + M z_g + m z_m) \dot{v} + (K_{ur} + M z_g + m z_m) U r + K_\delta \delta + \tau \end{aligned} \quad (8.10)$$

where

$$A_1 = \bar{A}_x = A_x + M z_g^2 + m z_m^2 = I_x - K_{\dot{p}} + M z_g^2 + m z_m^2 \quad (8.11)$$

I_x is the ship's inertia about the CG, $K_{\dot{p}}$ is the added mass and τ is the process disturbance. Note that the relation $\dot{\phi} = p$ is not valid in general, see Appendix A.1.2.

Note that (8.9a) equivalently can be written as

$$K_{\phi uu}U^2 - \rho g \nabla \overline{GM}_T = -k - Mg z_g - mg z_m \quad (8.12)$$

which are the restoring coefficients in the models (8.8) and (8.10), respectively. Here, we assume that the parameter k can be seen as representing the physical restoring properties of the ship, for instance, being dependent on factors such as the hull shape, while the term $Mg z_g + mg z_m$ represents the influence by the mass and its location on the restoring properties. The parameter k is thus, given a speed U , assumed to be fixed and independent of the loading condition. Hence, it is assumed that a big change in loading condition is captured by $Mg z_g + mg z_m$. Note that z_g and z_m for a typical surface vessel are negative and hence, increasing the distance from the center of rotation or increasing the mass would result in $-k - Mg z_g - mg z_m$ being positive which would result in the vessel being unstable. See Appendix A.2.1 for a definition and discussion of ship stability. The parameter d is in the same way, given a fixed speed U , assumed to be constant.

Disregarding that the model (8.10) is a model of a ship subsystem, given $\dot{v}$, r , δ and τ , the model (8.10) is a second order system similar to the approximate pendulum model described in Chapter 7 with the addition of the ship-related parameters, $K_{\dot{p}}$, $K_{\dot{v}}$, K_{ur} and K_δ , corresponding to the hydrodynamic derivatives and direct term from the rudder. The model (8.10) has the same issues as the model (7.10) of Section 7.2 since $\dot{v}$ is unknown. Again, the approaches enumerated in Section 7.3 can be used to deal with this issue. Similar to Section 7.3, neither $\dot{v}$ nor a model of it is known, and an indirect model can be formed by eliminating $\dot{v}$. The key to this elimination is the known relation between the measured tangential acceleration a_s defined in (A.43), the acceleration a_y and the variable $\dot{v}$.

Figure 8.2: A ship moving with constant forward speed on a circle without sideslip has tangential acceleration Ur pointing inwards.

The acceleration a_y of the ship in the world-fixed x - y -plane has two parts. The first contribution emanates from the sway motion and the second part is a contribution due to the angular velocity about the yaw axis. Figure 8.2 shows a ship moving on a circle with the forward speed U and turning with the angular velocity r . The resulting acceleration can, for instance, be derived using the equations for circular motion

$$\left. \begin{aligned} U &= yr \\ a_{y,r} &= yr^2 \end{aligned} \right\} \Rightarrow a_{y,r} = Ur \quad (8.13)$$

Furthermore, a ship in motion usually has a discrepancy between the course and the heading due to sideslip, and the sideslip gives the contribution

$$a_{y,\dot{v}} = \dot{v} \quad (8.14)$$

which means that the total sway acceleration is given by

$$a_y = a_{y,\dot{v}} + a_{y,r} = \dot{v} + Ur \quad (8.15)$$

The acceleration $\dot{v}$ is thus measured indirectly by the tangential acceleration a_s defined in (A.43) which combined with (8.15) give

$$a_s = z_s \ddot{\phi} + g\phi - a_y = z_s \ddot{\phi} + g\phi - \dot{v} - Ur \quad (8.16)$$

Solving (8.16) for $\dot{v}$ and substituting it into (8.10) give

$$\begin{aligned} A_2 \ddot{\phi} &= -(k - K_{\dot{v}}g)\phi - d\dot{\phi} \\ &\quad - (K_{\dot{v}} + Mz_g + mz_m)a_s + (K_{ur} - K_{\dot{v}})Ur + K_{\delta}\delta + \tau \end{aligned} \quad (8.17)$$

where

$$A_2 = A_x + Mz_g(z_g - z_s) + mz_m(z_m - z_s) - K_{\dot{v}}z_s \quad (8.18)$$

Since the surge speed U is assumed to be constant, this model can be further simplified by introducing the lumped parameter

$$K_r = (K_{ur} - K_{\dot{v}})U, \quad (8.19)$$

giving the surge–speed–free model

$$A_2 \ddot{\phi} = -(k - K_{\dot{v}}g)\phi - d\dot{\phi} - (K_{\dot{v}} + Mz_g + mz_m)a_s + K_r r + K_{\delta}\delta + \tau \quad (8.20)$$

There are a few properties of the suggested model (8.20) worth observing:

1. The parameters k , d and K_r all are dependent on the surge speed U but the dependency is not written out to ease notation.
2. If $K_{ur} = K_{\dot{v}}$, then r is not needed as an input since a_s contains all relevant information. This can also be seen in the model (8.10), where the coefficients of $\dot{v}$ and Ur would coincide if $K_{ur} = K_{\dot{v}}$.
3. The model (7.24) described in Section 7.3 is obtained if the parameters related to the ship, $K_{\dot{p}}$, $K_{\dot{v}}$, K_r and K_{δ} , are set to zero.
4. The surge speed U is assumed to be constant. If the speed is measured, it can be used in the identification process.

8.1.1 Center of Rotation Estimation

In Chapter 7, the x -axis of the b -frame coincided with the rotation axis. This is not necessarily true in this chapter and the next. Here, the x_b -axis of the b -frame is assumed to be parallel to the rotation axis but shifted in the z_b -direction, i.e.

$$z_g = \bar{z}_g + z_f, \quad z_m = \bar{z}_m + z_f \quad (8.21)$$

and this is illustrated in Figure 8.3. Note that (8.20) is really expressed in the center of rotation coordinate system and z_f will express the body-fixed coordinate system in relation to this. However, note that this will not change the estimation problem significantly and the introduction of the extra parameter z_f will be equivalent to working with the body-fixed system as if it coincides with the center of rotation coordinate system. The advantage of this formulation is the possibility to estimate the location of the rotation center in relation to the known body-fixed coordinate system, i.e. to estimate z_f when the true center of rotation is not known.

Figure 8.3: A sketch showing the shifted center of rotation in relation to the body-fixed system.

8.2 Model Analysis

This section presents the dependencies of the signals in the model (8.20). These dependencies are important to understand in order to make appropriate choices in the identification procedure as discussed in Chapter 4. The indirect model derived in Section 8.1 avoids the use of a complete model of the ship, while the use of an acceleration measurement solves the issue of the unknown input $\dot{v}$. This will however introduce new issues and an analysis is presented in this section. In Section 8.2.1, the signals' dependencies on the true input δ are discussed. The model (8.20) is also shown to share the important non-minimum phase (NMP) property of the complete model. Section 8.2.2 describes the signals' dependencies on the disturbance τ . It is shown that all signals are correlated with the disturbance and that certain properties of these signals are important for the choice of estimation method as discussed in Chapter 4.

8.2.1 The Signals' Dependency on the Input δ

The ship is assumed to be affected by two things, firstly the actuator, i.e. the rudder, and secondly, the disturbances acting on the ship. This section presents the behavior of the roll dynamics due to the rudder. Neglecting the process disturbance τ , the model (8.20) with $y_O = \dot{\phi}$ as the output can be written on the form

$$y_O = G(p)(a_s + F_r r + F_\delta \delta) \quad (8.22)$$

where the transfer functions are given by

$$G(p) = \frac{\beta_1 p}{p^2 + \alpha_1 p + \alpha_2}, \quad F_\delta = \frac{\gamma_1}{\beta_1}, \quad F_r = \frac{\kappa_1}{\beta_1} \quad (8.23)$$

and the parameters are given by

$$\begin{aligned} \alpha_1 &= \frac{d}{A_2}, & \alpha_2 &= \frac{k - K_{\dot{v}}g}{A_2}, & \beta_1 &= -\frac{K_{\dot{v}} + Mz_g + mz_m}{A_2}, \\ \gamma_1 &= \frac{K_\delta}{A_2} = -\frac{z_r C_2}{A_2} & \text{and} & & \kappa_1 &= \frac{K_r}{A_2} \end{aligned} \quad (8.24)$$

where A_2 is defined in (8.18).

The reference signal δ is assumed to be the only external user-controllable signal acting the system, which means that both a_s and r depends on δ . Similarly to the model (7.10) in Chapter 7, Proposition 7.1 can be applied to the model (8.10). Assuming that U is constant, the model (8.10) is equivalent to the model (8.20) in positive feedback with $F_y = (z_s p^2 + g)/p$ and with $-a_y$ as external signal with a_y given by (8.15). This gives the model

$$y_O = G(p)(F_y(p)y_O + (F_r - U)r - \dot{v} + F_\delta \delta) \quad (8.25)$$

and the structure that can be seen in Figure 8.4. Assuming that the transfer operator from δ to $\dot{v}$ is $G_{\delta\dot{v}}(p)$ and that the transfer operator from δ to r is $G_{\delta r}(p)$, (8.25) can be written

$$y_O = G(p)(F_y(p)y_O + [(F_r - U)G_{\delta r}(p) - G_{\delta\dot{v}}(p) + F_\delta]\delta) \quad (8.26)$$

Figure 8.4: As in the model (7.24) case, the model (8.10) is recovered if the acceleration measurement equation (A.43), together with (8.15), is substituted into the model (8.20).

which is equivalent to

$$y_O = \frac{G(p)[R_\delta(p) + F_\delta]}{1 - F_y(p)G(p)}\delta = G_0(p)[R_\delta(p) + F_\delta]\delta = G_{\delta y}(p)\delta \quad (8.27)$$

where

$$G_0(p) = \frac{G(p)}{1 - F_y(p)G(p)}, \quad (8.28)$$

$$R_\delta(p) = (F_r - U)G_{\delta r}(p) - G_{\delta \dot{v}}(p), \quad (8.29)$$

$$G_{\delta y}(p) = G_0(p)[R_\delta(p) + F_\delta] \quad (8.30)$$

and the structure can be seen in Figure 8.5. Hence, (8.30) contains two contributions and these are directly connected to the non-minimum phase behavior described in Appendix A.5.1. The contribution $F_\delta\delta$ is the torque acting directly on the rotation axis while $R_\delta(p)\delta$ corresponds to the contributions through the slower yaw-dynamics and the hydrodynamic forces. Thus, for a positive rudder angle, the direct term will initially give a negative roll angle that is eventually counteracted by the coupling to the other subsystems, see Appendix A.5.1.

Similarly, the transfer function from δ to a_s can be derived from Figure 8.5 and is

$$a_s = \frac{F_y(p)G(p)(F_\delta + F_rG_{\delta r}(p)) - (G_{\delta \dot{v}}(p) - UG_{\delta r}(p))}{1 - F_y(p)G(p)}\delta = G_{\delta a_s}(p)\delta \quad (8.31)$$

which means that the non-minimum phase behavior also affects the acceleration but the response is more complex due to the direct terms from $\dot{v}$ and r .

Figure 8.5: The true input to the system is δ and it also affects $\dot{v}$ and r . The dynamics in the transfer functions $G_{\delta \dot{v}}$ and $G_{\delta r}$ together with the direct term in F_δ gives the non-minimum phase behavior of the system.

8.2.2 Identification Issues – Correlation with τ

In the previous section, the signals' dependency of δ was investigated. From a system identification perspective it is also important to investigate the signals' dependency on the process disturbance τ . As mentioned in the previous section, the reference signal δ is assumed to be the only external user-controllable signal acting on the system, which implies that both a_s and r are dependent on δ . However, they are also dependent on y_O since the modes of the true system are coupled and thus correlated with the process disturbance τ . This means that there are similarities with identification in closed-loop and these dependencies are important for the choices in the identification procedure. To understand the coupling between the subsystems, the model (8.10) can be expressed in terms of the model (8.20) and the subsystems of the linearized system (8.3). This reveals important information about the dependencies between the measured signals.

The model (8.20) in Figure 8.6 with $y_O = \dot{\phi}$ as the output can be written on the form

$$y_O = G(p)(a_s + F_r r + F_\delta \delta + \tau) \quad (8.32)$$

where the transfer functions $G(p)$, F_r and F_δ are defined in (8.23). Assuming that U is constant, the model (8.10) is equivalent to the model (8.20) in positive feedback with $F_y = (z_s p^2 + g)/p$ defined in (7.30) and with $-a_y$ as external signal with a_y given by (8.15). This gives the structure that can be seen in Figure 8.7.

This is however not the whole truth since the process disturbance acting on the roll subsystem will also affect the sway and yaw motions due to the coupling between the subsystems. In the same way, disturbances acting directly on the yaw and sway subsystems will propagate to the roll subsystem, see Figure 8.8.

This can be seen by extracting the sway and yaw subsystem from (8.3). Given r , $\dot{\phi}$ and δ , the sway subsystem is, after a few manipulations, governed by

$$\begin{aligned} \dot{v} &= \frac{(Mz_g + mz_m + Y_{\dot{\phi}})p^2 + Y_{\phi}uuU^2}{(M+m-Y_{\dot{\phi}})p - Y_{|u|v}|U|} \dot{\phi} + \frac{p(Y_{\dot{\phi}}p + (Y_{ur} - M - m)U)}{(M+m-Y_{\dot{\phi}})p - Y_{|u|v}|U|} r + \frac{pC_2}{(M+m-Y_{\dot{\phi}})p - Y_{|u|v}|U|} \delta \\ &= \frac{C}{A} \dot{\phi} + \frac{D}{A} r + \frac{E}{A} \delta \end{aligned} \quad (8.33)$$

Figure 8.6: The model (8.20) represented in block form. Note that δ , a_s and r are dependent on y_O according to (8.3). Due to the coupling these signals will also be correlated with τ .

Figure 8.7: As in the model (7.24) case, the model (8.10) is recovered if the measurement equation (A.43), together with (8.15), is substituted into the model (8.20). This reveals more clearly the issues of using a_s as an input to the system, but the complete dependencies between $\dot{v}$, r , $\dot{\phi}$ and δ are not shown.

and similarly, given $\dot{v}$, $\dot{\phi}$ and δ , the yaw subsystem is described by

$$\begin{aligned} r &= \frac{N_{\dot{p}}p^2 + (N_p + N_{|u|p}|U|)p + N_{\phi u}|u|U|U|}{p(Az p - N_{|u|r}|U|)} \dot{\phi} + \frac{(N_{\dot{v}}p + N_{|u|v}|U|)}{p(Az p - N_{|u|r}|U|)} \dot{v} + \frac{p x_r C_2}{p(Az p - N_{|u|r}|U|)} \delta \\ &= \frac{F}{B} \dot{\phi} + \frac{G}{B} \dot{v} + \frac{H}{B} \delta \end{aligned} \quad (8.34)$$

The transfer functions from δ and y_O to $\dot{v}$ and r can be obtained by combining (8.33) and (8.34). These transfer functions have the following structures

$$\dot{v} = \frac{CB + DF}{AB - DG} y_O + \frac{DH + EB}{AB - DG} \delta \quad \text{and} \quad r = \frac{FA + GC}{AB - DG} y_O + \frac{GE + HA}{AB - DG} \delta \quad (8.35)$$

The dependency of the acceleration a_s on y_O can now be investigated by noting that

$$a_s = \frac{z_s p^2 + g}{p} y_O - \dot{v} - U r = \left(\overbrace{\frac{\bar{F}_y(p)}{p}}^{z_s p^2 + g - \frac{CB + DF}{AB - DG} - U \frac{FA + GC}{AB - DG}} \right) y_O + \delta\text{-dependent terms} \quad (8.36)$$

and the loop gain from τ to a_s contains a direct term since

$$\begin{aligned} \lim_{s \rightarrow \infty} G(s) \bar{F}_y(s) &= \\ \lim_{s \rightarrow \infty} \frac{\beta_1 s}{s^2} \left(\frac{z_s s^2}{s} - \frac{((Mz_g + mzm + Y_{\dot{p}}) \bar{A}_z + Y_{\dot{r}} N_{\dot{p}}) s^4}{((M + m - Y_{\dot{v}}) \bar{A}_z - Y_{\dot{r}} N_{\dot{v}}) s^3} - U \frac{(N_{\dot{p}}(M + m - Y_{\dot{v}}) + (Mz_g + mzm + Y_{\dot{p}})) s^3}{((M + m - Y_{\dot{v}}) \bar{A}_z - Y_{\dot{r}} N_{\dot{v}}) s^3} \right) &= \\ \lim_{s \rightarrow \infty} \frac{\beta_1}{s^2} \left(z_s s^2 - \frac{(Mz_g + mzm + Y_{\dot{p}}) \bar{A}_z + Y_{\dot{r}} N_{\dot{p}}}{((M + m - Y_{\dot{v}}) \bar{A}_z - Y_{\dot{r}} N_{\dot{v}})} s^2 - U \frac{N_{\dot{p}}(M + m - Y_{\dot{v}}) + (Mz_g + mzm + Y_{\dot{p}})}{((M + m - Y_{\dot{v}}) \bar{A}_z - Y_{\dot{r}} N_{\dot{v}})} s \right) &\neq 0 \end{aligned} \quad (8.37)$$

For the loop gain from τ to r there is no direct term since

$$\lim_{s \rightarrow \infty} G(s) F_r \frac{FA + GC}{AB - DG} = \lim_{s \rightarrow \infty} \frac{\beta_1 s}{s^2} \frac{(N_{\dot{p}}(M + m - Y_{\dot{v}}) + (Mz_g + mzm + Y_{\dot{p}})) s^3}{((M + m - Y_{\dot{v}}) \bar{A}_z - Y_{\dot{r}} N_{\dot{v}}) s^3} = 0 \quad (8.38)$$

Figure 8.8: The transfer function from δ to $\dot{\phi}$ viewed in terms of its subsystems. Since the focus is on the roll motion, the emphasis is placed on the model of the roll subsystem G_ϕ from (8.10) which is shown with its internal dependencies.

In conclusion, even though the indirect model avoids building a model of the entire ship, it introduces some new challenges:

1. The signals r , a_s and y_O are all correlated with the disturbance τ .
2. The loop gain from τ to a_s contains a direct term which might make the parameter estimator biased.

These challenges are consistent with the findings of Chapter 4. Finally, in addition to the process disturbance τ also the measurement noise for each signal has to be considered. If the model (8.20) is used for estimation, the needed signals, $\dot{\phi}$, a_s and r , are not known but measured with noise. Hence, the inputs will be noisy and the identification problem will be of errors-in-variables type.

8.3 Identifiability of the Roll Dynamics Subsystem

Compared to the model (7.10), the model (8.10) has three more parameters, $K_{\dot{v}}$, K_{ur} and K_δ , and two more inputs, r and δ . Hence, the added degrees of freedom given by the extra inputs will not make all the parameters identifiable and if a subset of the parameters is chosen, it is not obvious which combinations of parameters that can be identified simultaneously. In Section 8.3.1 it will be shown that if only a single dataset is used, then only a subset of the unknown parameters can be estimated simultaneously. In Chapter 7, the multi-stage approach could be used to estimate the parameters, where the nominal data set was used to estimate a first set of parameters and these parameter estimates were used in combination with a second dataset to estimate a second set of parameters. A linear approach could be used since linear constraints were sufficient to obey the original nonlinear parameterization. In contrast to the approach in Chapter 7, the linear sequential method cannot be used in this chapter due to the higher complexity in the relations between the parameters. Instead, the analysis in Section 8.3.2 will show what can be achieved if several datasets are used concurrently. These results lead to the development of the nonlinear joint method that was described in Section 5.2.4 and will be tailored to the indirect ship model in Section 8.5.

Note that the model (8.10) will be used in this section, but that we could as well have worked with the model (8.20), which has the same identifiability properties.

8.3.1 Limitations Using a Single Dataset

Consider the model

$$A_1 \ddot{\phi} = - (k + Mg z_g + mg z_m) \phi - d \dot{\phi} \\ + (K_{\dot{v}} + M z_g + m z_m) \dot{v} + (K_{ur} + M z_g + m z_m) U r + K_{\delta} \delta + \tau$$

defined in (8.10), with the output

$$y_O = [\dot{\phi} + e_1, a_s + e_2]^T = [\dot{\phi} + e_1, z_s \ddot{\phi} + g \phi - \dot{v} - U r + e_2]^T$$

defined by (A.41), (A.44) together with (8.15) and the input

$$u = [\dot{v}, r, \delta]^T \quad (8.39)$$

This model can be cast into a state-space form by choosing the states $x = [\phi, \dot{\phi}]^T$ and inserting (8.10), (8.15) and (8.21) into (A.44). The effects of disturbances are neglected in this section, i.e. $e_1 = e_2 = \tau = 0$. The state-space model is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k + Mg(\bar{z}_g + z_f) + mg(\bar{z}_m + z_f)}{A_1} & -\frac{d}{A_1} \end{bmatrix} x \\ + \begin{bmatrix} 0 & 0 \\ \frac{M(\bar{z}_g + z_f) + m(\bar{z}_m + z_f) + K_{\dot{v}}}{A_1} & \frac{M(\bar{z}_g + z_f) + m(\bar{z}_m + z_f) + K_{ur}}{A_1} \end{bmatrix} U \frac{K_{\delta}}{A_1} u \\ = A(\bar{\vartheta}_p) x + B(\bar{\vartheta}_p) u \quad (8.40a)$$

$$y_O = \begin{bmatrix} 0 & 1 \\ -\frac{k + Mg(\bar{z}_g + z_f) + mg(\bar{z}_m + z_f)}{\tilde{A}_1} + g & -\frac{d}{\tilde{A}_1} \end{bmatrix} x \\ + \begin{bmatrix} 0 & 0 \\ \frac{M(\bar{z}_g + z_f) + m(\bar{z}_m + z_f) + K_{\dot{v}} - \tilde{A}_1}{\tilde{A}_1} & \frac{M(\bar{z}_g + z_f) + m(\bar{z}_m + z_f) + K_{ur} - \tilde{A}_1}{\tilde{A}_1} \end{bmatrix} U \frac{K_{\delta}}{\tilde{A}_1} u \\ = C(\bar{\vartheta}_p) x + D(\bar{\vartheta}_p) u \quad (8.40b)$$

where (8.21) inserted in (8.11) gives

$$A_1 = A_x + M(\bar{z}_g + z_f)^2 + m(\bar{z}_m + z_f)^2, \quad (8.41)$$

$$\tilde{A}_1 = A_1 / (\bar{z}_s + z_f) \quad \text{and} \quad z_s = \bar{z}_s + z_f \quad (8.42)$$

To investigate the identifiability properties, the state-space model (8.40) was extended according to (2.62) and the parameters considered unknown are

$$\bar{\vartheta}_p = \left[\underbrace{M, \bar{z}_g}_{\bar{\vartheta}_{p,1}}, \underbrace{k, A_x, d}_{\bar{\vartheta}_{p,2}}, \underbrace{K_r, K_{\dot{v}}, K_{\delta}, z_f}_{\bar{\vartheta}_{p,3}}, \underbrace{m, \bar{z}_m}_{\bar{\vartheta}_{p,3}} \right]^T \quad (8.43)$$

The remaining parameters are assumed to be known. There are $n_x = 2$ states and $n_{\vartheta} = 11$ parameters, i.e. the extended nonlinear system has dimension 13. Since the analysis is performed with a symbolic software, it is assumed that parameters are non-zero and have distinctly different values. It is however worth noting that identifiability might be lost for certain numerical values of the parameters.

Given that the input u is persistently exciting, the maximum rank of the extended observability matrix $\mathcal{O}$ is 7 and hence, only $7 - n_x = 5$ parameters can be estimated simultaneously. The results of the identifiability analysis are summarized in Table 8.1. The implication of the results is that if the static parameters $\bar{\vartheta}_{\text{static}}$ and the center of rotation in relation to the body-fixed system z_f are known, then a sequential method can be used to first estimate the dynamic and hydrodynamic parameters $\vartheta_{\text{hydrodynamic}}$ and $\bar{\vartheta}_{\text{dynamic}}$ using the nominal dataset. The sequential method could still work with linear parameters, but it will require nonlinear constraints to obey the parameterization, which would make the estimation procedure more computationally heavy than the linear approach described in Chapter 7. Furthermore, this would restrict the usability of the algorithm since a lot of information has to be known about the ship. Note that extra sensors are required to estimate k in the first step, for instance, pressure sensors, see Appendix B.

From the subsystem point of view, the input to the roll dynamics subsystem is given by (8.39). However, it is assumed that the only external user-controllable signal affecting the ship is δ and that this signal will affect all subsystems. The variables $\dot{v}$ and r are thus uniquely determined by δ and the coupling to the other subsystems similarly to a_s in Section 7.3.1. In this section, it is assumed that the other subsystems and the couplings are of sufficient complexity to supply more information than δ does by itself, i.e. in contrast to a_s in Section 7.3.1, r and $\dot{v}$ contain information about the system due to the complexity in the feedback loops, see (8.35) in Section 8.2.2. For a discussion on complexity and identifiability in closed-loop system identification, see for instance, Gevers et al. (2009) or Bazanella et al. (2010).

8.3.2 Multi-dataset Identifiability

To create a system that would also suit a retrofit of older ships, the complexity of the system and the information needed about the ship should be kept to a minimum. Thus, to reduce the complexity of the method it would be preferable if only the parameter $\bar{\vartheta}_{p,1}$, i.e. the mass M and center of mass $\bar{z}_g$ in relation to the body-fixed system, was needed to be known from the first step and that the parameters $\bar{\vartheta}_{p,2}$ and $\vartheta_{\text{hydrodynamic}}$ could be estimated from nominal data. The only additional knowledge is the position $\bar{z}_s$ of the IMU in relation to the body-fixed system.

Table 8.1: The results of the identifiability analysis for the general case of the approximate model. The exceptions are: a) Any three-combination of $\{A_x, M, \bar{z}_g, m, \bar{z}_m, z_f\}$. b) Any combination that contains $\{k, K_r, K_{\dot{v}}\}$ and any two of $\{A_x, M, \bar{z}_g, m, \bar{z}_m, z_f\}$.

Case	# Parameters	Identifiable	Exceptions
$\delta \neq 0$	1	Yes	–
	2	Yes	–
	3	Yes	a
	4	Yes	a
	5	Yes	a,b

However, this is not possible if only the nominal dataset is used as illustrated in Section 8.3.1. The loaded dataset certainly also contains information about the unknown parameters and hence, the question is if there is enough information contained in both datasets to estimate all parameters at once in a joint approach?

This can be answered by introducing an artificial MIMO model where the channels are only related through the physical parameters. By introducing the states

$$\bar{x} = [\phi^n, \dot{\phi}^n, \phi^l, \dot{\phi}^l]^T \quad (8.44)$$

the output

$$\bar{y}_O = [y_O^n, y_O^l]^T \quad (8.45)$$

and the input

$$\bar{u} = [\dot{v}^n, r^n, \delta^n, \dot{v}^l, r^l, \delta^l]^T \quad (8.46)$$

the model can be written in the state-space form

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} A(\bar{\vartheta}_p^n) & 0 \\ 0 & A(\bar{\vartheta}_p^l) \end{bmatrix} \bar{x} + \begin{bmatrix} B(\bar{\vartheta}_p^n) & 0 \\ 0 & B(\bar{\vartheta}_p^l) \end{bmatrix} \bar{u} \\ \bar{y}_O &= \begin{bmatrix} C(\bar{\vartheta}_p^n) & 0 \\ 0 & C(\bar{\vartheta}_p^l) \end{bmatrix} \bar{x} + \begin{bmatrix} D(\bar{\vartheta}_p^n) & 0 \\ 0 & D(\bar{\vartheta}_p^l) \end{bmatrix} \bar{u} \end{aligned} \quad (8.47)$$

with

$$\bar{\vartheta}_p^n = \bar{\vartheta}_p|_{m=\bar{z}_m=0}, \quad \bar{\vartheta}_p^l = \bar{\vartheta}_p \quad (8.48)$$

and where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are defined in (8.40). Here, z^n denotes that a signal z belongs to the nominal dataset and z^l denotes that a signal z belongs to the loaded dataset. To understand which combinations of parameters that are possible to estimate, the unknown parameters are still given by

$$\bar{\vartheta}_p = \left[\underbrace{\bar{\vartheta}_{\text{static}}}_{\bar{\vartheta}_{p,1}}, \underbrace{\bar{\vartheta}_{\text{dynamic}}}_{\bar{\vartheta}_{p,2}}, \underbrace{\bar{\vartheta}_{\text{hydrodynamic}}}_{\bar{\vartheta}_{p,3}} \right]^T = \left[M, \bar{z}_g, k, A_x, d, K_r, K_{\dot{v}}, K_{\delta}, z_f, m, \bar{z}_m \right]^T$$

defined in (8.43) and the remaining parameters are assumed to be known. The connection between the channels are thus through the physical parameters.

Assuming that the inputs are persistently exciting, the maximum rank of the extended observability matrix $\mathcal{O}$ for the model (8.47) is 11. Since there are $n_x = 4$ states, at most 7 parameters can be estimated simultaneously. It is thus not sufficient to know $\bar{\vartheta}_{p,1}$, not even if the parameters $\bar{\vartheta}_{\text{static}}$ are assumed to be known, since there are too many unknown parameters to estimate.

Hence, more *a priori* information is needed. One solution is to introduce this information through a second calibration dataset that has a known additional mass m_c and center of mass $\bar{z}_c$ in relation to the body-fixed coordinate system. The set of parameters for this dataset is assumed to be given by

$$\bar{\vartheta}_p^c = \bar{\vartheta}_p|_{m=m_c, \bar{z}_m=\bar{z}_c} \quad (8.49)$$

Again, an extended model can be formed by introducing the states

$$\tilde{x} = [\phi^n, \dot{\phi}^n, \phi^c, \dot{\phi}^c, \phi^l, \dot{\phi}^l]^T \quad (8.50)$$

the output

$$\tilde{y}_O = [y_O^n, y_O^c, y_O^l]^T \quad (8.51)$$

and the input

$$\tilde{u} = [\dot{v}^n, r^n, \delta^n, \dot{v}^c, r^c, \delta^c, \dot{v}^l, r^l, \delta^l]^T \quad (8.52)$$

such that the model can be written in the state-space form as

$$\begin{aligned} \dot{\tilde{x}} &= \begin{bmatrix} A(\bar{\vartheta}_p^n) & 0 & 0 \\ 0 & A(\bar{\vartheta}_p^c) & 0 \\ 0 & 0 & A(\bar{\vartheta}_p^l) \end{bmatrix} \tilde{x} + \begin{bmatrix} B(\bar{\vartheta}_p^n) & 0 & 0 \\ 0 & B(\bar{\vartheta}_p^c) & 0 \\ 0 & 0 & B(\bar{\vartheta}_p^l) \end{bmatrix} \tilde{u} \\ \tilde{y}_O &= \begin{bmatrix} C(\bar{\vartheta}_p^n) & 0 & 0 \\ 0 & C(\bar{\vartheta}_p^c) & 0 \\ 0 & 0 & C(\bar{\vartheta}_p^l) \end{bmatrix} \tilde{x} + \begin{bmatrix} D(\bar{\vartheta}_p^n) & 0 & 0 \\ 0 & D(\bar{\vartheta}_p^c) & 0 \\ 0 & 0 & D(\bar{\vartheta}_p^l) \end{bmatrix} \tilde{u} \end{aligned} \quad (8.53)$$

where z^c denotes that a signal z belongs to the calibration dataset. The maximum rank of the extended observability matrix $\mathcal{O}$ of the model (8.53) is 15 which implies that a maximum of 9 parameters can be simultaneously estimated. All 9 combinations of parameters are identifiable except for any combinations containing all of $\{A_x, M, \bar{z}_g\}$ or any combination containing all of $\{k, K_r, K_{\dot{v}}\}$ and any two of $\{A_x, M, \bar{z}_g\}$. Most importantly, the desired combination of $\bar{\vartheta}_{p,2}, \vartheta_{\text{hydrodynamic}}$ and $\bar{\vartheta}_{p,3}$ is simultaneously identifiable using this approach.

8.4 Discretization Using Physical Parameters

As mentioned in the previous section, the connections between the different channels in (8.53) are through the physical parameters. In Chapter 7 it was possible to obey the original nonlinear physical parameterization by linear constraints. In this chapter, this is unfortunately not possible and due to this, a discrete-time model parameterized with the physical parameters is used. The model

$$A_2 \ddot{\phi} = -(k - K_{\dot{v}}g)\phi - d\dot{\phi} - (K_{\dot{v}} + Mz_g + mz_m)a_s + K_r r + K_{\delta}\delta + \tau$$

defined in (8.20) with the output $y_O = \dot{\phi}$ can be written on the transfer function form

$$y_O = G(\mathbf{p})(a_s + F_r r + F_{\delta}\delta + \tau)$$

defined in (8.32) where

$$G(\mathbf{p}) = \frac{\beta_1 \mathbf{p}}{\mathbf{p}^2 + \alpha_1 \mathbf{p} + \alpha_2}, \quad F_{\delta} = \frac{\gamma_1}{\beta_1}, \quad F_r = \frac{\kappa_1}{\beta_1}$$

are defined in (8.23) and where the parameters

$$\alpha_1 = \frac{d}{A_2}, \quad \alpha_2 = \frac{k - K_{\dot{v}}g}{A_2}, \quad \beta_1 = -\frac{K_{\dot{v}} + Mz_g + mz_m}{A_2},$$

$$\gamma_1 = \frac{K_\delta}{A_2} = -\frac{z_r C_2}{A_2} \quad \text{and} \quad \kappa_1 = \frac{K_r}{A_2}$$

are given by (8.24) with A_2 defined in (8.18). As in Section 7.3.3, the transfer function $G(p)$ can be discretized using the bilinear transform described in Section 5.2.2. Applying the bilinear transform to (8.24) gives

$$G_d(q) = \frac{\bar{\beta}_0(1 - q^{-2})}{1 + \bar{\alpha}_1 q^{-1} + \bar{\alpha}_2 q^{-2}}, \quad F_{\delta d} = \frac{\bar{\gamma}_0}{\bar{\beta}_0}, \quad F_{rd} = \frac{\bar{\kappa}_0}{\bar{\beta}_0} \quad (8.54)$$

with

$$\bar{\alpha}_1 = \frac{2\alpha_2 T^2 - 8}{A_2}, \quad \bar{\alpha}_2 = \frac{4 - 2\alpha_1 T + \alpha_2 T^2}{A_2}, \quad \bar{\beta}_0 = \frac{2\beta_1 T}{A_2}, \quad \bar{\kappa}_0 = \frac{2\kappa_1 T}{A_2}, \quad \bar{\gamma}_0 = \frac{2\gamma_1 T}{A_2} \quad (8.55)$$

and $\bar{A}_2 = 4 + 2\alpha_1 T + \alpha_2 T^2$. By introducing

$$\mu_t^T = [\mu_{1,t}, \mu_{2,t}, \mu_{3,t}] = [a_{s,t} - a_{s,t-2}, r_t - r_{t-2}, \delta_t - \delta_{t-2}] \quad (8.56)$$

the discrete-time model can be written as the linear regression

$$y_t = \varphi_t^T \bar{\vartheta} + \tau_t \quad (8.57)$$

where

$$\bar{\vartheta} = [\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\kappa}_0, \bar{\gamma}_0]^T \quad (8.58)$$

and

$$\varphi_t^T = [-y_{t-1}, -y_{t-2}, \mu_{1,t}, \mu_{2,t}, \mu_{3,t}] \quad (8.59)$$

Note that τ_t is a colored noise process. The discrete-time model can now be directly parameterized by the physical parameters if (8.24) is inserted into (8.55) which gives the nonlinear transformations

$$g_{\bar{\vartheta}}(\bar{\vartheta}_p) = \left[\begin{array}{c} \frac{2(k - K_{\dot{v}}g)T^2 - 8[A_x + m z_m(z_m - z_s) + M z_g(z_g - z_s) - K_{\dot{v}}z_s]}{\mathcal{A}_2(\bar{\vartheta}_p)} \\ \frac{-2dT + (k - K_{\dot{v}}g)T^2 + 4[A_x + m z_m(z_m - z_s) + M z_g(z_g - z_s) - K_{\dot{v}}z_s]}{\mathcal{A}_2(\bar{\vartheta}_p)} \\ -\frac{2T(K_{\dot{v}} + M z_g + m z_m)}{\mathcal{A}_2(\bar{\vartheta}_p)} \\ \frac{\mathcal{A}_2(\bar{\vartheta}_p)}{2TK_r} \\ \frac{\mathcal{A}_2(\bar{\vartheta}_p)}{2TK_\delta} \\ \mathcal{A}_2(\bar{\vartheta}_p) \end{array} \right] \quad (8.60)$$

where

$$\mathcal{A}_2(\bar{\vartheta}_p) = 2dT + (k - K_{\dot{v}}g)T^2 + 4[A_x + M z_g(z_g - z_s) + m z_m(z_m - z_s) - K_{\dot{v}}z_s] \quad (8.61)$$

The nonlinear transformation (8.60) can be used in estimation to reduce the variance either by directly inserting it into (8.57) or by introducing an equality constraint, i.e. $\bar{\vartheta} - g_{\bar{\vartheta}}(\bar{\vartheta}_p) = 0$, to obey the nonlinear parameterization.

8.5 Estimation of the Roll Dynamics

The estimation of the parameters

$$\tilde{\vartheta}_{p,2} = \underbrace{[k, A_x, d]}_{\bar{\vartheta}_{p,2}} \underbrace{[K_r, K_{\dot{v}}, K_{\delta}, z_f]}_{\vartheta_{\text{hydrodynamic}}} \underbrace{[m, \bar{z}_m]}_{\bar{\vartheta}_{p,3}}^T \quad (8.62)$$

of the model (8.20) fits into the joint identification approach described in Section 5.2. Firstly, the parameters

$$\tilde{\vartheta}_{p,1} = \underbrace{[M, \bar{z}_g, m_c, \bar{z}_c]}_{\bar{\vartheta}_{p,1}}^T \quad (8.63)$$

that need to be known a priori are estimated or calculated through some specific experiment. Furthermore, the other parameters, such as the position $\bar{z}_s$ of the IMU and the acceleration of gravity g , are assumed to be known. Then, the nominal and calibration datasets

$$Z_n = (y_{O,t}, \tilde{u}_t, \delta_t)_{t=1+t_n}^{N_n+t_n} \text{ and } Z_c = (y_{O,t}, \tilde{u}_t, \delta_t)_{t=1+t_c}^{N_c+t_c} \quad (8.64)$$

are collected. Finally, when a loaded dataset

$$Z_l = (y_{O,t}, \tilde{u}_t, \delta_t)_{t=1+t_l}^{N_l+t_l} \quad (8.65)$$

has been collected, the change in mass m and the change in center of mass $\bar{z}_m$ can be estimated simultaneously with the nominal and the hydrodynamic parameters according to Approach 5.2. The different starting times are marked using t_n , t_c and t_l to emphasize that these datasets are not collected at the same time.

Since the model (8.20) can be expressed as a linear regression according to (8.57) – (8.59) and the criterion function is quadratic in the linear parameters, the criterion function of (5.25) can be expressed as

$$\tilde{V}(\tilde{\vartheta}, \hat{\tilde{\vartheta}}_{p,1}, \tilde{\vartheta}_{p,2}, \tilde{Z}) = \sum_{i \in \{n,c,l\}} \|\Phi^i \tilde{\vartheta}_i - Y^i\|_{Q_i}^2 \quad (8.66)$$

where $\tilde{Z} = (Z_n, Z_c, Z_l)$, $\tilde{\vartheta} = [\bar{\vartheta}_n^T, \bar{\vartheta}_c^T, \bar{\vartheta}_l^T]^T$ are the discrete-time linear parameters defined in (8.58), Φ^i and Y^i are given by, for instance, (5.24), and $\|x\|_{Q_i}^2 = x^T Q_i x$, $Q_i \geq 0$, $i = \{n, c, l\}$ are weighting matrices. The constraint of (5.25) is given by

$$\tilde{\vartheta} - [g_{\vartheta}^T(\bar{\vartheta}_p^n), g_{\vartheta}^T(\bar{\vartheta}_p^c), g_{\vartheta}^T(\bar{\vartheta}_p^l)]^T = 0 \quad (8.67)$$

where $g_{\vartheta}(\bar{\vartheta}_p^i)$ are the transformations defined by (8.60). The discrete-time linear parameters can be viewed as auxiliary parameters in the optimization problem and the values of these are not necessarily interesting for the final solution. Note that the variance of the estimator is not increased by these auxiliary variables due to the constraints. These variables might even be helpful when solving the optimization problem since it can be seen as a form of lifting (Albersmeyer and Diehl, 2010).

Here, the discrete-time linear parameters $\bar{\vartheta}_i$ are eliminated from the problem by

combining (8.66) and (8.67). This give the criterion function

$$V(\hat{\vartheta}_{p,1}, \tilde{\vartheta}_{p,2}, \tilde{Z}) = \sum_{i \in \{n,c,l\}} \|\Phi^i g_{\vartheta}(\tilde{\vartheta}_p^i) - Y^i\|_{Q_i}^2 \quad (8.68)$$

that is directly parameterized with the physical parameters. Note that $\tilde{\vartheta}_{p,1}$, $i = \{n, c, l\}$ and $\tilde{\vartheta}_{p,2}$ is the union of the parameters in $\tilde{\vartheta}_p^i$. As already mentioned, the estimation problem becomes a nonlinear optimization problem which means that the solution can be sensitive to the initial condition. On the other hand, physical insight can be incorporated into the constraints on the parameters. Due to the structure of the problem, the criterion function (8.68) can be written on the form

$$V(\tilde{\vartheta}_{p,1}, \tilde{\vartheta}_{p,2}, \tilde{Z}) = \|\tilde{\Phi} \tilde{\vartheta}(\tilde{\vartheta}_{p,1}, \tilde{\vartheta}_{p,2}) - \tilde{Y}\|_{\tilde{Q}}^2 \quad (8.69)$$

by using

$$\tilde{\vartheta}(\tilde{\vartheta}_{p,1}, \tilde{\vartheta}_{p,2}) = \begin{bmatrix} g_{\vartheta}(\tilde{\vartheta}_p^n) \\ g_{\vartheta}(\tilde{\vartheta}_p^c) \\ g_{\vartheta}(\tilde{\vartheta}_p^l) \end{bmatrix}, \quad \tilde{\Phi} = \begin{bmatrix} \Phi^n & 0 & 0 \\ 0 & \Phi^c & 0 \\ 0 & 0 & \Phi^l \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} Y^n \\ Y^c \\ Y^l \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_n & 0 & 0 \\ 0 & Q_c & 0 \\ 0 & 0 & Q_l \end{bmatrix} \quad (8.70)$$

This is similar to a model of a MIMO system without coupling between the channels and where the difference is that the datasets could be of different lengths. Since the channels are decoupled, the choices of each individual Φ^i and Y^i follow from the discussion in Section 7.4.

8.5.1 The Iterative Joint IV Method

In Chapter 9, there will be extensive use of one particular version of Approach 5.2 where (5.25) is computed using the iterative extended instrumental variable (IV) method described in Section 2.5.2. To formalize this method, the creation of instruments for the initialization as well as for the iterative refinement will be discussed. To summarize, the method will finally be given in Algorithm 8.1.

The superscript i is either n , c or l and corresponds to the datasets defined in Section 8.5. To ease notation, $i = \{n, c, l\}$ is not explicitly written at all places and if the superscript i is used, it should be understood as for $i = \{n, c, l\}$.

In this section, the estimate (5.25) with the cost function given by (8.68) is computed with the extended IV method which corresponds to choosing

$$\Phi^i = \frac{1}{N_i} \begin{bmatrix} \zeta_1^i & \dots & \zeta_{N_i}^i \end{bmatrix} \begin{bmatrix} (\bar{\varphi}_1^i)^T \\ \vdots \\ (\bar{\varphi}_{N_i}^i)^T \end{bmatrix} \quad \text{and} \quad Y^i = \frac{1}{N_i} \begin{bmatrix} \zeta_1^i & \dots & \zeta_{N_i}^i \end{bmatrix} \begin{bmatrix} \bar{y}_{O,1}^i \\ \vdots \\ \bar{y}_{O,N_i}^i \end{bmatrix} \quad (8.71)$$

where ζ_t^i are the vectors of instruments, $\bar{y}_{O,t}^i = L^i(\mathbf{q})y_{O,t}^i$, $\bar{\varphi}_t^i = L^i(\mathbf{q})\varphi_t^i$ and $L^i(\mathbf{q})$ are stable prefilters. The choices of instruments and prefilters are essential in the IV estimator. Even though any instruments that fulfill the conditions in (2.44) will guarantee consistency, the variance properties of the estimator are greatly affected by this choice.

Assuming that the models are given on the form

$$y_{O,t}^i = G_d(\mathbf{q}, \bar{\vartheta}_p^i)(a_{s,t}^i + F_{r,d}(\bar{\vartheta}_p^i)r_t^i + F_{\delta,d}(\bar{\vartheta}_p^i)\delta_t^i) + H_{0,d}^i(\mathbf{q})e_t^i = (\varphi_t^i)^T g_\vartheta(\bar{\vartheta}_p^i) + \tau_t^i \quad (8.72)$$

where the regression vectors are given by

$$\varphi_t^i = [-y_{t-1}^i, -y_{t-2}^i, \mu_{1,t}^i, \mu_{2,t}^i, \mu_{3,t}^i]^T \quad (8.73)$$

the parameter vectors are defined in (8.60) and the process disturbances are

$$\tau_t^i = A_d(\mathbf{q}, \bar{\vartheta}_p^i)H_{0,d}^i(\mathbf{q})e_t^i \quad (8.74)$$

where $A_d(\mathbf{q}, \bar{\vartheta}_p^i)$ are the denominators of $G_d(\mathbf{q}, \bar{\vartheta}_p^i)$ and it is assumed that e_t^i are white discrete-time zero-mean noise signals. According to the previous work cited in Section 2.5.2, the optimal prefilters are given by the inverse noise models

$$L^i(\mathbf{q}, \bar{\vartheta}_p^i) = \frac{1}{A_d(\mathbf{q}, \bar{\vartheta}_p^i)H_{0,d}^i(\mathbf{q})} \quad (8.75)$$

and the optimal instruments are given by the filtered noise-free regression vectors

$$\zeta_t^i = L^i(\mathbf{q}, \bar{\vartheta}_p^i) [-\hat{y}_{O,t-1}^i, -\hat{y}_{O,t-2}^i, \hat{\mu}_{1,t}^i, \hat{\mu}_{2,t}^i, \hat{\mu}_{3,t}^i]^T \quad (8.76)$$

where $\hat{z}$ is the noise-free part of the signal z . Note that $\mu_{3,t}^i = \delta_t^i - \delta_{t-2}^i$ and it is thus independent of the process disturbance τ . However, these prefilters and instrument vectors are not implementable in reality since they require knowledge of the true system and perfect measurements. They can however be used as a guideline. Note that the inputs and outputs can be written

$$\hat{y}_{O,t}^i = G_{\delta y,d}^i(\mathbf{q}, \bar{\vartheta}_p^i)\delta_t^i, \quad \hat{a}_{s,t}^i = G_{\delta a,s,d}^i(\mathbf{q}, \bar{\vartheta}_p^i)\delta_t^i \quad \text{and} \quad \hat{r}_t^i = G_{\delta r,d}^i(\mathbf{q})\delta_t^i \quad (8.77)$$

and if the transfer functions are known, then the signals can be simulated using the external reference δ_t^i . These transfer functions were derived in Section 8.2.1 and are given by

$$G_{\delta y,d}^i(\mathbf{q}, \bar{\vartheta}_p^i) = G_{0,d}(\mathbf{q}, \bar{\vartheta}_p^i) [R_d^i(\mathbf{q}, \bar{\vartheta}_p^i) + F_{\delta,d}(\bar{\vartheta}_p^i)], \quad (8.78)$$

$$G_{\delta a,s,d}^i(\mathbf{q}, \bar{\vartheta}_p^i) = G_{0,d}(\mathbf{q}, \bar{\vartheta}_p^i) [F_{\delta,d}(\bar{\vartheta}_p^i) + F_{r,d}(\bar{\vartheta}_p^i)G_{\delta r,d}^i(\mathbf{q})] - \frac{G_{\delta \dot{v},d}^i(\mathbf{q}) - UG_{\delta r,d}^i(\mathbf{q})}{1 - F_{y,d}(\mathbf{q})G_d(\mathbf{q}, \bar{\vartheta}_p^i)}, \quad (8.79)$$

where

$$R_d^i(\mathbf{q}, \bar{\vartheta}_p^i) = (F_{r,d}(\bar{\vartheta}_p^i) - U)G_{\delta r,d}^i(\mathbf{q}) - G_{\delta \dot{v},d}^i(\mathbf{q}) \quad (8.80)$$

and

$$G_{0,d}(\mathbf{q}, \bar{\vartheta}_p^i) = \frac{G_d(\mathbf{q}, \bar{\vartheta}_p^i)}{1 - G_d(\mathbf{q}, \bar{\vartheta}_p^i)F_{y,d}(\mathbf{q})} \quad (8.81)$$

The method presented in this section is based on the method of Gilson et al. (2006) where the parameters are estimated in an iterative scheme. In each iteration, the vectors of instruments ζ_t^i are created by simulating the inputs and the outputs from the external reference by using the latest parameter estimates and the prefilters are updated by estimating a noise model from the residuals.

In the initializing iteration (0th iteration), the initial instruments are created from data. Firstly, blackbox models of the transfer functions in (8.77) are estimated using the datasets Z_i . These models are then simulated which gives the signals

$$\hat{y}_{O,t}^{i,0} = \hat{G}_{\delta y,d}^{i,0}(\mathbf{q})\delta_t^i, \quad \hat{a}_{s,t}^{i,0} = \hat{G}_{\delta a_s,d}^{i,0}(\mathbf{q})\delta_t^i \quad \text{and} \quad \hat{r}_t^{i,0} = \hat{G}_{\delta r,d}^{i,0}(\mathbf{q})\delta_t^i \quad (8.82)$$

The prefilters are given by

$$L^{i,0}(\mathbf{q}, \bar{\vartheta}_p^{i,0}) = 1 \quad (8.83)$$

and the instrument vectors are created according to

$$\zeta_t^{i,0} = \left[\hat{y}_{O,t}^{i,0} \cdots \hat{y}_{O,t-n_y+1}^{i,0}, \hat{\mu}_{1,t}^{i,0} \cdots \hat{\mu}_{1,t-n_{a_s}+1}^{i,0}, \hat{\mu}_{2,t}^{i,0} \cdots \hat{\mu}_{2,t-n_r+1}^{i,0}, \hat{\mu}_{3,t}^{i,0} \cdots \hat{\mu}_{3,t-n_\delta+1}^{i,0} \right]^T \quad (8.84)$$

The constants n_y , n_{a_s} , n_r and n_δ are the number of time lags (including the non-delay signal) included in $\zeta_t^{i,0}$ for the output y_O , the acceleration a_s , the angular velocity r and the rudder angle δ , respectively, i.e. $n_\delta = 0$ means that $\mu_{3,t}$ is not included in $\zeta_t^{i,0}$.

In the refining iterations (j^{th} iteration), firstly, $\hat{\vartheta}_p^{i,j}$ are estimated using the vectors of instruments and prefilters from the $j-1^{\text{th}}$ iteration. Preferably the transfer functions $G_{\delta y,d}^i(\mathbf{q}, \bar{\vartheta}_p^i)$ defined in (8.78), $G_{\delta a_s,d}^i(\mathbf{q}, \bar{\vartheta}_p^i)$ defined in (8.79) and $G_{\delta r,d}^i(\mathbf{q})$ should be used for simulation. However, $G_{\delta r,d}^i(\mathbf{q})$ and $G_{\delta \ddot{v},d}^i(\mathbf{q})$ are unknown. Furthermore, $\dot{v}^i$ are not measured which means that estimation of the transfer functions $G_{\delta \ddot{v},d}^i(\mathbf{q})$ are not possible. Instead the following transfer functions are used

$$\hat{G}_{\delta y,d}^{i,j}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) = G_{0,d}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) \left[\hat{R}_d^{i,j}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) + F_{\delta,d}(\hat{\vartheta}_p^{i,j}) \right], \quad (8.85)$$

$$\begin{aligned} \hat{G}_{\delta a_s,d}^{i,j}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) &= G_{0,d}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) \left[F_{\delta,d}(\hat{\vartheta}_p^{i,j}) + F_{r,d}(\hat{\vartheta}_p^{i,j}) \hat{G}_{\delta r,d}^{i,0}(\mathbf{q}) \right] \\ &+ \frac{U \hat{G}_{\delta r,d}^{i,0}(\mathbf{q})}{1 - F_{y,d}(\mathbf{q}) G_d(\mathbf{q}, \hat{\vartheta}_p^{i,j})}, \end{aligned} \quad (8.86)$$

where

$$\hat{R}_d^{i,j}(\mathbf{q}, \hat{\vartheta}_p^{i,j}) = (F_{r,d}(\hat{\vartheta}_p^{i,j}) - U) \hat{G}_{\delta r,d}^{i,0}(\mathbf{q}) \quad (8.87)$$

and $\hat{G}_{\delta r,d}^{i,0}(\mathbf{q})$ are the blackbox models estimated from data in the initialization step. The transfer functions (8.85), (8.86) and $\hat{G}_{\delta r,d}^{i,0}(\mathbf{q})$ are then simulated with δ_t^i as input, which gives the signals

$$\hat{y}_{O,t}^{i,j} = \hat{G}_{\delta y,d}^{i,j}(\mathbf{q})\delta_t^i, \quad \hat{a}_{s,t}^{i,j} = \hat{G}_{\delta a_s,d}^{i,j}(\mathbf{q})\delta_t^i \quad \text{and} \quad \hat{r}_t^{i,j} = \hat{G}_{\delta r,d}^{i,0}(\mathbf{q})\delta_t^i \quad (8.88)$$

Note that $\hat{r}_t^{i,j} = \hat{r}_t^{i,0}$, $\forall j$ which means that these signals only have to be simulated once. The prefilters are given by

$$L^{i,j}(\mathbf{q}, \bar{\vartheta}_p^{i,j}) = \frac{1}{\bar{H}_d^i(\mathbf{q}, \hat{\eta}^{i,j})} \quad (8.89)$$

where

$$\bar{H}_d^i(\mathbf{q}, \hat{\eta}^{i,j}) \quad (8.90)$$

are ARMA models estimated from the residuals

$$\varepsilon_t^{i,j} = y_{O,t}^i - (\varphi_t^i)^T g_\vartheta(\hat{\vartheta}_p^{i,j}) \quad (8.91)$$

and the instrument vectors are created according to

$$\zeta_t^{i,j} = L^{i,j}(\mathbf{q}, \bar{v}_p^{i,j}) \times \left[\hat{y}_{O,t}^{i,j} \cdots \hat{y}_{O,t-n_y+1}^{i,j}, \hat{\mu}_{1,t}^{i,j} \cdots \hat{\mu}_{1,t-n_{as}+1}^{i,j}, \hat{\mu}_{2,t}^{i,j} \cdots \hat{\mu}_{2,t-n_r+1}^{i,j}, \mu_{3,t}^i \cdots \mu_{3,t-n_\delta+1}^i \right]^T \quad (8.92)$$

The algorithm is summarized in Algorithm 8.1 and will be used for computation of the results in Chapter 9.

Algorithm 8.1 (The iterative joint IV method).

Here, z^i indicates that a signal z belongs to the dataset i . The superscript i is either n , c or l and corresponds to the datasets defined in Section 8.5. To ease notation, $i = \{n, c, l\}$ is not explicitly written at all places and if the superscript i is used, it should be understood as for $i = \{n, c, l\}$.

(A) Initialize:

- (a) Set initial value of $\bar{v}_p^{i,0}$
- (b) Create prefilters $L^{i,0}(\mathbf{q}, \bar{v}_p^{i,0})$ according to (8.83)
- (c) Create initial instruments
 - i. Estimate blackbox models of the transfer functions $\hat{G}_{\delta y,d}^{i,0}(\mathbf{q})$, $\hat{G}_{\delta a_s,d}^{i,0}(\mathbf{q})$ and $\hat{G}_{\delta r,d}^{i,0}(\mathbf{q})$ in (8.77)
 - ii. Simulate the signals $\hat{y}_{O,t}^{i,0}$, $\hat{a}_{s,t}^{i,0}$ and $\hat{r}_t^{i,0}$ according to (8.82)
 - iii. Create the instruments $\zeta_t^{i,0}$ according to (8.84)
- (d) Set $j = 1$

(B) Estimate parameters:

- (a) Compute $\hat{v}_{p,2}^j$ using Approach 5.2 with (8.68), (8.71), $L^{i,j-1}(\mathbf{q}, \hat{v}_p^{i,j-1})$ and $\zeta_t^{i,j-1}$
- (b) Estimate the parameters $\eta^{i,j}$ of the models $\bar{H}_d^i(\mathbf{q}, \eta^{i,j})$ from $\varepsilon_t^{i,j} = \bar{H}_d^i(\mathbf{q}, \eta^{i,j})e_t^i$ where $\varepsilon_t^{i,j}$ are the residual defined in (8.91)
- (c) Create the prefilters $L^{i,j}(\mathbf{q}, \hat{v}_p^{i,j})$ according to (8.89)
- (d) Simulate the signals $\hat{y}_{O,t}^{i,j}$, $\hat{a}_{s,t}^{i,j}$ and $\hat{r}_t^{i,j}$ according to (8.88)
- (e) Create the instruments $\zeta_t^{i,j}$ according to (8.92)

(C) Terminate: Increase j and go to Step B while

$$\|\hat{v}_{p,2}^j - \hat{v}_{p,2}^{j-1}\|_2 + \|\hat{\eta}^{n,j} - \hat{\eta}^{n,j-1}\|_2 + \|\hat{\eta}^{c,j} - \hat{\eta}^{c,j-1}\|_2 + \|\hat{\eta}^{l,j} - \hat{\eta}^{l,j-1}\|_2 \quad (8.93)$$

is above a threshold or as long as a maximum number of iterations is not reached. Otherwise, return $\hat{v}_{p,2} = \hat{v}_{p,2}^j$ and terminate.

8.6 Sensitivity to Initial Conditions

A challenging aspect of Algorithm 8.1 is that the proposed estimator is non-convex in the parameters and in this section, a brief simulation study will be presented to evaluate the estimated parameters' sensitivity to the initial conditions. To simplify the analysis, all disturbances were set equal to zero. For a discussion on the estimator's disturbance rejection, see Chapters 7 and 9. The system was assumed to be given by (8.3) and the parameters, given in Table 9.1, were chosen to resemble the parameters of the scale model in Chapter 9.

The datasets described in Section 8.3 were synthesized by simulating (8.3) with the masses and the centers of masses given by Table 9.1. The length of the datasets were chosen to be 60 seconds and they were sampled at 50 Hz. Figure 8.9 shows the nominal dataset as an example.

A Monte Carlo (MC) simulation with 10000 runs was performed to test the solution's sensitivity to the initial condition. In each run, the initial condition was sampled uniformly between the upper and lower bound given in Table 8.3. In all iterations, the instruments (8.92) were created using the constants $n_y = n_{a_s} = n_r = 16$ and $n_\delta = 2$.

Figure 8.9: Nominal data used in the simulation study of Section 8.6.

Table 8.2: The parameters used in the simulation of Section 8.6.

Description	Parameters and Values
General	$U = 1$, $M = 22.04$, $\bar{z}_g = 0$, $m_c = 0.2$, $\bar{z}_c = -0.172$, $m = 0.4$, $\bar{z}_m = -0.182$, $z_f = -0.028$, $\bar{z}_s = -0.2$, $g = 9.82$
Sway Dynamics	$Y_{\dot{v}} = -100$, $Y_{\dot{p}} = K_{\dot{v}}$, $Y_{\dot{r}} = -1$, $Y_{\phi uu} = 0$, $Y_{ u v} = 35$, $Y_{ur} = 0$
Roll Dynamics	$K_{\dot{v}} = 0.1$, $A_x = 0.1385$, $K_{\dot{r}} = 0$, $k = 10.38$, $d = 0.2067$, $K_{ur} = -0.4$ ($\Leftrightarrow K_r = -0.5$)
Yaw Dynamics	$N_{\dot{v}} = Y_{\dot{r}}$, $N_{\dot{p}} = K_{\dot{r}}$, $A_z = 3A_x$, $N_{ u r} = 1.4$, $N_{\phi u u } =$ $N_p = N_{ u p} = 0$

The means and standard deviations of the solutions were calculated and can be seen on the two last rows of Table 8.3. Five of the runs resulted in unrealistic solutions (that could be easily identified) and were not used in the calculations. Note that the approach was surprisingly robust for finding the mass m and the CM $\bar{z}_m$ despite the large variations in the other parameters. This was fortunate since these were the desired parameters. The other parameters were in some sense only estimated out of necessity for the estimator and should be treated with care due to the variations.

Figure 8.10 shows the cost function of the estimation problem as a function of the mass m and the CM $\bar{z}_m$ for one solution of the MC simulation (the other parameters were fixed to their estimated values). Note that the gradient is largest orthogonal to the dotted line and that there is a unique minimum (red cross). However, the cost function is quite flat close to the minimum and due to the gradient of the cost function, it is easier to detect that there has been a change in the loading condition than to separate the effect between the mass and the CM.

Table 8.3: Results for Monte Carlo simulation of Section 8.6. T: true, U/L: upper/lower bounds, M: mean and S: standard deviation.

	A_x	d	k	K_{δ}	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	0.1385	0.2067	10.380	-0.405	-0.5	0.1	-0.028	0.4	-0.182
U	100	100	100	0	0	10	1	1	0
L	0	0	0	1	10	0	-1	0	0.5
M	0.1261	0.2194	13.297	-0.4302	-0.5318	0.3396	-0.0403	0.4007	-0.1821
S	0.0296	0.0233	5.3579	0.0454	0.0584	0.4367	0.0225	0.0018	0.0004

Figure 8.10: A contour plot of the estimator's cost function as a function of m and $\bar{z}_m$ (with the other parameters fixed) for the simulation study in Section 8.6. Note that the gradient of the cost function is largest orthogonal to the dotted line.

9

Model Ship Experimental Verification

This chapter describes an experiment performed to collect data from a scale model of a ship and the estimation results obtained with the algorithm described in Section 8.5.1. Firstly, the experimental setup will be outlined with emphasis on the system overview. Secondly, the ship will be described in detail, including known physical quantities and calculation of the center of gravity. Thirdly, the experiments will be described and the data will be analyzed. Finally, estimation results for the input roll subsystem (8.20) using the ship data will be presented.

Figure 9.1: An overview of the model basin used in the experiments. The red piston on the left is part of the wave maker, see also Figure 9.2. The three cameras used for positioning are mounted on the towing carriage which can be seen on the far right and can be seen in more detail in Figure 9.3. In addition, the nominal dataset described in Table 9.3 is projected on to the picture.

9.1 Hardware - Experiment Overview

The experiments were performed in the small wave basin *Marine Cybernetics Laboratory* (MCLab) at the Norwegian University of Science and Technology (NTNU) in Trondheim. Figure 9.1 shows the basin which is $40 \times 6.45 \times 1.5$ m (length $\times$ width $\times$ depth). The basin had a wave maker that can be seen in the lower left of Figure 9.1 and in the right picture of Figure 9.2. There was also a towing carriage that can be used to drag ships along the basin. In these experiments, the towing carriage was stationary and it was only used for mounting the Oqus cameras used by the Qualisys real-time positioning system. The towing carriage can be seen in Figure 9.1 on the far right and in Figure 9.3.

An overview of the experimental setup can be seen in Figure 9.4. The lines correspond to information flow where the arrows represent the flow direction. The dashed line corresponds to bluetooth communication, dotted lines correspond to WIFI communication and continuous lines correspond to any wired communication or signal. The system had three major components, the ship, the Qualisys real-time positioning system and the human-machine interface. These were basically three computers communicating over an ethernet/WIFI network.

Figure 9.2: Left: An overview of the basin with the model ship to get a sense of scale. Note the IMU on the top of the ship and the markers attached at several points using aluminum tubing. Right: A close up picture of the wave maker seen to the left in Figure 9.1. The wave maker was of single paddle type and was driven by an electrical servo actuator (the red piston) that was controlled by a computer. The synthesized waves could be both regular and irregular.

Figure 9.3: A close up of the towing carriage seen on the right in Figure 9.1. In the experiments described in this chapter, the towing carriage was fixed and it was only used for access to the other side of the basin and for mounting the three Oqus cameras used by the Qualisys system. The cameras that were used can be seen on the far left, roughly in the middle and on the right.

Figure 9.4: An overview of the information flow in the experimental setup. The arrows indicate the flow direction, a dotted line corresponds to WIFI communication and a continuous line corresponds to any wired communication or signal. The system basically consisted of three computers communicating over ethernet and WIFI. The CompactRIO (cRIO) controller on the ship was used to control the motor and rudder, and to collect data from the IMU and the pulse encoders. The Qualisys system was used to track the position and orientation of the ship through the three Oqus cameras and the markers fitted to the ship. The external computer was used to receive input from the user, to display information about the ship and to log data.

In the ship, there was a cRIO Controller that collected data from the IMU and the pulse encoders. The IMU was an Xsens MTi-G that was equipped with a three-axis gyro, a three-axis accelerometer and a three-axis magnetometer, see Appendix A.6.1 for more information about the IMU. The cRIO also controlled the motor and the rudder of the ship. The controller ran the LabVIEW Real-Time operating system and communicated via a WIFI bridge to the external computer that ran LabVIEW. The external computer was used as a human-machine interface, plotting the ship's current status, logging data and taking commands from a Playstation 3 Joystick. The location of the ship was recorded using a Qualisys system with a computer running the Qualisys Track Manager (QTM) software and three Oqus cameras. This track information was sent to the external computer over ethernet.

The cameras track markers fitted to the ship and the QTM software calculates the position and orientation of the ship in real-time. A schematic view of the wave basin can be seen in Figure 9.5. Four markers had to be seen by at least two cameras for the QTM system to be able to calculate the ship's position and orientation. Due to this there was only a limited usable area that could be used to collect data. Furthermore, for certain positions and orientations, one or two of the markers were hidden behind the ship itself which created blind spots. Several different numbers of markers and positions were tested. The configuration that was chosen and used to collect the data can be seen in Figure 9.6. Since the QTM data was not used in the estimation procedure, this was however not a critical issue. Note that the usable area is larger than what the simplified field of view for the cameras implies.

Figure 9.5: A sketch of the model basin seen from the top. The (simplified) fields of view of the cameras are marked in green. There were certain blind spots since enough markers could not be seen by at least 2 cameras at all times. The usable area for collecting data is marked in yellow. Note that the usable area is larger than what the simplified field of view for the cameras implies.

Figure 9.6: Left: The ship seen from the front. Note that the change in wetted surface due to a roll motion should be small since the ship is square in shape. Right: The ship was very sensitive to the loading condition. The picture shows the massive roll angle resulting from an additional mass of 2 kg corresponding to a 9% increase in mass. Also note the five markers used by the Qualisys system. The fifth is barely visible behind the black box on the top.

Figure 9.7: The ship seen from the top. To protect the equipment, there were two hatches (white rings) that made the hull water proof. In the aft (left) hatch, the steering servo, the batteries and the pulse encoder measuring the steering angle are visible. In the fore (right) hatch, the CompactRIO-9022 (cRIO) is visible. The IMU (orange) can be seen in the center mounted on top of the bridge. Figure 9.6 shows the ship with the hatches closed and with a black protective box over the IMU.

Figure 9.8: A side view of the scale model that was used in the experiments. The IMU was located underneath the black and gray box on the top. Note the only actuators on the ship, the propeller and the rudder, in the lower right corner.

9.2 Model Ship

The scale model, hereafter called the ship, was a model of a fishing vessel called “Our Lass II” that is roughly 20 m long, see Figures 9.2, 9.7, 9.8 and 9.6. The ship was in scale 1:24 and the known physical quantities are listed in Table 9.1. The ship was made water proof by two hatches seen open in Figure 9.7 as two white rings and closed in Figure 9.6. The ship had its own propulsion and steering that were controlled by the onboard cRIO controller. The propeller and rudder can be seen in the lower right of Figure 9.8. The ship had three onboard sensors, the IMU, measuring angular velocities and linear accelerations, and two pulse encoders that were measuring the RPM of the motor and the steering angle, respectively. In Figure 9.7, the IMU can be seen in the center (orange box), the encoder measuring the steering angle can be seen to the left in the aft hatch (bronze colored wheels) and the cRIO can be seen in the front hatch. The cRIO communicates through a WIFI bridge which was hidden inside the white structure on top of the ship and the cRIO got its input from the Playstation 3 joystick through the external computer. The ship was powered by two batteries that can be seen in the aft hatch of Figure 9.7.

As can be seen in Figures 9.6 and 9.7, the ship is close to square in shape, which meant that for small changes in the roll angle, the wetted surface area should not change much (Journée and Massie, 2001). This implies that the restoring torque will be close to linear for a given forward speed, see Appendix A.2.1.

The position of the additional mass was chosen to be on top of the black box

Table 9.1: Model of “Our Lass II” in scale $1:24 = 1:\lambda_L$. Model parameters originating from Thys (2013).

Parameter	Model	Unit
Length between perpendiculars L_{pp}	0.8	m
Length at the waterline L_w	0.85	m
Beam B	0.3	m
Draft at COG D	0.144	m
Draft fore D_f	0.144	m
Draft aft D_a	0.144	m
Displacement Δ	22.04	kg
LCG from midship	0.125	m
$\overline{KG}$	0.146	m
$\overline{GM}_T$	0.018	m
Radius of Gyration in Roll K_{xx}/B	0.337	m
Radius of Gyration in Pitch K_{yy}/L	0.272	m
Radius of Gyration in Yaw K_{zz}/L	0.272	m

covering the IMU, see Figure 9.6. Since the distance from the center of rotation (> 20 cm) was large, the weights had quite a big impact on the roll dynamics and stability of the ship. In Figure 9.6, the result of adding an additional mass of 2 kg to the ship is shown. With this mass it was barely stable which resulted in a large constant roll angle. Due to this, the additional masses were chosen to be 0.200, 0.400 and 0.600 kg. These correspond to an increase of 0.91%, 1.81% and 2.72% in the total mass, respectively.

9.2.1 Calculation of the Ship’s Center of Gravity

As mentioned in Section 8.3, the static parameters $\bar{\vartheta}_{\text{static}}$ or at least the mass M and the center of mass $\bar{z}_g$, i.e. $\bar{\vartheta}_{p,1}$, have to be estimated in a first step. In this experiment, the mass M and the center of mass $\bar{z}_g$ were determined by a tilting experiment since the sensors needed for the approach in Appendix B were missing in the scale model.

The tilting experiment was based on the fact that the center of mass was shifting in the x_n -axis of the world-fixed coordinate system when the ship was tilted an angle α . This can be seen in Figure 9.9 where the level (upper) and tilted (lower) case are shown together with the coordinate systems.

The shift in mass was measured using a scale. As can be seen in Figure 9.9, the scale was used to measure the masses m_1 and $\bar{m}_1$ at the forward location and then the masses m_2 and $\bar{m}_2$ were calculated from the total mass M . Once these masses were known, the center of mass, i.e. $\bar{x}_g$ and $\bar{z}_g$, could be calculated using the torque equilibrium

$$0 = L_1 m_1 + L_2 m_2 - m \bar{x}_m - M \bar{x}_g \quad (9.1)$$

in the level case and the torque equilibrium

$$0 = \bar{m}_1(L_1 c_\alpha - L_3 s_\alpha) + \bar{m}_2(L_2 c_\alpha - L_3 s_\alpha) + m(\bar{z}_m s_\alpha - \bar{x}_m c_\alpha) + M(\bar{z}_g s_\alpha - \bar{x}_g c_\alpha) \quad (9.2)$$

in the tilted case, by solving for $\bar{x}_g$ and $\bar{z}_g$. Here, c_α and s_α means $\cos(\alpha)$ and $\sin(\alpha)$, respectively. To get several independent measurements, the tilting experiment was performed for several different additional masses m and centers of mass $\bar{z}_m$, and the result can be seen in Table 9.2. The results show that in average, the center of gravity is 1.8 mm behind and 7.3 mm below the reference lines.

9.3 Experiment Description

The data was collected in a free run experiments where the ship was untethered and running by its own power. The ship was manually controlled with the Playstation 3 joystick and the rudder angle was thus a true reference to the system that was independent of the other measured signals. Both short and long datasets were collected at constant speed. The shorter sets, one U- or S-turn each, were mainly collected to understand the dynamics and are around 20 seconds each. An example of a short dataset can be seen in Figure 9.10. The short test sequences were repeated multiple times for each loading case in Table 9.2. The longer runs did not follow any particular pattern and both short and long turns were performed while utilizing as much of the usable area of the basin as possible. Long datasets were also collected for each loading case in Table 9.2.

Table 9.2: The results from the center of gravity calculations in the tilting experiment. The parameters m and $\bar{z}_m$ refers to the additional mass. The masses m_1 and $\bar{m}_1$ are the measured quantities at the level and the tilted case, respectively. The masses m_2 and $\bar{m}_2$ are calculated as $M - m_1$ and $M - \bar{m}_1$, respectively. Dash indicates that the quantity is unknown or not applicable.

Load Case		Level		Tilted		Result	
m [kg]	$\bar{z}_m$ [m]	m_1 [kg]	m_2 [kg]	$\bar{m}_1$ [kg]	$\bar{m}_2$ [kg]	$\bar{x}_g$ [m]	$\bar{z}_g$ [m]
0	–	11.57	11.115	14.690	7.995	–0.00178	0.00728
0.200	–0.256	11.67	11.215	14.850	7.835	–0.00181	0.00734
0.400	–0.268	11.77	11.315	15.015	7.670	–0.00185	0.00728
0.600	–0.280	11.87	11.415	> 15	–	–	–
Mean						–0.00181	0.00730

Figure 9.9: The two steps in the tilting experiment. In the top figure, the ship is leveled such that the horizontal reference line (really the reference plane) is orthogonal to the direction of gravity and then the weight was measured using the scale drawn underneath the forward support. The procedure was repeated for the tilted case, except that an angle α was created between the reference line in the level case and the plane corresponding to the tilted case.

Figure 9.10: Two representative short datasets with U-turn maneuvers that are time synchronized to show the differences in behavior. The red lines correspond to the nominal dataset and the blue lines correspond to the dataset with a $m = 0.200$ kg additional load positioned $\bar{z}_m = -0.256$ m from the body-fixed coordinate system. Note the nonlinear behavior of the dynamics while turning or leaving the turn, for example, the difference in damping between times 10 to 13 s and times 15 to 18 s.

9.4 Analysis of Data

Figure 9.10 shows two representative datasets that are time-synchronized to show the difference between the nominal and the additional load dataset. The additional load was 0.200 kg and located -0.256 m from the body-fixed coordinate system. These datasets were collected without waves but there were still disturbances due to reflection of the bow-wave on the basin side walls. This corresponds to sea states 1 and 2 (rippled and smooth waves) (Fossen, 2011). There are several interesting

features that can be seen in the figure.

- There is a visible difference in the acceleration measurement between the datasets due to the additional mass, for instance, between times 10 to 14 s. This is more clear if the angular velocity about the roll axis is plotted against the acceleration measurement, see Figure 9.11. This difference is probably due to the larger stationary angle in the loaded case.
- The roll dynamics was nonlinear. For example, note that the damping was different between times 10 to 13 s and times 15 to 18 s. The reason for this difference is unclear. One explanation might be that a larger part of the hull is in contact with the water during a turn.
- The non-minimum phase behavior is visible both going into the turn and going out of the turn.
- The propeller speed decreased during the turn which indicates that the speed should also have decreased. This is verified by the decrease in sideways acceleration from times 10 to 14 s.

Figure 9.11: The angular velocity $\dot{\phi}$ about the x_b -axis plotted against the acceleration measurement a_s . Note that the angular velocities were roughly the same, but that the acceleration was increased during the turn when there was an additional load. Note also that the stationary point for the ship traveling straight is about (0,0) and that the stationary point during a turn is somewhere about (1,0). See also the acceleration measurement in Figure 9.10.

Table 9.3: A summary of the datasets used to estimate the parameters of the indirect ship model (8.20). The dataset number is only used for ease of notation. All datasets were sampled at 100 Hz, low-pass filtered and down-sampled to 50Hz before estimation. The corresponding datasets can be seen in the figures indicated in the last column. Note that the number of samples is before downsampling. See Table 9.2 for an explanation of the loading cases.

Load Case	Number	Length	Samples	Shown In
Nominal	1	63.2 s	6301	Figure 9.12 Figure 9.1
0.200 kg	2	36.1 s	3601	Figure 9.13
	3	51.1 s	5101	Figure 9.14
0.400 kg	4	65.5 s	6540	Figure 9.15

9.5 Estimation Results Under Moderate Sea Conditions

This section describes the results from estimation of the parameters in the indirect ship model (8.20) under moderate sea conditions. The experiments were executed with the wave maker turned off. However, there were clear process disturbances due to reflections of the bow waves on the sidewalls of the basin. Like in the previous section, this corresponds to sea states 1 and 2 (rippled and smooth waves) (Fossen, 2011). The data were collected according to the description in Section 9.3.

The results presented in this section were produced using the datasets listed in Table 9.3. In contrast to the short datasets shown in Figure 9.10, the long datasets show less variations in the damping between the stationary mode going straight and the stationary mode in turn. One explanation might be a lower average speed, and hence a smaller difference in speed entering and leaving the turn, since there are multiple turns after each other. In the short datasets, the ship had time to build up its speed over the whole length of the basin. The datasets in Table 9.3 were chosen since they were similar to each other, for example, having the same maximum rudder angle and similar lengths, and since there were no strong non-linear effects visible in the data. All four datasets were sampled at 100 Hz, the datasets were filtered through an FIR equiripple low-pass filter of order 20 with a cut-off frequency of 0.5 Hz and were down-sampled to 50Hz resulting in a data length roughly between 1 800–3 300 samples per dataset. The datasets can be seen in Figures 9.12 to 9.15. In addition, the nominal dataset can be seen projected onto the picture of the basin in Figure 9.1.

The datasets were combined into three different cases according to Table 9.4. In each case, the parameters

$$\tilde{\vartheta}_{p,1} = \underbrace{[M, \bar{z}_g, m_c, \bar{z}_c]}_{\bar{\vartheta}_{p,1}}^T$$

defined in (8.63) were assumed to be known.

Table 9.4: A summary of the combinations of datasets used for estimation. Columns 2–4 show the datasets used in the estimation. The last column shows where the results are summarized. The first and second row correspond to a table and figure showing the iterations.

Case	Nominal Dataset	Calibration Dataset	Loaded Dataset	Results In
1	Nominal – 1	0.200 kg – 2	0.200 kg – 3	Table 9.5 Figure 9.16
2	Nominal – 1	0.200 kg – 3	0.400 kg – 4	Table 9.8 Figure 9.18
3	Nominal – 1	0.200 kg – 2	0.400 kg – 4	Table 9.9 Figure 9.19

Several noise models were tried and the noise models used to produce the results in this section were

$$\bar{H}_d^i(\mathbf{q}, \hat{\eta}^i) = \frac{1}{1 + \bar{\alpha}_1 \mathbf{q}^{-1} + \bar{\alpha}_2 \mathbf{q}^{-2}} \quad (9.3)$$

where $1 + \bar{\alpha}_1 \mathbf{q}^{-1} + \bar{\alpha}_2 \mathbf{q}^{-2}$ is the denominator of $G_d(\mathbf{q}, \bar{\vartheta}_p^i)$. This corresponds to

$$L^i(\mathbf{q}, \bar{\vartheta}_p^i) = 1 \quad (9.4)$$

In the initial step, the transfer functions in (8.82) were estimated using ARX model structures with $n_a = 4$ and $n_b = 5$. In addition, the vectors of instruments (8.84) were created using the constants $n_y = 16$, $n_{a_s} = 16$, $n_r = 16$ and $n_\delta = 2$ for each dataset.

The initial conditions were in all cases assumed to be

$$\begin{aligned} \tilde{\vartheta}_{p,2} &= [k, A_x, d, K_r, K_{\dot{v}}, K_\delta, z_f, m, \bar{z}_m]^T \\ &= [9, 0.5, 0.1, -0.4, 1, -0.4, -0.018, 0, 0]^T \end{aligned} \quad (9.5)$$

In the refining steps, the noise models were assumed to be given by (9.3) and the inputs and the output were simulated using (8.88). The vectors of instruments (8.92) were created using the constants $n_y = 16$, $n_{a_s} = 16$, $n_r = 16$ and $n_\delta = 2$ for each dataset, i.e. the same as in the initial step.

Algorithm 8.1 was then applied to each case in Table 9.4 and the results can be seen in Tables 9.5, 9.8 and 9.9, and Figures 9.16, 9.18 and 9.19. Note that the estimated parameters were roughly the same in all three cases. As expected, Case 1 gave the best results, probably since both the calibration and the loaded dataset had the same additional mass. It should however be noted that in Cases 2 and 3 the largest relative error, 16.6%, corresponds to 66.5 gram error in the mass estimation. For more details on each case, see Sections 9.5.1 and 9.5.2.

Figure 9.12: Nominal data – Dataset 1. The red line corresponds to the end of Case 1(a) and the green line corresponds to the start of Case 1(b).

Figure 9.13: Loaded data 0.200 kg – Dataset 2. The red line corresponds to the end of Case 1(a) and the green line corresponds to the start of Case 1(b).

Figure 9.14: Loaded data 0.200 kg – Dataset 3. The red line corresponds to the end of Case 1(a) and the green line corresponds to the start of Case 1(b).

Figure 9.15: Loaded data 0.400 kg – Dataset 4.

Table 9.5: Estimation results with the datasets according to Case 1 in Table 9.4. The first column corresponds to: parameters (P), true values (T), initial values (I), iteration (numbers) and the relative error (E).

P	$\bar{A}_x$	d	k	K_δ	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	—	—	—	—	—	—	—	0.2000	-0.2560
I	0.5000	0.1000	9.0000	-0.4000	-0.4000	1.0000	-0.0180	0.0000	0.0000
0	0.1689	0.2982	9.0026	-0.4994	-2.1676	0.9385	-0.0278	0.1388	-0.3535
1	0.1578	0.2696	9.0056	-0.4798	-2.0079	0.9627	-0.0294	0.1867	-0.2739
2	0.1582	0.2683	9.0057	-0.4821	-2.0099	0.9604	-0.0294	0.1986	-0.2541
3	0.1583	0.2685	9.0057	-0.4822	-2.0101	0.9607	-0.0294	0.1997	-0.2530
4	0.1583	0.2685	9.0057	-0.4822	-2.0100	0.9607	-0.0294	0.1997	-0.2529
5	0.1583	0.2685	9.0057	-0.4822	-2.0100	0.9607	-0.0294	0.1997	-0.2529
6	0.1583	0.2685	9.0057	-0.4822	-2.0100	0.9607	-0.0294	0.1997	-0.2529
7	0.1583	0.2685	9.0057	-0.4822	-2.0100	0.9607	-0.0294	0.1997	-0.2529
8	0.1583	0.2685	9.0057	-0.4822	-2.0100	0.9607	-0.0294	0.1997	-0.2529
E	—	—	—	—	—	—	—	0.1338	1.2120 %

9.5.1 Case 1 - Equal Load in Calibration and Loaded Datasets

The results for Case 1 can be seen in Table 9.5 and Figure 9.16. As expected, this case had the best results, probably since the calibration dataset was similar to the loaded dataset, i.e. having the same weight and position. The estimated mass was less than 1 g from the true value and the position error was less than 4 mm.

For comparison, the constrained LS estimator, i.e. Approach 5.2 with the cost function given by (8.68) where Φ^i and Y^i are given by (5.23), was used to compute the estimate. The result of the LS estimator can be seen as L in Table 9.6 and there is quite a significant difference compared to the IV estimator.

More flexible prefilters were tested but gave worse results. In Figure 9.17 the normalized residuals are plotted together with the normalized and scaled absolute value of the rudder angle. It can be seen that there is a clear correlation between the large spikes and the turn entries (and also the exits if it is a short turn as it was around 3500). This indicated that there was unmodeled dynamics and this might have made a variance calculation misleading due to the deterministic components in the residuals.

Table 9.6: The result of the LS estimate with the datasets according to Case 1. The first column corresponds to: parameters (P), true values (T), initial values (I), solution (L) and the relative error (E).

P	$\bar{A}_x$	d	k	K_δ	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	—	—	—	—	—	—	—	0.2000	-0.2560
I	0.5000	0.1000	9.0000	-0.4000	-0.4000	1.0000	-0.0180	0.0000	0.0000
L	0.1810	0.2739	9.0125	-0.5828	-2.1185	0.7861	-0.0248	0.0574	-0.6474
E	—	—	—	—	—	—	—	71.2819	152.8834 %

Figure 9.16: Estimation results with the datasets according to Case 1 in Table 9.4. The parameter values are plotted as functions of the iterations. Blue corresponds to estimated values and if a true value is known, it is shown in red. The parameter values in each iteration are shown in Table 9.5.

Table 9.9: Estimation results with the datasets according to Case 3 in Table 9.4. The first column corresponds to: parameters (P), true values (T), initial values (I), iteration (numbers) and the relative error (E).

P	$\bar{A}_x$	d	k	K_δ	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	—	—	—	—	—	—	—	0.4000	-0.2680
I	0.5000	0.1000	9.0000	-0.4000	-0.4000	1.0000	-0.0180	0.0000	0.0000
0	0.1693	0.2973	8.9967	-0.4605	-2.1405	0.9585	-0.0282	0.2814	-0.3220
1	0.1569	0.2674	9.0021	-0.4351	-1.9960	0.9527	-0.0294	0.3299	-0.2698
2	0.1571	0.2674	9.0021	-0.4382	-1.9987	0.9518	-0.0294	0.3326	-0.2676
3	0.1571	0.2675	9.0021	-0.4383	-1.9995	0.9516	-0.0294	0.3335	-0.2671
4	0.1571	0.2675	9.0021	-0.4383	-1.9995	0.9516	-0.0294	0.3335	-0.2671
5	0.1571	0.2675	9.0021	-0.4383	-1.9995	0.9516	-0.0294	0.3335	-0.2671
6	0.1571	0.2675	9.0021	-0.4383	-1.9995	0.9516	-0.0294	0.3335	-0.2671
E	—	—	—	—	—	—	—	16.6312	0.3286 %

Table 9.10: The result of the LS estimate with the datasets according to Case 2 in Table 9.4. For comparison with IV estimator, see Table 9.8. The first column corresponds to: parameters (P), true values (T), initial values (I), solution (L) and the relative error (E).

P	$\bar{A}_x$	d	k	K_δ	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	—	—	—	—	—	—	—	0.4000	-0.2680
I	0.5000	0.1000	9.0000	-0.4000	-0.4000	1.0000	-0.0180	0.0000	0.0000
L	0.1882	0.2830	9.0108	-0.5827	-2.1800	0.8256	-0.0247	0.8602	-0.1221
E	—	—	—	—	—	—	—	115.0549	54.4553 %

Table 9.11: The result of the LS estimate with the datasets according to Case 3 in Table 9.4. For comparison with IV estimator, see Table 9.9. The first column corresponds to: parameters (P), true values (T), initial values (I), solution (L) and the relative error (E).

P	$\bar{A}_x$	d	k	K_δ	K_r	$K_{\dot{v}}$	z_f	m	$\bar{z}_m$
T	—	—	—	—	—	—	—	0.4000	-0.2680
I	0.5000	0.1000	9.0000	-0.4000	-0.4000	1.0000	-0.0180	0.0000	0.0000
L	0.1817	0.2802	9.0099	-0.5886	-2.1624	0.8185	-0.0251	0.5580	-0.1826
E	—	—	—	—	—	—	—	39.5013	31.8663 %

Figure 9.18: Estimation results with the datasets according to Case 2 in Table 9.4. The parameter values are plotted as functions of the iterations. Blue corresponds to estimated values and if a true value is known, it is shown in red. The parameter values in each iteration are shown in Table 9.8.

Figure 9.19: Estimation results with the datasets according to Case 3 in Table 9.4. The parameter values are plotted as functions of the iterations. Blue corresponds to estimated values and if a true value is known, it is shown in red. The parameter values in each iteration are shown in Table 9.9.

9.6 Sensor Bias Mitigation and Surge Speed Dependence

Note that no pre-processing, except for the lowpass-filtering, has been performed on the data. A positive side effect of using the IV method is that the biases in the measurements, in addition to the process disturbance and the measurement noises, are automatically taken care of. The disturbances acting on the data, i.e. the sensor biases $b_{i,t}$, $i = 1, 2, 3$, the process disturbance τ and the measurement noises $e_{i,t}$, $i = 1, 2, 3$, were quite important to consider and would introduce a bias in the estimator if they were neglected. This is indicated by the least-squares results presented in Sections 9.5.1 and 9.5.2 and shown in Tables 9.6, 9.10 and 9.11. As previously noted, there is quite a significant difference compared to the IV estimator which performed better in all of the cases. Note that a constant heel angle, i.e. that the mean of the roll angle is non-zero, only will be visible through the gravity-term in (7.14) and that this bias will be treated in the same way as a sensor bias.

Figure 9.20 shows an estimate of the surge speed for the nominal dataset. The estimate was obtained using an extended Kalman smoother with a continuous turn model and data from the Qualisys positioning system. There is a clear variation of the surge velocity that is correlated to the rudder angle which means that the assumption of constant velocity was violated. This might be one reason for the spikes in the residuals that can be seen in Figure 9.17. However, the method managed to estimate the parameters fairly well and was thus robust against these variations. Note that the surge speed estimate was not used to obtain the parameters.

Figure 9.20: Variations in surge speed in the nominal dataset. Red: The surge velocity in m/s estimated from Qualisys data. Gray: The scaled absolute value of the rudder angle. The area between the rudder angle and the x -axis is filled to make the plot easier to read.

10

Conclusions

In this thesis we have discussed system identification for unknown input problems and proposed a new indirect modelling framework for treating such problems systematically. These problems arise naturally in the engineering sciences but are also common if only a part of a larger system is estimated. The framework was based on the assumption that the effects of the unknown inputs were visible some in additional measurements. These measurements, called *direct* and *indirect input measurements*, were used to eliminate the unknown inputs from the model. The direct input measurements were assumed to be noisy measurements of some of the unknown inputs through known dynamics. The indirect input measurements, on the other hand, were assumed to be measured through possibly unknown dynamics, and also affected by other signals as well as noise. The resulting *indirect model* could be used together with the output and input measurements to estimate the desired information. The indirect model was also expressed in an dynamic network formulation to increase the clarity of the correlation structure between the signals and show properties of the indirect model. In addition to the derivation of the indirect model in Chapter 4 and estimation of it in Chapter 5, filtering and control using the indirect model were briefly discussed in Chapter 6.

The framework was applied to a graybox model of a ship in Chapters 7 to 9. In particular, a ship's roll dynamics was studied and it was assumed that only motion measurements from an inertial measurement unit (IMU) together with measurements of the rudder angle were available. To understand the basic properties of the proposed method, it was first applied to the approximate model (7.10) that despite its simplicity, shared the same properties as the roll dynamics of many vehicles, e.g. cars, trucks, trains and ships. It was shown that a more complete ship model could be used as a starting point to derive the generalized model (8.10) of the approximate model (7.10), with the yaw-rate r and rudder angle δ as additional inputs.

The effects of using the indirect input measurements was presented in Section 4.3. The indirect model was analyzed with respect to consistency, variance and identifiability. The analyses showed that the estimation problem was similar to a closed-loop errors-in-variable problem due to the use of the indirect input measurements. Furthermore, it was shown under which conditions a certain part of the dynamics could be recovered using a black-box model. More concrete analyses for the applications were presented in Chapters 7 and 8. Identifiability analyses were performed on both the approximate model (7.10) in Section 7.2.1, and the generalized model (8.10) in Section 8.3. In both cases it was shown that only a subset of the unknown parameters could be estimated simultaneously and that extra calibration datasets could be used to overcome some of these identifiability restrictions. The impact of disturbances was analyzed for the approximate model (7.10) in Section 7.3.2, and for the generalized model (8.10) in Section 8.2.2.

The multiple dataset approaches were introduced in Chapter 5. In Section 5.2.3 the multi-stage method was introduced. In this method, several datasets were used in a sequential manner to introduce more information in each step. The joint identification approach, in which several datasets were used simultaneously, was introduced in Section 5.2.4. The graybox formulation enabled fusion of data even though the data were collected for different system configurations. This approach is similar to merging segments of data from the same system configuration and it is certainly applicable also in other types of problems than the ones described in this thesis. The approaches were applied in Chapters 7 and 8 to circumvent the restrictions that were shown in the identifiability analyses. In Section 7.4 the multi-stage approach was applied to the indirect pendulum model. Because of the extra complexity of the indirect ship model, the joint identification approach was used in Section 8.5. An iterative closed-loop instrumental variable approach was formulated both for estimating the parameters in each step of the multi-stage method of Section 5.2.3 and for solving the joint identification problem in Section 5.2.4. The choice of the instrumental variable method was mainly due to the results of the disturbance analyses.

The experimental verification of the joint identification approach applied to the indirect model with data collected from a model basin was described in Section 9. The results were promising, but, there are still many possibilities for improvement and robustification.

A

Introduction to Ship Modelling

This chapter is an introduction to modelling of ships and the environmental forces acting on them. The interaction between the ship and the environment makes the modelling of forces and moments acting on the ship very complex. The model type used is dependent on its field of application and the major difference is in how the theories deal with hydrodynamic phenomena. The models are divided into *seakeeping theory*, i.e. motion due to waves of a ship with constant speed and course, and *maneuvering theory*, i.e. motion due to control inputs of a ship in calm water. Classically, the complete response of the ship is given by motion superposition of these models. An alternative is to use the *unified theory* that combines components from both theories with force superposition (Perez, 2005).

In this thesis, the focus is on separating the motion induced by actuators from the total motion without modelling the motions induced by the environmental disturbances. Here, we will hence concentrate on specific parts of the maneuvering theory models. Additionally, environmental disturbances will briefly be discussed with the aim of understanding the properties of these disturbances. For a more complete treatment covering modelling of environmental disturbances and ships, also covering seakeeping and unified theory, see for instance, Newman (1977), Faltinsen (1990), Journée and Massie (2001), Perez (2005) and Fossen (2011).

Here, we will consider models suitable for system identification and work with a simplified version of the model

$$\begin{aligned} \dot{\eta} &= J(\eta)\nu && \text{-- Kinematics: Appendix A.1.2} \\ M_{RB}\dot{\nu} + C_{RB}(\nu)\nu &= \tau && \text{-- Kinetics: Appendix A.1.3} \end{aligned} \quad (\text{A.1})$$

that was first described in Fossen (1991). The notation will be described in the following sections. The forces acting on the system are

$$\tau = \underbrace{\tau_{\text{hydro}}}_{\text{Appendix A.2}} + \underbrace{\tau_{\text{actuators}}}_{\text{Appendix A.3}} + \underbrace{\tau_{\text{environmental}}}_{\text{Appendix A.4}} \quad (\text{A.2})$$

and they will be described in the indicated sections. Furthermore, the ship's state will be measured by sensors described by

$$y = h(\eta, \nu) + e \quad \text{-- Sensors: Appendix A.6} \quad (\text{A.3})$$

A.1 Rigid-body Dynamics

In this thesis, the ship is assumed to be a non-flexible structure. This is a common assumption and for a treatment of flexible structures, see for instance, Drummen (2008). This is an idealization, but allows us to use *rigid-body dynamics*, which is the study of the motion of a rigid-body under influence from external forces. Dynamics can straightforwardly be divided into *kinematics*, the geometric description of the motion, and *kinetics*, the effect of forces acting on the body, see for instance, Thornton and Marion (2004), Spong and Hutchinson (2005) or Fossen (2011).

The pose of the ship can be described using three coordinates for the position, three coordinates for the rotation and the motion of the ship is fully specified by the pose and its time derivatives. These are called the generalized position and the generalized velocity, respectively, and we will get back to them shortly.

A.1.1 Reference Frames and Reference Points

A convenient way of relating the generalized positions and velocities is by the means of *reference frames*, one body-fixed frame attached to the ship and one inertial frame. These will be called the *b*-frame and the *n*-frame, respectively, and are defined according to Definition A.1 and Definition A.2. The sensor *s*-frame is also defined in Definition A.3 for later use.

Definition A.1 (*b*-frame). The *b*-frame has its origin o_b fixed to the ship, the x_b -axis pointing towards the bow (forward), the y_b -axis pointing to starboard (right) and the z_b -axis pointing downwards, see Figure A.1.

Definition A.2 (*n*-frame). The *n*-frame has its origin o_n fixed to earth. The *n*-frame is a north-east-down (NED) coordinate system, i.e. the x_n -axis pointing towards the north, the y_n -axis pointing towards the east and the z_n -axis pointing towards the center of the earth, see Figure A.1(a). Note that this is not truly an inertial frame, but for a ship moving locally it is a reasonable assumption (Fossen, 2011).

Figure A.1: The reference frames and reference points in Definitions A.1–A.9.

Figure A.2: A rectangular block is floating in water. The water plane is the intersection between the block and the water surface when the block is at rest. The centroid, or geometric center, is in this case in the center of the block.

Definition A.3 (s -frame). The s -frame has its origin o_s fixed to the ship, the x_s -axis pointing towards the bow (forward), the y_s -axis pointing to port (left) and the z_s -axis pointing upwards.

In order to further simplify the modelling, certain body-fixed reference points can be defined. These are defined as a vector $\mathbf{r}_i = [x_i, y_i, z_i]^T$ in the b -frame. The significance of these points will be clear in the coming sections. Here, they are simply enumerated in Definitions A.4 – A.9.

Definition A.4 (Center of Buoyancy (CB)). The center of buoyancy, described by $\mathbf{r}_b$, is the center of mass for the water displaced by the ship, see Figure A.1(b). Note that this point is dependent on the hull shape and the attitude of the ship, see also Appendix A.2.1.

Definition A.5 (Center of Flotation (CF)). The center of flotation, described by $\mathbf{r}_f$, is the centroid (geometric center) of the of water plane in calm water, see Figure A.1(b). An example for a rectangular block can be seen in Figure A.2.

Definition A.6 (Center of Gravity (CG)). Here it is assumed that the ship is operating in a constant and homogenous gravity field. Hence, the center of gravity, described by $\mathbf{r}_g$, coincides with the ship's center of mass (CM), see Figure A.1(b).

Definition A.7 (Center of Rotation (CR)). The center of rotation is the instantaneous point around which the ship rotates. Note that for a rigid-body in free space the rotation center will coincide with its CG. However, for a ship the center of rotation is shifted due to the added mass (read added inertia) due to its motion through water, see for instance Balcer (2004). In this thesis, however, the center of rotation is assumed to be fixed and equal to the CF.

Definition A.8 (Center of Origin (CO)). The center of origin, is the origin o_b of the b -frame.

Definition A.9 (Center of Sensor (CS)). The center of sensor, with the origin o_s , located with $\mathbf{r}_s$, is the center of the IMU described in Appendix A.6.1. See also Definition A.3.

Table A.1: The generalized positions and velocities used to describe the ship's position and orientation. Note that ϕ , θ and ψ are Euler angles and that the zyx convention is used (Fossen, 2011).

		Position	Velocity
Generalized		$\boldsymbol{\eta}$	$\boldsymbol{\nu}$
Linear	Surge	x	u
	Sway	y	v
	Heave	z	w
Angular	Roll	ϕ	p
	Pitch	θ	q
	Yaw	ψ	r

A.1.2 Kinematics

The kinematics can now be derived by relating the generalized position

$$\boldsymbol{\eta} = [\mathbf{p}^T, \boldsymbol{\Theta}^T]^T = [x, y, z, \phi, \theta, \psi]^T \quad (\text{A.4})$$

which is the translation and rotation of the b -frame with respect to the n -frame expressed in the n -frame, and the generalized velocity

$$\boldsymbol{\nu} = [\mathbf{v}^T, \boldsymbol{\omega}^T]^T = [u, v, w, p, q, r]^T \quad (\text{A.5})$$

which is the body-fixed linear velocities and body-fixed angular velocities expressed in the b -frame. Note that if the forward speed u is assumed to be constant, it is denoted by U . See Table A.1 for a description of the variables and Figure A.1(a) for a geometric description.

As already mentioned, the generalized positions and velocities have a geometric relation that is independent of the forces acting on the rigid-body. The relation is

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \quad (\text{A.6})$$

where

$$\mathbf{J}(\boldsymbol{\eta}) = \begin{bmatrix} \mathbf{R}(\boldsymbol{\Theta}) & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{T}(\boldsymbol{\Theta}) \end{bmatrix} \quad (\text{A.7})$$

is a transformation matrix with the rotation matrix

$$\mathbf{R}(\boldsymbol{\Theta}) = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\theta + c_\psi s_\theta s_\phi & s_\psi s_\theta + c_\psi c_\theta s_\phi \\ s_\psi c_\theta & c_\psi c_\theta + s_\psi s_\theta s_\phi & -c_\psi s_\theta + s_\psi c_\theta s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \quad (\text{A.8})$$

describing the relation between the linear speeds in the b -frame and the linear speeds in the n -frame (Fossen, 2011, Ch. 2). Here, $\mathbf{0}_{x \times y}$ is a x by y matrix with all elements equal to zero, $s_x = \sin(x)$, $c_x = \cos(x)$ and $t_x = \tan(x)$. Furthermore, the transformation matrix

$$\mathbf{T}(\boldsymbol{\Theta}) = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi/c_\theta & c_\phi/c_\theta \end{bmatrix} \quad (\text{A.9})$$

describes the angular velocity transformation from b - to n -frame (Fossen, 2011, Ch. 2). Note that ϕ , θ and ψ are Euler angles and that the rotation matrix has been created by multiplying the principal rotation matrices for the z , y and x axes in that order, see for instance, Thornton and Marion (2004), Spong and Hutchinson (2005) or Fossen (2011), for a thorough treatment of transformation matrices.

A.1.3 Kinetics

The kinetics, i.e. effects of forces acting on the rigid-body, can be derived, for instance, by using Euler-Lagrange equations or the Newton-Euler formulation. Here, the derivation is omitted and the interested reader is referred to Fossen (1991), Thornton and Marion (2004), Perez (2005), Spong and Hutchinson (2005) or Fossen (2011). The kinetics can be expressed as

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau} \quad (\text{A.10})$$

which is the vectorial representation described in Fossen (1991). Here $\mathbf{M}_{RB}$ is the rigid-body inertia matrix, $\mathbf{C}_{RB}(\boldsymbol{\nu})$ represents the centripetal and Coriolis terms, and $\boldsymbol{\tau}$ is a vector of generalized forces, all expressed in the b -frame.

The matrices $\mathbf{M}_{RB}$ and $\mathbf{C}_{RB}(\boldsymbol{\nu})$ are calculated for the fixed point CO that does not necessarily have to coincide with the CG. This is convenient since the CG will move with different loading conditions. The vector relating CO and CG is denoted $\mathbf{r}_g$ and is defined in Definition A.6. Here, the skew-symmetric matrix

$$\mathbf{S}(\mathbf{x}) = -\mathbf{S}(\mathbf{x})^T = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (\text{A.11})$$

is used to simplify the notation. Note that $\mathbf{S}(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$, i.e. the vector cross-product, see Fossen (2011, p. 20). For motion about CO the inertia matrix is given by

$$\mathbf{M}_{RB} = \begin{bmatrix} \bar{M}\mathbf{I}_{3 \times 3} & -\bar{M}\mathbf{S}(\mathbf{r}_g) \\ \bar{M}\mathbf{S}(\mathbf{r}_g) & \mathbf{I}_g - \bar{M}\mathbf{S}^2(\mathbf{r}_g) \end{bmatrix} = \begin{bmatrix} \bar{M}\mathbf{I}_{3 \times 3} & -\bar{M}\mathbf{S}(\mathbf{r}_g) \\ \bar{M}\mathbf{S}(\mathbf{r}_g) & \mathbf{I}_b \end{bmatrix} \quad (\text{A.12})$$

where $\mathbf{I}_{x \times x}$ is the x by x identity matrix, $\bar{M}$ is the total mass of the ship including additional loads such as cargo, $\mathbf{I}_b$ is the inertia matrix around CO and $\mathbf{I}_g$ is the inertia matrix around CG. The inertia matrix has a unique representation and satisfies $\mathbf{M}_{RB} = \mathbf{M}_{RB}^T > 0$ and $\dot{\mathbf{M}}_{RB} = \mathbf{0}_{6 \times 6}$. The matrix $\mathbf{C}_{RB}(\boldsymbol{\nu})$ is however not unique due to the cross-terms, for example, $K_{ur}ur = C_1(u)r = C_2(r)u$. One representation is

$$\mathbf{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \bar{M}\mathbf{S}(\boldsymbol{\omega}) & -\bar{M}\mathbf{S}(\boldsymbol{\omega})\mathbf{S}(\mathbf{r}_g) \\ \bar{M}\mathbf{S}(\mathbf{r}_g)\mathbf{S}(\boldsymbol{\omega}) & -\mathbf{S}(\mathbf{I}_b\boldsymbol{\omega}) \end{bmatrix} \quad (\text{A.13})$$

which is independent of the linear velocities $\mathbf{v}$. Which representation that is suitable depends on the intended use of the model, see Fossen (2011, p. 53–56).

A.2 Forces and Moments Due to Water Interaction

The forces acting on a ship due to interaction with the water are very complex. A common assumption is that the forces due to different phenomena can be modeled individually and that linear superposition can be used (Faltinsen, 1990, 2005). In this thesis, the forces acting on the ship are assumed to be given by a general non-linear function which will here be expressed by a series expansion. The forces are usually split into hydrostatics, which is the restoring forces, and hydrodynamics, which is the added inertia and damping. The forces and moments in (A.2) due to interaction with water are assumed to be given by

$$\tau_{\text{hydro}} = \overbrace{-M_A \dot{\nu} - C_A(\nu)\nu - D(\eta, \nu)}^{\text{Hydrodynamics}} - \overbrace{g(\eta)}^{\text{Hydrostatics}} = -f(\eta, \dot{\nu}, \nu) - g(\eta) \quad (\text{A.14})$$

which combined with the rigid-body kinetics gives the resulting nonlinear model

$$M\dot{\nu} + C(\nu)\nu + D(\eta, \nu) + g(\eta) = \tau_{\text{actuators}} + \tau_{\text{environmental}} \quad (\text{A.15})$$

that is of mass–spring–damper type and where the coefficients are assumed to be constant. Here, the inertia and Coriolis matrices are

$$M = M_{RB} + M_A \quad (\text{A.16})$$

and

$$C(\nu) = C_{RB}(\nu) + C_A(\nu) \quad (\text{A.17})$$

respectively.

A.2.1 Hydrostatics

The restoring forces of a vessel are given by the buoyancy and gravitational forces acting on the ship. At equilibrium, all floating bodies will have the CG and CB on a vertical line. Note that for most stable floating objects, such as ships, the CG will be above the CB (Lautrup, 2004). The restoring torque is expressed in terms of the so called righting arm. In order to understand what the righting arm is, consider Figure A.3 that shows a box-shaped ship seen from the front. When the ship rolls around the CF, the rolling motion will result in the CB being shifted away from the ship's vertical axis. This shift, from the old to the new CB, creates a restoring torque that can be written as

$$\rho \nabla \overline{GZ}(\phi) \quad (\text{A.18})$$

where ∇ is the displaced water volume, g is the acceleration of gravity and ρ is the density of water. The righting arm $\overline{GZ}(\phi)$ is the distance between the center of gravity CG and a vertical line through the new CB, see Figure A.3.

It can be shown that for small roll angles ϕ , the righting arm can be described using the metacenter M_T . The metacenter M_T is the point at which a vertical line through a new CB crosses the line that passes through the old CB and the CG when the ship is tilted an infinitesimal small roll angle. Note that M_T is a constant and a purely geometric property. The distance between the centre of gravity CG and the metacenter M_T is called the metacenter height $\overline{GM}_T$. The shift of the CB will thus for small angles result in the restoring torque

$$\rho \nabla \overline{GZ}(\phi) \approx \rho \nabla \overline{GM}_T \sin(\phi) \approx \rho \nabla \overline{GM}_T \phi \quad (\text{A.19})$$

Figure A.3: A boxed-shaped ship seen from the front. The ship has been perturbed from its equilibrium which shifts the CB away from the original position. This shift creates a restoring torque. For small angles ϕ , the righting arm is proportional to $\overline{GM}_T \sin(\phi)$.

In Figure A.4 a sketch of a righting arm function is shown. Initially, the righting arm will increase as ϕ gets larger and then decrease until the righting arm becomes zero. If the roll angle ϕ increases after that point, the ship will capsize, see Lautrup (2004, p. 48–55) or Journée and Massie (2001) for a detailed discussion about ship stability. Note that the position of the metacenter M_T , the righting arm $\overline{GZ}(\phi)$ and the metacentric height $\overline{GM}_T$ may vary depending on a number of factors, such as hull shape and loading conditions.

In this thesis, the total restoring forces are assumed to be

$$\mathbf{g}(\boldsymbol{\eta}) = [0, 0, \rho g C_z, \rho \nabla \overline{GM}_T \phi, \rho \nabla \overline{GM}_L \theta, 0]^T \quad (\text{A.20})$$

where $\overline{GM}_L$ is the metacentric height in the pitch direction.

Figure A.4: The righting arm is a function of the roll angle ϕ . For small angles ϕ , the righting arm is proportional to the roll angle, i.e. $\overline{GZ}(\phi) \approx \overline{GM}_T \phi$. As the roll angle ϕ increases further, the righting arm will eventually become zero which will result in the ship capsizing (Lautrup, 2004, p. 48–55).

A.2.2 Maneuvering Hydrodynamics

Hydrodynamic forces and moments acting on a ship have two effects.

1. When a ship accelerates or decelerates in water, the water must separate around the ship and then join behind it. This change of momentum of the water can be modeled as a virtual *added mass*. As an example, consider that you try to move your arm through water. This will be more difficult than doing the same movement in air.
2. When a ship moves through water a certain damping, for instance, due to friction, will affect it.

The hydrodynamic forces and moments are modeled differently depending on the application. Here, the general nonlinear function $\mathbf{f}(\boldsymbol{\eta}, \dot{\boldsymbol{\nu}}, \boldsymbol{\nu})$ is described using a series expansion. This expansion can, for instance, be split into terms depending on the derivatives of the body-fixed velocities (added mass), terms that are quadratic in the body-fixed velocities (added Coriolis-centripetal forces), terms depending linearly on the body-fixed velocities (potential/linear damping) and modulus terms (nonlinear damping). An example from Perez (2005, p. 83) of a model of hydrodynamic forces in roll is

$$\begin{aligned}
 \tau_p = & K_{\dot{v}}\dot{v} + K_{\dot{p}}\dot{p} + \dots && \text{Added mass} \\
 & + K_{ur}ur && \text{Coriolis due to added mass} \\
 & + K_{\dot{p}}p + \dots && \text{Linear damping} \\
 & + K_{|u|v}|u|v + K_{|v|v}|v|v + K_{|r|v}|r|v + \dots \Big\} && \text{Nonlinear damping} \\
 & + K_{\phi|uv}|\phi|uv| + K_{\phi|ur}|\phi|ur| + \dots &&
 \end{aligned} \tag{A.21}$$

where the coefficients are assumed to be constant. In general, the function can be written on the form

$$\mathbf{f}(\boldsymbol{\eta}, \dot{\boldsymbol{\nu}}, \boldsymbol{\nu}) = \mathbf{M}_A \dot{\boldsymbol{\nu}} + \mathbf{C}_A(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\eta}, \boldsymbol{\nu}) \tag{A.22}$$

where the added mass inertia matrix

$$\mathbf{M}_A = - \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix} \tag{A.23}$$

and the added mass Coriolis matrix $\mathbf{C}_A(\boldsymbol{\nu})$ can be shown to have the same properties as the rigid-body equivalents (Ross, 2008).

A complete example will be given in Appendix A.5. For a thorough treatment of hydrodynamic forces and moments for maneuvering models, see, for instance, Blanke and Christensen (1993), Perez (2005), Ross (2008) or Fossen (2011).

A.3 Actuators

There are in principle two things affecting a ship, the environmental disturbances and the actuators, e.g. propellers, rudders or thrusters. These can be modeled with different levels of accuracy and complete models are very complex due to the interaction with the water, see for instance, Perez (2005) or Faltinsen (2005). In this section, simple linear models of a rudder and a propeller will be given. Note that most actuators found on a ship can be modeled as either a rotating propeller, i.e. with changing direction of thrust, or as a control surface acting like a fin.

A.3.1 Rudder

A rudder will create a rudder force L and a drag force D if the deflection angle is nonzero. These forces are in general dependent on, for instance, the forward speed u , the sway velocity v and the yaw rate r . Here, we assume that the forces created by the rudder are

$$\boldsymbol{\tau}_r = [-C_1|\delta| \quad C_2\delta \quad 0 \quad -z_r C_2\delta \quad 0 \quad x_r C_2\delta]^T \quad (\text{A.24})$$

where $C_1, C_2 > 0$ are constants and $\mathbf{r}_r = [x_r, y_r, z_r]^T$ is the vector describing the rudder lever arm relative to the CG. A positive rudder angle δ is defined in the b -frame and is thus equivalent to a left turn. The x -component x_r is more commonly known as the *Lateral Center of Gravity* (LCG) in ship literature.

A.3.2 Propeller

A propeller creates thrust by moving water in one direction. The main propeller of a ship mainly creates a negative thrust in the x_b -direction of the b -frame. Due to misalignment and since it is mounted below the CG, it will also create forces and moments in other directions. However, in this thesis, it is assumed that these effects are negligible and that the created force is proportional to the propeller RPM ω_p . The model is

$$\boldsymbol{\tau}_p = [C_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \omega_p \quad (\text{A.25})$$

A.4 Environmental Disturbances

There are three major environmental forces acting on a marine vessel, namely currents, wind and waves. The wave-induced forces can be divided into first- and second-order effects. The first-order effects induce a zero-mean oscillatory motion and the second-order effects are seen as a nonzero slowly varying drift.

The energy content of waves is described by using a power spectral density (PSD) function usually called a wave spectrum. Waves develop from wind that is forming small waves that grow until they break up. If the wind has been near stationary for a long time, the waves are considered to be fully developed (statistically stable) and the PSD contains a peak at lower frequencies. Because of changing wind strengths and directions, complex multi-modal directional wave spectra can be observed.

There are a number of common standard spectrum formulas that have been developed over a long time. Which family of spectra to use depend on the situation. They are usually dependent on one or two parameters that describe the sea-state. One example is the JONSWAP PSD (Fossen, 2011, p. 216) given by

$$S(\omega) = 0.2H_s^2\omega_0^4\omega^{-5} \exp(-1.25\omega_0^4\omega^{-4}) \gamma^{\exp\left(\frac{\omega_0-\omega}{\sqrt{2}\omega_0\sigma}\right)} \quad (\text{A.26})$$

where H_s is the significant wave height, ω_0 is the modal frequency and γ is a constant that is usually chosen to be approximately 3.3. The parameter σ is 0.07 if $\omega < \omega_0$ and $\sigma = 0.09$ otherwise (Fossen, 2011, p. 206). An example of the JONSWAP PSD when $\gamma = 3.3$, $H_s = 1$ and $\omega_0 = 0.4$ can be seen in Figure A.5.

For simulation purposes, the JONSWAP PSD can be approximated by letting white noise pass through a linear shaping filter, for instance,

$$V(s) = \frac{K_0 s}{s^2 + 2\zeta\omega_a s + \omega_a^2} \quad (\text{A.27})$$

where the parameters can be chosen such that the PSD $|V(\omega)|^2$ best represents the original spectrum (Fossen, 2011, p. 216).

Now, consider a ship moving with constant forward speed U and constant heading. Since the ship is moving, the perceived wave frequency, called the encounter frequency, is dependent on the forward speed. The encounter frequency can be

Figure A.5: The red curve shows an example of a JONSWAP power spectral density function. The blue curve shows a linear approximation that can be used for simulation purposes.

Figure A.6: The perceived wave frequency, i.e. the encounter frequency ω_e , is dependent on the forward speed U , the actual wave frequency and the direction of the waves relative to the ship's heading. This is due to the Doppler effect.

derived from Figure A.6 where the relative speed between the ship and the waves is $c - U \cos(\beta)$ which gives the encounter frequency

$$\omega_e = \left| \omega_0 - \omega_0^2 \frac{U}{g} \cos(\beta) \right| \quad (\text{A.28})$$

where ω_0 is the wave frequency, U is the forward speed, β is the encounter angle, g is the acceleration of gravity and the absolute value is used since a frequency cannot be negative (Fossen, 2011, p. 210). The encounter angle is 0 when the waves are following the ship from behind and is increasing clockwise. It should be noted that since the perceived frequencies are different for a moving ship, this will not only change the frequency range but also the shape of the wave spectrum (Perez, 2005, p. 36). For more details, see for instance, Faltinsen (2005) or Fossen (2011).

The induced motions from waves are due to the varying pressure on the ship's hull and hence the hull geometry is needed to calculate the response given the wave spectrum and direction. Assuming a linear relation, the transfer functions from wave spectrum to forces on the ship are called force response amplitude operators.

The implications are that the disturbance is colored, since the PSDs describing the energy content of the waves are narrow-banded or at least not flat, and that the peak frequency is dependent on ship pose and speed.

A.5 Surge-Sway-Roll-Yaw Maneuvering Model

The approach used in this thesis concerns the roll motion. Hence, the surge-sway-yaw maneuvering model extensively covered in the ship modelling literature is inadequate. Instead, a model similar to the four degree-of-freedom surge-sway-roll-yaw, maneuvering model with rudder input developed and discussed in Blanke and Christensen (1993) and Perez (2005) is presented in this section. The difference is mainly that the parameterization here is chosen to avoid identifiability issues.

With some abuse of notation, by neglecting the coupling to pitch and heave, the dynamic model defined in (A.6) and (A.10) to (A.13) can be written as

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \quad (\text{A.29})$$

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}$$

where

$$\mathbf{J}(\boldsymbol{\eta}) = \begin{bmatrix} c_\psi & -c_\phi s_\psi & 0 & 0 \\ s_\psi & c_\phi c_\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c_\phi \end{bmatrix}, \quad (\text{A.30})$$

$$\mathbf{M}_{RB} = \begin{bmatrix} \bar{M} & 0 & 0 & -\bar{M}\bar{y}_g \\ 0 & \bar{M} & -\bar{M}\bar{z}_g & \bar{M}\bar{x}_g \\ 0 & -\bar{M}\bar{z}_g & \bar{I}_x & 0 \\ -\bar{M}\bar{y}_g & \bar{M}\bar{x}_g & 0 & \bar{I}_z \end{bmatrix}, \quad (\text{A.31})$$

$$\mathbf{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -\bar{M}r & \bar{M}\bar{z}_gr & -\bar{M}\bar{x}_gr \\ \bar{M}r & 0 & -\bar{M}\bar{y}_gp & -\bar{M}\bar{y}_gr \\ -\bar{M}\bar{z}_gr & \bar{M}\bar{y}_gp & 0 & 0 \\ \bar{M}\bar{x}_gr & \bar{M}\bar{y}_gr & 0 & 0 \end{bmatrix}, \quad (\text{A.32})$$

$\boldsymbol{\eta} = [x, y, \phi, \psi]^T$ and $\boldsymbol{\nu} = [u, v, p, r]^T$. Note that the 3×4 and 4×3 elements of $\mathbf{C}_{RB}(\boldsymbol{\nu})$ are 0 due to the chosen representation. The ship's total mass is assumed to be $\bar{M}$ located at $\mathbf{r}_g = \bar{\mathbf{r}}_g = [\bar{x}_g, \bar{y}_g, \bar{z}_g]^T$ and $\bar{I}_x$ and $\bar{I}_z$ are the inertias expressed in the b -frame. The forces acting on the ship are described in the earlier sections of this chapter. These forces are

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\text{hydro}} + \boldsymbol{\tau}_{\text{actuators}} + \boldsymbol{\tau}_{\text{environmental}} \quad (\text{A.33})$$

Firstly, the hydrodynamic forces and moments acting on the ship are described using the theory presented in Appendix A.2, i.e.

$$\boldsymbol{\tau}_{\text{hydro}} = -\mathbf{M}_A\dot{\boldsymbol{\nu}} + \bar{\mathbf{f}}(\boldsymbol{\eta}, \boldsymbol{\nu}) - \mathbf{g}(\boldsymbol{\eta}) \quad (\text{A.34})$$

where $\bar{\mathbf{f}}(\boldsymbol{\eta}, \boldsymbol{\nu})$ is the Coriolis terms combined with the nonlinear damping and $\mathbf{M}_A$ is the added mass matrix. The added mass matrix is assumed to be

$$\mathbf{M}_A = - \begin{bmatrix} X_{\dot{u}} & 0 & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{p}} & Y_{\dot{r}} \\ 0 & K_{\dot{v}} & K_{\dot{p}} & K_{\dot{r}} \\ 0 & N_{\dot{v}} & N_{\dot{p}} & N_{\dot{r}} \end{bmatrix} \quad (\text{A.35})$$

which is defined in (A.23) but where the coupling to surge has been neglected due to the assumption that the surge motion has much slower dynamics. The nonlinear hydrodynamic damping and Coriolis affects are assumed to be described by

$$\bar{\mathbf{f}}(\boldsymbol{\nu}, \boldsymbol{\eta}) = \begin{bmatrix} X_{|u|u}|u|u \\ Y_{|u|v}|u|v + Y_{ur}ur + Y_{v|v|v}|v| + Y_{v|r|v}|r| + Y_{r|v|r}|r|v| \\ + Y_{\phi|uv}|\phi|uv| + Y_{\phi|ur}|\phi|ur| + Y_{\phi uu}\phi u^2 \\ K_{|u|v}|u|v + K_{ur}ur + K_{v|v|v}|v| + K_{v|r|v}|r| + K_{r|v|r}|r|v| \\ + K_{\phi|uv}|\phi|uv| + K_{\phi|ur}|\phi|ur| + K_{\phi uu}\phi u^2 + K_{|u|p}|u|p \\ + K_{p|p}|p|p + K_{pp}p + K_{\phi\phi\phi}\phi^3 \\ N_{|u|v}|u|v + N_{|u|r}ur + N_{r|r|r}|r| + N_{r|v|r}|r|v| \\ + N_{\phi|uv}|\phi|uv| + N_{\phi u|r}|\phi u|r| + N_{\phi|uv}|\phi|uv| + N_{pp}p \\ + N_{|p|p}|p|p + N_{|u|p}|u|p + N_{\phi u|u}|\phi u|u| \end{bmatrix} \quad (\text{A.36})$$

where the coefficients are assumed to be constant (Perez, 2005, p. 83). Assuming that the roll angle ϕ is small, the hydrostatic forces and moments are given by

$$\mathbf{g}(\boldsymbol{\eta}) = \begin{bmatrix} 0 & 0 & \rho g \nabla \overline{GM}_T \phi & 0 \end{bmatrix}^T \quad (\text{A.37})$$

defined in (A.20).

Secondly, the only actuator acting on the ship is assumed to be the simplified rudder described in Appendix A.3.1 and a constant force acting as the forward propulsion. The actuator contribution is

$$\boldsymbol{\tau}_{\text{actuators}} = \begin{bmatrix} C_3\omega_p - C_1\delta & C_2\delta & -z_r C_2\delta & x_r C_2\delta \end{bmatrix}^T = \mathbf{R}(\delta) \quad (\text{A.38})$$

Finally, the environmental forces acting on the ship are assumed to have the properties described in Appendix A.4 but are otherwise unknown.

The total model is

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \\ \mathbf{M}\dot{\boldsymbol{\nu}} &= -\mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \bar{\mathbf{f}}(\boldsymbol{\nu}, \boldsymbol{\eta}) - \mathbf{g}(\boldsymbol{\eta}) + \mathbf{R}(\delta) + \boldsymbol{\tau}_{\text{environmental}} \end{aligned} \quad (\text{A.39})$$

where the inertia matrix is

$$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A = \begin{bmatrix} \bar{M} - X_{\dot{u}} & 0 & 0 & -\bar{M}\bar{y}_g \\ 0 & \bar{M} - Y_{\dot{v}} & -\bar{M}\bar{z}_g - Y_{\dot{p}} & \bar{M}\bar{x}_g - Y_{\dot{r}} \\ 0 & -\bar{M}\bar{z}_g - K_{\dot{v}} & \bar{A}_x & -K_{\dot{r}} \\ -\bar{M}\bar{y}_g & \bar{M}\bar{x}_g - N_{\dot{v}} & -N_{\dot{p}} & \bar{A}_z \end{bmatrix} \quad (\text{A.40})$$

Here, the parameters $\bar{A}_x = \bar{I}_x - K_{\dot{p}}$ and $\bar{A}_z = \bar{I}_z - N_{\dot{r}}$ have been introduced to avoid identifiability issues.

A.5.1 Non-minimum Phase Behavior

A surface vessel will generally show a non-minimum phase behavior in the response from the rudder angle δ to the roll angle ϕ . This can be explained by the location of the rudder and the coupling between the different degrees of freedom. Figure A.7(a) shows the rudder location in relation to the CF and the CG. When a positive rudder angle δ is applied, the ship will initially lean inwards (left). This is due to the rudder's location below the CG and that the roll dynamics is faster than the other subsystems, see Figure A.7(b). Eventually, coupling from the slower dynamics, especially the yaw subsystem, and hydrodynamic forces will create a torque in the opposite direction resulting in the ship leaning outwards, see Figure A.7(c) (Perez, 2005, p. 150).

Figure A.7: The non-minimum phase behavior from the rudder angle δ to the roll angle ϕ can be explained with interaction between the subsystems. Here, F_r and F_H are the forces induced by the rudder and hydrodynamics, respectively.

A.6 Sensors

In this section, the inertial measurement unit (IMU) and its measurement equations, which are used extensively throughout the thesis, will be described.

A.6.1 Inertial Measurement Unit

The ship motion is assumed to be observed with a three degrees-of-freedom strap-down inertial measurement unit (IMU) and a schematic view of the IMU's position can be seen in Figure A.8. Firstly, the IMU measures the angular velocity

$$y_{1,t} = \dot{\phi}_t + e_{1,t} \quad (\text{A.41})$$

about the x_s -axis in the s -frame, at discrete times t , where $\dot{\phi}_t$ is the sampled angular velocity and $e_{1,t}$ is discrete-time white zero-mean measurement noise.

Secondly, the observed tangential acceleration at any point a fixed distance D from the CR due to the rotation about the x_s -axis in the s -frame is

$$a = D\ddot{\phi} \quad (\text{A.42})$$

The tangential acceleration sensed by the IMU is thus

$$a_s = z_s\ddot{\phi} + g\phi - a_y \quad (\text{A.43})$$

assuming that the roll angle ϕ is small, where the first term is the contribution from the angular acceleration, the second term is due to gravity and the third term is the acceleration of the CR in the y_n -axis in the n -frame. The parameter $-z_s$ is the distance from the CR to the origin of the IMU coordinate system.

Figure A.8: The 3DOF IMU is assumed to measure the tangential acceleration a_s , and the angular velocities about the x_s - and the z_s -axes. Note that the IMU's s -frame has a different orientation than the b -frame.

The IMU measures this tangential acceleration

$$y_{2,t} = a_{s,t} + e_{2,t} \quad (\text{A.44})$$

at discrete times t , where the first term is (A.43) after sampling and the second term is discrete-time white zero-mean measurement noise.

Finally, the IMU measures the angular velocity

$$y_{3,t} = -r_t + e_{3,t} \quad (\text{A.45})$$

about the z_s -axis in the s -frame, at discrete times t , where r_t is the sampled system's angular velocity and $e_{3,t}$ is discrete-time white zero-mean measurement noise.

The IMU that has been used in the experiments can be seen in Figure A.9. It is an Xsens MTi-G, equipped with a three-axis accelerometer, a three-axis gyro, a three-axis magnetometer and a GPS unit. Only a few of the axes will be used to collect data. The y -axis of the accelerometer corresponds to the tangential acceleration (A.44), the x -axis of the gyro corresponds to the angular velocity (A.41) while the z -axis of the gyro corresponds to the angular velocity (A.45).

Figure A.9: A Xsens MTi-G. The silver socket is the data-connection and the golden socket is the connection for the GPS antenna. Note the coordinate system (enhanced in software) on top of the IMU with the x -axis pointing inwards and left, the y -axis pointing downwards and left, and the z -axis pointing upwards.

B

Calculating the Mass and Center of Mass Through Ballasting

The identifiability analysis in Section 7.2.1 showed that it is only possible to estimate three-combinations of parameters given that the acceleration a_y of the center of rotation is persistently exciting. As a first step, a static estimation procedure can be used to estimate the mass M , the center of mass z_g and the spring coefficient k . These can be used together with IMU data from the nominal case to detect and estimate mass changes without the need to redo the static procedure.

In the harbor, or any other place where the dynamic pressure is not influencing the pressure sensors, the pressure sensors can be used to get an estimate of the draft and angle of the ship in relation to the waterline. These measurements can

Figure B.1: The mass and position of a ship in mechanical equilibrium before and after ballasting. Note that the second case only has to be another stationary angle ϕ which does not have to be zero.

be used in combination with the geometric shape of the hull and the ballast system to get an estimate of the mass and the CM. Assuming that the ballast system's tanks have a known weight and position, the mass and the CM can be found using the fact that the system is in mechanical equilibrium. Moment equilibrium about the CF of Figure B.1(a) gives

$$\bar{M}g\bar{H}\sin(\phi + \phi_\delta) - \bar{m}_2g\bar{h}_2\cos(\phi + \bar{\phi}_2) + \bar{m}_1g\bar{h}_1\cos(\bar{\phi}_1 - \phi) - \rho g \nabla \overline{GM}_T \sin(\phi) = 0 \quad (\text{B.1})$$

and secondly, force equilibrium in the world-fixed z -axis gives

$$(\bar{M} + \bar{m}_1 + \bar{m}_2)g - \rho g \nabla = 0 \quad (\text{B.2})$$

where the distance to the unknown CM is calculated using the y and z position, i.e. $\bar{H} = \sqrt{\bar{y}_g^2 + \bar{z}_g^2}$. However, there are three unknowns (y_g , z_g and $\bar{M}$) and two equations. Shifting the weight using the ballast system gives a new moment equilibrium and the needed third equation

$$\bar{M}g\bar{H}\sin(\phi_\delta) - m_2gh_2\cos(\phi_2) + m_1gh_1\cos(\phi_1) = 0 \quad (\text{B.3})$$

As an example, assume a box shaped ship and a small change in the angle due to ballasting. This gives

$$\overline{GM}_T = \frac{I_T}{\nabla} - \overline{BG} = \frac{I_T}{\nabla} - \sqrt{(H_{b,y} - y_g)^2 + (H_{b,z} - z_g)^2} \quad (\text{B.4})$$

where $I_T = \frac{1}{12}B^3L$, $\nabla = LBD$ and $\overline{BG}$ is the distance between center of buoyancy (known) and center of gravity (unknown) (Fossen, 2011; Perez, 2005). The CM can now be calculated by

$$y_g = \frac{S_{xz}}{M} = \frac{-\bar{M}\bar{H}\sin(\phi_\delta) - m_1h_1\cos(\phi_1) + m_2h_2\cos(\phi_2)}{\bar{M} + m_1 + m_2} \quad (\text{B.5})$$

and

$$z_g = \frac{S_{xy}}{M} = \frac{\bar{M}\bar{H}\cos(\phi_\delta) + m_1h_1\sin(\phi_1) + m_2h_2\sin(\phi_2)}{\bar{M} + m_1 + m_2} \quad (\text{B.6})$$

where S_{xy} and S_{xz} are the first moments of area. Equation (B.1) - (B.6) can be solved for $\bar{M}$, $\bar{x}_g$ and $\bar{z}_g$, for instance, by using some numerical method. The total mass is given by

$$M = \bar{M} + m_1 + m_2 \quad (\text{B.7})$$

the distance to the CM is

$$H = \sqrt{y_g^2 + z_g^2} \quad (\text{B.8})$$

and the spring coefficient is given by

$$k = \rho g \nabla \overline{GM}_T - MgH \quad (\text{B.9})$$

Bibliography

- K. Abed-Meraim, W. Qiu, and Y. Hua. Blind system identification. *Proceedings of the IEEE*, 85(8):1310–1322, Aug 1997. ISSN 0018-9219. doi: 10.1109/5.622507. Cited on pages 3 and 27.
- J. Albersmeyer and M. Diehl. The lifted Newton method and its application in optimization. *SIAM Journal on Optimization*, 20(3):1655–1684, 2010. Cited on page 126.
- K. Aljanaideh, A. Ali, M. Holzel, S. Kukreja, and D. Bernstein. Sensor-to-sensor identification of hammerstein systems. In *Proceedings of the 51st IEEE Conference on Decision & Control*, pages 2846–2851, Maui, Hawaii, USA, 2012. doi: 10.1109/CDC.2012.6426385. Cited on page 28.
- M. Anguelova. *Observability and identifiability of nonlinear systems with applications in biology*. Phd thesis, Chalmers University of Technology, Gothenburg, Sweden, 2007. Cited on page 23.
- K. Åström and B. Wittenmark. *Adaptive Control*. Dover Books on Electrical Engineering. Dover Publications, 2008. ISBN 9780486462783. Cited on page 2.
- L. Balcer. Location of ship rolling axis. *Polish Maritime Research*, pages 3–7, 2004. Cited on page 165.
- A. S. Bazanella, M. Gevers, and L. Miškovic. Closed-loop identification of MIMO systems: A new look at identifiability and experiment design. *European Journal of Control*, 16(3):228–239, 2010. ISSN 0947-3580. Cited on pages 22, 60, 93, and 122.
- R. Bellman and K. Åström. On structural identifiability. *Mathematical Biosciences*, 7(3–4):329 – 339, 1970. ISSN 0025-5564. doi: 10.1016/0025-5564(70)90132-X. Cited on pages 22 and 60.
- M. Blanke and A. C. Christensen. Rudder-roll damping autopilot robustness to sway-yaw-roll couplings. In *Proceedings of the 10th Ship Control Systems Symposium*, Ottawa, Canada, 1993. Cited on pages 170 and 174.

- A. Bryson and L. Henrikson. Estimation using sampled data containing sequentially correlated noise. *Journal of Spacecraft and Rockets*, 5(6):662–665, 1968. doi: 10.2514/3.29327. Cited on page 78.
- A. Bryson and D. Johansen. Linear filtering for time-varying systems using measurements containing colored noise. *IEEE Transactions on Automatic Control*, 10(1):4–10, Jan 1965. ISSN 0018-9286. doi: 10.1109/tac.1965.1098063. Cited on page 78.
- A. M. D’Amato, A. J. Brzezinski, M. S. Holzel, J. Ni, and D. S. Bernstein. Sensor-only noncausal blind identification of pseudo transfer functions. In *Proceedings of the 15th IFAC Symposium on System Identification*, pages 1698–1703, Saint-Malo, France, July 2009. Cited on pages 3, 28, 44, and 62.
- A. Dankers, P. M. J. V. den Hof, X. Bombois, and P. S. C. Heuberger. Predictor input selection for two stage identification in dynamic networks. In *2013 European Control Conference (ECC)*, pages 1422–1427, July 2013a. Cited on page 34.
- A. Dankers. *System Identification in Dynamic Networks*. Phd thesis, Delft University of Technology, The Netherlands, 2014. Cited on pages 12, 28, 34, 55, and 63.
- A. Dankers, P. Van den Hof, X. Bombois, and P. Heuberger. Identification of dynamic models in complex networks with prediction error methods: Predictor input selection. *IEEE Transactions on Automatic Control*, 61(4):937–952, April 2016. ISSN 0018-9286. doi: 10.1109/TAC.2015.2450895. Cited on pages 12, 33, and 34.
- A. G. Dankers, P. M. J. Van den Hof, and P. S. C. Heuberger. Predictor input selection for direct identification in dynamic networks. *52nd IEEE Conference on Decision and Control*, Dec 2013b. doi: 10.1109/cdc.2013.6760589. Cited on page 34.
- C. Devriendt and P. Guillaume. Identification of modal parameters from transmissibility measurements. *Journal of Sound and Vibration*, 314(1-2):343–356, Jul 2008. ISSN 0022-460X. doi: 10.1016/j.jsv.2007.12.022. Cited on page 44.
- S. Diop and M. Fliess. Nonlinear observability, identifiability, and persistent trajectories. In *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, England, 1991. doi: 10.1109/CDC.1991.261405. Cited on page 22.
- R. Diversi, R. Guidorzi, and U. Soverini. Kalman filtering in extended noise environments. *IEEE Transactions on Automatic Control*, 50(9):1396–1402, Sep 2005. ISSN 0018-9286. doi: 10.1109/tac.2005.854627. URL <http://dx.doi.org/10.1109/TAC.2005.854627>. Cited on pages 76 and 78.
- R. Diversi, R. Guidorzi, and U. Soverini. Algorithms for optimal errors-in-variables filtering. *Systems & Control Letters*, 48(1):1–13, Jan 2003. ISSN 0167-6911. doi: 10.1016/s0167-6911(02)00243-8. Cited on page 76.

- I. Drummen. *Experimental and Numerical Investigation of Nonlinear Wave-Induced Load Effects in Containerships considering Hydroelasticity*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2008. Cited on page 164.
- O. M. Faltinsen. *Sea loads on ships and offshore structures*. Cambridge University Press, Cambridge, 1990. ISBN 0521458706. Cited on pages 163 and 168.
- O. M. Faltinsen. *Hydrodynamics of High-Speed Marine Vehicles*. Cambridge University Press, 2005. ISBN 9780521845687. Cited on pages 5, 168, 171, and 173.
- U. Forssell. *Closed-loop identification: Methods, theory, and applications*. PhD thesis, Linköping University, Linköping, Sweden, 1999. Cited on pages 19 and 99.
- U. Forssell and C. Chou. Efficiency of prediction error and instrumental variable methods for closed-loop identification. In *Proceedings of the 37th IEEE Conference on Decision and Control*, pages 1287–1288 vol.2, Tampa, Florida, USA, 1998. Cited on pages 19 and 66.
- U. Forssell and L. Ljung. Closed-loop identification revisited. *Automatica*, 35(7): 1215 – 1241, 1999. ISSN 0005-1098. doi: 10.1016/S0005-1098(99)00022-9. Cited on pages 12 and 63.
- T. I. Fossen. *Nonlinear modelling and control of underwater vehicles*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 1991. Cited on pages 163 and 167.
- T. I. Fossen. *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley, 2011. ISBN 9781119991496. Cited on pages 4, 5, 144, 146, 163, 164, 166, 167, 170, 172, 173, and 180.
- T. Fujiwara and T. Haraguchi. Roll motion of Ro-Ro passenger ship with flooded vehicle deck. *International Journal of Offshore and Polar Engineering*, 15(2): 109–116, 2005. Cited on page 3.
- P. Funk, E. Olsson, M. Bengtsson, and N. Xiong. Case-based experience reuse and agents for efficient health monitoring, preventive and corrective actions. In R. B. R. Uday Kumar, Aditya Parida, editor, *Proceedings of the 19th Congress on Condition Monitoring and Diagnostic Engineering Management*, pages 445–453. Division of Operation and Maintenance Engineering, Luleå University of Technology, June 2006. Cited on page 3.
- H. Garnier and L. Wang. *Identification of Continuous-time Models from Sampled Data*. Advances in Industrial Control. Springer, 2008. ISBN 9781848001619. Cited on page 62.
- H. Garnier and P. C. Young. The advantages of directly identifying continuous-time transfer function models in practical applications. *International Journal of Control*, 87(7):1319–1338, 2014. doi: 10.1080/00207179.2013.840053. Cited on page 30.

- M. Gevers, A. Bazanella, X. Bombois, and L. Miskovic. Identification and the information matrix: How to get just sufficiently rich? *IEEE Transactions on Automatic Control*, 54(12):2828–2840, Dec 2009. ISSN 0018-9286. doi: 10.1109/TAC.2009.2034199. Cited on page 122.
- M. Gevers and A. S. Bazanella. Identification in dynamic networks: Identifiability and experiment design issues. In *Proceedings of the 54th IEEE Conference on Decision and Control (CDC)*, Osaka, Japan, December 2015. doi: 10.1109/cdc.2015.7402842. Cited on page 22.
- M. Gilson and P. Van den Hof. Instrumental variable methods for closed-loop system identification. *Automatica*, 41(2):241–249, 2005. Cited on pages 19 and 66.
- M. Gilson, H. Garnier, P. J. Young, P. Van den Hof, et al. A refined IV method for closed-loop system identification. In *Proceedings of the 14th IFAC Symposium on System Identification*, pages 903–908, Newcastle, Australia, 2006. Cited on page 128.
- M. Gilson, H. Garnier, P. C. Young, and P. M. Van den Hof. Optimal instrumental variable method for closed-loop identification. *IET Control Theory & Applications*, 5(10):1147–1154, 2011. Cited on pages 19, 20, 66, and 67.
- G. Golub and C. Van Loan. *Matrix Computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, 1996. ISBN 9780801854149. Cited on pages 71 and 98.
- R. Guidorzi, R. Diversi, and U. Soverini. Optimal errors-in-variables filtering. *Automatica*, 39(2):281–289, Feb 2003. ISSN 0005-1098. doi: 10.1016/S0005-1098(02)00200-5. Cited on page 76.
- M. Hautus. Strong detectability and observers. *Linear Algebra and its Applications*, 50:353 – 368, 1983. ISSN 0024-3795. doi: 10.1016/0024-3795(83)90061-7. Cited on page 45.
- R. Hermann and A. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, 1977. Cited on page 23.
- M. Hou and P. Müller. Disturbance decoupled observer design: a unified viewpoint. *Automatic Control, IEEE Transactions on*, 39(6):1338–1341, Jun 1994. ISSN 0018-9286. doi: 10.1109/9.293209. Cited on pages 44 and 45.
- M. Hou and R. Patton. Optimal filtering for systems with unknown inputs. *Automatic Control, IEEE Transactions on*, 43(3):445–449, Mar 1998. ISSN 0018-9286. doi: 10.1109/9.661621. Cited on pages 44 and 45.
- T. Iseki and D. Terada. Bayesian estimation of directional wave spectra for ship guidance system. In *Proceedings of the International Offshore and Polar Engineering Conference*, volume 4, pages 577–582, 2001. Cited on page 5.
- J. M. J. Journée and W. W. Massie. Offshore Hydromechanics, 2001. Lecture notes

- on offshore hydromechanics for Offshore Technology students, code OT4620. Cited on pages 140, 163, and 169.
- Y. Jung and M. Enqvist. Estimating models of inverse systems. In *Proceedings of the 52nd IEEE Conference on Decision & Control*, Firenze, Italy, 2013. doi: 10.1109/CDC.2013.6761022. Cited on pages 61 and 78.
- T. Kailath. *Linear Systems*. Information and System Sciences Series. Prentice-Hall, 1980. ISBN 9780135369616. Cited on page 69.
- C. Källström, K. Åström, N. Thorell, J. Eriksson, and L. Sten. Adaptive autopilots for tankers. *Automatica*, 15(3):241 – 254, 1979. ISSN 0005-1098. doi: 10.1016/0005-1098(79)90042-6. Cited on page 13.
- P. Lancaster and M. Tismenetsky. *The Theory of Matrices: With Applications*. Computer science and applied mathematics. Academic Press, 1985. ISBN 9780124355606. Cited on page 51.
- B. Lautrup. *Physics of Continuous Matter: Exotic and Everyday Phenomena in the Macroscopic World*. Taylor & Francis, 2004. ISBN 9780750307529. Cited on pages 168 and 169.
- J. Linder. *Graybox Modelling of Ships Using Indirect Input Measurements*. Linköping Studies in Science and Technology. Thesis 1681. Linköping University, Sweden, 2014. Not cited.
- J. Linder and M. Enqvist. On Indirect Input Measurements. In *Proceedings of the 17th IFAC Symposium on System Identification*, Beijing, China, October 2015. Cited on page 51.
- J. Linder and M. Enqvist. Identification of systems with unknown inputs using indirect input measurements. *International Journal of Control*, 2016a. doi: 10.1080/00207179.2016.1222557. Cited on page 100.
- J. Linder and M. Enqvist. Identification and Prediction in Dynamic Networks with Unobservable Nodes. Technical Report LiTH-ISY-R-3095, Department of Electrical Engineering, Linköping University, Linköping, 2016b. Not cited.
- J. Linder and S. Lindkvist. Interactive multiobjective optimization with application to hot rolling mills. Master’s thesis, Chalmers tekniska högskola, Gothenburg, Sweden, 2011. Cited on page 7.
- J. Linder, M. Enqvist, and F. Gustafsson. A closed-loop instrumental variable approach to mass and center of mass estimation using IMU data. In *Proceedings of the 53rd IEEE Conference on Decision & Control*, Los Angeles, CA, USA, December 2014a. Cited on page 100.
- J. Linder, M. Enqvist, F. Gustafsson, and J. Sjöberg. Identifiability of physical parameters in systems with limited sensors. In *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, August 2014b. Cited on page 100.

- J. Linder, M. Enqvist, T. I. Fossen, T. A. Johansen, and F. Gustafsson. Online Estimation of Ship's Mass and Center of Mass Using Inertial Measurements. In *Proceedings of the 10th IFAC Conference on Manoeuvring and Control of Marine Craft (MCMC)*, Copenhagen, Denmark, August 2015a. Not cited.
- J. Linder, M. Enqvist, T. I. Fossen, T. A. Johansen, and F. Gustafsson. Modeling for IMU-based Online Estimation of Ship's Mass and Center of Mass. In *Proceedings of the 10th IFAC Conference on Manoeuvring and Control of Marine Craft (MCMC)*, Copenhagen, Denmark, August 2015b. Not cited.
- J. Linder, M. Enqvist, F. Gustafsson, and J. Sjöberg. Righting arm model determination, February 18 2016. WO Patent App. PCT/EP2014/067,460. Not cited.
- L. Ljung. *System Identification: Theory for the User (2nd Edition)*. Prentice Hall, 1999. ISBN 0136566952. Cited on pages 2, 3, 9, 15, 17, 18, 19, 21, 60, 62, 73, 90, 93, 95, and 99.
- L. Ljung and T. Glad. On global identifiability for arbitrary model parametrizations. *Automatica*, 30(2):265 – 276, 1994. ISSN 0005-1098. doi: 10.1016/0005-1098(94)90029-9. Cited on pages 22 and 86.
- K. Mahata and H. Garnier. Identification of continuous-time errors-in-variables models. *Automatica*, 42(9):1477 – 1490, 2006. ISSN 0005-1098. doi: 10.1016/j.automatica.2006.04.012. Cited on page 62.
- N. Maia, J. Silva, and A. Ribeiro. The transmissibility concept in multi-degree-of-freedom systems. *Mechanical Systems and Signal Processing*, 15(1):129–137, Jan 2001. ISSN 0888-3270. doi: 10.1006/mssp.2000.1356. Cited on page 44.
- I. Markovsky and B. De Moor. Linear dynamic filtering with noisy input and output. *Automatica*, 41(1):167–171, Jan 2005. ISSN 0005-1098. doi: 10.1016/j.automatica.2004.08.014. Cited on page 76.
- J. Newman. *Marine Hydrodynamics*. Wei Cheng Cultural Enterprise Company, 1977. ISBN 9780262140263. Cited on page 163.
- U. D. Nielsen. Response-based estimation of sea state parameters – influence of filtering. *Ocean Engineering*, 34(13):1797 – 1810, 2007. ISSN 0029-8018. doi: 10.1016/j.oceaneng.2007.03.002. Cited on page 5.
- U. D. Nielsen. The wave buoy analogy - estimating high-frequency wave excitations. *Applied Ocean Research*, 30(2):100–106, 2008. Cited on page 5.
- U. D. Nielsen and T. Iseki. The wave buoy analogy: Analysis of synthetic data by bayesian modelling. *ASME Conference Proceedings*, 2010(49125):301–310, 2010. doi: 10.1115/OMAE2010-20530. Cited on page 5.
- U. D. Nielsen and T. Iseki. A study on parametric wave estimation based on measured ship motions. *Nihon Kokai Gakkai Ronbunshu*, 2011. ISSN 0388-7405. Cited on page 5.

- R. Penrose. A generalized inverse for matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51:406–413, 7 1955. ISSN 1469-8064. doi: 10.1017/S0305004100030401. Cited on page 45.
- T. Perez. *Ship Motion Control: Course Keeping and Roll Stabilisation Using Rudder and Fins*. Springer-Verlag London Limited, 2005. ISBN 9781846281570. Cited on pages 5, 163, 167, 170, 171, 173, 174, 175, 176, and 180.
- C. Phillips, J. Parr, and E. Riskin. *Signals, Systems, and Transforms*. Pearson/Prentice Hall, 2008. ISBN 9780131989238. Cited on page 10.
- R. Pintelon and J. Schoukens. *System Identification: A Frequency Domain Approach*. Wiley, 2012. ISBN 9781118287392. Cited on page 9.
- A. Ross. *Nonlinear Manoeuvring Models for Ships: A Lagrangian Approach*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2008. Cited on page 170.
- J. Sanchez, W. A. Rogers, A. D. Fisk, and E. Rovira. Understanding reliance on automation: effects of error type, error distribution, age and experience. *Theoretical Issues in Ergonomics Science*, 15(2):134–160, Sep 2011. ISSN 1464-536X. doi: 10.1080/1463922x.2011.611269. Cited on page 3.
- J. Schoukens and R. Pintelon. *Identification of Linear Systems: A Practical Guideline to Accurate Modeling*. Pergamon Press, Inc., Elmsford, NY, USA, 1991. ISBN 0-08-040734-X. Cited on page 9.
- R. Shumway and D. Stoffer. *Time Series Analysis and Its Applications: With R Examples*. Springer Texts in Statistics. Springer, 2010. ISBN 9781441978653. Cited on pages 2 and 27.
- J. Sjöberg, S. Lindkvist, J. Linder, and A. Daneryd. Interactive multiobjective optimization for the hot rolling process. In *Proceedings of the 51st IEEE Conference on Decision & Control*, Maui, Hawaii, USA, December 2012. Not cited.
- J. Sjöberg, S. Lindkvist, J. Linder, and J. Öhr. Interactive multiobjective optimization for a grab-shift unloader crane. In *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, August 2014. Not cited.
- S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*. Wiley, 2005. ISBN 9780470011676. Cited on page 3.
- T. Söderström. System identification for the errors-in-variables problem. *Transactions of the Institute of Measurement and Control*, 34(7):780–792, Sep 2011. ISSN 1477-0369. doi: 10.1177/0142331211414616. Cited on pages 40, 43, 62, and 67.
- T. Söderström and P. Stoica. *Instrumental Variable Methods for System Identification*. Lecture Notes in Control and Information Sciences. Springer-Verlag, 1983. ISBN 9783540128144. Cited on page 19.

- T. Söderström. Identification of stochastic linear systems in presence of input noise. *Automatica*, 17(5):713 – 725, 1981. ISSN 0005-1098. doi: 10.1016/0005-1098(81)90018-2. Cited on page 90.
- T. Söderström. Errors-in-variables methods in system identification. *Automatica*, 43(6):939 – 958, 2007. ISSN 0005-1098. doi: 10.1016/j.automatica.2006.11.025. Cited on pages 3, 28, 40, 43, and 62.
- T. Söderström and P. Stoica. *System Identification*. Prentice Hall, 1989. ISBN 9780138812362. Cited on pages 9, 19, 67, 93, and 102.
- T. Söderström, L. Wang, R. Pintelon, and J. Schoukens. Can errors-in-variables systems be identified from closed-loop experiments? *Automatica*, 49(2):681–684, 2013. Cited on pages 28 and 63.
- M. Spong and S. Hutchinson. *Robot Modeling and Control*. Wiley, 2005. ISBN 9780471649908. Cited on pages 164 and 167.
- A. Tangirala. *Principles of System Identification: Theory and Practice*. CRC Press, 2014. ISBN 9781439896020. Cited on page 108.
- E. Tannuri, J. Sparano, A. Simos, and J. Da Cruz. Estimating directional wave spectrum based on stationary ship motion measurements. *Applied Ocean Research*, 25(5):243–261, 2003. Cited on pages 4 and 5.
- S. Thornton and J. Marion. *Classical Dynamics of Particles and Systems*. Brooks/Cole, Cengage Learning, 2004. ISBN 9780534408961. Cited on pages 164 and 167.
- M. Thys. *Theoretical and experimental investigation of a free running fishing vessel at small frequency of encounter*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2013. Cited on page 141.
- C. Tzeng, G. Goodwin, and S. Crisafulli. Feedback linearization design of a ship steering autopilot with saturating and slew rate limiting actuator. *International Journal of Adaptive Control and Signal Processing*, 13(1):23–30, 1999. Cited on page 13.
- P. M. Van Den Hof and R. J. Schrama. An indirect method for transfer function estimation from closed loop data. *Automatica*, 29(6):1523–1527, Nov 1993. ISSN 0005-1098. doi: 10.1016/0005-1098(93)90015-L. Cited on pages 39 and 43.
- P. M. Van den Hof, D. K. De Vries, and P. Schoen. Delay structure conditions for identifiability of closed loop systems. *Automatica*, 28(5):1047–1050, Sep 1992. ISSN 0005-1098. doi: 10.1016/0005-1098(92)90161-8. Cited on page 55.
- P. Van den Hof, A. Dankers, P. Heuberger, and X. Bombois. Identification of dynamic models in complex networks with prediction error methods, basic methods for consistent module estimates. *Automatica*, 49(10):2994 – 3006, 2013. ISSN 0005-1098. doi: 10.1016/j.automatica.2013.07.011. Cited on pages 12 and 32.

- É. Walter. *Identifiability of State Space Models: With Applications to Transformation Systems*. Lecture Notes in Biomathematics. Springer-Verlag Berlin, 1982. ISBN 9783540115908. Cited on page 22.
- H. Weerts, A. Dankers, and P. Van den Hof. Identifiability in dynamic network identification. In *Proceedings of the 17th IFAC Symposium on System Identification*, volume 48, pages 1409 – 1414, Beijing, China, October 2015. doi: 10.1016/j.ifacol.2015.12.330. Cited on page 60.

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