An infinitesimal characterization of nonlinear contracting interference functions

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An infinitesimal characterization of nonlinear contracting interference functions

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***Abstract*— Contractive interference functions are a subclass of the standard interference functions used in the design and analysis of distributed power control algorithms for wireless networks. Their peculiarity is that for the resulting positive system the existence and global asymptotic stability of a unique positive equilibrium point is guaranteed. In this paper we give an infinitesimal characterization of nonlinear contractive interference functions in terms of the spectral radius of the Jacobian linearization at any point in the positive orthant. The condition we obtain, that the spectral radius is always less than 1, extends to the nonlinear case an equivalent property of linear interference functions, and leads to a Jacobian characterization similar to the one commonly used in contraction analysis of nonlinear systems.**

I. Introduction

The field of power control of wireless networks is concerned with the design of closed-loop systems achieving certain quality-of-service requirements while minimizing the power consumption of the mobile equipment of the end-users [3], [4]. Although this problem, which was a major issue on CDMA (Code-Division Multiple Access)-based standards such as 3G, is now bypassed by orthogonal frequency allocation schemes, it remains an interesting case study for control theoreticians, as a model for solving in a distributed way a class of problems prosaically referred to as "cocktail-party conversation" scenarios, i.e., interactions among agents enabling them to communicate with other agents while beating the interference created by the communications of the other agents (and keeping the volume as low as possible), see [4], Ch. 1. Such a setting has generated a considerable amount of control-oriented research in recent years [3], [19], [5], [13], [14], [7], [22]. Given that power is a nonnegative variable, the right context in which to place this literature is distributed modeling and control of positive systems. In particular, an interesting question is to understand what kind of interference functions (describing the interference induced on the receiver of a user by the transmissions of the other users) leads naturally to simple, linear or nonlinear schemes converging to a positive equilibrium point. A class of such functions, called standard interference functions, was identified in [9], [21]. In the linear case, provided that a solution exists, a standard interference function can solve the problem in an optimal way in both the continuous-time and the discrete-time case. Linear standard interference functions have a rich mathematical structure, based essentially on the Perron-Frobenius theorem [3], [16]. In particular, existence of a solution can be expressed as the spectral radius of the interference function being less than 1, condition which can also be associated to contractivity of the resulting system towards the equilibrium point. For nonlinear standard interference functions, however, such simple characterization of existence of an equilibrium point is not readily available, and it must be checked on a case-by-case basis.

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To avoid issues connected with existence, in [8], [7] a novel class of interference functions in introduced, denoted contractive interference functions. For them, existence of a single positive equilibrium point is guaranteed and so is convergence to it in the entire positive orthant. The notion of contraction used in [8], [7] is inspired by [2], and it is for instance not clear how it relates to other notions of contractivity of widespread use in nonlinear dynamics, such as [12].

The aim of this paper is to give an infinitesimal characterization of the class of nonlinear contractive interference functions, in terms of the linear tangent system at any point in $\mathbb{R}_+^n$. In particular, we show that the Jacobian has to have a spectral radius less than 1 at any point in $\mathbb{R}_+^n$. Such an infinitesimal condition can be easier to verify that those required in the original formulation of [8], [7]. In both continuous-time and discrete-time, this allows to conclude that the dynamical system corresponding to a contractive interference function is contractive also in the sense of [12], thereby reconciliating two of the notions of contractivity normally used in control theory.

As another consequence of our characterization, we show that (positive) concave increasing functions belong to the class of contractive interference functions, provided they satisfy an easily verifiable condition in the origin. This type of functions is related to our recent work [20] but also to a large body of literature on cooperative systems [18], [17], [11] never mentioned in the context of power control algorithms. This widens considerably the range of known positive systems for which the contractivity property (with its guaranteed existence and uniqueness of the attractor) can be used, especially for applications beyond wireless networks.

The rest of the paper is organized as follows. After recalling some background material in Section II, in Section III the various classes of interference functions are described in detail. In Section IV our infinitesimal characterization of contractive interference functions is given. It is then used in Section V to provide conditions under which interconnections of concave functions form contractive interference functions.

II. Notation and background material

A matrix $A \in \mathbb{R}^{n\times n}$ is said Hurwitz stable if all its eigenvalues $\lambda_i(A)$, $i = 1, \dots, n$, have $\mathrm{Re}[\lambda_i(A)] < 0$. A is nonnegative (hereafter $A \geqslant 0$) if all its entries a_{ij} are nonnegative. It is a Metzler matrix if $a_{ij} \geqslant 0 \ \forall\, i \neq j$. A is said irreducible if there does not exist a permutation matrix Π such that $\Pi^T A \Pi$ is block triangular. The spectral radius of A, denoted $\rho(A)$, is the smallest real positive number such that $\rho(A) \geqslant |\lambda_i(A)|$, $i = 1, \dots, n$.

Theorem 1 *(Perron-Frobenius theorem, see [1]) Let $A \geqslant 0$ be irreducible. Then $\rho(A)$ is an eigenvalue of A of multiplicity 1 and the corresponding eigenvector v is positive.*

Given the vector $v > 0$, let $\|\cdot\|_\infty^v$ denote the weighted l_∞ norm: $\|x\|_\infty^v = \max_i \frac{|x_i|}{v_i}$. Let $\|\cdot\|_\infty^v$ denote also the corresponding induced matrix norm: $\|A\|_\infty^v = \max_{x\neq 0} \frac{\|Ax\|_\infty^v}{\|x\|_\infty^v}$.

Proposition 1 *([2], Prop. 2.6.6.) Let $A \geqslant 0$. For every $\epsilon > 0$ $\exists\, v > 0$ such that $\rho(A) \leqslant \|A\|_\infty^v \leqslant \rho(A) + \epsilon$.*

Recall also the submultiplicativity property of any matrix norm:

$$\|AB\| \leq \|A\|\|B\| \qquad \forall\, A,\, B \in \mathbb{R}^{n\times n}. \tag{1}$$

Given $W \subset \mathbb{R}^n$, a mapping $T : W \to W$ is called a *c-contraction* (i.e., a contraction of modulus c) if $\exists\, c \in [0,\, 1)$ such that

$$\|T(x_1) - T(x_2)\| \leq c\|x_1 - x_2\| \qquad \forall\, x_1,\ x_2 \in W \tag{2}$$

where $\|\cdot\|$ is some norm, see [2], Chapter 3.

Proposition 2 *([2], Prop. 3.1.1.) If W is closed and the mapping $T : W \to W$ is c-contractive then T has a unique fixed point $x^* \in W$.*

If the map is linear and the norm is the weighted l_∞ norm, then we have the following.

Proposition 3 *([2], Prop. 2.6.6 and Cor. 2.6.1) Consider $A \geqslant 0$. Then $\exists\ v > 0$ such that $\|A\|_\infty^v < 1$ if and only if $\rho(A) < 1$. Furthermore, if A irreducible and $v > 0$ is the Perron-Frobenius eigenvector, then $\rho(A) = \|A\|_\infty^v$.*

In [12], a variational approach to contraction is proposed, based on the analysis of the linear tangent system at any point. Given a nonlinear differentiable autonomous system

$$\dot{x} = f(x) \tag{3}$$

where $f : W \to W$, $W \subset \mathbb{R}^n$, if $F(x) = \frac{\partial f}{\partial x}(x)$ is the Jacobian linearization at any x (not necessarily an equilibrium point) and $\delta x = \delta x(t)$ is the infinitesimal displacement between neighbouring trajectories at a certain time instant t, then for δx one can define the linear tangent system

$$\frac{d}{dt}\delta x = F(x)\delta x.$$

Consequently, the infinitesimal Euclidean distance between two neighbouring trajectories $\|\delta x\| = \delta x^T \delta x$ obeys the following ODE

$$\frac{d}{dt}\|\delta x\|^2 = \delta x^T (F(x) + F(x)^T)\delta x.$$

The norm $\|\delta x\|$ contracts if $F(x) + F(x)^T$ is uniformly negative definite over the entire W. In [12] this notion is generalized to include more general Riemannian metrics. For us a trivial constant "Riemannian" metric equal to the identity matrix is however enough on the entire tangent bundle $\mathcal{T}X$ (which below will be identified with $\mathbb{R}_+^n$, as also $W = \mathbb{R}_+^n$), meaning that simple convergence arguments such as the Krasovskii method [10] are enough. See also [15] for related material. Contracting metrics lead to the following convergence result.

Theorem 2 *Consider the system* (3)*. If the norm $\|\delta x\|$ contracts over the entire W and an equilibrium point exists in W, then it is unique and all trajectories converge to it.*

An analogous concept holds for the discrete-time system

$$x(t+1) = f(x(t)). \tag{4}$$

Infinitesimal contraction for (4) corresponds to $F(x)^T F(x) - I$ being uniformely negative definite over the entire W, see [12] for the details.

For nonlinear functions we will use the following Jacobian characterization of irreducibility. A function $f : W \to \mathbb{R}_+^n$ is said *irreducible* if its Jacobian matrix $F(x) = \frac{\partial f}{\partial x}(x)$ is irreducible everywhere in W.

A function $f : W \to \mathbb{R}^n$ is said non-decreasing in W if $x_1 \leqslant x_2$ implies $f(x_1) \leqslant f(x_2)\ \forall\, x_1,\, x_2 \in W$. It is said increasing if in addition $x_1 < x_2$ implies $f(x_1) < f(x_2)$. $f : W \to \mathbb{R}^n$ is said *concave* if

$$f(\alpha x_1 + (1-\alpha)x_2) \geqslant \alpha f(x_1) + (1-\alpha) f(x_2) \tag{5}$$

$\forall\, x_1,\, x_2 \in W$ and $\forall\, 0 \leqslant \alpha \leqslant 1$. It is said *strictly concave* if the inequality in (5) is strict in $0 < \alpha < 1$ $\forall\, x_1,\, x_2 \in W$, $x_{1,i} \neq x_{2,i}$, $i = 1, \dots, n$. Clearly, f strictly concave and non-decreasing means f increasing.

III. Standard and contractive interference functions

Definition 1 *A function $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is called a* standard interference function *if for all $x \in \mathbb{R}_+^n$ the following properties are satisfied:*

1) *(positivity) $\phi(x) > 0$;*
2) *(increasing) If $x_1 \leqslant x_2$, then $\phi(x_1) \leqslant \phi(x_2)$;*
3) *(scalability) $\forall\, \alpha > 1$, $\alpha\phi(x) > \phi(\alpha x)$.*

Several classes of standard interference functions are given in the literature on power control of wireless networks, see

[21]. A classical example is the linear interference function $\phi^\ell(x)$ of components

$$\phi_i^\ell(x) = \beta_i \frac{\sum_{j\neq i} g_{ij} x_j + \xi_i}{g_{ii}}, \qquad i = 1, \dots, n \qquad (6)$$

where β_i is the desired Signal-to-Interference-and-Noise Ratio (SINR), $g_{ij} \in \mathbb{R}_+$ is the channel gain between user j and the receiver of user i (here associated with the same index) and ξ_i is the background noise at the receiver of user i.

Standard interference functions $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ such as (6) are used to construct iterative schemes converging to a positive equilibrium point [9], [21]. For instance in discrete-time, the iteration is given by

$$x(t+1) = \phi(x(t)). \qquad (7)$$

By construction $x(0) \geq 0 \implies x(t) \geq 0$ i.e., (7) is a positive system. If $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is an irreducible standard interference function and it has a fixed point x^* (i.e., $\exists\, x^*$ such that $x^* = \phi(x^*)$), then it is shown in [21] that this fixed point is unique and the iteration (7) converges to x^* for any $x(0) > 0$.

In continuous-time, the system one considers is

$$\dot{x} = K(-x + \phi(x)) \qquad (8)$$

where $K = \mathrm{diag}(k_1, \dots, k_n)$, $k_i > 0$. Also the system (8) is a positive system by construction. If $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is an irreducible standard interference function and it has a fixed point x^* (i.e., also here $\exists\, x^*$ such that $x^* = \phi(x^*)$), then x^* must be the unique equilibrium point and it must be a global attractor (in $\mathbb{R}_+^n$).

In both discrete-time and continous-time, existence of the fixed point x^* is however not guaranteed, see [8], [7] for counterexamples. It is the lack of a guaranteed existence of a fixed point that has motivated the authors of [8], [7] to seek a more stringent class of interference functions than those given by Definition 1.

Definition 2 *A function $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is said to be a* c-contractive interference *function if the following properties are satisfied $\forall\, x \in \mathbb{R}_+^n$:*

1) *(positivity) $\phi(x) > 0$;*
2) *(increasing) If $x_1 \leqslant x_2$, then $\phi(x_1) \leqslant \phi(x_2)$;*
3) *(contractivity) $\exists$ a constant $c \in [0, 1)$ and a vector $\nu > 0$ such that $\forall\, \epsilon > 0$*

$$\phi(x + \epsilon\nu) \leqslant \phi(x) + c\epsilon\nu. \qquad (9)$$

Contractive interference functions define contraction mappings in the weighted l_∞ norm. From Proposition 2, $\phi(x^*) = x^*$ always admits a unique solution in $int(\mathbb{R}_+^n)$. Hence (2) in the weighted l_∞ norm becomes

$$\|\phi(x_1) - \phi(x_2))\|_\infty^v \leqslant c\|x_1 - x_2\|_\infty^v \qquad \forall\, x_1,\, x_2 \in \mathbb{R}_+^n.$$

In [8], [7] stability of x^* and a characterization of the rate of convergence are shown in both discrete-time and continuous-time.

Theorem 3 *([8], Thm. 1 and [7], Thm. 3) If an interference function $\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is c-contractive and irreducible, then for any initial condition $x(0) > 0$,*

1) *The discrete-time system* (7) *is asymptotically stable and converges to x^* as*

$$\|x(p) - x^*\|_\infty^v \leq c^p \|x(0) - x^*\|_\infty^v.$$

2) *The continuous-time system* (8) *is asymptotically stable for any $k_i > 0$. The solution $x(t)$ of* (8) *satisfies*

$$\|x(t) - x^*\|_\infty^v \leq \|x(0) - x^*\|_\infty^v e^{-k_{min}(1-c)t}, \ \ t \geqslant 0.$$

In [8] it is shown that many practical interference functions and all the examples in [21] are c-contractive. In particular, the linear interference function (6) is c-contractive.

IV. A SPECTRAL CONDITION FOR NONLINEAR CONTRACTIVITY

If we consider a system like (7) or (8), then contractive interference functions guarantee existence, uniqueness and asymptotic stability of a positive equilibrium point [8], [7].

Let us consider the linear interference function (6). It can be rewritten in matrix form as

$$\phi^\ell(x) = Mx + \mu$$

where $M = (m_{ij})$ has entries

$$m_{ij} = \begin{cases} \beta_i \frac{g_{ij}}{g_{ii}} & \text{fi } j \neq i \\ 0 & \text{if } j = i \end{cases}$$

and $\mu_i = \beta_i \frac{\xi_i}{g_{ii}}$. Since $\phi^\ell(x)$ is c-contracting, combining Proposition 3 and Theorem 3, $\exists\, v > 0$ such that $\|M\|_\infty^v < 1$, and hence $\rho(M) < 1$. If in addition M (i.e., $G = (g_{ij})$) is also irreducible then it is also $\rho(M) = \|M\|_\infty^v < 1$ with v the Perron-Frobenius eigenvector.

However, when $\phi(x) \ : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is nonlinear, a characterization as the one given in Proposition 3 is no longer available. The following theorem shows that for (differentiable) contractive interference functions an analogous characterization can be obtained in terms of the spectral radius of the Jacobian $\Phi(x) = \frac{\partial\phi(x)}{\partial x}$.

Theorem 4 *If $\phi(x) \ : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a C^1 c-contractive interference function, then*

$$\rho\left(\Phi(x)\right) < 1 \qquad \forall x \in \mathrm{int}(\mathbb{R}_+^n).$$

Proof: From the second condition of Definition 2, we have that the following holds

$$\Phi(x) = \begin{pmatrix} \nabla\phi_1(x) \\ \vdots \\ \nabla\phi_n(x) \end{pmatrix} \geqslant 0.$$

Given that in the contractivity condition (9), $\epsilon > 0$ can be chosen arbitrarily, we can make the choice $\epsilon = \gamma/\|\nu\|$, $\gamma > 0$. For every $\gamma > 0$ and with our choice of ϵ, the inequality in (9), which in components reads

$$\phi_i(x + \epsilon\nu) \leqslant \phi_i(x) + c\epsilon\nu_i,$$

becomes

$$\phi_i\left(x+\gamma\frac{\nu}{\|\nu\|}\right)\leqslant\phi_i(x)+c\gamma\frac{\nu_i}{\|\nu\|}.$$

Rearranging the inequality using the unit vector $u=\nu/\|\nu\|$ (note that $u>0$ since $\nu>0$ by assumption), we have

$$\frac{\phi_i(x+\gamma u)-\phi_i(x)}{\gamma}\leqslant cu_i.$$

From the differentiability of ϕ_i in x and taking the limit for $\gamma\to 0$ we obtain the following

$$\lim_{\gamma\to 0}\frac{\phi_i(x+\gamma u)-\phi_i(x)}{\gamma}=\nabla\phi_i(x)\cdot u,$$

which is the directional derivative along u. The contractivity condition is then

$$\nabla\phi_i(x)\cdot u\leqslant cu_i.$$

The previous inequality holds for all $i=1,\dots,n$, so it yields

$$\Phi(x)u\leqslant cu. \tag{10}$$

Since the Jacobian matrix $\Phi(x)$ is non-negative $\forall x\in\mathbb{R}^n_+$, it must be

$$\rho\left(\Phi(x)\right)\geqslant 0$$

and there must exist a non-zero left eigenvector $w\in\mathbb{R}^n_+$, $w\neq 0$ such that

$$w^T\Phi(x)=\rho\left(\Phi(x)\right)w^T.$$

From this last equation, together with the inequality in (10), we have

$$\rho\left(\Phi(x)\right)w^Tu\leqslant cw^Tu.$$

Since $u>0$ and $w^T\geqslant 0, w\neq 0$, we have that w^Tu is a real positive value. Furthermore c is a positive constant less that 1. The inequality is then

$$\rho\left(\Phi(x)\right)<1$$

which concludes the proof. ∎

For increasing functions which are irreducible, a converse result is also true.

Theorem 5 *If $\phi(x)\ :\mathbb{R}^n_+\to\mathbb{R}^n_+$ is a C^1 increasing function such that the following hold $\forall\, x\in\mathbb{R}^n_+$:*

1) $\phi(x)>0$,
2) $\rho\left(\Phi(x)\right)\leqslant\zeta_0<1$,
3) $\phi(x)$ irreducible

then $\phi(x)$ is a c-contractive interference function.

Proof: Consider Taylor's approximation

$$\phi(x+\epsilon\nu)=\phi(x)+\Phi(x)\epsilon\nu+\eta(\epsilon\nu).$$

Now let $\nu=\gamma v_0$ with $\gamma>0$ and v_0 is the right Perron-Frobenius eigenvector of $\Phi(x)$ corresponding to the eigenvalue $\rho\left(\Phi(x)\right)$. The last equation becomes

$$\phi(x+\epsilon\nu)=\phi(x)+\epsilon\rho\left(\Phi(x)\right)\gamma v_0+\eta\left(\epsilon\gamma v_0\right). \tag{11}$$

From Definition 2, ϕ is c-contractive if for some $c\in[0,1)$ (9) holds. If we define c as

$$c=\zeta_0+\frac{1-\zeta_0}{2} \tag{12}$$

for some ζ_0 such that

$$\rho\left(\Phi(x)\right)\leqslant\zeta_0<1, \tag{13}$$

from (11) we can guarantee the contractivity condition if

$$\epsilon\rho\left(\Phi(x)\right)\gamma v_0+\eta\left(\epsilon\gamma v_0\right)\leqslant\left(\zeta_0+\frac{1-\zeta_0}{2}\right)\epsilon\gamma v_0. \tag{14}$$

Clearly $1>c>\zeta_0$. Inequality (14) is then

$$\gamma\left(\left(c-\rho\left(\Phi(x)\right)\right)\epsilon v_0+\frac{\eta\left(\epsilon\gamma v_0\right)}{\gamma}\right)\geqslant 0. \tag{15}$$

By assumption for $\rho\left(\Phi(x)\right)$ we have that $c-\rho\left(\Phi(x)\right)>0$ is a positive scalar for all $x\in\mathbb{R}^n_+$. From Taylor's approximation $\eta\to 0$ if $\gamma\to 0$, and inequality (15) is satisfied if $v_0>0$, that is if $\Phi(x)$ is irreducible. In conclusion, under conditions (13) and $\Phi(x)$ irreducible for all $x\in\mathbb{R}^n_+$, ϕ is contractive, i.e. (9) holds with $\nu=\gamma v_0$, for an appropriate $\gamma>0$ and c defined as is (12). ∎

Combining Proposition 2 and Theorem 3 we have the following spectral characterization for existence, uniqueness and stability of a positive equilibrium point.

Corollary 1 *Consider $\phi\ :\ \mathbb{R}^n_+\to\mathbb{R}^n_+$ a C^1 increasing function. Assume $\phi(x)>0\ \forall\, x\in\mathbb{R}^n_+$ and $\phi(x)$ irreducible $\forall\, x\in\mathbb{R}^n_+$. If $\rho(\Phi(x))\leqslant\zeta_0<1\ \forall\, x\in\operatorname{int}(\mathbb{R}^n_+)$, then both the continuous-time system* (8) *and the discrete-time system* (7) *admit a unique positive equilibrium point $x^*\in\operatorname{int}(\mathbb{R}^n_+)$ which is asymptotically stable with domain of attraction $\mathcal{A}(x^*)\supset\mathbb{R}^n_+$.*

Proof: From Theorem 5, ϕ is a c-contractive interference function, hence from Proposition 2 it has a unique equilibrium point x^*. For it, then, Theorem 3 implies asymptotic stability of x^*. ∎

A consequence of Theorem 4 is that the contractivity notion of Theorem 2 can be readily verified for c-contractive interference functions.

Corollary 2 *If $\phi:\mathbb{R}^n_+\to\mathbb{R}^n_+$ a C^1 c-contractive interference function, then for both the continuous-time system* (8) *and the discrete-time system* (7) *the norm $\|\delta x\|$ is contracting over $\mathbb{R}^n_+$.*

Proof: Consider the continuous-time case. From Theorem 4, $\rho(\Phi(x))<1\ \forall\, x\in\mathbb{R}^n_+$, hence $-I+\Phi(x)$ is a Metzler matrix which is strictly diagonally dominant (it is a negated M-matrix). It follows that $-2I+\Phi(x)^T+\Phi(x)$ is negative definite $\forall\, x\in\mathbb{R}^n_+$. Similarly, in discrete-time, since $\rho(\Phi(x))<1$, combining (1) with Proposition 1, one gets, for $\epsilon>0$ sufficiently small

$$\begin{aligned}\rho(\Phi(x)^T\Phi(x))&\leqslant\|\Phi(x)^T\Phi(x)\|^v_\infty\\&\leqslant\|\Phi(x)^T\|^v_\infty\|\Phi(x)\|^v_\infty\leqslant(\rho(\Phi(x))+\epsilon)^2<1,\end{aligned}$$

from which it follows that $\Phi(x)^T\Phi(x)-I$ is negative definite everywhere in $\mathbb{R}^n_+$. ∎

V. Contractive interference functions from concave interactions

Consider a distributed system on a strongly connected graph $\mathcal{G}$ determined by a weighted nonnegative adjacency matrix $G=[g_{ij}]$. Assume that a node j exerts the same form of influence on all its neighbours, up to a scaling constant which corresponds to the weight g_{ij} of the edge connecting j with i. If $\psi_j(x_j):\mathbb{R}_+\to\mathbb{R}_+$ is the functional form of the interaction from node j to all its neighbours, then in continuous-time we can write the system (8) as

$$\frac{dx}{dt}=K(-x+\phi(x))=K(-x+M\psi(x)+p), \qquad (16)$$

where $M=K^{-1}G$, $p\in\mathbb{R}^n_+$ is a vector of additive positive constants, and $\psi(x)=\left[\psi_1(x_1)\ldots\psi_n(x_n)\right]^T$. We assume that each $\psi_j(x_j)$ is increasing and strictly concave. One example is the following "Michaelis-Menten" functional [6], mutuated from biochemical reaction theory:

$$\psi_j(x_j)=\frac{x_j}{\theta_j+x_j}, \qquad (17)$$

where in this case the slope at $x_j=0$ is $\frac{1}{\theta_j}$.

The Jacobian of $\phi(x)$ is therefore

$$\Phi(x)=M\frac{\partial\psi(x)}{\partial x}=M\begin{bmatrix}\frac{\partial\psi_1(x_1)}{\partial x_1} & & \\ & \ddots & \\ & & \frac{\partial\psi_n(x_n)}{\partial x_n}\end{bmatrix}.$$

Theorem 6 *Let $\psi_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\ldots,n$, be C^1 strictly concave, non-decreasing, and such that $\psi_i(0)=0$. If $M\geqslant 0$ irreducible, then*

$$\rho(\Phi(x_1))>\rho(\Phi(x_2)) \qquad \forall\, x_1,\,x_2\in\mathbb{R}^n_+,\ x_1<x_2. \qquad (18)$$

Proof: ψ_j strictly concave and non-decreasing implies that $\frac{\partial\psi_j(x_j)}{\partial x_j}\geqslant 0$ is decreasing (since $\frac{\partial^2\psi_j(x_j)}{\partial x_j^2}<0$). For $0<x_1<x_2$ then

$$\frac{\partial\psi(x_1)}{\partial x}>\frac{\partial\psi(x_2)}{\partial x}\geqslant 0.$$

Therefore $\Phi(x_1)=M\frac{\partial\psi(x_1)}{\partial x}\geqslant M\frac{\partial\psi(x_2)}{\partial x}=\Phi(x_2)$. Since $\Phi(x)$ is Metzler $\forall\,x\in\mathbb{R}^n_+$ and irreducible $\forall\,x\in\mathbb{R}^n_+$, necessarily $\Phi(x_1)\neq\Phi(x_2)$, hence we can apply Theorem 2.14 of [1] from which it follows that $\rho(\Phi(x_1))>\rho(\Phi(x_2))$. ∎

Corollary 3 *Under the same hypothesis as Theorem 6, then*

$$\rho(\Phi(0))>\rho(\Phi(x)) \qquad \forall\,x\in\operatorname{int}(\mathbb{R}^n_+). \qquad (19)$$

Combining Theorem 5 with Corollary 3, we have the following.

Theorem 7 *Let $\psi_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\ldots,n$, be C^1 strictly concave, non-decreasing, and such that $\psi_i(0)=0$. Assume $M\geqslant 0$ irreducible, and consider*

$$\phi(x)=M\psi(x)+p, \quad p>0. \qquad (20)$$

If $\rho(\Phi(0))<1$ then $\phi(x)$ is a c-contractive interference function.

Proof: By construction, M irreducible implies $\phi(x)$ irreducible everywhere in $\mathbb{R}^n_+$. Since $p>0$ and $\psi_i(0)=0$, $\phi(0)>0$. From Corollary 3, if $\rho(\Phi(0))<1$, then $\rho(\Phi(x))<1$ everywhere in $\mathbb{R}^n_+$. Hence Theorem 5 holds, and the claim follows. ∎

Consequently Corollary 1 holds for our concave ψ, as stated in the following Corollary.

Corollary 4 *Let $\psi_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\ldots,n$, be C^1 strictly concave, non-decreasing, and such that $\psi_i(0)=0$. Assume $M\geqslant 0$ irreducible. Consider the interference functions* (20). *If $\rho(\Phi(0))<1$, then both the continuous-time system* (8) *(i.e.,* (16)*) and the discrete-time system* (7) *admit a unique positive equilibrium point $x^*\in\operatorname{int}(\mathbb{R}^n_+)$ which is asymptotically stable with domain of attraction $\mathcal{A}(x^*)\supset\mathbb{R}^n_+$.*

For instance, when the functional forms (17) are used, if $\theta_i\geqslant 1\ \forall\,i=1,\ldots,n$, then the sufficient condition of Corollary 4 becomes $\rho(M)<1$. Since $M=K^{-1}G$, such a condition can always be attained by properly choosing the gains k_i.

VI. Conclusions

Many practical applications that can be modeled as positive (nonlinear) systems benefit from guaranteed convergence properties to an equilibrium point. Similarly, many algorithms are based on the convergence of an iterated numerical scheme, even nonlinear. The aim of this paper is to contribute to the understanding of what kind of "structural" properties make a positive system provably convergent, without having to rely on specific parameter values to support such a conclusion. For nonlinear system such a problem is much more difficult than for their linear counterpart. The conditions we have obtained in this paper are simple enough to be interesting in a broad range of applicative contexts.

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