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On the connectedness of the branch locus of moduli space of hyperelliptic Klein surfaces with one boundary

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To the memory of Mika Seppälä.

1 Introduction

Moduli space of Klein surfaces is the set of dianalytic structures on a given topological compact surface (possibly non-orientable, with boundary), and it has a natural topology given as quotient of Teichmüller space. The aim of this work is a better understanding of some topological properties of moduli spaces. Connectedness is specially important for subsets of moduli spaces because it allows the deformation of structures with given types.

F. Klein conjectured, and M. Seppälä showed, that the set of real Riemann surfaces is a connected subspace of the moduli space [**?**, **?**, **?**]. To prove this result M. Seppälä uses the connectedness of the locus of hyperelliptic real Riemann surfaces.

The moduli space has an orbifold structure whose singular locus consists of surfaces with non-trivial automorphism [**?**]. Branch loci of moduli spaces

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of Riemann surfaces are connected only for few genera [?] (see also [?]); this is in contrast to the case of moduli spaces of orientable Klein surfaces whose branch loci are connected [?] (see also [?] for the branch loci of moduli spaces of Riemann surfaces considered as Klein surfaces). Bujalance et al. [?] have recently shown that the branch loci of moduli spaces of non-orientable surfaces without boundary and low genus is connected (compare with [?] and [?] for the case Riemann surfaces).

In this work we study the connectedness of the hyperelliptic branch locus of Klein surfaces with one boundary component. We show that the hyperelliptic branch locus of orientable Klein surfaces with one boundary component is connected and we prove that it is disconnected in the non-orientable case. In this last case we characterize the connected components of the hyperelliptic branch locus in terms of topological types of actions of automorphisms.

Finally, we show that the hyperelliptic branch locus for non-orientable Klein surfaces of topological genus 2 with two boundary components is an example of connected branch loci.

2 Klein surfaces and non-euclidean crystallographic groups

A Klein surface X is a compact surface (may be non-orientable and with boundary) endowed with a dianalytic structure, that is to say a class of atlases where the transition maps are analytic or anti-analytic maps of $\mathbb{C}$ (see [?, ?, ?]). Klein surfaces are important in the study of real algebraic curves [?, ?].

The topological type of X is given by $t = (h, \pm, k)$ where h is the genus, $+$ if X is orientable and $-$ if X is non-orientable and k is the number of connected components of the boundary. The integer $\varepsilon h + k - 1$, where $\varepsilon = 2$ if there is a $+$ sing in t and $\varepsilon = 1$ if there is a $-$ sign in t, is the *algebraic genus* of X.

A non-Euclidean crystallographic group or NEC group Γ is a discrete subgroup of the group $\mathrm{Aut}^{\pm}\mathbb{D}$ of conformal and anticonformal automorphisms of the unit disc $\mathbb{D}$ of $\mathbb{C}$ and in this paper we shall assume that the orbit space $\mathbb{D}/\Gamma$ is compact. When the NEC group Γ does not contain any orientation-reversing automorphism of $\mathbb{D}$, we say that Γ is a Fuchsian group.

The algebraic structure of Γ and the geometric and topological structures of the quotient orbifold $\mathbb{D}/\Gamma$ are given by the signature:

$$s(\Gamma) = (h; \pm; [m_1, ..., m_r]; \{(n_{1,1}, ..., n_{1,r_1}), ..., (n_{k,1}, ..., n_{k,r_k})\}). \qquad (1)$$

See [?, ?, ?]. The orbit space $\mathbb{D}/\Gamma$ is an orbifold with underlying surface of genus h, having $r \geq 0$ cone points and k boundary components, each with $r_i \geq 0$ corner points, $i = 1, ..., k$. The signs $''+''$ and $''-''$ correspond to orientable and non-orientable quotient surfaces respectively. The integers m_i are called the proper periods of Γ and they are the orders of the cone points of $\mathbb{D}/\Gamma$. The brackets $(n_{i,1}, ..., n_{i,r_i})$ are the period cycles of Γ. The integers $n_{i,j}$ are the link periods of Γ and the orders of the corner points of $\mathbb{D}/\Gamma$. The group Γ is isomorphic to the fundamental group of the orbifold $\mathbb{D}\Gamma$.

Given an NEC group Γ, the subgroup Γ^+ consisting of the orientation-preserving elements of Γ is called the canonical Fuchsian subgroup of Γ.

A group Γ with signature (**??**) has a canonical presentation with generators :

1. Hyperbolic generators: $a_1, b_1, ..., a_h, b_h$ if $\mathbb{D}/\Gamma$ is orientable; or glide reflection generators: $d_1, ..., d_h$ if $\mathbb{D}/\Gamma$ is non-orientable,
2. Elliptic generators: $x_1, ..., x_r$,
3. Connecting generators (hyperbolic or elliptic transformations): $e_1, \ldots, e_k$
4. Reflection generators: $c_{i,j}$, $1 \leq i \leq k, 1 \leq j \leq r_i + 1$.

And relators:

1. $x_i^{m_i}, i = 1, ..., r$,
2. $c_{i,j}^2$,
3. $(c_{i,j-1}c_{i,j})^{n_{i,j}}$, $j = 1, ..., r_i$,
4. $e_i^{-1}c_{i,r_i}e_i^{-1}c_{i,0}, i = 1, ..., k$,
5. The long relation:

$x_1...x_re_1...e_ka_1b_1a_1^{-1}b_1^{-1}...a_hb_ha_h^{-1}b_h^{-1}$ or $x_1...x_re_1...e_kd_1^2...d_h^2$, according to whether $\mathbb{D}/\Gamma$ is orientable or not.

The hyperbolic area of the orbifold $\mathbb{D}/\Gamma$ coincides with the hyperbolic area of an arbitrary fundamental region of Γ and it is equal to:

$$\mu(\Gamma) = 2\pi(\varepsilon h - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{r_i}(1 - \frac{1}{n_{i,j}})), \tag{2}$$

where $\varepsilon = 2$ if there is a $''+''$ sign and $\varepsilon = 1$ otherwise. If Γ' is a subgroup of Γ of finite index then Γ' is an NEC group and the following Riemann-Hurwitz formula holds:

$$[\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma). \tag{3}$$

An NEC (Fuchsian) group Γ without elliptic elements is called a surface NEC (Fuchsian) group and it has signature $(h; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\})$. Any Klein surface X with algebraic genus ≥ 2 can be represented as the orbit space $X = \mathbb{D}/\Gamma$, with Γ an NEC surface group. If a finite group G is isomorphic to a group of automorphisms of X then there exists an NEC group Δ and an epimorphism $\theta : \Delta \to G$ with $\ker(\theta) = \Gamma$. The NEC group Δ is the lifting of G to the universal covering $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$.

3 Topological classification of automorphisms of Klein surfaces

Two automorphisms f and g of a Klein surface X are topologically equivalent if f and g are conjugated by a homeomorphism of X. The topological types of automorphisms are the topological equivalence classes. The topological types of automorphisms are described using topological invariants (see [**?**], [**?**] and [**?**]). Here we present the topological types automorphisms of Klein surfaces of primer order.

Assume that X is a Klein surface, with algebraic genus ≥ 2, and let $\varphi : X \to X$ be an order p automorphism; where $p > 2$ is a prime. The topological type of φ is given by the rotation indices for the fixed points of φ and the rotation angles of setwise invariant boundary components. If φ has r fixed points and leaves setwise invariant s boundary components, the topological type of φ is described by the following data $\theta = (p; \{n_1, ..., n_r\}, \{m_1, \ldots, m_s\})$ where $1 \leq n_i, m_i \leq p-1$. Observe that in the case of non-orientable surfaces the data n_i and $p - n_i$ and m_i and $p - m_i$ define topologically equivalent automorphisms. The n_i (respectively m_i) means that there is a fixed point of φ (respectively a boundary component of X) where locally φ acts topologically as a rotation with angle $2\pi n_i/p$ (resp. $2\pi m_i/p$). The surface X can be uniformized by a group Γ with signature

$$(g; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\})$$

and the fact of admitting an automorphism of topological type

$$\theta = (p; \{n_1, ..., n_r\}, \{m_1, ..., m_b\})$$

implies that there is an NEC group Δ with signature

$$(h; +; [p, \overset{r}{\ldots}, p]\{(-), \overset{b}{\ldots}, (-), (-), \overset{\frac{k-b}{p}}{\ldots}, (-)\})$$

an epimorphism $\omega_\theta : \Delta \to C_p = \langle \alpha \rangle$ such that $\Gamma = \ker \omega_\theta$ and if x_j, c_l, e_l are a set of canonical generators of Δ , must be:

$$\omega_\theta(x_j) = \alpha^{n_j}, \omega_\theta(e_j) = \alpha^{m_j}, \omega_\theta(e_w) = 1, \text{for } w > b, \omega_\theta(c_l) = 1$$

Note that for Riemann-Hurwitz formula p divides $k-b$.

Assume that X is a Klein surface and let $\iota : X \to X$ be an involution. The topological invariants for ι, see [**?**], are mainly related with the set $\mathrm{Fix}(\iota)$ of fixed points of ι. The set $\mathrm{Fix}(\iota)$ consists of:

(a) a finite number of r isolated points,

(b) a finite number of simple closed curves that we shall call *ovals*. Ovals will be called twisted or untwisted accordingly to whether they have Möbius band or annular neighbourhoods respectively. Let q^+ be the number of untwisted ovals and q^- be the number of twisted ones.

(c) a finite number of chains, which we define now. A *chain* of length s_i (we shall consider s_i always to be even) is a set C of $s_i/2$ disjoint arcs properly embedded in X (i. e. the ends of each component of C are in the boundary of X) such that for each boundary component B of X, either $C \cap B = \varnothing$ or $C \cap B$ consists of two distinct points. Chains may also be twisted or untwisted. The natural definition of the to types of chains is obtained by filling the holes of X with discs, see page 462 of [**?**]. Let t^+ and t^- be the number of untwisted and twisted chains respectively.

The extra information that we shall need to determine ι up topological equivalence is

(d) the number r_b of setwise fixed boundary components which contain no points of $\mathrm{Fix}(\iota)$

(e) the orientability of $X/\langle\iota\rangle$, where $\langle\iota\rangle$ is the cyclic group of order two generated by ι,

(f) two homological invariants in the case when $\mathrm{Fix}(\iota) = \varnothing$ and that will not be necessary in our work; therefore we will omit them.

All the above information can be presented in a symbol

$$\theta = (2; \pm; r, r_b; q^+, q^-; \{s_1, ..., s_{t^+}\}, \{s_1, ..., s_{t^-}\})$$

the sign $+$ is used if $X/\langle\iota\rangle$ is orientable and the $-$ sign is used if $X/\langle\iota\rangle$ is non-orientable (see [**?**], [**?**]).

By using uniformization by NEC groups X can be uniformized by a group Γ with signature $(g; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\})$ and the fact of admitting an involution with topological invariants $(2; \pm; r, r_b; q^+, q^-; \{s_1, ..., s_{t^+}\}, \{s_1, ..., s_{t^-}\})$ implies that $\Gamma = \ker \omega_\theta$ where $\omega_\theta : \Delta \to C_2 = \langle\iota\rangle$, Δ has signature

$$(h; \pm; [2, \overset{r}{\ldots}, 2]\{(-), \overset{l+r_b}{\ldots}, (-), (-), \overset{q^+ + q^-}{\ldots}, (-), (2, \overset{s_1}{\ldots}, 2), \overset{t^+ + t^-}{\ldots}, (2, \overset{s_t}{\ldots}, 2)\}),$$

and if either $a_i, b_i, x_i, e_j, c_i, c_{i,j}$ or $d_i, x_i, e_j, c_i, c_{i,j}$ is a set of canonical generators of Δ, must be:

$\omega_\theta(x_i) = \iota$
$\omega_\theta(e_j) = 1$ $j = 1, ..., l$ and $\omega_\theta(e_j) = \iota$ $j = l, ..., l + r_b$
$\omega_\theta(c_i) = 1,\ 1 \leq i \leq l + r_b$
$\omega_\theta(e_j) = 1$ $j = l + r_b + 1, ..., l + r_b + q^+$ and
$\omega_\theta(e_j) = \iota$ $j = l + r_b + q^+ + 1, ..., l + r_b + q^+ + q^-$
$\omega_\theta(c_i) = \iota, l + r_b + 1 \leq i \leq l + r_b + q^+ + q^-;$
$\omega_\theta(e_j) = 1$ $j = l + r_b + q^+ + q^- + 1, ..., l + r_b + q^+ + q^- + t^+$ and
$\omega_\theta(e_j) = \iota$ $j = l + r_b + q^+ + q^- + t^+ + 1, ..., l + r_b + q^+ + q^- + t^+ + t^-$
$\omega_\theta(c_{i,j}) = 1, \omega_\theta(c_{i,j+1}) = \iota.$

4 Moduli spaces

Let s be a signature of NEC groups (**??**) and $\mathcal{G}$ be an abstract group isomorphic to the NEC groups with signature s. We denote by $R(s)$ the set of monomorphisms $r : \mathcal{G} \to \mathrm{Aut}^{\pm}(\mathbb{D})$ such that $r(\mathcal{G})$ is an NEC group with signature s. The set $R(s)$ has a natural topology given by the topology of $\mathrm{Aut}^{\pm}(\mathbb{D})$. Two elements r_1 and $r_2 \in R(s)$ are said to be equivalent, $r_1 \sim r_2$, if there exists $g \in \mathrm{Aut}^{\pm}(\mathbb{D})$ such that for each $\gamma \in \mathcal{G}$, $r_1(\gamma) = g r_2(\gamma) g^{-1}$. The space of classes $\mathbf{T}(s) = R(s)/\sim$ is called the Teichmüller space of NEC groups with signature s (see [**?**]). The Teichmüller space $T(s)$ is homeomorphic to $\mathbb{R}^{d(s)}$ where

$$d(s) = 3(\varepsilon h - 1 + k) - 3 + (2\sum_{i=1}^{r} m_i + \sum_{i=1}^{k}\sum_{j=1}^{s_i} n_{ij}).$$

The modular group $\mathrm{Mod}(\mathcal{G})$ of $\mathcal{G}$ is the quotient $\mathrm{Mod}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Inn}(\mathcal{G})$, where $\mathrm{Inn}(\mathcal{G})$ denotes the inner automorphisms of $\mathcal{G}$. The moduli space for NEC groups with signature s is the quotient $\mathcal{M}_s = \mathbf{T}(s)/\mathrm{Mod}(\mathcal{G})$ endowed with the quotient topology and we shall denote $\pi_s : \mathbf{T}(s) \to \mathcal{M}_s$. Hence $\mathcal{M}_s$ is an orbifold with fundamental orbifold group $\mathrm{Mod}(\mathcal{G})$.

If s is the signature of a surface group uniformizing surfaces of topological type t, then we denote by $\mathbf{T}(s) = \mathbf{T}_t$ and $\mathcal{M}_s = \mathcal{M}_t$ the Teichmüller and the moduli space of Klein surfaces of topological type t.

Let $\mathcal{G}$ and $\mathcal{G}'$ be groups isomorphic to NEC groups with signatures s and s' respectively. The inclusion mapping $\alpha : \mathcal{G} \to \mathcal{G}'$ induces an embedding $\mathbf{T}(\alpha) : \mathbf{T}(s') \to \mathbf{T}(s)$ defined by $[r] \mapsto [r \circ \alpha]$. See [**?**].

If a finite group G is isomorphic to a group of automorphisms of Klein surfaces with topological type $t = (g, \pm, k)$, then the action of G is determined by an epimorphism $\theta : \mathcal{D} \to G$, where $\mathcal{D}$ is an abstract group isomorphic to NEC groups with a given signature s and $\ker(\theta)$ is isomorphic to an abstract surface group $\mathcal{G}$ of signature $(g; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\})$. Then there is an inclusion $\alpha : \mathcal{G} \to \mathcal{D}$ and an embedding $\mathbf{T}(\alpha) : \mathbf{T}(s) \to \mathbf{T}_t$. The continuous map $\mathbf{T}(\alpha)$ induces a continuous map $\mathcal{M}_s \to \mathcal{M}_t$. Therefore the set $\mathcal{B}_t^{G,\theta}$ of points in $\mathcal{M}_t$ corresponding to surfaces having a group of automorphisms isomorphic to G and with a fixed action θ is connected.

5 Non-orientable Klein surfaces with one boundary component

By the last section, the set $\mathcal{B}_{(g,-,1)}^{Hyp,G,\theta}$ of points in $\mathcal{M}_{(g,-,1)}$ corresponding to hyperelliptic surfaces having a group of automorphisms isomorphic to $G \supsetneq \langle \varphi \rangle$, where φ is the hyperelliptic involution an with a fixed action θ, is a connected set. The set

$$\mathcal{B}_{(g,-,1)}^{K,Hyp} = \cup_{G,\theta} \mathcal{B}_{(g,-,1)}^{Hyp,G,\theta}$$

consist of the points in $\mathcal{M}_{(g,-,1)}$ that are hyperelliptic and with automorphisms different from the hyperelliptic involution and the identity.

We are interested in the connectedness of $\mathcal{B}_{(g,-,1)}^{K,Hyp}$.

Theorem 1 *$\mathcal{B}_{(g,-,1)}^{K,Hyp}$ is disconnected and has $\frac{g}{2} + 1$ connected components for g even and $\frac{g+1}{2}$ connected components for g odd.*

Proof. A Klein surface X of genus g is said to be hyperelliptic if there is an involution φ of X such that $X / \langle \varphi \rangle$ has algebraic genus 0. When X is non-orientable and with one boundary component; the involution φ has g isolated interior fixed points and an arc of fixed points with ends on the boundary (a chain following [**?**]).

In terms of uniformization groups there is an NEC group Δ with signature $(0; [2^g]; \{(2, 2)\})$ and an index two surface subgroup Γ of Δ uniformizing X.

We have:

$$\mathcal{B}_{(g,-,1)}^{K,Hyp} = \cup \mathcal{B}_{(g,-,1)}^{Hyp,G,\theta} = \pi_t(\cup_{s \prec (0;[2^g];\{(2,2)\}),\theta} i_*(\mathbf{T}_s)),$$

where $s \prec (0;[2^g];\{(2,2)\})$ means signatures of NEC groups containing groups with signature $(0;[2^g];\{(2,2)\})$ and $\theta : \mathcal{D} \to G$ are epimorphisms such that $\ker\theta$ is isomorphic to the groups with signature $(g;-;[-];\{(-)\})$ and $\mathcal{D}$ is an abstract group isomorphic to NEC groups with signature s.

By Theorem 6.3.3 in [**?**], a group G of automorphisms of a hyperelliptic non-orientable Klein surface X with one boundary component and automophisms different from the hyperelliptic involution is $C_2 \times C_2$.

Let us give now a geometrical reason of the above fact. The quotient orbifold $X/\langle\varphi\rangle$ is a disc with g conic points of order 2, two corner points of angle $\pi/2$ dividing the topological boundary of the disc in two arcs: one arc consisting of points with non-trivial isotropy groups (the projection of the chain); the other arc corresponds to the projection of the boundary component of X. Since φ is central in $\mathrm{Aut}(X)$ then $\mathrm{Aut}(X)/\langle\varphi\rangle$ acts on the orbifold $X/\langle\varphi\rangle$ and then each arc in the topological boundary of $X/\langle\varphi\rangle$ must be preserved. Since the arcs admit only actions of C_2, then $\mathrm{Aut}(X)/\langle\varphi\rangle \cong C_2$ and $\mathrm{Aut}(X) \cong C_2 \times C_2$.

The possible signatures of NEC groups Λ containing the group Δ as a subgroup of order two are:

$$(0;[2^r];\{(2,\overset{s}{\ldots},2)\});\ \text{with } 2r+s=g+3$$

Given a group Λ as above, there is a unique epimorphism $\theta_r : \Lambda \to \Lambda/\Gamma = \mathrm{Aut}(X) \cong C_2 \times C_2 = \langle a,b\rangle$ such that $\theta_r^{-1}(\langle a\rangle)$ has signature $(0;[2^g];\{(2,2)\})$ (i.e. the element a represents the hyperelliptic involution in $X = \mathbb{D}/\Gamma$).

Using a canonical presentacion of Λ, the epimorphism θ_r is:

$$\begin{gathered}
\theta_r(x_i) = a, i = 1,...,r \\
\theta_r(e) = a \text{ if } r \equiv 1 \bmod 2 \text{ and } \theta_r(e) = \mathrm{id} \text{ if } r \equiv 0 \bmod 2 \\
\theta_r(c_0) = a \\
\theta_r(c_1) = \mathrm{id} \\
\theta_r(c_2) = b \\
\theta_r(c_3) = ab \\
\theta_r(c_4) = b \\
\theta_r(c_5) = ab \\
\ldots \\
\theta_r(c_{s-1}) = b, \text{ if } s \equiv 1 \bmod 2 \text{ and } \theta_r(c_{s-1}) = ab, \text{ if } s \equiv 0 \bmod 2 \\
\theta_r(c_s) = a
\end{gathered}$$

For each signature we have an action of $C_2 \times C_2$, since there are no bigger groups of automorphisms each epimorphism or action is maximal and defines a connected component of $\mathcal{B}^{K,Hyp}_{(g,-,1)}$. ■

In the next result we describe the topological type of the automorphisms of the Klein surfaces in each component of the hyperelliptic branch locus $\mathcal{B}^{K,Hyp}_{(g,-,1)}$.

Proposition 2 *Let $p \circ i_*(\mathbf{T}_{(0;[2^r];\{(2,\overset{s}{\ldots},2)\})})$ be a connected component of $\mathcal{B}^{K,Hyp}_{(g,-,1)}$. Given $X \in p \circ i_*(\mathbf{T}_{(0;[2^r];\{(2,\overset{s}{\ldots},2)\})})$, $\mathrm{Aut}(X)$ contains:*

- *the hyperelliptic involution φ with g isolated fixed points and one chain, the chain is twisted if g is odd and untwisted if g is even. Following the notation in Section 3 the topological type of the hyperelliptic involution: $(2;+;g,0;0,0;\{2\},\{0\})$ if g is even and $(2;+;g,0;0,0;\{0\},\{2\})$ if g is odd.*

- *an involution α with the following topological type: $(2;\pm;1,0;\frac{g-1}{2}-r,0;\{0\},\{2\})$ for g odd and $(2;\pm;0,0;\frac{g}{2}-r-1,1;\{0\},\{2\})$ for g even; that is, the involution α has $\frac{g-1}{2}-r$ untwisted ovals, an isolated fixed point and one twisted chain, for g odd; $\frac{g}{2}-r$ ovals, only one of them twisted (if there are ovals), and one twisted chain, for g even. Finally $X/\langle\alpha\rangle$ is non-orientable if $r>0$ and $X/\langle\alpha\rangle$ is orientable if $r=0$,*

- *finally, an involution $\alpha\varphi$ with topological type: $(2;-;0,0;\frac{g+1}{2}-r-2,2;\{0\},\{0\})$ for g odd and $(2;-;1,0;\frac{g}{2}-r-1,1;\{0\},\{0\})$ for g even; that is, the involution α has $\frac{g+1}{2}-r$ ovals, exactly two of them twisted (if there are ovals), for g odd; but $\frac{g}{2}-r$ ovals, one of them twisted (if there are ovals) and one isolated fixed point, for g even. The quotient $X/\langle\alpha\varphi\rangle$ is always non-orientable.*

Proof. Let $X \in p \circ i_*(\mathbf{T}_{(0;[2^r];\{(2,\overset{s}{\ldots},2)\})})$. Consider the monodromy $\theta_r : \Lambda \to \Lambda/\Gamma = \mathrm{Aut}(X) \cong C_2 \times C_2 = \langle a,b\rangle$ defined in the proof of Theorem **??**. Where a represents the hyperelliptic involution φ and b represents the involution α. Appying [**?**] and [**?**], we have that:

$\theta_r^{-1}(\langle a\rangle)$ has signature $(0;[2^g];\{(2,2)\})$,

- for g odd:

$\theta_r^{-1}(\langle b\rangle)$ has signature $(r;-;[2];\{(-)\overset{\frac{g-1}{2}-r}{\ldots}(-)(2,2)\})$ with $r>0$ and $(0;+;[2];\{(-)\overset{\frac{g-1}{2}}{\ldots}(-)(2,2)\})$ with $r=0$,

$\theta_r^{-1}(\langle ab\rangle)$ has signature $(r+1;-;[-];\{(-)\overset{\frac{g+1}{2}-r}{\ldots}(-)\})$.

- for g even:

$\theta_r^{-1}(\langle b\rangle)$ has signature $(r;-;[-];\{(-)\overset{\frac{g}{2}-r}{\ldots}(-)(2,2)\})$ with $r>0$ and $(0;+;[2];\{(-)\overset{\frac{g-1}{2}}{\ldots}(-)(2,2)\})$ with $r=0$,

$\theta_r^{-1}(\langle ab\rangle)$ has signature $(r+1;-;[2];\{(-)\overset{\frac{g}{2}-r}{\ldots}(-)\})$.

The twisted property for chains and ovals is determined by the image of the connecting generator for the NEC groups. For instance for the case g odd and the involution α the twisted chain is produced by

$$\begin{gathered}\theta_r(c_0)=a\\ \theta_r(c_1)=\mathrm{id}\\ \theta_r(c_2)=b\\ \theta_r(c_3)=ab\end{gathered}$$

■

Remark 3 *For genus $g=2$ the disconnectedness of the hyperelliptic branch locus of non-orientable Klein surfaces may be deduced from [**?**].*

6 The branch locus for the moduli of hyperelliptic orientable Klein surfaces with one boundary component

Now we shall study the connectedness of the set $\mathcal{B}_{(g,+,1)}^{K,Hyp}$ that consist of the points in $\mathcal{M}_{(g,+,1)}$ that are hyperelliptic and with automorphisms different from the hyperelliptic involution and the identity.

Proposition 4 *The hyperelliptic branch locus $\mathcal{B}_{(g,+,1)}^{K,Hyp}$ is connected.*

Proof. Consider a surface X in the hyperelliptic branch locus. Consider the hyperelliptic involution φ. The quotient $X/\langle\varphi\rangle$ is a disc with $2g+1$ conic points of order 2 (see [**?**]). Let G be the automorphism group of X, the quotient group $G/\langle\varphi\rangle$ is cyclic or dihedral, see [**?**] (remark that $G/\langle\varphi\rangle$ acts on a disc).

Let $\mathcal{B}_{(g,+,1)}^{Hyp,G,\theta}$ be the connected subset of $\mathcal{B}_{(g,+,1)}^{K,Hyp}$ given by the surfaces with automorphism group containing the group G (where G contains the hyperelliptic involution) acting in a fixed topological way given by θ.

We have: $\mathcal{B}_{(g,+,1)}^{K,Hyp}=\cup\mathcal{B}_{(g,+,1)}^{Hyp,G,\theta}=\pi_t(\cup_{s\prec(0;[2^{2g+1}];\{(-)\}),\theta}i_*(\mathbf{T}_s))$, where $s\prec(0;[2^{2g+1}];\{(-)\})$ means signatures of NEC groups containing groups with signature $(0;[2^{2g+1}];\{(-)\})$ and $\theta : \mathcal{D} \to G$ are epimorphisms such that $\ker\theta$ is isomorphic to the groups with signature $(g;+;[-];\{(-)\})$ and $\mathcal{D}$ is an abstract group isomorphic to NEC groups with signature s. Let $\mathcal{B}(s) = \pi_t(\cup_\theta i_*(\mathbf{T}_s))$, where $s \prec (0;[2^{2g+1}];\{(-)\})$ and θ runs over all epimorphism from groups with signature s.

Thus

$$\begin{aligned}\mathcal{B}^{Hyp,G,\theta}_{(g,+,1)} = &(\cup_{p|2g}\mathcal{B}(0;+;[2p,2,\overset{\frac{2g}{p}}{\ldots},2],\{(-)\})\cup \\ &\cup(\cup_{p|2g+1}\mathcal{B}(0;+;[p,2,\overset{\frac{2g+1}{p}}{\ldots},2],\{(-)\})\cup \\ &(\cup_{2r+s=2g+3}\mathcal{B}(0;+;[2,\overset{r}{\ldots},2]\{(2,\overset{s}{\ldots},2)\}).\end{aligned}$$

The following monodromy (essentially unique) $\theta_p : \Delta \to D_p \times C_2 = \langle s,t\rangle \times \langle a\rangle$, where Δ has signature $(0;+;[2,\overset{\frac{2g+1-p}{2p}}{\ldots},2],\{(p,2,2,2)\})$ and θ_p is defined by

$$\begin{gathered} x_i \to a \\ e \to a \text{ or id (according to the parity of } g/p) \\ c_0 \to s \\ c_1 \to t \\ c_2 \to ta \\ c_3 \to \text{id} \\ c_4 \to s \end{gathered}$$

yields that $\mathcal{B}(0;+;[2,\overset{\frac{2g+1-p}{2p}}{\ldots},2],\{(p,2,2,2)\}) = \mathcal{B}^{Hyp,,D_p\times C_2,\theta_p}_{(g,+,1)} \subset \mathcal{B}(0;+;[p,2,\overset{\frac{2g+1}{p}}{\ldots},2],\{(-)\})$ $\cap\mathcal{B}(0;+;[2,\overset{g}{\ldots},2],\{(2,2,2)\}) \neq \varnothing$, the hyperelliptic involution of $\mathbb{D}/\ker\theta_p$ is represented by a.

Similarly $\mathcal{B}(0;+;[2,\overset{\frac{g}{p}}{\ldots},2],\{(2p,2,2)\}) = \mathcal{B}^{Hyp,D_{2p},\theta'_p}_{(g,+,1)} \subset \mathcal{B}(0;+;[2p,2,\overset{\frac{2g}{p}}{\ldots},2],\{(-)\})$ $\cap\,\mathcal{B}(0;+;[2,\overset{g}{\ldots},2],\{(2,2,2)\}) \neq \varnothing$,

Now for the points in $\mathcal{B}^{K,Hyp}_{(g,+,1)}$ having anticonformal involutions we consider the following monodromies:

1. The monodromy $\theta_{-3} : \Delta \to D_4 = \langle s,t\rangle$, where Δ has signature

$(0;+;[2,\overset{k}{\ldots},2],\{(4,2,\overset{g+2-2k}{\ldots},2)\})$, defined by

$$\begin{gathered}x_i \to (st)^2\\ e \to (st)^2 \text{ or id (according to the parity of } k)\\ c_0 \to s\\ c_1 \to t\\ c_2 \to t(st)^2\\ c_3 \to t\\ c_4 \to t(st)^2\\ c_5 \to t\\ \ldots\\ c_{g+1-2k} \to t \text{ or } t(st)^2\\ c_{g+2-2k} \to \text{id}\\ c_{g+3-2k} \to s\end{gathered}$$

(the hyperelliptic involution φ of $\mathbb{D}/\ker\theta_{-3}$ is represented by $(st)^2$).
This yields:

$\mathcal{B}^{Hyp,D_4,\theta_{-3}}_{(g,+,1)} \subset \mathcal{B}(0;+;[2,\overset{2k}{\ldots},2],\{(2,\overset{2g+3-4k}{\ldots},2)\}) \cap \mathcal{B}(0;+;[2,\overset{g}{\ldots},2],\{(2,2,2)\}) \neq \varnothing$

2. The monodromy $\theta_{-5} : \Delta \to D_4 = \langle s,t\rangle$, where Δ has signature $(0;+;[2,\overset{k}{\ldots},2],\{(4,2,\overset{g+2-2k}{\ldots},2)\})$, defined by

$$\begin{gathered}x_i \to (st)^2\\ e \to (st)^2 \text{ or id (according to the parity of } k)\\ c_0 \to s\\ c_1 \to t\\ c_2 \to t(st)^2\\ c_3 \to t\\ c_4 \to t(st)^2\\ c_5 \to t\\ \ldots\\ c_{g-1-2k} \to t \text{ or } t(st)^2\\ c_{g-2k} \to \text{id}\\ c_{g+1-2k} \to s\\ c_{g+2-2k} \to t\\ c_{g+3-2k} \to s\end{gathered}$$

yields: $\mathcal{B}^{Hyp,D_4,\theta_{-5}}_{(g,+,1)} \subset \mathcal{B}(0;+;[2,\overset{2k}{\ldots},2],\{(2,\overset{2g+1-4k}{\ldots},2)\}) \cap \mathcal{B}(0;+;[2,\overset{g-1}{\ldots},2],\{(2,2,2,2,2)\}) \neq \varnothing$

3. Finally the monodromy $\theta_{-3,-5} : \Delta \to D_{4g} = \langle s, t\rangle$ where Δ has signature $(0; +; [-], \{(4g, 2, 2, 2)\})$ defined by:

$$c_0 \to s$$
$$c_1 \to t$$
$$c_2 \to \mathrm{id}$$
$$c_3 \to s(st)^{2g}$$

where φ is now represented by $(st)^{2g}$ produces: $\mathcal{B}_{(g,+,1)}^{Hyp,D_{4g},\theta_{-3,-5}}$ $=\mathcal{B}(0; +; [-], \{(4g, 2, 2, 2)\}) \subset$ $\mathcal{B}(0; +; [2, \overset{g}{\ldots}, 2], \{(2, 2, 2)\})$
$\cap\, \mathcal{B}(0; +; [2, \overset{g-1}{\ldots}, 2], \{(2, 2, 2, 2, 2)\}) \neq \varnothing$

Hence $\mathcal{B}_{(g,+,k)}^{KH}$ is connected. ■

7 Non-orientable Klein surfaces with two boundary components

In this section we show, as an example, that the hyperelliptic branch locus of non-orientable Klein surfaces with two boundary components is connected

Proposition 5 $\mathcal{B}_{(2,-,2)}^{K,Hyp}$ *is connected*

Proof. The signature of the NEC groups uniformizing the quotient orbifold $X/\langle\varphi\rangle$ is: $(0; [2, 2]; \{(2, 2, 2, 2)\})$. The topological type of φ is given by:

$$(2; +; 2, 0; 0, 0; \{4\}, \{-\})$$

that is to say φ has two fixed points and an untwisted chain of length 4.

The automorphisms of X necessarily have order two, since such a automorphism will induce an automorphism of a disc with boundary is divided in four arcs alternately bicoloured (this bicoloration is given by the projection of the boundary components and points in the chain φ).

If X has an automorphism ψ different from φ the possible types of actions of groups $\langle\varphi, \psi\rangle$ produce the following signatures of NEC groups Γ such that $X/\langle\varphi, \psi\rangle = \mathbb{D}/\Gamma$ with some epimorphism $\theta : \Gamma \to C_2 \times C_2$ and $\mathbb{D}/\ker\theta = X$:

1. $(0; [2, 2]; \{(2, 2)\})$
2. $(0; [2]; \{(2, 2, 2, 2)\})$
3. $(0; [-]; \{(2, 2, 2, 2, 2, 2)\})$

Let now describe the equivalence classes of epimorphisms $\theta : \Gamma \to C_2 \times C_2 = \langle a, s\rangle$.

1. For signature $(0;[2,2];\{(2,2)\})$ there is only one class:

$$\theta_2: \quad x_1 \to a, x_2 \to s, e \to as, c_0 \to a, c_1 \to \mathrm{id}, c_2 \to a$$

2. For signature $(0;[2];\{(2,2,2,2)\})$ there are two classes θ_1 and θ_1' defined by:

$$\theta_1: \quad x \to a, e \to a, c_0 \to a, c_1 \to \mathrm{id}, c_2 \to a, c_3 \to s, c_4 \to a$$

$$\theta_1': \quad x \to a, e \to a, c_0 \to a, c_1 \to \mathrm{id}, c_2 \to s, c_3 \to \mathrm{id}, c_4 \to a$$

For signature $(0;[-];\{(2,2,2,2,2,2)\})$ there are two classes θ_0 and θ_0' defined by:

$$\theta_0: \quad c_0 \to a, c_1 \to \mathrm{id}, c_2 \to a, c_3 \to s, c_4 \to sa, c_5 \to s, c_6 \to a$$

$$\theta_0': \quad c_0 \to \mathrm{id}, c_1 \to a, c_2 \to \mathrm{id}, c_3 \to s, c_4 \to as, c_5 \to s, c_6 \to \mathrm{id}$$

We have: $\mathcal{B}^{K,Hyp}_{(2,-,2)} = \cup \mathcal{B}^{Hyp,G,\theta}_{(2,-,2)} = \pi_{(2,-,2)}(\cup_{s \prec (0;[2,2];\{(2,2,2,2)\}),\theta} i_*(\mathbf{T}_s))$, where $s \prec (0;[2,2];\{(2,2,2,2)\})$ means signatures 1, 2, 3 above and θ is θ_i, $i = 0,1,2$ or θ_j', $j = 0,1$. We shall denote the connected set $\pi_{(2,-,2)}(\cup_{s \prec (0;[2,2];\{(2,2,2,2)\}),\theta} i_*(\mathbf{T}_s))$ by $\mathcal{B}^{Hyp,\theta}_{(2,-,2)}(s)$.

Consider the monodromies $\theta_{0',1} : \Delta \to C_2 \times C_2 \times C_2 = \langle s,t,a \rangle$ (the hyperelliptic involution φ is represented by a) and $\theta_{0,1'}' : \Delta \to C_2 \times C_2 \times C_2$, with Δ of signature $(0;[-]\{(2,2,2,2,2)\})$ defined by:

$$\theta_{0',1}: \quad c_0 \to atc_1 \to a, c_2 \to \mathrm{id}, c_3 \to sa, c_4 \to s, c_5 \to t$$

$$\theta_{01'}: \quad c_0 \to t, c_1 \to \mathrm{id}, c_2 \to a, c_3 \to sa, c_4 \to s, c_5 \to t$$

We have (see [**?**]):

$$\mathcal{B}^{Hyp,C_2\times C_2\times C_2,\theta_{0'1}}_{(2,-,2)} \subset \mathcal{B}(0;[2,2];\{(2,2)\}) \cap \mathcal{B}^{Hyp,C_2\times C_2,\theta_1}_{(2,-,2)} \cap \mathcal{B}^{Hyp,C_2\times C_2,\theta_{0'}}_{(2,-,2)} \neq \varnothing$$

and

$$\mathcal{B}^{Hyp,C_2\times C_2\times C_2,\theta_{01'}}_{(2,-,2)} \subset \mathcal{B}(0;[2,2];\{(2,2)\}) \cap \mathcal{B}^{Hyp,C_2\times C_2,\theta_{1'}}_{(2,-,2)} \cap \mathcal{B}^{Hyp,C_2\times C_2,\theta_0}_{(2,-,2)} \neq \varnothing$$

Then $\mathcal{B}^{K,Hyp}_{(2,-,2)}$ is connected. ■

Remark 6 *Remark that the groups of automorphisms of hyperelliptic non-orientable Klein surfaces with two boundary components and even genus are, as for genus two:* $C_2, C_2 \times C_2$ *and* $C_2 \times C_2 \times C_2$*. The subspace of* $\mathcal{B}^{K,Hyp}_{(g,-,2)}$ *provided by NEC groups with signature* $(0;+;[2,\overset{\frac{g}{2}+1}{\ldots},2],\{(2,2)\})$ *is connected and cuts all the other equisymmetric subspaces.*

The groups of automorphisms of hyperelliptic non-orientable Klein surfaces with two boundary components of odd genus are: $C_2, C_4, C_2 \times C_2$ *and* D_4. *Again, the subspace of* $\mathcal{B}^{K,Hyp}_{(g,-,2)}$ *provided by NEC groups with signature* $(0;+;[4,2,\overset{\frac{g-1}{2}}{\ldots},2],\{(2,2)\})$ *(corresponding to surfaces with a unique topological class of actions of* C_4*) is connected and cuts all the other equisymmetric subspaces.*

The hyperelliptic branch locus of non-orientable Klein surfaces with two boundary components are connected.

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