On the relation between conservation and dual consistency for summation-by-parts schemes

Jan Nordström and Fatemeh Ghasemi

Journal Article

N.B.: When citing this work, cite the original article.

Original Publication:

Jan Nordström and Fatemeh Ghasemi, On the relation between conservation and dual consistency for summation-by-parts schemes, Journal of Computational Physics, 2017. 344, pp.437-439.

<http://dx.doi.org/10.1016/j.jcp.2017.04.072>

Copyright: Elsevier

<http://www.elsevier.com/>

Postprint available at: Linköping University Electronic Press

<http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-137545>

On the relation between conservation and dual consistency for summation-by-parts schemes

Jan Nordström & Fatemeh Ghasemi

*Department of Mathematics, Computational Mathematics, Linköping University,
SE-581 83 Linköping, Sweden*

`jan.nordstrom@liu.se, fatemeh.ghasemi@liu.se`

1. Introduction

We consider initial boundary value problems discretized by summation-by-parts (SBP) operators together with the simultaneous approximation term (SAT) technique. In [1, 2, 3, 4], it was shown that dual consistent discretizations on SBP-SAT form, leads to superconvergent linear functionals. For multi-block/element SBP-SAT based schemes [5, 6, 7, 8, 9], conservation is one of the most important issues. In this note, we show that dual consistency and conservation are equivalent concepts for linear conservation laws.

2. The continuous problem

Consider the two coupled linear conservation laws

$$\begin{aligned} u_t + f_L(u)_x &= 0, & -1 \leq x \leq 0, \quad t > 0, \\ v_t + f_R(v)_x &= 0, & 0 \leq x \leq 1, \quad t > 0, \\ f_L(u(0, t)) &= f_R(v(0, t)), & x = 0, \quad t > 0, \end{aligned} \tag{1}$$

augmented with initial conditions. The m component long solution vectors u and v are smooth and continuous, also across the interface. The subscripts L, R refer to the left and right spatial intervals, respectively. For clarity and ease of presentation, we ignore the boundary conditions at $x = \pm 1$.

The coupled problem (1) can be written more compactly as

$$\begin{aligned} w_t + F(w)_x &= 0, & -1 \leq x \leq 1, \quad t > 0, \\ f_L(u(0, t)) &= f_R(v(0, t)), & x = 0, \quad t > 0, \end{aligned} \tag{2}$$

where the flux is given by

$$F(w) = \begin{cases} f_L(u), & -1 \leq x \leq 0 \\ f_R(v), & 0 \leq x \leq 1, \end{cases} \quad \text{and} \quad w = \begin{cases} u, & -1 \leq x \leq 0 \\ v, & 0 \leq x \leq 1. \end{cases}$$

For future reference we define the inner product as $(g, h)_a^b = \int_a^b g^T h \, dx$.

2.1. Conservation

We multiply the equation (1) with the transpose of a smooth vector function $\varphi(x)$ with compact support in $-1 \leq x \leq 1$. Integration in space yields

$$\begin{aligned} \frac{d}{dt}(\varphi, w)_{-1}^1 &= (\varphi_x, f_L(u))_{-1}^0 - \varphi^T f_L(u)|_{-1}^0 + (\varphi_x, f_R(v))_0^1 - \varphi^T f_R(v)|_0^1 \\ &= (\varphi_x, f_L(u))_{-1}^0 + (\varphi_x, f_R(v))_0^1. \end{aligned} \quad (3)$$

The relation (3) show that the problem (2) is a conservation law.

2.2. The dual problem

The dual problem (see [2]) associated to (1), (2) is

$$\begin{aligned} \theta_\tau - F(\theta)_x &= 0, & -1 \leq x \leq 1, \quad \tau > 0, \\ f_L(\phi(0, t)) &= f_R(\psi(0, t)), & x = 0, \quad \tau > 0, \end{aligned} \quad (4)$$

augmented with homogeneous initial conditions. In (4), $\theta = \phi \in [-1 \leq x \leq 0]$ and $\theta = \psi \in [0 \leq x \leq 1]$. The boundary conditions at $x = \pm 1$ are ignored, and similiary to the primal problem, (4) can be shown to be conservative.

3. The semi-discrete approximation

We discretize the derivatives by using difference operators on SBP form,

$$(D_L \otimes I)\mathbf{f}_L \approx (f_L)_x, \quad (D_R \otimes I)\mathbf{f}_R \approx (f_R)_x \quad (5)$$

where $\otimes$ denotes the Kronecker product [10] and I is the $m \times m$ identity matrix. In (5), $D_{L,R} = P_{L,R}^{-1}Q_{L,R}$ where the matrices $P_{L,R}$ are symmetric positive definite, $Q_{L,R}$ are almost skew-symmetric and satisfy the SBP property $Q_{L,R} + Q_{L,R}^T = \text{diag}(-1, 0, \dots, 0, 1)$.

The semi-discrete SBP-SAT formulation of (2) for $\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T$ is

$$\mathbf{w}_t + P^{-1}(Q + \Sigma)\mathbf{F} = 0, \quad \Sigma = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & -\Sigma_L & \Sigma_L & \\ & & \Sigma_R & -\Sigma_R & \\ & & & \ddots & 0 \end{bmatrix}, \quad (6)$$

where $\mathbf{F} = [\mathbf{f}_L, \mathbf{f}_R]^T$, $Q = \text{diag}(Q_L, Q_R) \otimes I$ and $P = \text{diag}(P_L, P_R) \otimes I$.

3.1. Conservation of the semi-discrete approximation

Let the discrete scalar product for the two vectors $\mathbf{g}$ and $\mathbf{h}$ be defined by $(\mathbf{g}, \mathbf{h})_P = \mathbf{g}^T P \mathbf{h}$. Multiplying (6) by $\boldsymbol{\varphi}^T P$, with $\boldsymbol{\varphi} = [\boldsymbol{\varphi}_L, \boldsymbol{\varphi}_R]^T$, where $\boldsymbol{\varphi}$ is a projection of the smooth function φ onto the grid. By using the SBP property of Q_L and Q_R , and $(\boldsymbol{\varphi}_L)_N = (\boldsymbol{\varphi}_R)_0 := \boldsymbol{\varphi}_I$ this leads to

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\varphi}, \mathbf{w})_P &= -\boldsymbol{\varphi}^T(Q + \Sigma)\mathbf{F} = (D_L \boldsymbol{\varphi}_L)^T P_L \mathbf{f}_L + (D_R \boldsymbol{\varphi}_R)^T P_R \mathbf{f}_R \\ &\quad + \boldsymbol{\varphi}_I^T (\Sigma_L - \Sigma_R - I_m)[f_L(\mathbf{u}_N) - f_R(\mathbf{v}_0)], \end{aligned}$$

where $\mathbf{u}_N$ and $\mathbf{v}_0$ represent $u(0, t)$ and $v(0, t)$, respectively. The scheme (6) is *conservative* (compare with (3)) if

$$\Sigma_L = I_m + \Sigma_R. \quad (7)$$

3.2. Dual Consistency

The discrete problem (6) can be rewritten as

$$\mathbf{w}_t + L\mathbf{F} = 0, \quad L = P^{-1}(Q + \Sigma). \quad (8)$$

The discrete dual problem for $\boldsymbol{\theta} = [\boldsymbol{\phi}, \boldsymbol{\psi}]^T$ corresponding to (8) is (see [2])

$$\boldsymbol{\theta}_\tau + P^{-1}(L)^T P \mathbf{F} = 0. \quad (9)$$

Substituting L into (9), and using the SBP property of Q_L and Q_R , leads to

$$\boldsymbol{\theta}_\tau - P^{-1}(Q - \tilde{\Sigma})\mathbf{F} = 0, \quad \tilde{\Sigma} = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & I_m - \Sigma_L & \Sigma_R & & \\ & & \Sigma_L & -I_m - \Sigma_R & & \\ & & & & \ddots & 0 \end{bmatrix}. \quad (10)$$

Note that $P^{-1}Q\mathbf{F}$ approximates $F(\theta)_x$ in (4). If $P^{-1}\tilde{\Sigma}\mathbf{F}$ impose the dual interface conditions in (4), then the scheme (10) is *dual consistent* [1, 2].

4. The relation between conservation and dual consistency

We are now ready to state the main result.

Proposition 1. *The scheme (6) is conservative if and only if it is dual consistent.*

Proof. The vector $P^{-1}\tilde{\Sigma}\mathbf{F}$ impose the dual interface conditions, if $I_m - \Sigma_L = -\Sigma_R$, which is equal to the conservation condition (7). By inserting the conservation condition (7) into (10) we find that $P^{-1}\tilde{\Sigma}\mathbf{F}$ impose the dual interface conditions, which implies dual consistency of (10). $\square$

5. References

- [1] J. E. Hicken, D. W. Zingg, Superconvergent functional estimates from summation-by-parts finite-difference discretizations, *SIAM Journal on Scientific Computing* 33 (2011) 893–922.
- [2] J. Berg, J. Nordström, Superconvergent functional output for time-dependent problems using finite differences on summation-by-parts form, *Journal of Computational Physics* 231 (2012) 6846–6860.
- [3] J. E. Hicken, D. W. Zingg, Dual consistency and functional accuracy: a finite-difference perspective, *Journal of Computational Physics* 256 (2014) 161–182.
- [4] J. Berg, J. Nordström, On the impact of boundary conditions on dual consistent finite difference discretizations, *Journal of Computational Physics* 236 (2013) 41–55.
- [5] M. H. Carpenter, J. Nordström, D. Gottlieb, A stable and conservative interface treatment of arbitrary spatial accuracy, *Journal of Computational Physics* 148 (1999) 341–365.
- [6] M. H. Carpenter, J. Nordström, D. Gottlieb, Revisiting and extending interface penalties for multidomain summation-by-parts operators, *Journal of Scientific Computing* 45 (2010) 118–150.
- [7] J. Nordström, J. Gong, E. van der Weide, M. Svärd, A stable and conservative high order multi-block method for the compressible Navier-Stokes equations, *Journal of Computational Physics* 228 (2009) 9020–9035.
- [8] G. J. Gassner, A skew-symmetric discontinuous galerkin spectral element discretization and its relation to SBP-SAT finite difference methods, *Journal of Scientific Computing* 35 (2013) 1233–1253.
- [9] D. Funaro, D. Gottlieb, A new method of imposing boundary conditions in pseudospectral approximations of hyperbolic equations, *Mathematics of Computation* 51 (1988) 599–613.
- [10] R. A. Horn, C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.