Spectra, current flow, and wave-function morphology in a model $\mathcal{PT}$ -symmetric quantum dot with external interactions

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In this paper we use numerical simulations to study a two-dimensional (2D) quantum dot (cavity) with two leads for passing currents (electrons, photons, etc.) through the system. By introducing an imaginary potential in each lead the system is made symmetric under parity-time inversion ($\mathcal{PT}$ symmetric). This system is experimentally realizable in the form of, e.g., quantum dots in low-dimensional semiconductors, optical and electromagnetic cavities, and other classical wave analogs. The computational model introduced here for studying spectra, exceptional points (EPs), wave-function symmetries and morphology, and current flow includes thousands of interacting states. This supplements previous analytic studies of few interacting states by providing more detail and higher resolution. The Hamiltonian describing the system is non-Hermitian; thus, the eigenvalues are, in general, complex. The structure of the wave functions and probability current densities are studied in detail at and in between EPs. The statistics for EPs is evaluated, and reasons for a gradual dynamical crossover are identified.

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I. INTRODUCTION

Introducing $\mathcal{PT}$ -symmetric non-Hermitian [1,2] Hamiltonians gives rise to a number of new phenomena not seen in Hermitian quantum mechanics [3,4]. One of the most striking differences occurs when degeneracies are considered. For Hermitian Hamiltonians two states are degenerate if they have the same eigenvalue and their eigenstates are linearly independent. However, for non-Hermitian Hamiltonians the crossing of two eigenvalues, under the variation of some parameter, may give rise to eigenstates that differ by only a phase factor [5,6]. To study this phenomenon we thus analyze a parameter-dependent Hamiltonian $H = H(iV_I)$, where V_I is a real parameter. For $\mathcal{PT}$ -symmetric Hamiltonians it is natural to let V_I emulate the gain and loss in the system. By varying this parameter, let us say by gradually increasing it, it is possible to cause two states to coalesce at some critical value of V_I ; at this point the eigenvalues become equal and form an exceptional point (EP). For a $\mathcal{PT}$ -symmetric Hamiltonian the eigenvalues are real for values of V_I less than this specific value; in this regime the symmetry is still unbroken. At the EP, where the real eigenvalues meet, the symmetry breaking sets in, and the eigenvalues become complex and the complex conjugate of one another [4] on further parametric increase. EPs of this kind may be studied experimentally, and there is a rapidly increasing number of studies related to fundamental science and technology. For example, there are studies with pump-induced lasers [7,8], microwave cavities and wires [9–17], LRC circuits [18], exciton-polariton billiards [19], and more [20,21].

Here we consider a specific system, a model $\mathcal{PT}$ -symmetric quantum dot described by the Hamiltonian $H(V_I)$ with a tuning parameter for controlling gain and loss. Our aim is thus to

numerically perform a detailed study of EPs in a realistic two-dimensional (2D) cavity in contact with two reservoirs for input and output (gain and loss), for example, when a finite bias is applied. The roles of the reservoirs and leads are here emulated by imaginary potentials iV_I . In a number of previous studies parametrized analytic models with a small number of states were used (see, e.g., [8,22–27]). Such matrix models give clear insight into the principal analytic features of non-Hermitian Hamiltonians and EPs. They do not, however, describe the global features and effects of EPs. We also present simulations of the probability current flow, which is a novel aspect of this kind of physics. Such a current could be measured in a $\mathcal{PT}$ -symmetric microwave billiard since it corresponds to the Poynting vector [28].

A numerical study like this one shows many of the known phenomena related to non-Hermitian Hamiltonians such as width bifurcation, resonance trapping, similarities with dynamical phase transitions (DPTs), and the appearance of noninteracting states (NISs). With many states there is an interplay between all these phenomena, and therefore, each individual phenomenon is less distinct than in analytical studies with a small number of states. Dynamical phase transitions are caused by the influence of EPs; this influence is observed as an enhancement of transmission through the dot [22]. As shown in [29,30], the transmission is anticorrelated to the phase rigidity, which is given by

$$r_i = \frac{\langle \psi_i^* | \psi_i \rangle}{\langle \psi_i | \psi_i \rangle}, \quad (1)$$

where the denominator $\langle \psi_i | \psi_i \rangle$ is the standard Hermitian inner product. To understand the numerator we must take into account that the Hamiltonian is non-Hermitian. For a non-Hermitian Hamiltonian the left and right eigenfunctions form a biorthogonal set. If the Hamiltonian is symmetric, i.e., if $H^T = H$, where H^T denotes the transpose of H , the

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FIG. 1. The structure of the cavity and the grid used for the computation. Arbitrary units are used, but in a realistic case this could be a 22×21 cm electromagnetic cavity or a 2.2×2.1 μm quantum dot [33]. The imaginary potential is applied in the filled regions at the end of the leads.

biorthogonal relation is [5]

$$\langle \psi_i^* | \psi_j \rangle = \delta_{ij}. \quad (2)$$

When the wave functions are real, the phase rigidity is equal to 1 since $\langle \psi_i^* | \psi_i \rangle = \langle \psi_i | \psi_i \rangle$; by enforcing this and Eq. (2) the phase is fixed. But as an EP is approached the phase rigidity goes to zero as $\langle \psi_i | \psi_i \rangle \rightarrow \infty$. Phase rigidity can therefore be seen as a measure of how far the states are from being orthogonal [5]. Related to the phase rigidity is another quantity that gives a more direct measure of the mixing between two states. By expressing ψ_j in a basis $\{\psi_i^0\}$ we define the mixing coefficients as

$$\psi_j = \sum_i b_{ij} \psi_i^0. \quad (3)$$

The basis is chosen to be the eigenfunctions for the system with $V_I = 0$. As ψ_j and ψ_i approach an EP, the mixing becomes stronger, and at the EP $|b_{ij}| \rightarrow \infty$ [24].

As mentioned, the phase rigidity is anticorrelated with the transmission. Therefore, if the phase rigidity goes to zero for all states at a certain value of the tuning parameter, the entire system undergoes a sharp DPT. However, if states are isolated from the leads of the system and hence from the imaginary potential, their interaction with other states is negligible. For this reason such states will never form EPs and therefore always have real eigenvalues. As a consequence, these states

behave as bound states since they have infinite lifetime and are termed noninteracting states. They are changed so little as the imaginary potential is changed that one cannot expect that these are part of the phase transition. Thus, because of the spread of interactions, we rather describe this as a dynamical crossover (DC) rather than a DPT. The formation of noninteracting states as above is clearly visualized in [31,32] and in Fig. 7 below, which shows how the wave function resides in the interior of the structures.

This paper is organized as follows. In Sec. II we recall the physics of open quantum systems with imaginary potential and with a special focus on wave-function morphology, flow patterns, and DCs. The numerical method used is also described. The results are shown in Sec. III, and finally, conclusions are presented in Sec. IV.

II. QUANTUM DOT WITH IMAGINARY POTENTIAL

A. Model of an electron dot

The two-dimensional model quantum dot (cavity) we have in mind is shown in Fig. 1; it is designed from the real semiconductor nanodevice described in [33]. We thus consider a dot that is embedded in between source and drain regions (electron reservoirs) and assume that there is an interaction in terms of leads between these and the states within the dot. By applying a small voltage bias between the reservoirs the particles will tunnel or diffuse via the leads from the source to the drain. Alternatively, we may assume that the transport of particles between the two reservoirs via the dot takes place because of a difference in temperatures in the two regions. If we assume that the dot is nearly closed, i.e., the interaction between the dot and reservoirs is weak and hence the current is small, we may emulate the system in the following way. Thus, we do away with the reservoirs and replace them by imaginary potentials in the regions of leads (see recent work on interacting Bose-Einstein condensates [34]). Effectively, there will thus be gain and loss in the system, i.e., input and output of particles, as an imaginary potential is used [35]. This potential is induced in the leads as marked in Fig. 1 and is, to summarize, used to replace, for example, infinite leads or a sea of scattered wave functions [36]. Here we focus on a semiconductor quantum dot and the Schrödinger equation. The description is, however, generic and may be applied to, for example, microwave billiards [37,38].

FIG. 2. (a) The energies and (b) widths for state numbers 20 to 40.

FIG. 3. (a) The energy of states 39 and 40; the states coalesce and then stay that way until they separate. (b) During the coalesced period the widths bifurcate. (c) The fact that the eigenvalues are complex makes it possible to study their trajectories in the complex plane. These trajectories intersect at an EP. (d) The phase rigidity of the two states; where the phase rigidity is zero, we have an EP.

To motivate the use of an imaginary potential, one may consider the two-dimensional Schrödinger equation for the wave function $\Phi(x, y, t)$:

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Phi + V \Phi, \quad (4)$$

where $V(x, y) = V_R(x, y) + iV_I(x, y)$. In this case V_R is a real confining potential which equals zero inside the cavity and infinity at the boundaries, giving a “particle-in-a-box” structure. The second part of the potential, V_I , is also real, but multiplied by i , it gives the imaginary potential mentioned above. This Hamiltonian is no longer Hermitian, but by enforcing an extra condition it can be made $\mathcal{PT}$ symmetric. This extra condition restricts the potential to $V(-x) = V^*(x)$, where the asterisk (*) denotes the complex conjugation and $(-x)$ denotes inversion through the symmetry axis of the cavity [38]. This means that gain and loss in the system are equal as long as the $\mathcal{PT}$ symmetry is unbroken. For a real potential the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (5)$$

where $\rho = |\Phi|^2$ and the current density is

$$\mathbf{J}(x, y, t) = \frac{i\hbar}{2m} (\Phi \nabla \Phi^* - \Phi^* \nabla \Phi). \quad (6)$$

An extra term arises when an imaginary potential is introduced as in [35],

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = \frac{2V_I}{\hbar} \rho. \quad (7)$$

Since the potential $V(x, y)$ is independent of time, we may write the wave function and current density as

$$\Phi(x, y, t) \rightarrow \psi(x, y) \exp(-iEt/\hbar), \quad (8)$$

$$\mathbf{J}(x, y, t) \rightarrow \mathbf{j}(x, y) \exp[2\text{Im}(E)t/\hbar]. \quad (9)$$

FIG. 4. The mixing coefficients of states 39 and 40. The coefficients $|b_{39,39}|$ and $|b_{40,40}|$ begin at 1, while $|b_{39,40}|$ and $|b_{40,39}|$ begin at zero.

FIG. 5. The wave function of state 39 is shown for different values of V_I . For $V_I = 0$ the Hamiltonian is Hermitian so the wave function is real in (a). At the first EP the wave function is complex; the real part is shown in (b), and the imaginary is shown in (c). In the coalesced region, where the width is largest, the real and imaginary parts of the wave function are given in (d) and (e), respectively. For the second EP, the real part is shown in (f), and the imaginary part is shown in (g). Beyond the second EP, when the states have separated, the real part appears as in (h), and the imaginary appears as in (i). Colors are as follows: red (lighter gray) and blue (darker gray) show positive and negative regions of the wave functions; white lines define nodal lines. Axis as in Fig. 1.

By Eq. (7) $V_I > 0$ can be regarded as the creation of probability, which may be interpreted as gain. In the same way, $V_I < 0$ can be regarded as loss, so the condition $V(-x) = V^*(x)$ is equivalent to $V_I(-x) = -V_I(x)$, which means equal gain and loss.

By putting $\hbar^2/2m = 1$ in Eq. (4) and studying the time-independent wave function ψ the Schrödinger equation becomes

$$\nabla^2 \psi - i V_I \psi + E \psi = 0. \quad (10)$$

This may be solved as an eigenvalue problem with E as the eigenvalue, which in general may be complex. As discussed in Sec. I, changing the gain and loss may break the $\mathcal{PT}$ symmetry, and that is also the case here. At certain V_I at which interactions

between close-by levels are large the symmetry breaks; that is, the states encounter an EP, and E becomes complex from there on. The complex eigenvalue for state number j may be written as

$$E_j = \mathcal{E}_j - \frac{i}{2} \Gamma_j, \quad (11)$$

where $\mathcal{E}_j$ is the energy and Γ_j is the width [22]. Assume that state j has coalesced with state i . At the EP the eigenvalues are equal, $E_j = E_i$; as V_I continues to increase, the energies will stay degenerate, $\mathcal{E}_j = \mathcal{E}_i$, but the widths bifurcate, $\Gamma_j = -\Gamma_i$, until the second EP is reached and the states separate again [3]. The widths are pushed back together because of resonance trapping, i.e., the creation of very broad states that eat the widths of the other states. This

FIG. 6. The wave function at $V_l = 0$ for state 40 is shown in (a). For the first EP, the real part of the wave function is shown in (b), and the imaginary part is shown in (c); note that these panels are equal to -1 times Figs. 5(c) and 5(b) (unchanged), respectively. In the coalesced region where the width is largest the real part appears as in (d), and the imaginary part appears as in (e); note that these are just the $\mathcal{PT}$ transformations of Figs. 5(d) and 5(e). For the second EP, the real part is shown in (f), and the imaginary part is shown in (g); also, these states differ by just a factor i from the corresponding figures for state 39. When the states have separated again, beyond the second EP, the real part appears as in (h), and the imaginary part appears as in (i). Colors are as in Fig. 5. Axis as in Fig. 1.

phenomenon has been studied in [39–41] and experimentally in [42,43]. We will see that this is closely connected to a DC.

Let us now look closer at the region between two EPs; in this region we will adopt the notation $E_j = \mathcal{E} - \frac{i}{2}\Gamma$ and $E_i = \mathcal{E} + \frac{i}{2}\Gamma$. To understand the physical meaning of these complex eigenvalues we calculate the relative population probability for states i and j in this region. Let $|\Phi(t)\rangle$ be the right solution to the Schrödinger equation; the Hamiltonian is symmetric, so the left solution is $\langle\Phi^*(t)|$. The spatial wave functions are biorthogonal, so the normalization is given by Eq. (2). Now, the left and right solutions of Eq. (4) are time dependent, but since the Hamiltonian is time independent, they may be written

as

$$|\Phi_j(t)\rangle = e^{-iE_j t/\hbar} |\psi_j\rangle, \quad (12)$$

$$\langle\Phi_j^*(t)| = \langle\psi_j^*| e^{iE_j^* t/\hbar} \quad (13)$$

for state j and

$$|\Phi_i(t)\rangle = e^{-iE_i t/\hbar} |\psi_i\rangle, \quad (14)$$

$$\langle\Phi_i^*(t)| = \langle\psi_i^*| e^{iE_i^* t/\hbar} \quad (15)$$

for state i . These states may now be joined to the relative population probability of each state,

$$\langle \Phi_j^*(t) | \Phi_j(t) \rangle = e^{-\Gamma t/\hbar}, \quad (16)$$

$$\langle \Phi_i^*(t) | \Phi_i(t) \rangle = e^{\Gamma t/\hbar}, \quad (17)$$

which means that state j has a decay rate $\propto \Gamma$, while i has a growth rate $\propto \Gamma$.

If the system consists of fermions, we have now reached a limitation of the model. Since fermions need to obey Pauli's exclusion principle, it is in this case nonphysical to have exponential population growth since the states cannot be infinitely populated. However, for bosons [44] and the other wave analogs this is possible.

B. Dynamical phase transitions

The notion of DPT is usually considered in the context of non-Hermitian Hamiltonians describing open systems [22], and we will in the beginning do the same but in the last paragraph connect this discussion to our specific $\mathcal{PT}$ -symmetric system. Consider a simple non-Hermitian Hamiltonian of the form

$$H = H_0 - i\alpha W W^\dagger, \quad (18)$$

where H_0 is the Hermitian Hamiltonian of a closed system, W describes the coupling between the channels and the eigenstates of H_0 , and α is a real positive control parameter. Let H_0 describe the N discrete states of the closed system, and assume that the system can have $K < N$ channels. Then W is a $K \times N$ matrix in which each element W_{kn} describes the coupling of state n to channel k . For small α the $W W^\dagger$ term in Eq. (18) can be seen as a small perturbation, so H is expected to have N almost real eigenvalues. On the other hand, for large α the H_0 term can be seen as a perturbation compared to $W W^\dagger$, and we therefore obtain K eigenvalues with a large imaginary part. This may be regarded as a transition between two phases with α as a control parameter. Dynamical transitions with the Hamiltonian in Eq. (18) have been studied in, e.g., [45,46] and used in [42,43] to interpret the experimental data. In [45,46] the number of channels is $K = 1$, and it has been shown that the transition really can be regarded as a second-order phase transition in the sense of thermodynamics in the limit $N \rightarrow \infty$. Although a finite N does not mathematically give a proper phase transition, the physical features of a phase transition are obtained for a finite N .

If α is allowed to be complex [46] or the system is allowed to have more than one channel [29,30], there is no longer a specific value of α at which the phase transition occurs but rather a finite interval over which the transition occurs. This transition can be observed as a conductance peak. The physical reason for this transition is fundamentally the presence of EPs. Near EPs the phase rigidity is less than 1, so the resonance states of the system will start to align with the scattering states outside the system. This alignment is mediated by the $W W^\dagger$ term. In the critical parameter range we have many EPs, so many states are aligned with the scattering states outside, which allows for traveling modes in the system. These traveling modes explain the high conductance.

FIG. 7. The probability density for state 37 at $V_I = 0$; the state is localized inside the cavity. Colors are as in Fig. 5. Axis as in Fig. 1.

In our specific case with the $\mathcal{PT}$ -symmetric Hamiltonian $H(V_I)$ describing the structure in Fig. 1, there are two leads, so the number of open channels is 2 times the number of accessible transverse modes.

C. Computation

The computations are performed with the finite-difference method using MATLAB. To solve the obtained eigenvalue problem, Eq. (10), we write it in the form

$$H\bar{\psi} = -E\bar{\psi}, \quad (19)$$

where $\bar{\psi}$ is a column vector containing the wave function in every point shown in Fig. 1 and H is the matrix representation of the Hamiltonian obtained by finite differences. By relabeling the elements in $\bar{\psi}$ such that they carry two indices, one representing the x coordinate and one representing the y coordinate, a general matrix element inside the dot is written as

$$H_{nk} = \frac{\psi_{n-1,k} + \psi_{n,k-1} - 4\psi_{n,k} + \psi_{n+1,k} + \psi_{n,k+1} - i\hbar^2 V_I(n,k)}{\hbar^2}, \quad (20)$$

FIG. 8. The phase rigidity for state 37; because of the localization of this state it is hardly affected by the change in V_I .

FIG. 9. The wave function for state 22 is shown in (a), and that for state 23 is shown in (d). As V_I increases, the states get pushed out from the cavity; (b) and (c) show the real and imaginary parts of the wave function for state 22 at $V_I = 0.8$, respectively. In (e) and (f) the corresponding case is shown for state 23. Colors are as in Fig. 5. Axis as in Fig. 1.

where h is the spacing between the grid points. The imaginary potential V_I is only nonzero in the regions marked in Fig. 1. The boundary conditions are chosen to be Dirichlet, $\psi = 0$, everywhere except at the end of the leads, where the Neumann condition, $\partial\psi/\partial x = 0$, is chosen since we want the system to be open. The basis used in Eq. (3) is calculated with the same boundary conditions but with $V_I \equiv 0$ in each matrix element. To generate the grid the method presented in [47] is used, and to obtain the matrix H_{nk} and to implement the boundary conditions methods from [48] are used. For computational efficiency and to minimize the memory usage sparse matrices are used, and the predefined routine EIGS is used to solve Eq. (19).

III. RESULTS

A selection of characteristic states is examined more thoroughly, but some general aspects are also presented. The energy spectrum for states 20 to 40, i.e., $\mathcal{E}_j$ for $j = 20, 21, \dots, 40$, with V_I ranging from 0 to 1.2 (in an appropriate choice of units) is shown in Fig. 2(a) together with the widths $\Gamma_j/2$ in Fig. 2(b). Lower states are not considered since they are below the threshold of the first channel and hence unaffected by V_I . In all the figures arbitrary units are assumed.

A. Coalesced states

To begin with, consider states 39 and 40. These are the uppermost states in Fig. 2(a); the energies are more clearly

FIG. 10. The energy-level spacing distribution for two values of V_I . In (a) $V_I = 0$, and in (b) $V_I = 0.4$; the number of states with level spacing equal to zero has increased for $V_I = 0.4$ compared to $V_I = 0$, indicating a dynamical crossover.

FIG. 11. The energy of states 30, 31, and 32 (and the collective state) is shown in (a). The width of these states is shown in (b), and the trajectories are shown in (c). From the mixing coefficients $|b_{31,30}|$ and $|b_{32,30}|$ in (d) we observe that although state 30 is real in the entire parameter region, it interacts with states 31 and 32 when they are close to each other.

shown in Fig. 3(a) together with the widths in Fig. 3(b). At first these states are separated, but as V_I increases, they approach each other, and eventually, they coalesce in an EP. This point has many characteristic features; the trajectories of the eigenvalues in the complex plane intersect [Fig. 3(c)], the phase rigidity is zero [Fig. 3(d)], and the mixing coefficients b_{ij} in Eq. (2) go to infinity (Fig. 4). Note that the phase rigidity increases between the two EPs and reaches its maximum at maximum width bifurcation. We note that we are not able to reproduce the results in [24,26,49], in which the phase rigidity almost reaches 1 between the EPs. When $V_I = 0$ the system is Hermitian, so both the eigenvalues and eigenfunctions are real; these wave functions are shown in Fig. 5(a) for state 39 and Fig. 6(a) for state 40. When $V_I > 0$, the two states begin to mix, as seen from the mixing coefficients, which all increase. State 40 obtains a small imaginary perturbation resembling state 39 for sufficiently small $V_I \neq 0$ and vice versa for state 39; this is summarized in the following expressions:

$$\psi_{40}(V_I > 0) \approx \psi_{40}(V_I = 0) + i\lambda\psi_{39}(V_I = 0), \quad (21)$$

$$\psi_{39}(V_I > 0) \approx \psi_{39}(V_I = 0) + i\lambda\psi_{40}(V_I = 0), \quad (22)$$

where λ is a small number. As the EP is approached, however, the mixing becomes stronger, and at the same time the phase rigidity becomes smaller. Finally, at the EP the states are multiples of each other; that is, they differ only by a phase factor. By studying Figs. 5(b), 5(c), 6(b), and 6(c) the phase

factor can be determined to be

$$\psi_{40} = i\psi_{39} \quad (23)$$

at the first EP. After the first EP, the two states stay coalesced, and the real and imaginary parts of the wave functions are of the same magnitude. In this region the eigenvalues are the complex conjugate of one another [Fig. 3(b)], and the wave functions are related by a $\mathcal{PT}$ transformation given by

$$\psi_{40} = \mathcal{PT}(\psi_{39}), \quad (24)$$

as seen in Figs. 5(d), 5(e), 6(d), and 6(e).

FIG. 12. The wave function for state 30 at $V_I = 0$. Colors are as in Fig. 5. Axis as in Fig. 1.

FIG. 13. The wave function for state 31 with different values of V_I . The case when $V_I = 0$ is shown in (a); the real and imaginary parts of the wave function at the EP are shown in (b) and (c), respectively. Colors are as in Fig. 5. Axis as in Fig. 1.

Eventually, states 39 and 40 separate again at a second EP with a further increase of V_I . As before the states are multiples of each other,

$$\psi_{40} = i\psi_{39}, \quad (25)$$

as seen in Figs. 5(d), 5(e), 6(d), and 6(e). When the second EP is passed with a further increase of V_I , the wave functions regain much of their former structure [Figs. 5(h), 5(i), 6(h), and 6(i)], with the difference that they are now $i^2 = -1$ times Eqs. (21) and (22). The real part of the states beyond the two EPs is now dominant, and the imaginary part may be considered a small perturbation.

B. Noninteracting states and dynamical crossover

As Fig. 2(a) shows, there are states that do not coalesce with other states. So even though these states have energies above the threshold, which means that the system is open for these energies, they have real eigenvalues. These states form NISs. The reason for this is that they have very weak interaction with the imaginary potential. An example of such a state is state 37, shown in Fig. 7, where $|\psi_{37}|^2$ is shown at $V_I = 0$. As seen, the state is localized inside the cavity and is only weakly affected by V_I , as Fig. 8 shows. The small dip in phase rigidity observed in Fig. 8 appears in the parameter range in which we observe most EPs [Fig. 2(a)]. So EPs not only affect the states in which they appear but actually affect states in their surroundings. The appearance of many EPs in a finite parameter range indicates that the system undergoes a dynamical crossover in this regime.

As discussed in Sec. II B the notion of dynamical phase transitions is especially clear for an open system modeled by Eq. (18). For the $\mathcal{PT}$ cavity studied here, as just observed, not all states participate, so we call the phenomenon in this case a dynamical crossover. But in line with the model in Sec. II B the number of collective states is the same as the number of open channels. The cavity has two openings, so when the energy reaches above the threshold of the first transverse mode in the leads, two channels are opened. However, because of the $\mathcal{PT}$ symmetry both channels are not decay channels; rather, just one is a decay channel, while the other is for growing states. Because of this only one of the collective states that are formed when a channel is opened is very broad; the other is, instead, a strongly growing state.

The first transverse modes in the leads appear for energies just below that of state 22 [Fig. 2(a)]. Therefore, states 22 and 23 form the first collective states. These states are broad and strongly growing [Fig. 2(b)], and their wave functions show strong overlap with the imaginary potential (Fig. 9). The formation of these states pushes the other states back into the cavity by resonance trapping [Fig. 2(b)]. At sufficiently large V_I the states that coalesced into EPs will separate and obtain real eigenvalues. So for these large V_I both the NISs and the previously coalesced states have real eigenvalues, and only the collective states have complex eigenvalues. The system has crossed over between two phase, the first phase (small V_I) with only real eigenvalues and the second (large V_I) in which the number of states with complex eigenvalues is equal to the number of open channels.

FIG. 14. The wave function for state 32 with different values of V_I . The case when $V_I = 0$ is shown in (a). The real and imaginary parts of the wave function at the EP are shown in (b) and (c), respectively. Colors are as in Fig. 5. Axis as in Fig. 1.

FIG. 15. The probability current density and nodal lines, $\text{Re}(\psi) = 0$ (black) and $\text{Im}(\psi) = 0$ (gray), for states (a) 39 and (b) 40 at the first EP. The structure of the current density is equal for the two states because the wave functions differ by only a phase, which also leads to the fact that the black and gray lines are interchanged between the panels. Axis as in Fig. 1.

In Fig. 2(a) only 20 states are visualized, so to see that the density of coalesced states is globally increased around $V_I \sim 0.4$ we study the level spacing distribution of 800 states (Fig. 10). These figures show a clear increase of states with small energy spacing, which is due to a higher number of coalesced states.

C. Coalesced states and NISs

Since only some states coalesce, there will be situations where coalesced states interact with NISs, e.g., states 30 (NIS), 31, and 32 in Fig. 2(a). The energy of these states is shown in Fig. 11(a); as seen there, states 31 and 32 coalesce, and the coalesced state intersects state 30. There is also a fourth state in Fig. 11(a); this state is the collective state of states 22 and 23.

In Fig. 11(b) the widths of the states are shown; states 32 and 31 bifurcate at the EP, while the width for state 30 stays equal to zero. We observe that around $V_I = 0.7$ state 30 intersects states 31 and 32 (in energy). At this point the difference in energy between the states is zero, while the difference in width is ~ 0.025 . Since the difference in width is greater than the difference in energy, the states feel each other and interact. This interaction is observed in the mixing coefficients $|b_{31,30}|$

and $|b_{32,30}|$ [Fig. 11(d)] where an effect is clearly shown around $V_I = 0.7$. Note that state 30 mixes somewhat stronger with 31 than 32, this is because states 30 and 31 are, on average, closer than states 30 and 32 over this parameter range. The trajectories of states 30, 31, and 32 are shown in Fig. 11(c).

Even though state 30 is a NIS, we have observed that it interacts with neighboring states. From the wave function of this state (Fig. 12) we note a difference from the previously studied NIS, state 37 in Fig. 7. The wave function of state 30 has an overlap with the imaginary potential and hence has some interaction with the other states; this differs from state 37, which is almost totally localized to the interior of the structure. Since it is the imaginary potential that mediates the interaction between the states, this explains why state 30 has a stronger interaction with its neighbors than state 37, even though both are NISs. The mixing of states 31 and 32 resembles the mixing of states 39 and 40. For small V_I (before the EP) state 32 obtains state 31 as a small imaginary perturbation and vice versa [compare with Eqs. (21) and (22)]. At the EP the states are multiples of each other with the same factor i as states 39 and 40 (Figs. 13 and 14):

$$\psi_{32} = i\psi_{31}. \quad (26)$$

FIG. 16. The probability current density and nodal lines (black for the real part and gray for the imaginary part) for states (a) 39 and (b) 40 at V_I in the coalesced region. Here the currents are mirrored through the x axis since the states are a $\mathcal{PT}$ transformation of each other. State 39 has a flow out of the cavity and is therefore the state that decays, while state 40 has a flow in and is therefore the growing state. Axis as in Fig. 1.

FIG. 17. The probability current density and nodal lines (black for the real part and gray for the imaginary part) for states (a) 39 and (b) 40 at the second EP. At this point the current density has regained its structure at the first EP, and the black and gray lines are interchanged between the panels, which shows that the states differ by only a phase i . Axis as in Fig. 1.

D. Probability current density

As discussed in the Introduction, the observed DC can be observed as increased conductance. The conductance can be obtained from the probability current density, Eq. (6), and the current density is also (in the microwave analogs) a direct measurable quantity given by the Poynting vector [28]. We have studied the current for states 39 and 40. At the first EP the currents have the same structure (Fig. 15). This is because the states at this point differ by only a phase, $\psi_{40} = i\psi_{39}$, and the current, Eq. (6), is clearly independent of a global phase. This phase difference is observed in the nodal lines, which are the lines where $\text{Re}(\psi) = 0$ or where $\text{Im}(\psi) = 0$, as they are interchanged between Figs. 15(a) and 15(b). The nodal lines intersect at nodal points, $\psi(x, y) = 0$, around which the current density circulates in the form of a vortex [50]. Much of the current structure can be obtained by studying only the nodal lines and nodal points [51]. The nodal points may, e.g., also be used to determine if an open quantum system is chaotic, as done in [52], and they give a qualitative indication of the transmission through the cavity [53].

In the coalesced region (Fig. 16) we have the complex-conjugate eigenvalues, and the states differ by a $\mathcal{PT}$ transformation. In this case it is state 40 that is the growing state, while state 39 is the decaying one. This is seen in Fig. 16 as the current corresponding to state 40 is flowing into the cavity while the current for state 39 flows out. This is a reflection of the $\mathcal{PT}$ relation between the states. This relation implies that $\mathbf{j}_{40} = \mathcal{P}(\mathbf{j}_{39})$. At the second EP the currents for the two states have the same structure (Fig. 17). Note that this structure resembles the structure at the first EP.

IV. CONCLUSIONS

In this paper we have studied a $\mathcal{PT}$ -symmetric 2D quantum cavity with almost 5000 interacting states. The system has an imaginary potential in each of the leads that emulates, e.g., infinite leads or a sea of scattered states. Exceptional points are created in this system when the energy is high enough that the first transverse mode in the leads is accessible. Just above the threshold for each transverse mode two collective states

are created, one growing and one decaying. The exponential growing mode is nonphysical for fermions since it would, at some time, be populated with enough fermions to violate the Pauli principle. But for bosons and the classical analogs there is no problem. Above the first threshold many EPs are created [Fig. 2(a)], but not all states participate in this process; some states have real eigenvalues in the entire parameter range and form NISs. These states stay real because they are localized mainly inside the cavity (Fig. 7). In the parameter region where many EPs are created the system undergoes a dynamical crossover. It goes from a phase with only real eigenvalues to a phase in which only the collective modes (two for each open transverse mode) still have complex eigenvalues. The system reaches this phase thanks to the collective states, which, due to resonance trapping [Fig. 2(b)], decrease the width of the states that created EPs and force them to eventually become real.

We have done a detailed study of the wave functions at different parameter values and observed that two states differ by only a factor i at the EPs (Figs. 5, 6, 13, and 14). The EPs generate a DC between the two dynamical phases. We avoid the term dynamical phase transition because of the existence of NISs that do not participate in the crossover. However, the NISs are not entirely unaffected by their surroundings; we show in Fig. 11(d) that there is a weak mixing between a NIS and its neighboring states with a maximum where they are degenerate in energy.

At the EPs the states differ by only a phase, and in the coalesced region they differ by a $\mathcal{PT}$ transformation. This relation implies that the probability current density of the two states at the EP is equal (Figs. 15 and 17), while they are the mirror image of each other in the coalesced region (Fig. 16). Experimental verification of such phenomena should be within reach, for example, confirmation of dynamical crossovers.

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