

On Social Choice in Social Networks

Ema Becirovic

Master of Science Thesis in Electrical Engineering
On Social Choice in Social Networks

Ema Becirovic

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Supervisor: **Marcus Karlsson**
ISY, Linköpings universitet

Examiner: **Erik G. Larsson**
ISY, Linköpings universitet

*Division of Communication Systems
Department of Electrical Engineering
Linköping University
SE-581 83 Linköping, Sweden*

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To my friends and family.

Sammanfattning

Kollektiva beslut blir en del av vardagen när grupper av människor står inför val. Vi anpassar ofta våra personliga övertygelser med hänsyn till våra vänner. Vi är naturligt beroende av lyckan hos dem som står oss nära.

I det här exjobbet undersöker vi en befintlig *empatimodell* som används för att välja en vinnare från en uppsättning alternativ genom att använda poängbaserade omröstningsprocedurer. Vi visar att en liten modifikation av modellen är tillräcklig för att kunna använda överlägsna omröstningsprocedurer som bygger på parvisa jämförelser av alternativen.

Sammanfattningsvis visar vi att det i grunden inte finns någon anledning att använda poängbaserade omröstningsprocedurer i de föreslagna modellerna, eftersom ett mer önskvärt resultat uppnås genom att använda de överlägsna omröstningsprocedurerna.

Abstract

Social choice becomes a part of everyday life when groups of people are faced with decisions to make. We often adjust our personal beliefs with the respect to our friends. We are inherently dependent on the happiness of those near us.

In this thesis, we investigate an existing *empathy model* that is used to select a winner in a set of alternatives by using scoring winner selection methods. We show that a slight modification of the model is enough to be able to use superior winner selection methods that are based on pairwise comparisons of alternatives.

We show that there is essentially no reason to use scoring winner selection methods in the models proposed as a more desirable result is achieved by using superior winner selection methods.

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Notation

MATHEMATICAL NOTATION

Notation	Description
$(\cdot)^\top$	Matrix transpose
$\mathbf{1}_m$	Vector of ones with dimension m
I_m	Identity matrix with dimension $m \times m$
\otimes	The Kronecker product
$\mathcal{G}(\mathcal{V}, \mathcal{E})$	Graph
\mathcal{V}	Set of nodes
\mathcal{E}	Set of edges
\mathcal{C}	Set of candidates
\mathcal{S}	Smith set
n_v	Number of nodes
n_e	Number of edges
n_c	Number of candidates
d	Degree
d^{in}	In-degree
d^{out}	Out-degree

ABBREVIATIONS

Abbreviation	Description
DD	Decision disagreement
SNAP	Stanford Network Analysis Platform

1

Introduction

Social choice and group decision making have been a part of our society for a very long time. There are many reasons to collectively make a decision as a group. There could be a political election where a nation chooses its leader for the subsequent years, or there could be a group of friends choosing what pizza to order for dinner. Regardless of what the circumstances for making a decision are, there are many different ways of electing a winning alternative.

Groups of people and their relation to each other can be represented in social networks. The networks represent connections, either social or professional. In recent years, with the emergence of social media, the consciousness of these networks has grown and accessing the networks has become easier through social media platforms. More and more of our interaction with each other occur on these platforms. However, we tend to create so called *echo chambers* where we only interact with people that have the same beliefs as we do, which might legitimise opinions that, in the population as a whole, are not supported.

1.1 Motivation

This thesis will focus on how to combine social choice theory and network theory. How can the underlying structure of a network be used when voting? Although the principal focus of the thesis is humans voting in an election, it is not necessary to exclusively consider this scenario. The method proposed in this thesis can be used with agents in artificial intelligence systems, where the goal for the agents is to make a group decision on a set of alternatives.

Consider the case when a group of friends are going out to eat. One of the people is allergic to food served at most of the places, but there are two restaurants where they serve food that the person with the allergy can eat. However, those two restaurants are not popular among the others in the group. When the

group votes, and every person in the group is honest, they will select a restaurant where the person with the allergy is not able to choose any food that they could eat. That person will then complain that they are hungry and in the end nobody will be happy. In this case, it would have been beneficial for the group to consider the empathy the friends in the group had towards each other.

The thesis work has a starting-point in the empathetic model proposed by Salehi-Abari and Boutilier [11]. In their model, each node, representing a person, in the network accounts for their friends' preferences in addition to their own. This is done by having each person state their empathy for the other people in the network in addition to stating their preferences.

1.2 Problem Statements

The thesis work will consist of two parts. The first is reproduction of the results of [11]. Afterwards, a modification of the original approach is studied such that other winner selection methods can be used. The winner selection methods used in [11] are plurality and Borda count which are inferior to other winner selection methods. This subject is discussed in Section 2.2.

Our goal is to answer the following questions:

1. How can we modify the empathy models such that we can use other, superior winner selection methods than plurality and Borda count?
2. Will the same arguments for convergence and fixed-point solutions hold as for the original models?
3. Will the winner selection methods maintain their properties in the empathy models?

1.3 Limitations

As the duration of the thesis and the availability of real-world data is limited, only synthetic data will be used.

In the thesis only four different winner selection methods are studied, where two of them are from [11]. The models provided in Chapter 3 can be used with other winner selection methods that determine the winner with the help of the pairwise comparison graph.

Only scale-free empathy networks and only one distribution of preference profiles is studied.

1.4 Thesis Outline

Chapter 2 Chapter 2 will introduce the theory used in the later parts of the thesis. More specifically, some theory about networks and graphs, voting and some theory about the empathy models will be discussed.

Chapter 3 In chapter 3, an extension of the original empathy models will be presented, such that different and more competent winner selection methods can be used.

Chapter 4 Chapter 4 will present the simulations performed and the results acquired.

Chapter 5 In chapter 5 the results of the thesis will be discussed. The results are also discussed in a broader context.

Chapter 6 Chapter 6 will conclude the thesis and answer the questions stated in the problem statement. Some future work will also be presented.

2

Theoretical Background

This chapter focuses on providing the reader with useful theory concerning the thesis. In the first section, some network theory is introduced. Some light definition of concepts are touched on and a method of generating a network with real world properties is brought up. The next section is on voting and introduces the winner selection methods used throughout the thesis and some requirements that are usually put on winner selection methods. Finally, *empathy* is discussed, and the definition proposed by Salehi-Abari and Boutilier in [11] is introduced.

2.1 Networks

Networks of people, or things, are usually represented as graphs with *nodes* that are connected by *edges*. The nodes represent people and the edges represent the relation between them.

We denote graphs as pairs $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, where $\mathcal{V}(\mathcal{G})$ is the set of nodes, or vertices, and $\mathcal{E}(\mathcal{G})$ the set of edges. We let $n_v(\mathcal{G})$, where “v” stands for vertex (or voter, later in the thesis), be the number of nodes in graph \mathcal{G} . Similarly, we let $n_e(\mathcal{G})$ be the number of edges in graph \mathcal{G} . In this thesis, when only referring to one graph, we will exclude the \mathcal{G} from the aforementioned notations and they will simply be \mathcal{V} , \mathcal{E} , n_v and n_e .

We label the specific nodes in a graph by the integers $1, \dots, n_v$. We can now identify every edge by a pair (i, j) for an edge from node i to node j . If a node i has an edge to itself, (i, i) , we call that edge a *self-edge*.

In an *undirected graph* the edges represent a mutual relation between the nodes. Therefore, there is no difference between the edge (i, j) and the edge (j, i) , $(i, j) = (j, i)$. However, in a *directed graph* the relation is directed from the first node to the second node in the edge pair, which means that the edge (i, j) is not necessarily the same as the edge (j, i) . *Multi-graphs* are graphs that can have

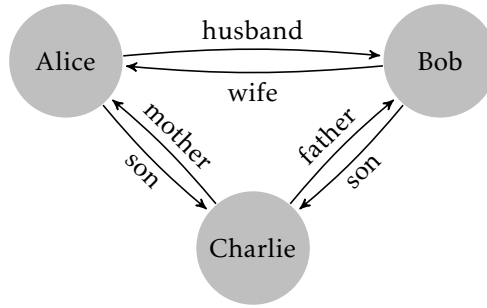


Figure 2.1: A simple directed graph showing the relationship between family members.

more than one edge between two nodes. We will not consider such graphs in this thesis.

A *weighted graph* is a graph where each edge has a number assigned to it. The number, or *edge weight*, is a quantitative measure of the relation between the nodes. There are many examples of what the edge weight might represent. For example, in a graph where the nodes represent cities, the edge weights can represent distance, in kilometres, between the cities. In another example, with the same nodes, the edge weights represent capacity of the roads, or how many cars can use the roads simultaneously, between the cities.

When illustrating graphs, we draw the nodes as circles with the node labels within the nodes. The edges in an undirected graph is simply a line, while the edges in a directed graph are arrows from the start node to the end node. The edge labels, or edge weights, are placed such that there is no ambiguity between them.

An example of a directed graph can be seen in Figure 2.1. The graph represents the members of a family as nodes and their relations as edges, where the specific relation is noted as an edge label.

Graphs can also be represented by an *adjacency matrix*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n_v} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n_v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_v,1} & a_{n_v,2} & \cdots & a_{n_v,n_v} \end{pmatrix}$$

where

$$a_{i,j} = \begin{cases} 1, & \text{if there is an edge from node } j \text{ to node } i \\ 0, & \text{otherwise.} \end{cases}$$

If the graph is weighted, $a_{i,j}$ is the edge weight from node j to node i . Adjacency matrices are always square matrices of dimension $n_v \times n_v$. From the definition, an adjacency matrix of an undirected graph is always symmetric, while this is not necessarily the case for a directed graph.

The *degree*, d_i , of node i in an unweighted, undirected network is defined by

$$d_i = \sum_{j=1}^{n_v} a_{i,j} = \sum_{j=1}^{n_v} a_{j,i}.$$

Nodes in directed graphs have two degrees, one in-degree, d^{in} , and one out-degree, d^{out} . Those degrees are defined by

$$d_i^{\text{in}} = \sum_{j=1}^{n_v} a_{i,j}$$

and

$$d_j^{\text{out}} = \sum_{i=1}^{n_v} a_{i,j}.$$

A *path* is a sequence of nodes with edges in between them. For directed graphs, the edges in a path must be directed in the correct direction. Neither the nodes nor the edges in a path are allowed to be repeated. For example in the graph in Figure 2.1 there are two paths from Alice to Charlie: Alice $\xrightarrow{\text{son}}$ Charlie and Alice $\xrightarrow{\text{husband}}$ Bob $\xrightarrow{\text{son}}$ Charlie.

2.1.1 Generating Synthetic Networks

According to Barabási in [3, Ch. 4], most often, the degree distribution of naturally occurring networks follows a power law. That is, the network has the following property

$$p_d \sim d^{-b},$$

where p_d is the fraction of nodes having degree d and b is the degree exponent. The degree exponent is different for every real world network. Two examples of these kinds of networks are the internet ($b = 3.42$) and network of actors ($b = 2.12$) [3, Ch. 4]. This property can also be applied to directed networks. In directed networks the following expressions hold for the in-degree distribution and the out-degree distribution, respectively:

$$p_{d^{\text{in}}} \sim d^{\text{in}-b^{\text{in}}},$$

$$p_{d^{\text{out}}} \sim d^{\text{out}-b^{\text{out}}}.$$

Networks which follow a power law are called *scale-free networks*.

One method of generating scale-free undirected networks is to use the Barabási-Albert model [3, Ch. 5]. The model creates a network by connecting new nodes to existing nodes in the network with a probability proportional to the current degree of the nodes in the network. More formally, we start a new network with n_0 nodes. It does not matter how these nodes are connected, although each of the nodes has to have at least one edge. We continue by, at each time step, connecting k nodes to the existing nodes in the network. The probability that a new node

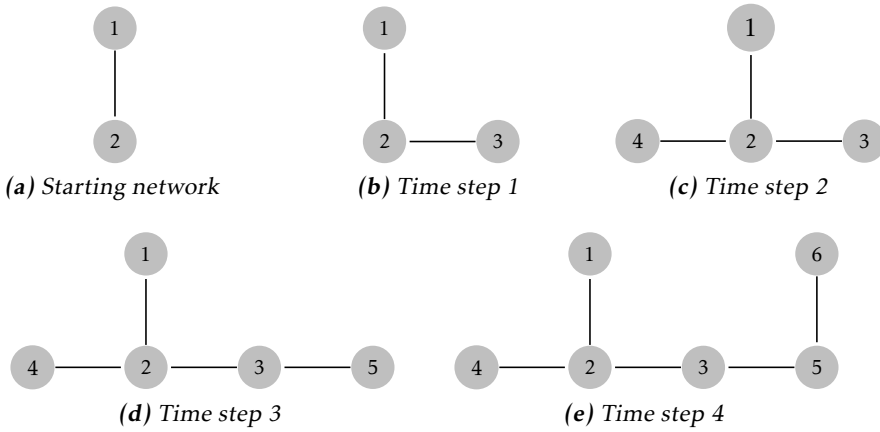


Figure 2.2: One example of a network generated with the Barabási-Albert model, with $n = 6$, $k = 1$ and $n_0 = 2$.

connects to the node i is $\frac{d_i}{\sum_j d_j}$. We repeat connecting k new nodes until we reach the target number of nodes, n , in the network. An example of a network, at each time step, generated with the Barabási-Albert model can be seen in Figure 2.2.

2.2 Voting

Voting occurs naturally when a group of people are forced to make a group decision. It arises in different constellations of people with different objectives: in a whole nation when deciding who will lead the country or in a family when deciding what film to watch on a Friday evening.

Most of the time a vote is cast on a single candidate, which means that every person voting simply states the alternative which they like the best, which is a naive way of voting. Instead of this, we will from now on consider ranking of the candidates to be votes. We will denote that candidate A is “ranked higher than” candidate B, or candidate A is “preferred to” candidate B, with the following:

$$A > B \text{ or } B < A.$$

If candidate A and candidate B are ranked equally, or the voter is indifferent to the two candidates, we denote it with¹

$$A \sim B \text{ or } B \sim A.$$

Sometimes we denote that a specific person i has a certain preference by $>_i$, $<_i$ or \sim_i . When considering votes in this thesis, we assume that all preferences are

¹Earlier \sim denoted “similarity”, while here it denotes indifference between candidates. These two uses for \sim are considered so different that the usage of the same symbol will not confuse the reader.

Table 2.1: Example of a preference schedule.

3	A > B > C
5	B > C > A
2	C > B > A

transitive. This means that if a person i has ranked $A \succ_i B$ and $B \succ_i C$, then $A \succ_i C$, i.e.

$$\left. \begin{array}{l} A \succ_i B \\ B \succ_i C \end{array} \right\} \Rightarrow A \succ_i C.$$

We call a preference *complete* if no candidates are rated equally.

We do not require the users to score the alternatives as scoring is usually very subjective to each user. Different people will probably skew their scores in some direction. One might enjoy and be entertained by a film but realise that it is not well made and that the acting is not the best and therefore rate the film with a lower score than one actually believes.

When more than one vote is cast we will denote them in a *preference schedule*. The preference schedule is more compact than simply listing every voter's preference. An example of a preference schedule can be seen in Table 2.1 where the leftmost column is the number of votes with the preference given in the rightmost column.

Pairwise comparison is a direct comparison between pairs of candidates in the candidate set. When computing the pairwise comparison between two candidates we do not take any of the other candidates into consideration. The pairwise comparison is computed by calculating the sum of the number of votes that prefer the first candidate to the second candidate and vice-versa. The candidate which is preferred by the most votes is said to be the pairwise comparison winner with a margin of the difference between the two summations. An example where pairwise comparison is illustrated can be seen in Example 2.1.

Example 2.1: Pairwise comparison

Consider the following preference schedule

3	A > B > C
5	B > C > A
2	C > B > A

To determine the pairwise comparison of candidate A and B, we first remove candidate C from the preference schedule

3	A > B
5	B > A
2	B > A

Candidate A is ranked higher than candidate B in 3 of the votes, while candidate B is ranked higher than candidate A in $5 + 2 = 7$ of the votes. We then say

that the pairwise comparison between candidate A and B is $7 - 3 = 4$ in favour of candidate B.

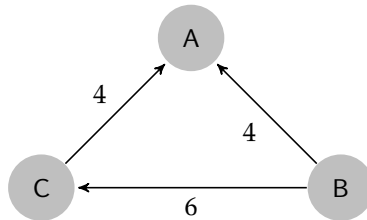
A *pairwise comparison graph* is a graph with the candidates as nodes and the pairwise comparison as the edge weights between candidates. An example can be seen in Example 2.2.

Example 2.2: Pairwise comparison graph

Consider the preference schedule

$$\begin{array}{l} 3 \quad A > B > C \\ 5 \quad B > C > A \\ 2 \quad C > B > A \end{array}$$

As stated in Example 2.1, the pairwise comparison between candidates A and B is 4 in favour of B. We can compute the pairwise comparison of the two remaining candidate pairs; B wins the pairwise comparison against C, with a margin of 6, and C wins the pairwise comparison against candidate A, with a margin of 4. The relations between the candidates can be seen in the figure below.



The adjacency matrix of the above graph is

$$A = \begin{pmatrix} 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}.$$

We can partition the set of candidates, \mathcal{C} , into two sets, \mathcal{X} and \mathcal{Y} . The first set, \mathcal{X} , is the set of candidates where every candidate wins the pairwise comparison against the candidates in the second set, \mathcal{Y} . The smallest set \mathcal{X} is called the *Smith set*. We denote the Smith set by \mathcal{S} .

Example 2.3: Smith set

Consider the following preference schedule which we are familiar with from Example 2.1

$$\begin{array}{l} 3 \quad A > B > C \\ 5 \quad B > C > A \\ 2 \quad C > B > A \end{array}$$

B wins the pairwise comparison against A and C. Therefore, the Smith set is composed of candidate B, $\mathcal{S} = \{B\}$. Notice that C wins the pairwise comparison against A. We could partition the candidates in two sets $\mathcal{X} = \{B, C\}$ and $\mathcal{Y} = \{A\}$ so that every candidate in set \mathcal{X} wins the pairwise comparison against every candidate in \mathcal{Y} . However, \mathcal{X} will not be the Smith set as there exists a set that is smaller, namely $\{B\}$.

A *Condorcet candidate* is a candidate that wins every pairwise comparison. This occurs when the Smith set only contains one candidate, the Condorcet candidate, as in Example 2.3.

2.2.1 Winner Selection Methods

In this section some *winner selection methods* will be presented. That is, methods of how the votes cast by the voters determine which of the candidates has won the election.

Plurality

The *plurality* winner selection method, sometimes called first-past-the-post, is the simplest voting method. The winner is the candidate which has received the highest ranking on most votes [4, Ch. 1]. One advantage of the plurality method is that the votes cast are not required to be rankings, but simply every voter's most preferred candidate. This makes the method simple and fast to be used in casual group decisions. The members of parliament in the House of Commons in the United Kingdom are elected with a plurality winner selection method [1]. An example of determining the winner with the plurality method can be seen in Example 2.4.

Example 2.4: Plurality

Recall the following preference schedule

$$\begin{array}{l} 3 \quad A > B > C \\ 5 \quad B > C > A \\ 2 \quad C > B > A \end{array}$$

Candidate A has the highest ranking on 3 of the votes. Candidate B has the highest ranking on 5 of the votes. Candidate C has the highest ranking on 2 of

the votes. Because $5 > 3 > 2$, candidate B is the winner.

Borda Count

The *Borda count* winner selection method is based on assigning points to the candidates according to their ranking. The candidate ranked highest on a vote gets $n_c - 1$ points, where n_c is the number of candidates. Furthermore, the candidate ranked the second highest gets $n_c - 2$ points and so on. This is repeated for every vote and then summed up. The candidate with the highest Borda score is the winner [4, Ch. 1]. Note that Borda count and plurality will give the same results when there are only two candidates. The Borda count method can be found as a political voting system in the Pacific Islands [10]. In Example 2.5, an example of determining the winner with the Borda count winner selection method is presented.

Example 2.5: Borda count

Recall the preference schedule

$$\begin{array}{l} 3 \quad A > B > C \\ 5 \quad B > C > A \\ 2 \quad C > B > A \end{array}$$

In this example $n_c = 3$. Candidate A has the Borda score $3 \cdot 2 + 5 \cdot 0 + 2 \cdot 0 = 6$. Candidate B has the Borda score $3 \cdot 1 + 5 \cdot 2 + 2 \cdot 1 = 15$. Candidate C has the Borda score $3 \cdot 0 + 5 \cdot 1 + 2 \cdot 2 = 9$. Because $15 > 9 > 6$, candidate B is the winner.

Beatpath

The *beatpath* method, also called the Schulze method after Markus Schulze [13], selects winners by comparing the capacity (the smallest edge weight of a path) of the widest path (the path with the largest capacity) between candidates in the pairwise comparison graph [4, Ch. 6]. The candidates are compared in pairs, X and Y. For every pair, the capacity of the widest path between the candidates, in both directions, is computed. If the capacity from candidate X to Y is bigger than the capacity from candidate Y to X, candidate Y is taken out from the set of possible winners. If the capacities are equal, both candidates stay in the set of possible winners. This step is computed until no more candidates can be removed from the winning set. The beatpath method is used in Swedish party Piratpartiet [6].

Example 2.6: Beatpath

Consider a preference schedule

3	A > B > C > D
2	A > D > C > B
2	D > A > C > B
1	D > B > A > C
2	D > B > C > A

We start by looking at the pairwise comparisons for each candidate pair. A is ranked over B in $3 + 2 + 2 = 7$ of the votes and B is ranked over A in $1 + 2 = 3$ of the votes. The pairwise comparison between candidate A and B is 4 in favour of A. We do the same for the other candidate pairs. From this information we make a graph with edges from the winning candidates of the pairwise comparison to the losing candidates with edge weights.

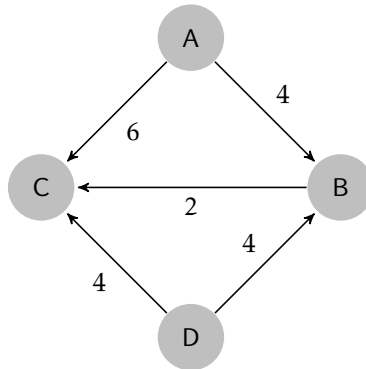


Figure 2.3: Example 2.6: Pairwise comparison graph.

To determine the winner we look at the candidate in pairs and compare the capacity of the widest path between nodes. First, we look at candidate D and C. The capacity of the widest path from D to C is 4, $D \xrightarrow{4} C$. Notice that there is another path from D to C, $D \xrightarrow{4} B \xrightarrow{2} C$, but that path has capacity 2 which is less than 4. There is no path from C to D in our graph. This means that C is a losing candidate. We remove C from the winning set but keep it in the graph.

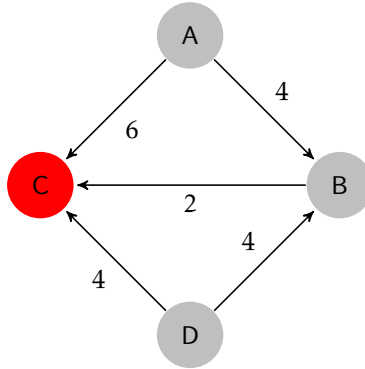


Figure 2.4: Example 2.6: Pairwise comparison graph with candidate C marked with red to show that it is not a beatpath winner.

We continue to compare the capacity of the widest path from each candidate pair until we have removed all losers from our set. This leaves us with our winner(s), in this case candidates A and D.

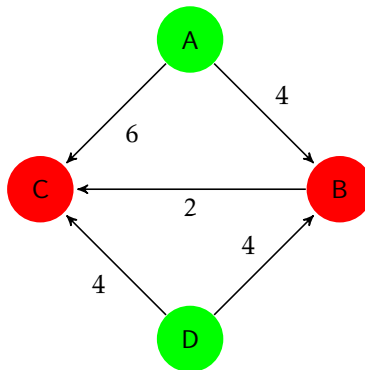


Figure 2.5: Example 2.6: Pairwise comparison graph with the winning beatpath candidates, A and D, in green and the losing beatpath candidates, B and C, in red.

Kemeny-Young

The *Kemeny-Young* winner selection method is a maximum likelihood winner selection method which selects the ranking that is most likely to be the “true” ranking. This is done under the assumption that each voter has some probability $p > \frac{1}{2}$ that they will rank two candidates according to the true ranking and will order the two candidates in the wrong order otherwise. The winner is selected by selecting a ranking that minimises the distance to the votes in the ballot.

The Kemeny distance, which is the distance that should be minimised, between two rankings is defined as the number of candidate pairs where the pairwise ranking differs between the two rankings. This distance is computed between all possible rankings and every vote in the set of votes. The winning ranking is the ranking which has the smallest distance to the votes. As we get a whole ranking instead of only a winner, we take the candidate that is ranked highest in the resulting ranking and consider it the winner [17].

One can also view this method as maximising the number of agreements of candidate pairs between the rankings and the votes. In that case, the information in the adjacency matrix of the pairwise comparison graph can be used.

Example 2.7: Kemeny-Young

Consider the preference schedule

3 $A > B > C$
 5 $B > C > A$.
 2 $C > B > A$

We have six ($n_c! = 3!$) possible rankings which are all stated in Table 2.2.

Table 2.2: Example 2.7: All possible rankings with three candidates.

$A > B > C$
 $A > C > B$
 $B > A > C$
 $B > C > A$
 $C > A > B$
 $C > B > A$

We start by computing the distance for the first ranking, $A > B > C$, to the votes in the ballot. The distance to the vote $A > B > C$ is zero as the vote is equal to the proposed ranking. The distance to $B > C > A$ is 2 as there is two candidate pairs that are not equal to the ranking $A > B > C$, namely pairs (A, B) and (A, C). Finally, the distance to $C > B > A$ is 3. This gives us a total distance of $3 \cdot 0 + 5 \cdot 2 + 2 \cdot 3 = 16$ for the first ranking. We continue to calculate the distance for each ranking. The resulting distances can be seen in Table 2.3.

Table 2.3: Example 2.7: Distances for rankings.

Ranking	Distance
$A > B > C$	16
$A > C > B$	22
$B > A > C$	12
$B > C > A$	8
$C > A > B$	18
$C > B > A$	14

The winning ranking is $B > C > A$ with the smallest distance of 8 which makes candidate B the winner of the election.

2.2.2 Criteria for a Voting System

When designing or using voting systems, we often have some criteria for the winner selection methods. In this section we will present some of them and discuss whether the aforementioned winner selection methods fulfil them or not. The latter can be found in condensed form in Table 2.4.

When reading this section, one might think that these criteria seem obvious or trivial. However, there is no such thing as a perfect voting system; a voting system can not fulfil all criteria [2]. This means that we have to omit some of the criteria. The criteria presented in this section are only a selection of all the requirements. Most of the reasoning follows the reasoning in [4]. Also, some more sophisticated requirements are added.

One Person, One Vote

One person, one vote is the concept that every voter's opinion is equal and taken into account when selecting the winner. This means that all that should matter when deciding on a winner is how many voters that have stated a certain ranking and not *which* of the voters that stated it [4, Ch. 1]. This is a cornerstone in a democracy.

A *dictatorship* is when one voter has the total say in what candidate wins the election. A dictatorship does not fulfil the one person, one vote requirement.

Plurality, Borda count, beatpath and Kemeny-Young all fulfil the requirement of one person, one vote. All votes in the preference schedule and the amount of votes all affect the decision of winners.

Independence of Candidate Names

We also require a winner selection method to be *independent of candidate names*. This means that *Alice* should not have an advantage over *Bob* because of her name [4, Ch. 2].

Plurality, Borda count, beatpath and Kemeny-Young are all independent of candidate names. This criterion might be obvious, but there exist winner selection methods which do not fulfil it, e.g. sequential comparison [4, Ch. 2].

No-weak-spoiler Criterion

Let us introduce a so called *weak spoiler*. A weak spoiler is a candidate that is not in the Smith set, which when removed from the ballot makes the winner selection method select another winner than the previously selected winner. If such a candidate never exists in a winner selection method, we say that the winner selection method satisfies the *no-weak-spoiler* criterion [4, Ch. 5].

Table 2.4: Fulfilled criteria for the voting systems.

	Winner selection methods			
	Plurality	Borda count	Beatpath	Kemeny-Young
One person, one vote	✓	✓	✓	✓
Independence of candidate names	✓	✓	✓	✓
No-weak-spoiler criterion	✗	✗	✓	✓
Condorcet criterion	✗	✗	✓	✓
Monotonicity	✓	✓	✓	✓
Independence of irrelevant alternatives	✗	✗	✗	✗
Local independence of irrelevant alternatives	✗	✗	✗	✓
Independence of Clones	✗	✗	✓	✗

Neither plurality nor Borda count satisfy the no-weak-spoiler criterion which can be seen in Example 2.8. However, the beatpath and Kemeny-Young winner selection methods do.

For the beatpath method, in the pairwise comparison graph, there is an edge from every candidate in the Smith set to every non-Smith candidate. Therefore, no non-Smith candidate has a path from it to a Smith candidate, which means that the winner must be a candidate from the Smith set. Also, a beating path from the winner to any other Smith candidate does not go through a non-Smith candidate. By removing a non-Smith candidate, the paths from the winner to any other Smith candidate do not change. Therefore, the winner will stay the winner and the beatpath method fulfils the no-weak-spoiler criterion [4, Ch. 6].

Also, for the Kemeny-Young method, it is beneficial to rank the Smith candidates above all of the non-Smith candidates, which means that a Smith candidate will be the winner. The winner will stay the same as the ranking that had the smallest distance, without the removed candidate, will still have the smallest distance to the votes.

Example 2.8: No-weak-spoiler

Consider the following preference schedule

4	$A > B > C$
4	$B > A > C$
3	$C > B > A$
2	$A > C > B$

Our aim is to prove, with this counter example, that neither plurality nor Borda count fulfil the no-weak-spoiler criterion. We start by computing the winners with the two winner selection methods.

Using plurality, candidate A has $4 + 2 = 6$ points, candidate B has 4 points and candidate C has 3 points. This means that candidate A is the winner with the plurality method.

Using Borda count, candidate A has $4 \cdot 2 + 4 \cdot 1 + 3 \cdot 0 + 2 \cdot 2 = 16$ points, candidate B has $4 \cdot 1 + 4 \cdot 2 + 3 \cdot 1 + 2 \cdot 0 = 15$ points and candidate C has $4 \cdot 0 + 4 \cdot 0 + 3 \cdot 2 + 2 \cdot 1 = 8$ points. This means that candidate A is a winner with the Borda count method.

We continue by determining the Smith set. Both candidate A and B win the pairwise comparison against candidate C with a margin of $4 + 4 - 3 + 2 = 7$ and $4 + 4 - 3 - 2 = 3$, respectively. Candidate B wins the pairwise comparison against A with a margin of $-4 + 4 + 3 - 2 = 1$, which means that the Smith set is $S = \{B\}$. Notice that candidate B is also a Condorcet candidate as it is the only candidate in the Smith set.

Next, we remove a non-Smith candidate, for example C from the votes. This gives us a new preference schedule

4 A > B
4 B > A
3 B > A
2 A > B

We compute the winners with plurality and Borda count, which in this case with two candidates will give us the exact same answer. Candidate A will get $4 + 2 = 6$ points and candidate B will get $4 + 3 = 7$ points. This gives us a new winner, namely candidate B. This notifies us that candidate C, that we removed from the votes, was a spoiler. Thus, neither plurality nor Borda count fulfil the no-weak-spoiler criterion.

Condorcet Criterion

As mentioned in Section 2.2, a Condorcet candidate is a candidate that wins every pairwise comparison. The property we desire a voting system to have is that if a Condorcet candidate exists, it must be a winner. Such a voting system is called *Condorcet fair* [4, Ch. 2]. We can see that neither Borda count nor plurality fulfils this requirement in Example 2.8. As a matter of fact, fulfilment of the no-weak-spoiler criterion implies fulfilment of the Condorcet criterion and failure to fulfil the Condorcet criterion implies failure to fulfil the no-weak-spoiler criterion [4, Ch. 7]. The beatpath method fulfils this criterion as the Condorcet candidate does not have any incoming edges in the pairwise comparison graph. Thus, there is no path that ends at the Condorcet candidate that can compete with the outgoing paths [13]. The Kemeny-Young method also fulfils the Condorcet criterion as the distance will be the smallest with the Condorcet candidate ranked highest [17].

Monotonicity

The requirement of *monotonicity* states that if we change a ballot in favour of the winner, the winner will never become a loser [4, Ch. 7]. Plurality, Borda count, beatpath and Kemeny-Young all fulfil this requirement.

In a plurality election, by changing a vote in favour of the winner we can achieve two things: that the winner gets a higher score and one of the other candidates gets a lower score or that the scores do not change. Both of these scenarios do not change the winner in the election [5, Ch. 2].

In Borda count, when we change a vote in favour of the winner, the winner will get a higher Borda count score, and one of the other candidates will get a lower score. Therefore, the winner will stay the winner [5, Ch. 2].

Changing a vote in favour of the winner will actualize one of two scenarios in the pairwise comparison graph. If there was an edge between the winner and the candidate which the winner surpassed on the changed vote, that edge weight will increase. Otherwise, if the edge was in the other direction, the edge weight

will decrease. In a beatpath scenario this will not affect the winner, which will stay the same [4, Ch. 7].

When using the Kemeny-Young winner selection method and changing the ballot in favour of the winner, the distance to the ranking which selected the winner will decrease which will cause the winner to stay the same [7].

The *instant run-off method* is a method that violates the monotonicity requirement [9].

Independence of Irrelevant Alternatives

Independence of irrelevant alternatives is a criterion proposed by Kenneth Arrow [2]. The criterion is that a candidate's ranking in regards to another candidate should only be affected by the pairwise comparison of the two candidates, not by other candidates' pairwise comparisons [4, Ch. 12]. For example, say that we have a ballot and a winner selection method such that candidate X is the winner. If a voter decides that they preferred candidate Z to candidate Y and changes their vote accordingly, while keeping the relative ranking of X to Y and Z, the winner should not change from X to another candidate. Neither plurality, Borda count, beatpath nor Kemeny-Young fulfil this requirement. This is shown in Example 2.9.

Example 2.9: Independence of irrelevant alternatives

Consider the following preference schedule

3	A > B > C
2	B > C > A
2	C > A > B

If the winner is selected with the plurality method, candidate A would win with 3 points against candidate B's 2 points and candidate C's 2 points. Candidate A would also win if we were using Borda count, with 8 points against candidate B's 7 points and candidate C's 6 points.

We draw the pairwise comparison graph to decide the beatpath winner.

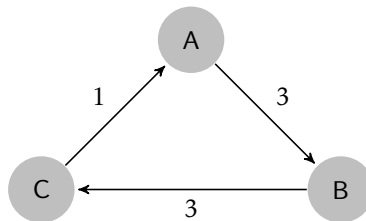


Figure 2.6: Example 2.9: Pairwise comparison graph.

Candidate A is the winner as it beats both candidate B and candidate C with the capacity 3 against 1 ($A \xrightarrow{3} B$ against $B \xrightarrow{3} C \xrightarrow{1} A$ and $A \xrightarrow{3} B \xrightarrow{3} C$ against $C \xrightarrow{1} A$).

Finally, we compute the Kemeny-Young winner which, according to Table 2.5, is candidate A.

Table 2.5: Example 2.9: Distances and rankings for Kemeny-Young example.

Ranking	Distance
A > B > C	8
A > C > B	11
B > A > C	11
B > C > A	10
C > A > B	10
C > B > A	13

First, we prove that the plurality, beatpath and Kemeny-Young methods fail the independence of irrelevant alternatives criterion. Two of the voters decide that they did not like B more than C and change their votes so that the updated preference schedule with the changes in bold text becomes:

3	A > B > C
2	C > B > A .
2	C > A > B

With this change, candidate C becomes a Condorcet candidate, and all winner selection methods that satisfy the Condorcet criterion will select C as a winner. Thus, the beatpath and Kemeny-Young winner selection methods, which fulfil the Condorcet criterion, fail the independence of irrelevant alternatives requirement as they previously selected candidate A as the winner.

Plurality also fails the independence of irrelevant alternatives as now the winner is candidate C with 4 points against candidate A with 3 points and candidate B with 0 points.

To prove that Borda count also fails the independence of irrelevant alternatives the original ballot is changed differently. Three voters now change their votes so that candidate C is ranked higher than candidate B. The new preference schedule is

3	A > C > B
2	B > C > A .
2	C > A > B

We conclude that the Borda count winner is candidate C with 9 points against candidate A's 8 points and candidate B's 4 points. Thus, plurality, Borda count, beatpath and Kemeny-Young have all failed the independence of irrelevant alternatives requirement.

Local Independence of Irrelevant Alternatives

As concluded in Example 2.9 both the beatpath and Kemeny-Young winner selection methods, which have the property that they fulfil the Condorcet criterion, fail the independence of irrelevant alternatives requirement. This is unfortunate since both requirements are desirable.

Arrow's theorem states that the winner selection method that satisfies transitivity ($A > B$ and $B > C$ implies that $A > C$), unanimity (if candidate A is preferred to candidate B by every voter, candidate A should be ranked above candidate B in the election) and independence of irrelevant alternatives, is dictatorship [2]. This means that we must omit one of the four requirements, transitivity, unanimity, independence of irrelevant alternatives and non-dictatorship.

Instead of the independence of irrelevant alternatives, there is a more relaxed requirement called the *local independence of irrelevant alternatives*. The requirement is that the following statements must hold: if the winner is eliminated, the candidate that was ranked second wins and if the lowest ranked candidate is eliminated, the winner must not change [17]. The Kemeny-Young winner selection method fulfils this requirement [17], while neither the plurality, Borda count nor beatpath winner selection methods does.

Independence of Clones

The criterion proposed by Tideman in [14] as the *independence of clones* says that addition or elimination of very similar candidates, or clones, should not affect the outcome of the election.

If one of the clones win with the original ballot one of the clones should also win if a clone is added or eliminated. Similarly, if a candidate outside of the clone set wins, it should also win when clones are added or eliminated.

Neither plurality, Borda count nor Kemeny-Young satisfy this criterion, as can be seen in Example 2.10. However, the beatpath method fulfils this requirement. When adding clones we do not affect the pairwise comparison graph for the original candidates and the added clones will have the same relation to all other candidates as the cloned candidate had. This means that we can group the clones and regard them as one when comparing them to other alternatives. If the cloned candidate was the winner in the original ballot, one of the clones will win the new one. If one of the non-cloned candidates was the winner in the original ballot it will beat all clones (and all other candidates) the same way that it beat the cloned candidate in the original ballot. Therefore, beatpath fulfils the requirement of independence of clones. A more rigorous proof can be found in [13].

Example 2.10: Independence of clones

We start with an example where A wins both with plurality and with Borda count. The preference schedule is

- 4 A > B
- 3 B > A

Suppose that the winner selection method in use is plurality, and that candidate B wants to add clones so that their chance to win the election is higher. Candidate B then nominates another candidate A_2 which is similar to candidate A. The resulting preference schedule after the added candidates looks like

$$\begin{array}{l} 2 \quad A > A_2 > B \\ 2 \quad A_2 > A > B \quad . \\ 3 \quad B > A > A_2 \end{array}$$

After the added clones, candidate B wins the election ($3 > 2 = 2$), and the plurality winner selection method has failed the independence of clones criterion.

Now, instead of using plurality method, we use the Borda count method. Candidate B still wants to win, so they nominate a friend, B_2 who is very similar to B. Candidate B does not care if they or their friend wins the election, the most important thing is that A does not win. The preference schedule is now

$$\begin{array}{l} 4 \quad A > B > B_2 \\ 3 \quad B > B_2 > A \quad . \end{array}$$

Candidate A now has the Borda count score $4 \cdot 2 + 3 \cdot 0 = 8$, B has $4 \cdot 1 + 3 \cdot 2 = 10$ and B_2 has $4 \cdot 0 + 3 \cdot 1 = 3$ which makes candidate B the winner, and the Borda count method has also failed the independence of clones criterion.

To prove that the Kemeny-Young method does not fulfil the independence of clones criterion, we look at the following preference schedule

$$\begin{array}{l} 2 \quad A > B > C \\ 3 \quad B > C > A \quad . \\ 2 \quad C > A > B \end{array}$$

We first calculate the winner without adding clones. The distances are presented in Table 2.6.

Table 2.6: Example 2.10: Distances and rankings for Kemeny-Young example, without clones.

Ranking	Distance
$A > B > C$	10
$A > C > B$	13
$B > A > C$	11
$B > C > A$	8
$C > A > B$	10
$C > B > A$	11

We can see that the ranking with the least distance to the votes is $B > C > A$, which means that candidate B is the winner. Next, three clones of candidate B are added to the ballot. The updated preference schedule looks like:

- 2 $A > B > B_2 > B_3 > B_4 > C$
- 3 $B > B_2 > B_3 > B_4 > C > A$.
- 2 $C > A > B > B_2 > B_3 > B_4$

We first convince ourselves that the winning ranking has all the clones in the order $B > B_2 > B_3 > B_4$ directly after each other. This is true because all distances between a clone and any other non-clone candidate is the same for all clones. Moreover, the clones are ranked the same in all votes, and ranking them in that way will not contribute with distance to the voters. With the previous statements we look at all the feasible rankings and their distances to the votes. These are stated in Table 2.7.

Table 2.7: Example 2.10: Distances and rankings for Kemeny-Young example, with clones.

Ranking	Distance
$A > B > B_2 > B_3 > B_4 > C$	25
$A > C > B > B_2 > B_3 > B_4$	37
$B > B_2 > B_3 > B_4 > A > C$	29
$B > B_2 > B_3 > B_4 > C > A$	26
$C > A > B > B_2 > B_3 > B_4$	34
$C > B > B_2 > B_3 > B_4 > A$	38

We see that the ranking with the smallest distance is $A > B > B_2 > B_3 > B_4 > C$ which makes candidate A the winner and thus the requirement of independence of clones is broken for the Kemeny-Young method.

2.3 Empathy

The notion of empathy in a voting context is proposed by Salehi-Abari and Boutilier in [11]. Each person in the social network has a certain *empathy* towards their friends and acquaintances which is essentially how dependent others happiness is for them to be happy or satisfied.

We model the set of voters and their empathy as a graph with the voters as nodes and the empathy as directed weighted edges. The edges are directed which means that the empathy does not have to be mutual. If voter i has empathy towards voter j there is an edge from voter i to voter j in the network.

There are three requirements on the empathy networks. The first requirement, *normalization*, is that all outgoing empathy must sum to one:

$$\sum_{j=1}^{n_v} a_{j,i} = 1, \forall i \in \mathcal{V}.$$

This is analogous to the one person, one vote criterion stated in Section 2.2.2. Each person can give away equally much empathy as every other person in the network. The next requirement, *non-negativity*, is that there can not be negative empathy, that is, edge weights can not be negative. In a social context this means that either you care about someone, or you do not. You can not require that another person's vote is worth less. The last requirement, *positive self-weight*, is that every node in the network must have a strictly positive self-edge, that is, a person can not give away their whole vote to the other people in the network. These requirements are also helpful (and necessary) when proving that the methods used converge.

Salehi-Abari and Boutilier [11] introduce three *empathetic models*: the *intrinsic empathetic model*, the *local empathetic model* and the *global empathetic model*. The models differs in the way that the voters consider their neighbours. In the intrinsic model, voters do not allow for their neighbours to affect their decision in the election, they only state what their intrinsic opinion is. This is identical to ordinary voting. In the local model, the voters only take the intrinsic opinion of their closest neighbours into account. Lastly, in the global model, the voters take the total opinion of their neighbours into account.

We let $u_j^I(c)$ denote the intrinsic *utility* of voter $j \in \mathcal{V}$ and candidate $c \in \mathcal{C}$. The utility is the plurality or Borda score from a specific voter. We let \mathbf{A} be the adjacency matrix for the empathy network, with elements $a_{i,j}$. We can now express the resulting utility for each user by

$$u_j(c) = u_j^I(c) \quad (2.1)$$

for the intrinsic model,

$$u_j(c) = \sum_{k \in \mathcal{V}} a_{k,j} u_k^I(c), \forall j \in \mathcal{V} \quad (2.2)$$

for the local model and

$$u_j(c) = a_{j,j} u_j^I(c) + \sum_{\substack{k \in \mathcal{V} \\ k \neq j}} a_{k,j} u_k(c), \forall j \in \mathcal{V} \quad (2.3)$$

for the global model.

By denoting the utilities with a vector \mathbf{u} with voter j 's utility at position j in the vector,

$$\mathbf{u}(c) = \left(u_1(c) \quad u_2(c) \quad \dots \quad u_{n_v}(c) \right)^T$$

and denoting \mathbf{D} as the main diagonal of adjacency matrix \mathbf{A} , such that

$$d_{j,j} = a_{j,j}, \quad \forall j \in \mathcal{V},$$

we can more compactly express the above equations with the following:

$$\mathbf{u}(c) = \mathbf{u}^I(c) \quad (2.4)$$

for the intrinsic model,

$$\mathbf{u}(c) = \mathbf{A}^\top \mathbf{u}^I(c) \quad (2.5)$$

for the local model and

$$\mathbf{u}(c) = (\mathbf{A}^\top - \mathbf{D})\mathbf{u}(c) + \mathbf{D}\mathbf{u}^I(c) \quad (2.6)$$

for the global model.

With the requirements on the network proposed earlier (non-negativity, normalization and positive self-weight), we can write (2.6) iterative as

$$\mathbf{u}^{(t+1)}(c) = (\mathbf{A}^\top - \mathbf{D})\mathbf{u}^{(t)}(c) + \mathbf{D}\mathbf{u}^I(c) \quad (2.7)$$

or as a fixed-point solution

$$\mathbf{u}(c) = (\mathbf{I}_{n_v} - \mathbf{A}^\top + \mathbf{D})^{-1} \mathbf{D}\mathbf{u}^I(c), \quad (2.8)$$

where \mathbf{I}_{n_v} is an identity matrix of dimension n_v . The proofs of the existence of the fixed-point solution and convergence can be found in [11, Appendix B].

Let ω be a vector with elements ω_j indicating the *weight* of voter $j \in \mathcal{V}$. Let $\mathbf{1}_{n_v}$ be a column vector of ones with length n_v (matrix of dimension $n_v \times 1$). Then

$$\omega^\top = \begin{cases} \mathbf{1}_{n_v}^\top, & \text{in the intrinsic model} \\ \mathbf{1}_{n_v}^\top \mathbf{A}^\top, & \text{in the local model} \\ \mathbf{1}_{n_v}^\top (\mathbf{I} - \mathbf{A}^\top + \mathbf{D})^{-1} \mathbf{D}, & \text{in the global model.} \end{cases} \quad (2.9)$$

The Borda score, and similarly the plurality score, of candidate c will be given by $s(c) = \omega^\top \mathbf{u}^I(c) = \mathbf{1}_{n_v}^\top \mathbf{u}(c)$, in [11] this is called the *social welfare* of candidate c . The winner of the election will be the candidate with the highest score,

$$c^* = \operatorname{argmax}_{c \in \mathcal{C}} s(c).$$

An example of determining the plurality and Borda count winners in an empathetic setting can be seen in Example 2.11.

Example 2.11: Empathy

Consider an example with an empathy network as in Figure 2.7. The voters' preferences are stated in proximity of their node.

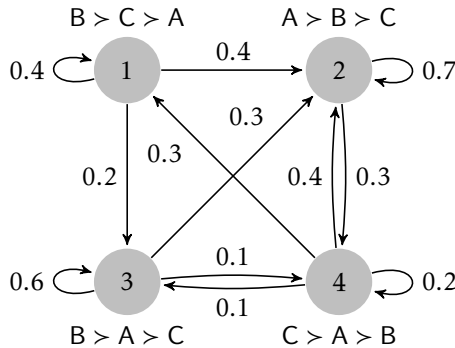


Figure 2.7: Example 2.11: The empathy network for Example 2.11.

The adjacency matrix of the empathy network is

$$\mathbf{A} = \begin{pmatrix} 0.4 & 0 & 0 & 0.3 \\ 0.4 & 0.7 & 0.3 & 0.4 \\ 0.2 & 0 & 0.6 & 0.1 \\ 0 & 0.3 & 0.1 & 0.2 \end{pmatrix}.$$

The intrinsic utilities of the voters, \mathbf{u}^I , are

$$\mathbf{u}_{\text{Plurality}}^I(\text{A}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_{\text{Plurality}}^I(\text{B}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{u}_{\text{Plurality}}^I(\text{C}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

when using plurality and

$$\mathbf{u}_{\text{Borda}}^I(\text{A}) = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_{\text{Borda}}^I(\text{B}) = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \text{ and } \mathbf{u}_{\text{Borda}}^I(\text{C}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

when using Borda count as the winner selection method.

We compute the weights² in the three different empathetic models

$$\boldsymbol{\omega}_{\text{Intrinsic}}^\top = \mathbf{1}_4^\top,$$

$$\boldsymbol{\omega}_{\text{Local}}^\top = \mathbf{1}_4^\top \mathbf{A}^\top = (0.7 \quad 1.8 \quad 0.9 \quad 0.6),$$

$$\boldsymbol{\omega}_{\text{Global}}^\top = \mathbf{1}_4^\top (\mathbf{I}_4 - \mathbf{A}^\top + \mathbf{D})^{-1} \mathbf{D} = (0.64 \quad 2.04 \quad 0.91 \quad 0.41).$$

We compute the plurality scores for each candidate and empathy model. These are gathered in Table 2.8. From the table we can see that candidate B is the winner when using the intrinsic model while candidate A is the winner when using both the local and the global model.

Table 2.8: Example 2.11: The plurality score for each candidate and empathy model.

	A	B	C
Intrinsic	1	2	1
Local	1.8	1.6	0.6
Global	2.0374	1.5575	0.4051

We perform the same computation for the Borda scores. The resulting scores can be seen in Table 2.9. Here, just as in the plurality example, candidate B is the winner when using the intrinsic model while A is the winner when using both the local and the global model.

²The weights are rounded to two decimal points.

Table 2.9: Example 2.11: The Borda score for each candidate and empathy model.

	A	B	C
Intrinsic	4	5	3
Local	5.1	5	1.9
Global	5.3943	5.1524	1.4533

3

Aggregating Pairwise Comparisons Using Empathy

The empathy models are a way to model people's consideration for each other in a voting setting. However, in [11], the only winner selection methods discussed are plurality and Borda count and the empathy models are presented in such a way that only scoring winner selection methods are applicable.

From Table 2.4 we see that plurality and Borda count are inferior to beatpath and Kemeny-Young: they simply fulfil less of the requirements we put on voting systems. The question then arises, "Can we apply the beatpath or Kemeny-Young winner selection methods to the empathetic models proposed in [11]?"

The Kemeny-Young and beatpath winner selection methods are not based on giving scores in the same way that plurality and Borda count are, they instead rely on the pairwise comparison graphs of the votes. We want to adapt the empathy models so that we can aggregate the pairwise comparison graphs and use the superior winner selection methods.

We will first show how this is done by reasoning in Example 3.1 and after that we will formalise the results.

Example 3.1

In order to get a good understanding of what the empathy models, and especially the global model, is, we illustrate what occurs after each time step in the global empathy model. We consider each person's vote to be a one vote preference schedule. In each time step the preference schedules are updated. We study the empathy network in Figure 3.1.

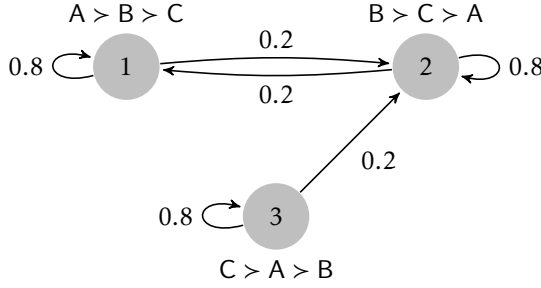


Figure 3.1: Example 3.1: Empathy network used to study the consequences of an iteration of the preference schedule in the global empathy model.

In Figure 3.1, each of the voters has a self-weight of 0.8 and an empathy of 0.2 towards their neighbour. At each time step, the voters will consider 0.8 of their intrinsic vote and 0.2 of their neighbour's vote from the previous time step. The votes can be added in a preference schedule, such that every voter will have a preference schedule for each time step. The preference schedules of each voter can then be summed up to create a preference schedule for the whole community.

In Table 3.1 the preference schedule of each voter after each time step is presented. The left most column in each preference schedule is now expressing the fraction of the votes with the specific preference rather than the absolute number of votes with the specific preference. It can be seen that the size of the preference schedules grows rapidly. Note that the preference schedules can be further simplified as the same vote is stated several times in each preference schedule. However, this is omitted to show the growth of the preference schedules.

From Table 3.1 we see that the total preference schedule will converge to

$$\begin{array}{l} \frac{31}{30} \quad A > B > C \\ \frac{35}{30} \quad B > C > A \\ \frac{24}{30} \quad C > A > B \end{array} .$$

If we compute the global weight of the nodes,

$$\omega_{\text{Global}}^{\top} = \mathbf{1}_3^{\top} (\mathbf{I}_3 - \mathbf{A}^{\top} + \mathbf{D})^{-1} \mathbf{D} = \left(\frac{31}{30} \quad \frac{35}{30} \quad \frac{24}{30} \right),$$

we can see that the weight of voter 1, which had the preference $A > B > C$, is the same as the fraction of votes with the same preference as voter 1. This is also true for the other voters. This means that, at the point of convergence, each voter's contribution to the community's preference schedule is the voter's intrinsic preference schedule weighted by their global weight.

After studying this example, we know that we can aggregate preference schedules in the network, instead of aggregating utilities. From Section 2.2, we know that preference schedules can be expressed as pairwise comparison graphs, and more specifically adjacency matrices of pairwise comparison graphs. Thus, we

Table 3.1: Example 3.1: A table showing how each iteration of the global empathy model affects the preference schedule of each voter and the total preference schedule of all the voters from the network in Figure 3.1. The infinity symbol (∞) is used to denote the point of convergence.

Iteration	Voter 1	Voter 2	Voter 3	Total
0	1 A > B > C	1 B > C > A	1 C > A > B	1 A > B > C 1 B > C > A 1 C > A > B
1	0.8 A > B > C 0.2 B > C > A	0.8 B > C > A 0.2 A > B > C	0.8 C > A > B 0.2 B > C > A	1 A > B > C 1.2 B > C > A 0.8 C > A > B
2	0.8 A > B > C 0.16 B > C > A 0.04 A > B > C	0.8 B > C > A 0.16 A > B > C 0.04 B > C > A	0.8 C > A > B 0.16 B > C > A 0.04 A > B > C	1.04 A > B > C 1.16 B > C > A 0.8 C > A > B
3	0.8 A > B > C 0.16 B > C > A 0.032 A > B > C 0.008 B > C > A	0.8 B > C > A 0.16 A > B > C 0.032 B > C > A 0.008 A > B > C	0.8 C > A > B 0.16 B > C > A 0.032 A > B > C 0.008 B > C > A	1.032 A > B > C 1.168 B > C > A 0.8 C > A > B
...
7	0.8 A > B > C 0.16 B > C > A 0.032 A > B > C 0.0064 B > C > A 0.00128 A > B > C 0.000256 B > C > A 0.0000512 A > B > C 0.0000128 B > C > A	0.8 B > C > A 0.16 A > B > C 0.032 B > C > A 0.0064 A > B > C 0.00128 B > C > A 0.000256 A > B > C 0.0000512 B > C > A 0.0000128 A > B > C	0.8 C > A > B 0.16 B > C > A 0.032 A > B > C 0.0064 B > C > A 0.00128 A > B > C 0.000256 B > C > A 0.0000512 A > B > C 0.0000128 B > C > A	1.0333312 A > B > C 1.1666688 B > C > A 0.8 C > A > B
...
∞	$\frac{25}{30}$ A > B > C $\frac{5}{30}$ B > C > A	$\frac{5}{30}$ A > B > C $\frac{25}{30}$ B > C > A	$\frac{1}{30}$ A > B > C $\frac{5}{30}$ B > C > A $\frac{24}{30}$ C > A > B	$\frac{31}{30}$ A > B > C $\frac{35}{30}$ B > C > A $\frac{24}{30}$ C > A > B

can aggregate adjacency matrices of pairwise comparison graphs instead of aggregating utilities.

We denote the adjacency matrix of voter j 's intrinsic pairwise comparison graph with P_j^I . In this example, the different intrinsic adjacency matrices are

$$P_1^I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, P_2^I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } P_3^I = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $P_j^{(t)}$ be the adjacency matrix of the pairwise comparison graph of user j at iteration t and P_j the adjacency matrix of the pairwise comparison graph at the point of convergence. We let $P_{\text{tot}}^{(t)}$ be the sum of all adjacency matrices of the pairwise comparison graphs of all voters at time t , $P_{\text{tot}}^{(t)} = \sum_{j \in \mathcal{V}} P_j^{(t)}$. And for the converged case, $P_{\text{tot}} = \sum_{j \in \mathcal{V}} P_j$. In this example,

$$P_{\text{tot}} = \frac{1}{30} \begin{pmatrix} 0 & 0 & 0 \\ 20 & 0 & 0 \\ 20 & 30 & 0 \end{pmatrix} + \frac{1}{30} \begin{pmatrix} 0 & 20 & 20 \\ 0 & 0 & 0 \\ 0 & 30 & 0 \end{pmatrix} + \frac{1}{30} \begin{pmatrix} 0 & 0 & 28 \\ 20 & 0 & 18 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{30} \begin{pmatrix} 0 & 20 & 48 \\ 40 & 0 & 18 \\ 20 & 60 & 0 \end{pmatrix},$$

where preference schedules at the point of convergence from which the adjacency matrices of the pairwise comparison graphs are taken from, can be found in Table 3.1. As we can see, P_{tot} does not look like the adjacency matrices of pairwise comparison graphs that we are used to, as P_{tot} has edges in both directions between candidate pairs. We define $P = \max(0, P_{\text{tot}} - P_{\text{tot}}^T)$, where the max operator is performed on each element in the matrix. P would in this case become

$$P = \frac{1}{30} \begin{pmatrix} 0 & 0 & 28 \\ 20 & 0 & 0 \\ 0 & 42 & 0 \end{pmatrix},$$

which certainly looks like an adjacency matrix of a pairwise comparison graph except that the elements are not integers. However, if the empathies are rational the elements would also become rational such that they could be normalised to look like an adjacency matrix of a pairwise comparison graph.

Formalising the reasoning from Example 3.1,

$$P_j^{(t+1)} = a_{j,j} P_j^I + \sum_{\substack{k \in \mathcal{V} \\ k \neq j}} a_{k,j} P_k^{(t)} \quad (3.1)$$

will converge to some P_j , $\forall j \in \mathcal{V}$, which summed together becomes P_{tot} :

$$P_{\text{tot}} = \sum_{j \in \mathcal{V}} P_j.$$

We let

$$\bar{\mathbf{P}} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_{n_v} \end{pmatrix} \text{ and } \bar{\mathbf{P}}^I = \begin{pmatrix} \mathbf{P}_1^I \\ \mathbf{P}_2^I \\ \vdots \\ \mathbf{P}_{n_v}^I \end{pmatrix}$$

be matrices of dimension $n_v n_c \times n_c$, in which we have “stacked” the adjacency matrices of the pairwise comparison graphs for each user. The superscript I indicates that the intrinsic preference is considered. It is noteworthy to mention that in Equation 3.1 it is only the elements of the matrix that are iterated. According to Theorem 3.2, we can now express Equation 3.1 in vector form that resembles Equation 2.6.

Theorem 3.2. *The following three expressions are equivalent.*

$$\mathbf{P}_j^{(t+1)} = a_{j,j} \mathbf{P}_j^I + \sum_{\substack{k \in \mathcal{V} \\ k \neq j}} a_{k,j} \mathbf{P}_k^{(t)}, \quad \forall j \in \mathcal{V}, \quad (3.2)$$

$$\bar{\mathbf{P}} = (\mathbf{D} \otimes \mathbf{I}_{n_c}) \bar{\mathbf{P}}^I + ((\mathbf{A}^\top - \mathbf{D}) \otimes \mathbf{I}_{n_c}) \bar{\mathbf{P}} \quad (3.3)$$

and

$$\bar{\mathbf{P}} = \left(\left((\mathbf{I}_{n_v} - \mathbf{A}^\top + \mathbf{D})^{-1} \mathbf{D} \right) \otimes \mathbf{I}_{n_c} \right) \bar{\mathbf{P}}^I \quad (3.4)$$

where \otimes is the Kronecker product. The proof of this theorem can be found in Section A.1.

According to Theorem 3.2 we can rewrite Equation 3.2 as Equation 3.4 where we can apply the same arguments as before for the fixed point solution. The proof of existence of $(\mathbf{I}_{n_v} - \mathbf{A}^\top + \mathbf{D})^{-1}$ can be found in [11, Appendix B].

A second way to express \mathbf{P}_{tot} is

$$\mathbf{P}_{\text{tot}} = (\mathbf{1}_{n_v}^\top \otimes \mathbf{I}_{n_c}) \bar{\mathbf{P}}.$$

Note that we can still express \mathbf{P}_{tot} as a sum of the intrinsic preferences weighted with the aggregated weights,

$$\mathbf{P}_{\text{tot}} = \sum_{j \in \mathcal{V}} \omega_j \mathbf{P}_j^I,$$

as a consequence of Theorem 3.2. Therefore we can use the weights from Equation 2.9 to calculate \mathbf{P}_{tot} and avoid the more complicated calculations of the Kronecker product. The definition of \mathbf{P} is still the same:

$$\mathbf{P} = \max(0, \mathbf{P}_{\text{tot}} - \mathbf{P}_{\text{tot}}^\top).$$

4

Results

In this section, some numerical results and the methods used to obtain them will be presented.

4.1 Method

First, the results from [11] were reproduced. This was done by implementing the empathy models in C++. To be able to easier handle the networks, the Stanford Network Analysis Platform (SNAP) was used [8]. For linear algebra computations, the Armadillo library was used [12].

Next, the beatpath and Kemeny-Young winner selection methods were implemented. To find the widest path in the pairwise comparison graph, a modified version of Dijkstra's algorithm was used. The winning Kemeny-Young ranking was found by using exhaustive search.

4.2 Simulation Set-Up

There are no real data sets (that we have been able to find) that has a network with preferences. There are data sets with networks and there are data sets with preferences, but no data sets that combine the two. Therefore, it has been necessary to generate synthetic data.

Synthetic votes were generated with a uniform distribution, meaning that all preferences were equally probable, which is called *impartial culture*. Generating votes with impartial culture gives the highest probability of cycles in the pairwise comparison graph, the worst case scenario [15]. When the pairwise comparison graph has cycles we do not have a Condorcet candidate.

The synthetic networks are generated with the Barabási-Albert method [3, Ch. 5]. As the Barabási-Albert method generates undirected networks and the empathy model requires directed networks, the networks must be converted to directed networks. This is done by adding one directed edge in each direction where there was one undirected edge. In addition, a self-edge is added at every node.

The edge weights are added such that every node has a self-edge with edge weight α . The rest of the edges get an equal edge weight such that the sum of every outgoing edge from every node, including the self-edge, sums to one.

The parameters used in the simulations were chosen to be similar as in [11] to simplify comparing the results. The number of voters, the number of candidates and the value of α , unless stated differently, are 1000, 5 and 0.25, respectively.

The metric used, when comparing empathy models and winner selection methods, is decision disagreement (DD). The DD is the percentage of the simulations where the two methods compared select a different winner. There is no true decision to compare with which is why we compare all the different decisions to one and other. Decision disagreement is one of the metrics used in [11].

Each time one of the methods selects multiple winners, which with 1000 voters, 5 candidates, the network generated with the modified Barabási-Albert method and $\alpha = 0.25$ happens less than 0.1% of the time, the simulation was re-run.

In [11], there are two metrics called *relative social welfare loss* and *normalized social welfare loss* which are metrics quantifying the decrease of the Borda or plurality score when picking the “wrong” candidate. Those metrics can be used due to Borda count and plurality being scoring winner selection methods. Neither the beatpath nor the Kemeny-Young winner selection method have equivalent metrics, therefore those metrics are not treated in this thesis.

4.3 Confirmation of Simulation Set-Up

The results produced here are produced such that they could easily be compared to the results produced in [11]. We generate 100 different empathy networks with the modified Barabási-Albert method with $\alpha = 0.25$ and for each empathy network we generate 100 preference profiles with the impartial culture distribution. The simulation is similar to the one for [11, Table 2], except that more networks were generated (100 instead of 50) and more preference profiles were generated (100 instead of 50) resulting in more instances (10000 instead of 2500). The results in Table 4.1a closely resembles those in [11, Table 2].

From Table 4.1 we see that no matter which winner selection method we choose, the DD will be approximately equal when comparing the empathy models.

The interpretation of this simulation is, if we were to assume the wrong empathy model, in how many of the cases would we select the wrong winner? Let us say that we are using the beatpath method, and we are assuming the network to be a local empathetic network but the voters are actually considering a global

empathy, we would select the wrong winner 12.88% of the times.

Table 4.2 is interpreted as, in how many of the cases would we select a different winner depending on which voting method we are using? We can see that Table 4.2a, Table 4.2b and Table 4.2c do not differ so much. This is because the problem is not dependent on the empathetic model. This test could have been performed on whichever set of empathy models. The only thing the empathy model will change is the distribution of votes. It will not be uniform any longer.

Table 4.1 and Table 4.2 are symmetrical; it does not matter if we are comparing A vs B or B vs A, it will give us the same result.

Table 4.1: Percentage of decision disagreement (DD) between different empathy models using four different winner selection methods. 100 networks with 100 different preference profiles on each. $\alpha = 0.25$, $n_v = 1000$, $n_c = 5$.

<i>(a) Borda count</i>			
	Intrinsic	Local	Global
Intrinsic	—	58.56	50.14
Local	58.56	—	12.27
Global	50.14	12.27	—

<i>(b) Plurality</i>			
	Intrinsic	Local	Global
Intrinsic	—	60.60	53.61
Local	60.60	—	10.71
Global	53.61	10.71	—

<i>(c) Beatpath</i>			
	Intrinsic	Local	Global
Intrinsic	—	60.43	53.08
Local	60.43	—	12.88
Global	53.08	12.88	—

<i>(d) Kemeny-Young</i>			
	Intrinsic	Local	Global
Intrinsic	—	60.50	53.15
Local	60.50	—	12.99
Global	53.15	12.99	—

Table 4.2: Percentage of decision disagreement (DD) between different winner selection methods using the three different empathy models. 100 networks with 100 different preference profiles on each. $\alpha = 0.25$, $n_v = 1000$, $n_c = 5$.

<i>(a) Intrinsic</i>				
	Borda count	Plurality	Beatpath	Kemeny-Young
Borda count	—	45.67	18.39	19.15
Plurality	45.67	—	47.02	47.13
Beatpath	18.39	47.02	—	3.05
Kemeny-Young	19.15	47.13	3.05	—

<i>(b) Local</i>				
	Borda count	Plurality	Beatpath	Kemeny-Young
Borda count	—	42.2	21.80	22.81
Plurality	42.25	—	40.92	41.01
Beatpath	21.80	40.92	—	4.76
Kemeny-Young	22.81	41.01	4.76	—

<i>(c) Global</i>				
	Borda count	Plurality	Beatpath	Kemeny-Young
Borda count	—	42.46	21.57	22.84
Plurality	42.46	—	42.38	42.41
Beatpath	21.57	42.38	—	4.73
Kemeny-Young	22.84	42.41	4.73	—

4.4 Varying the Self-Weight

Next, we inspect what happens when we vary the self-weight. We analyse how varying the self-weight affects the DD using the different winner selection methods and empathy models.

In the simulations, we vary the self-weight in steps of 0.1 from 0.1 to 0.9. We do not consider the case where $\alpha = 1$ as, in the case where the self-weights are all 1, the local and global models are the same as the intrinsic model. 100 networks were generated with the modified Bárabasi-Albert method for each α . On these networks 100 preference profiles were generated with impartial culture.

In Figure 4.1 the decision disagreement between the different empathy methods is plotted. The point $\alpha = 1$ is added in the figure as it is known that the DD is 0% between all the models. From the figure we can see that the DD is quite high for low values of the self-weight, but decreases with increasing self-weight.

The fact that the local vs global DD is smaller, in comparison the intrinsic vs local DD and the intrinsic vs global DD, shows us that the local model is a good first order approximation for the global model, as stated in [11]. It is interesting to note that the intrinsic model is more similar to the global model than the local model. The reason for this is unclear.

Comparing the different winner selection methods shows that there is no significant difference between them in how they are affected by the self-weight. The DD between the different empathy models follows the same trend, no matter which winner selection method is used. The curves are a bit jagged which is due to the amount of simulations run and the sparseness of the values for the self-weight. With more simulations and more dense values of the self-weight, the curves will be smoother.

In Figure 4.2 the DD between different winner selection methods in the different empathy models is studied. We can see that neither using different empathy models nor varying the self-weight gives us any significant difference in DD between the winner selection methods.

We see that the plurality winner differs from the Borda winner, the beatpath winner and the Kemeny-Young winner in approximately 45% of the cases. The Borda winner differs from the beatpath winner and the Kemeny-Young winner in approximately 20% of the cases. The beatpath winner differs from the Kemeny-Young winner in approximately 5% of the cases.

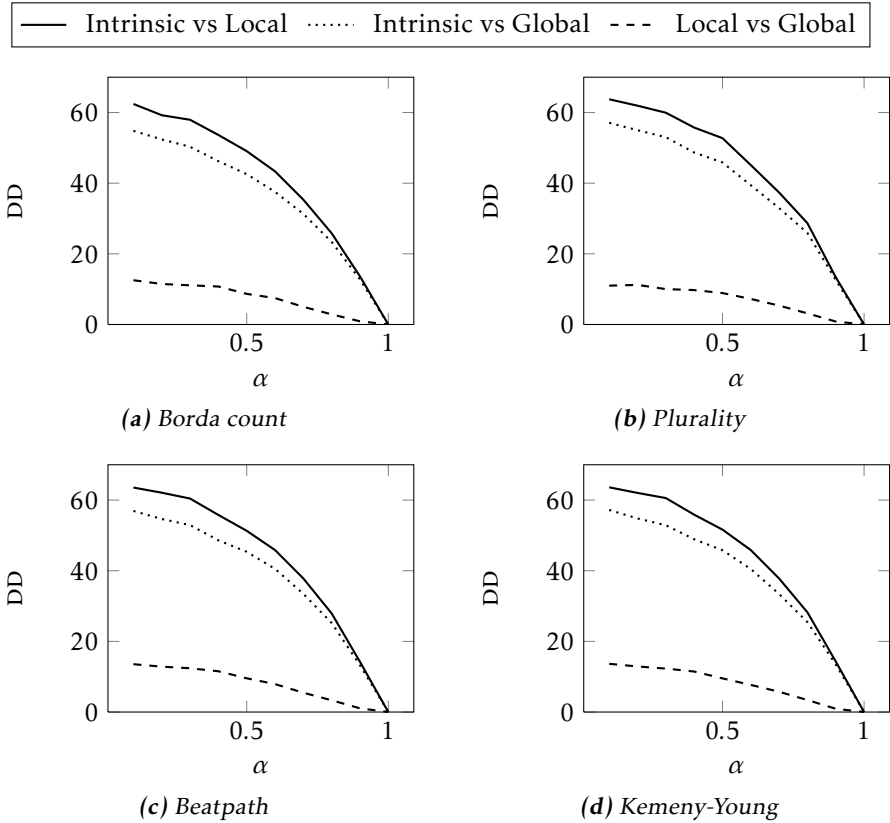


Figure 4.1: Percentage of decision disagreement (DD) between different empathy models using four different winner selection methods. Varying α , $n_v = 1000$, $n_c = 5$. For each α , 100 networks were generated, for each network 100 preference schedules were generated.

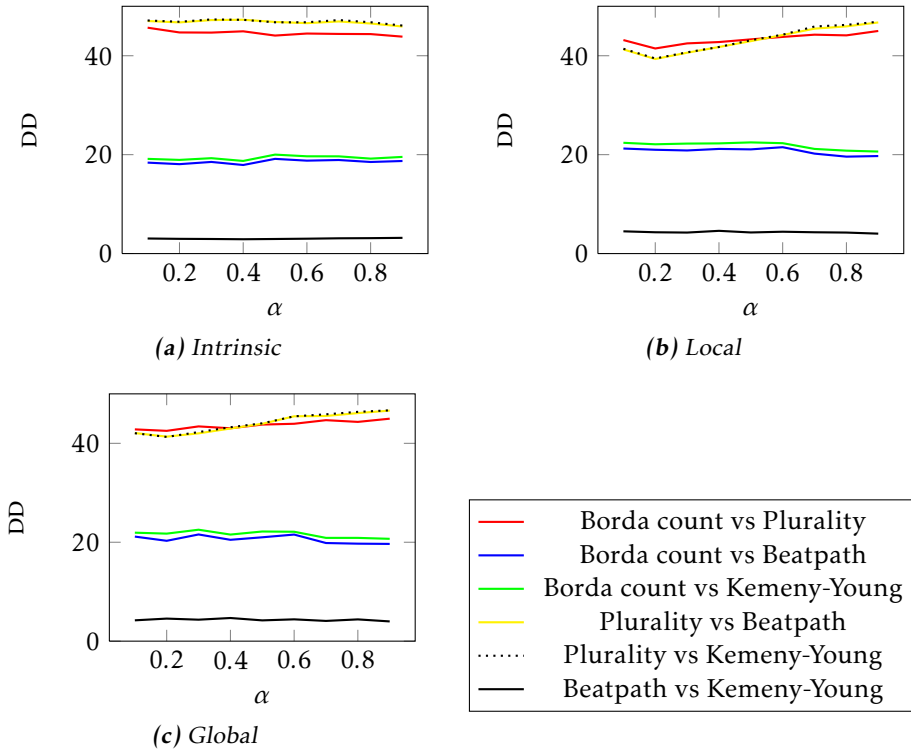


Figure 4.2: Percentage of decision disagreement (DD) between different winner selection methods using the three empathy models. Varying α , $n_v = 1000$, $n_c = 5$. For each α , 100 networks were generated, for each network 100 preference schedules were generated.

4.5 Emergence of “Dictatorship” in Certain Networks

An interesting phenomenon happens when the empathy networks have a certain structure. When a person in the network receives a lot of empathy while at the same time having a large empathy towards itself, it can sometimes become a “dictator”.

Consider an empathy network as the one in Figure 4.3. We can see that voters 1 and 2 have high empathy towards the third voter. This will result in that the third voter has a large weight.

In the local model,

$$\omega_{\text{Local}}^{\top} = \mathbf{1}_3^{\top} \mathbf{A}^{\top} = (1 \quad 1 \quad 1) \begin{pmatrix} 0.1 & 0 & 0.9 \\ 0 & 0.1 & 0.9 \\ 0 & 0 & 1 \end{pmatrix} = (0.1 \quad 0.1 \quad 2.8),$$

and the global model,

$$\begin{aligned} \omega_{\text{Global}}^{\top} &= \mathbf{1}_3^{\top} (\mathbf{I}_3 - \mathbf{A}^{\top} + \mathbf{D})^{-1} \mathbf{D} \\ &= (1 \quad 1 \quad 1) \begin{pmatrix} 1 & 0 & -0.9 \\ 0 & 1 & -0.9 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (1 \quad 1 \quad 1) \begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0.1 \quad 0.1 \quad 2.8), \end{aligned}$$

voter 3 has a weight of 2.8, while both voters 1 and 2 have a weight of 0.1. Because of the difference of the weights and the fact that voter 3 has a weight that is more than half of the total weight, it will neither matter how the voters 1 and 2 vote nor which winner selection method is used when deciding the winner; voter 3 will essentially decide the outcome of the election by themselves.

For a winner selection method to have a dictator in a regular voting setting, is an undesirable trait. However, in this scenario this is acceptable as the voters also have stated their empathy towards the other voters and collectively choose who

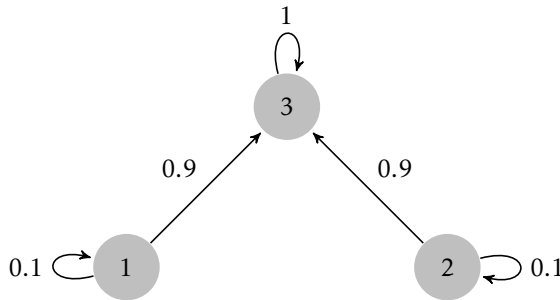


Figure 4.3: An empathy network with a “dictator”.

becomes the “dictator”. Unfortunately, the emergence of the “dictator” will result in loss of many of the traits we desire in a voting system, such as Condorcet fairness, if we consider the original preference profile. If we consider the weighted votes, all the original criteria fulfilled by the voting systems will be preserved.

5

Discussion

In this chapter the results and the methods used are discussed. We discuss the results in a broader perspective than in Chapter 4.

5.1 Results

From the results we can deduce that each time we have preferences instead of votes with only one candidate selected we should use the beatpath or the Kemeny-Young winner selection method as these winner selection methods fulfil more criteria. When we have many candidates it would be a good idea to use the beatpath method as the Kemeny-Young method is NP-hard compared to the beatpath method which is in P [5, Ch. 4]. From Figure 4.2 we see that the beatpath method and the Kemeny-Young method only differ in about 5% of the time which might not be enough to justify solving an NP-hard problem.

There is no reason to ever use Borda count. However, plurality might be used if there is only one candidate selected by every voter. It is easier to collect those votes than the votes with complete preferences.

The results obtained are as expected as the empathy models do not take the winner selection methods into account.

5.2 Method

It is unfortunate that there is no real world empathy networks with preferences. With regards to this, the methods used to generate synthetic networks and preference profiles are adequate.

The scale-free networks are true to real-world networks. However, for the sake of completeness, it might have been a good idea to also look at networks gen-

erated differently, e.g. Erdős–Rényi networks [3, Ch. 3], to see how it would affect the results obtained. For example, in a very tight-knit friend group, where the network is complete and the empathy weight are all equal, the empathy model used would not affect the outcome as all the voters would have weight 1 regardless of which model was used. The situations where the people have different self-weights and different empathy towards different neighbours is not studied, presumably it will give quite a bit of differences in the results. This is a possibility for future work.

While the scale-free networks are true to reality, the distribution of preferences, impartial culture, is not. It is a worst case scenario as it gives the highest probability of cycles in the pairwise comparison graph, which means that it gives the highest probability of not having a Condorcet candidate. By studying elections where we do not have a Condorcet candidate we will study the differences of the Condorcet fair winner selection methods, which are the most relevant cases to study. If we would have studied a different distribution of the preferences, comparing the Condorcet fair winner selection methods, beatpath and Kemeny-Young, would have yielded a lower decision disagreement.

The lack of real world data is a problem for analysing the empathy models. It is legitimate as this type of computations have not been used before. There is data from recommendation systems with so called trust networks, where some users trust recommendations from trusted users more than from others [16]. However, the trust networks are binary while the empathy in the networks we considered can be more weighted. Also, the recommendations are far from complete, all users have not rated every item, which means that it is harder to adapt the recommendations to votes.

One big difference from [11] is that there is no numerical way for the beatpath and Kemeny-Young methods to measure how much worse it is to elect the wrong candidate compared to the correct winner.

By running more realisations than in [11] the results obtained are a little more trustworthy.

5.3 The Results in a Broader Context

The work has a clear societal connection as it is directly applicable to be used in group decision making.

One important aspect to be considered if a voting system using empathy would be implemented in reality is how to make it anonymous for the users as they rate each other by stating their empathy towards one another. If the results and votes were made public, as they usually are in elections, one must consider the possibility for people to reverse engineer the votes to find out who cast which vote.

It would be good if metadata from social media like Facebook could be used. Likes, shares and other activities could be used to deduce the empathy users have towards each other. The manipulation of votes and empathy in this setting would in this case have to be studied deeper to prevent easy manipulation of the

outcome of the voting.

Another application aspect would be in a system of multiple artificial agents that make some group decision. The ethical aspects must be considered such that the agents do not come to a decision that humans would not take.

6

Conclusion

In this chapter we will respond to the problem statement as well as conclude the thesis and discuss future work. We recall the problem statement from Section 1.2.

1. How can we modify the empathy models such that we can use other, superior winner selection methods than plurality and Borda count?
2. Will the same arguments for convergence and fixed-point solutions hold as for the original models?
3. Will the winner selection methods maintain their properties in the empathy models?

We answered the first question in Chapter 3 where we show that aggregating the elements in the adjacency matrix of the pairwise comparison graphs works in the same way as aggregating utilities.

In Chapter 3 we also show that the same arguments for convergence and fixed point solutions hold when aggregating other things than utilities, which answers the second question.

The third question can be interpreted in different ways. In Section 4.5 we show that the requirements of the winner selection methods on the original preference do not hold after aggregation. However, if we consider the aggregated preference profile, the criteria are fulfilled.

6.1 Future Work

Currently, we are assuming that the voters have the same empathy towards their friends for every candidate. In certain cases the voters might not have the same empathy depending on which alternative is voted on. For example, if a group

of friends are deciding what to eat and one of them has gluten intolerance the friends might have a high empathy for that person when bread is in question but not if they are deciding about other foods. If their preference schedule looked like

$$\begin{array}{l} 4 \quad \text{Bread} > \text{Vegetables} > \text{Meat} \\ 1 \quad \text{Meat} > \text{Vegetables} > \text{Bread} \end{array}$$

the four people that preferred bread would still rather eat vegetables than meat and would only have empathy towards the person with gluten intolerance in the question about bread. This is an aspect that could be studied in future work.

Another aspect to study in future work is if the empathy would vary over time, how would that affect the models?

Manipulation of votes and the outcome of elections is well studied in social choice theory. However, it is not clear how easy or hard it is to manipulate the empathy networks to give other winners.

The most important areas for future work is to study how applicable these models are in real-life and whether the voters are satisfied with the results or not.

Appendix

A

Proofs

A.1 Proof of Theorem 3.2

We start by proving that

$$P_j = a_{j,j}P_j^I + \sum_{\substack{k=1 \\ k \neq j}}^{n_v} a_{k,j}P_k, \quad \forall j \in \mathcal{V}$$

can be written as

$$\bar{P} = (D \otimes I_{n_c}) \bar{P}^I + ((A^\top - D) \otimes I_{n_c}) \bar{P}.$$

$$\begin{aligned}
\bar{P} &= (D \otimes I_{n_c}) \bar{P}^I + ((A^\top - D) \otimes I_{n_c}) \bar{P} \iff \\
\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_v} \end{pmatrix} &= \begin{pmatrix} d_{1,1} I_{n_c} & \cdots & d_{1,n_v} I_{n_c} \\ \vdots & \ddots & \vdots \\ d_{n_v,1} I_{n_c} & \cdots & d_{n_v,n_v} I_{n_c} \end{pmatrix} \begin{pmatrix} P_1^I \\ P_2^I \\ \vdots \\ P_{n_v}^I \end{pmatrix} + \begin{pmatrix} (a_{1,1} - d_{1,1}) I_{n_c} & \cdots & (a_{n_v,1} - d_{1,n_v}) I_{n_c} \\ \vdots & \ddots & \vdots \\ (a_{1,n_v} - d_{n_v,1}) I_{n_c} & \cdots & (a_{n_v,n_v} - d_{n_v,n_v}) I_{n_c} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_v} \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} I_{n_c} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n_v,n_v} I_{n_c} \end{pmatrix} \begin{pmatrix} P_1^I \\ P_2^I \\ \vdots \\ P_{n_v}^I \end{pmatrix} \\
&+ \begin{pmatrix} 0 & a_{2,1} I_{n_c} & \cdots & a_{n_v,1} I_{n_c} \\ a_{1,2} I_{n_c} & & & \vdots \\ \vdots & & \ddots & a_{n_v,n_v-1} I_{n_c} \\ a_{1,n_v} I_{n_c} & \cdots & a_{n_v-1,n_v} I_{n_c} & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_v} \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} I_{n_c} P_1^I + \sum_{\substack{k=1 \\ k \neq 1}}^{n_v} a_{k,1} I_{n_c} P_k \\ a_{2,2} I_{n_c} P_2^I + \sum_{\substack{k=1 \\ k \neq 2}}^{n_v} a_{k,2} I_{n_c} P_k \\ \vdots \\ a_{n_v,n_v} I_{n_c} P_{n_v}^I + \sum_{\substack{k=1 \\ k \neq n_v}}^{n_v} a_{k,n_v} I_{n_c} P_k \end{pmatrix} = \begin{pmatrix} a_{1,1} P_1^I + \sum_{\substack{k=1 \\ k \neq 1}}^{n_v} a_{k,1} P_k \\ a_{2,2} P_2^I + \sum_{\substack{k=1 \\ k \neq 2}}^{n_v} a_{k,2} P_k \\ \vdots \\ a_{n_v,n_v} P_{n_v}^I + \sum_{\substack{k=1 \\ k \neq n_v}}^{n_v} a_{k,n_v} P_k \end{pmatrix} \iff \\
\iff P_j &= a_{j,j} P_j^I + \sum_{\substack{k=1 \\ k \neq j}}^{n_v} a_{k,j} P_k, \quad \forall j \in \mathcal{V} \quad \blacksquare
\end{aligned}$$

Next, we show that

$$\bar{P} = (D \otimes I_{n_c}) \bar{P}^I + ((A^\top - D) \otimes I_{n_c}) \bar{P}.$$

can be written as

$$\bar{P} = \left(\left((I_{n_v} - A^\top + D)^{-1} D \right) \otimes I_{n_c} \right) \bar{P}^I.$$

$$\begin{aligned}
& \bar{P} = (D \otimes I_{n_c}) \bar{P}^I + ((A^\top - D) \otimes I_{n_c}) \bar{P} \\
I_{n_v n_c} \bar{P} - ((A^\top - D) \otimes I_{n_c}) \bar{P} &= (D \otimes I_{n_c}) \bar{P}^I \iff \\
(I_{n_v n_c} - ((A^\top - D) \otimes I_{n_c})) \bar{P} &= (D \otimes I_{n_c}) \bar{P}^I \iff \\
(I_{n_v} \otimes I_{n_c} - ((A^\top - D) \otimes I_{n_c})) \bar{P} &= (D \otimes I_{n_c}) \bar{P}^I \iff \\
((I_{n_v} - (A^\top - D)) \otimes I_{n_c}) \bar{P} &= (D \otimes I_{n_c}) \bar{P}^I \iff \\
\bar{P} &= ((I_{n_v} - (A^\top - D)) \otimes I_{n_c})^{-1} (D \otimes I_{n_c}) \bar{P}^I \iff \\
\bar{P} &= ((I_{n_v} - (A^\top - D))^{-1} \otimes I_{n_c}^{-1}) (D \otimes I_{n_c}) \bar{P}^I \iff \\
\bar{P} &= (((I_{n_v} - A^\top + D)^{-1} D) \otimes I_{n_c}) \bar{P}^I \quad \blacksquare
\end{aligned}$$

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