A noncommutative catenoid

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Abstract

Noncommutative geometry generalizes many geometric results from such fields as differential geometry and algebraic geometry to a context where commutativity cannot be assumed. Unfortunately there are few concrete non-trivial examples of noncommutative objects. The aim of this thesis is to construct a noncommutative surface $C_h$ which will be a generalization of the well known surface called the catenoid. This surface will be constructed using the Diamond lemma, derivations will be constructed over $C_h$ and a general localization will be provided using the Ore condition.

Keywords:
noncommutative, catenoid, noncommutative geometry, diamond lemma, noncommutative algebra, ore condition

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Chapter 1

Introduction

The field of noncommutative geometry is a fairly young subject that covers the non-commutative case of various geometric results from a wide range of subjects, such as differential geometry, algebraic geometry and others. There are many established results in this field, but unfortunately not many examples that demonstrates these results has been given.

Non-commutative geometry is an extension to classical geometry which extends the concept of spaces in a natural way based primarily on quantum mechanics and other areas of physics [1]. Surfaces derived from classical geometry are unfortunately hard to come by. In this thesis an attempt to at least give one such surface is presented. This is done in order to demonstrate the connection between noncommutative geometry and classical geometry.

A strategy for constructing non-commutative surfaces will be demonstrated by constructing one possible non-commutative version of the well known surface called the catenoid.

1.1 Overview

Surfaces are, in terms of algebraic geometry, represented by so called algebras of functions (see Definition 2.3) which is a set of elements associated with three operations, addition, multiplication and scalar multiplication. Commutative surfaces has the assumption that multiplication is commutative (i.e. $xy = yx$), whereas non-commutative surfaces does not. The catenoid, a commutative sur-
face will be used as basis for attempting to construct a non-commutative sur-
face.

First, algebras and other concepts that are necessary to perform the construction
are introduced. After that the concept of “reductions” and “normal forms” are
introduced. An algebra $\mathcal{C}_\hbar$ will be constructed using a set of reductions based on
the properties of the catenoid such that all elements in the algebra has a normal
form. Formally speaking, each element in $\mathcal{C}_\hbar$ will represent all possible ways to
represent it, but will be written in one way only (the normal form).

The construction of the algebra $\mathcal{C}_\hbar$ is complete at the end of Chapter 4. In Chap-
ter 5 the algebra is extended to include some fundamental geometric properties,
 such as the non-commutative analogue of derivatives and the non-commutative
counterpart to reciprocal functions, thus making it a surface.
Chapter 2

Algebraic structures

The goal of this chapter is to introduce some fundamental structures used throughout this thesis. First, in Section 2.1 semigroups and algebras are defined. In Section 2.3 The most important structure, the free algebra is introduced, a special type of algebra that is the most general form of algebras.

2.1 Semigroups and algebras

The mathematical field of abstract algebra studies so called algebraic structures. An algebraic structure is a set paired with one or more operators over the set, where the operators satisfy a set of axioms.

An example of this would be the algebraic structure of $\mathbb{R}$ paired with addition and multiplication. Addition and multiplication both satisfies associativity and commutativity.

For all intents of this thesis it is sufficient to study the algebraic structures semigroups, fields and algebras.

**Definition 2.1.** A semigroup is a pair $(G, \cdot)$, with a set $G$ and a binary operator $\cdot : G \times G \rightarrow G$ such that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in G \text{ (associative)}$$

A semigroup is a very simple structure. For the purposes of this thesis one
should view semigroups as a building block for constructing more complex structures.

Note that \( xy = yx \) (commutativity) cannot be assumed for semigroups in general. The only property that can be assumed is associativity, since it is the only axiom over the semigroup operator.

**Definition 2.2.** A **field** is a 3-tuple \((K, +, \cdot)\) with a set \(K\), two binary operators \(+ : K \times K \rightarrow K\) (called addition) and \(\cdot : K \times K \rightarrow K\) (called multiplication), for which the following properties holds:

(i) \((a + b) + c = a + (b + c)\) and \((a \cdot b) \cdot c = a \cdot (b \cdot c)\) for \(a, b, c \in K\).

(ii) \(a + b = b + a\) and \(a \cdot b = b \cdot a\) for \(a, b \in K\).

(iii) There exists elements \(0, 1 \in K\) where \(0 \neq 1\) such that \(0 + a = a\) and \(1 \cdot a = a\).

(iv) For any \(a \in K\) there exists two elements \(-a, a^{-1} \in K\) such that \(a + (-a) = 0\) and \(a \cdot a^{-1} = 1\).

(v) \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\) for any \(a, b, c \in K\).

A field is a very important structure. It is a structure that encapsulates the properties of real and complex numbers and generalizes them. Therefore it is in some sense the most “natural” structure that is presented in this thesis.

Fields will not be used extensively in this thesis. They will primarily function as building blocks together with semigroups to construct the certain types of algebras that are used in this thesis.

**Definition 2.3.** An **algebra** is a 3-tuple \((k, A, \cdot)\) where \(k\) is a field, \(A\) is a vector space over \(k\) and \(\cdot : A \times A \rightarrow A\) is a binary operator such that

(i) \((x + y) \cdot z = (x \cdot z) + (y \cdot z)\) for all \(x, y, z \in A\).

(ii) \(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\) for all \(x, y, z \in A\).

(iii) \((ax) \cdot (by) = (ab)(x \cdot y)\) for all \(x, y \in A\) and \(a, b \in k\).

An algebra is called a **commutative algebra** if it also holds that \(x \cdot y = y \cdot x\).

To aid in the understanding of algebras an example based on linear algebra is presented. Consider all \(2 \times 2\) matrices \(M_{2,2}\). It is an algebra \((k, M_{2,2}, \cdot)\) where \(k = \mathbb{R}\) (shorthand for \(k = (\mathbb{R}, +, \cdot)\)) and \(\cdot\) is matrix multiplication since given
matrices $A, B, C \in M_{2,2}$ and constants $a, b \in \mathbb{R}$ it holds that
\[
(A + B)C = AC + BC \\
A(B + C) = AB + AC \\
(aA)(bB) = (ab)AB.
\]

What is important to note here is that algebras are not necessarily a concept separated from other fields such as linear algebra. Algebras are rather another way to perceive objects. The purpose of introducing algebras are to capture common properties between types of objects and derive new techniques and theorems that can be applied in a more general setting.

Continuing the example of $M_{2,2}$ above, note that the subset $M'$ of $M_{2,2}$ that consists of all matrices of the form
\[
\begin{pmatrix}
x \\
0 \\
0 
\end{pmatrix}
\]
for all $x \in \mathbb{R}$ also satisfies the definition of an algebra. But what is especially important about this set is that matrix addition and multiplication are closed under $M'$. That is
\[
\begin{pmatrix}
x \\
0 \\
0 
\end{pmatrix} \left( \begin{pmatrix}
y \\
0 \\
0 
\end{pmatrix} + \begin{pmatrix}
z \\
0 \\
0 
\end{pmatrix} \right) = \begin{pmatrix}
xy + xz \\
z \\
0 
\end{pmatrix} \in M'.
\]

This demonstrates that $M_{2,2}$, which is an algebra can contain smaller algebras that maintain the operators and the structure of $M_{2,2}$ but is still its own algebra. These smaller types of algebras are called subalgebras.

**Definition 2.4.** Let $\mathcal{A}$ be an algebra and let $\mathcal{A}_0 \subseteq \mathcal{A}$. If $\mathcal{A}_0$ is an algebra, then it is a subalgebra of $\mathcal{A}$.

### 2.2 Quotient algebras

In order to understand the result in Section 3.2 the concept of quotient algebra is necessary. A quotient algebra is the algebra of all equivalence classes under some congruence. A congruence is an equivalence relation that is closed under addition and multiplication.
Definition 2.5. Given an algebra $\mathcal{A} = (k, A, \cdot)$ a congruence $E$ over $\mathcal{A}$ is an equivalence relation $\sim$ over $A$ such that if $a_1 \sim a_2$ and $b_1 \sim b_2$ then $a_1 \cdot b_1 \sim a_2 \cdot b_2$ and $a_1 + b_1 \sim a_2 + b_2$ for all $a_1, a_2, b_1, b_2 \in A$. Let $[a]$ denote the equivalence class generated by $a$ under $E$.

A typical congruence relation is given by modular arithmetic, namely the congruent modulo over some $n$.

Take for example

$$17 \equiv 7 \pmod{10}$$
$$4 \equiv 4 \pmod{10}$$

Then

$$17 \cdot 4 = 68 \equiv 8 \pmod{10},$$
$$7 \cdot 4 = 28 \equiv 8 \pmod{10},$$
$$17 + 4 = 21 \equiv 1 \pmod{10},$$
$$7 + 4 = 11 \equiv 1 \pmod{10}.$$

So $17 \cdot 4$ is congruent to $7 \cdot 4$ and $17 + 4$ is congruent to $7 + 4$.

Take a partition of an algebra $\mathcal{A}$ such that every element in each class in the partition is equivalent according to some congruence $\sim$. Then the quotient algebra can be viewed as the algebra where each element is an equivalence class of $\mathcal{A}$ under $\sim$.

Definition 2.6. For an algebra $\mathcal{A}$ over the vector space $A$ and the field $k$, with a congruence $E$ let $A/E$ be the set of all equivalence classes in $A$ induced by $E$, and define

$$[a] \cdot [b] = [a \cdot b], \forall a, b \in A$$

Then the algebra $\mathcal{A}/E = (k, A/E, \cdot)$ is called a quotient algebra of $\mathcal{A}$.

Consider $x, y \in \mathcal{A}$ such that $x \sim y$. To further ones intuition about quotient algebras it would be beneficial to study what $x$ and $y$ have in common. For example, in the case of congruent modulo over $n$, $x$ and $y$ have the same remainder under division with $n$. That is $x = k_1 n + r$ and $y = k_2 n + r$.

The quotient algebra $\mathcal{A}/E$ maps $x$ and $y$ to the same equivalence class since they have equal remainders. The quotient part of the elements are mapped to zero, leaving only the remainders. In this example, the quotient part consists of all elements divisible by $n$. 
In order to generalize this concept beyond congruence modulo \( n \), the set of all “quotient” parts of elements (note that they do not have to be quotients in the normal sense) has to be captured in a meaningful way. The concept of ideals captures this. Ideals are subalgebras that no matter what element it is multiplied with it will always result in a element that is in the ideal.

**Definition 2.7.** Let \( I \) be a subalgebra of an algebra \( \mathcal{A} \).

(i) \( I \) is called a *left ideal* if \( ax \in I \) for all \( x \in I \) and \( a \in \mathcal{A} \).

(ii) \( I \) is called a *right ideal* if \( xa \in I \) for all \( x \in I \) and \( a \in \mathcal{A} \).

(iii) \( I \) is called a *two-sided ideal* if it is both a left and a right ideal.

From this definition, it follows that given \( x, z \in \mathcal{A} \) and \( y \in I \) where \( I \) is a two-sided ideal of \( \mathcal{A} \), the element \( xyz \in I \). But more importantly, if for \( a, b \in \mathcal{A} \) it holds that \( a - b \in I \), then \( a - b = xyz \) for arbitrary \( x, z \in \mathcal{A} \) and \( y \in I \). Therefore it is easy to draw parallels between congruence modulo \( n \) and ideals. For congruence modulo \( n \) we had that \( n = km + r \), where \( r \) is the remainder of \( n \) divided by \( m \). For ideals \( a = xyz + b \), so \( b \) can be seen as the “remainder” of \( a \) divided by \( y \in I \).

If a congruence that mimics the behaviour of congruence modulo is constructed, the connection to quotient algebras is immediately available.

Take \( a = k_1n + r_1 \) and \( b = k_2n + r_2 \) where \( r_i < n \). According to the desired behaviour, they are congruent if and only if \( r_1 = r_2 \). Another way to put it is, “They are congruent if and only if \( n \) divides \( a - b \)”. This concept is generalized in the following definition.

**Definition 2.8.** For a free algebra \( \mathcal{A} \) with an ideal \( I \), \( \mathcal{A}/I \) is shorthand for the quotient algebra given by the congruence \( \sim \) such that

\[
a \sim b \iff a - b \in I.
\]

(2.1)

To demonstrate how this type of quotient algebra may look, consider the algebra of all polynomials with real coefficients \( \mathcal{P} \), defined as \( \mathcal{P} = (\mathbb{R}, \mathcal{A}, \cdot) \) where \( \mathcal{A} \) is the vector space spanned by all elements \( x^k, k \geq 0 \).

Take \( I = \{ p \in \mathcal{P} \text{ such that } p \text{ is divisible by } (x^2 + 1) \} \). Then \( I \) is a two-sided ideal since \( p \cdot q \) and \( q \cdot p \) both still contains a factor divisible by \( x^2 + 1 \) for all \( q \in I, p \in \mathcal{P} \). That is, for \( p = p'(x^2 + 1) + r \) and \( q = q'(x^2 + 1) \), it holds that \( p \cdot q = (p'(x^2 + 1) + r) \cdot q'(x^2 + 1) \) which of course is divisible by \( (x^2 + 1) \) for all \( p', q', r \in \mathcal{P} \).
Now consider the congruence \( \sim \) defined such that it satisfies (2.1) for \( I \). Then for any \( q \in P \) its corresponding equivalence class \([q]_\sim\) is given by all \( p \in P \) such that \( p - q \in I \), i.e. \([q]_\sim\) is given by all polynomials \( p \) such that \( p - q = r(x^2 + 1) \Leftrightarrow p = q + r(x^2 + 1) \) for any \( r \in P \). So the quotient algebra \( P/I \) is equivalent to the algebra of all polynomials modulo \( x^2 + 1 \).

### 2.3 Free algebras

The aim of this section is to define the primary type of algebra studied in this thesis. The intuition for this type of structure is based around *concatenation* of formal symbols.

Concatenation is the operation of joining two sequences of symbols into one long sequence of symbols. One simple example of this would be the following.

Given a set of symbols \( \{a, b, c\} \) two sequences of these symbols, say \( abc \) and \( cba \) can be constructed. Then the concatenation of these sequences will be the sequence \( abccba \).

The first construction needed is the following.

**Definition 2.9.** For a set \( X = \{X_1, X_2, \ldots, X_n\} \) the *free semigroup* \langle X \rangle is the semigroup over the set of all possible sequences of elements in \( X \) (including the empty sequence denoted by 1) with its binary operator, called *concatenation* defined as

\[
(X_{i_1}X_{i_2} \cdots X_{i_p}) \cdot (X_{j_1}X_{j_2} \cdots X_{j_q}) = X_{i_1}X_{i_2} \cdots X_{i_p}X_{j_1}X_{j_2} \cdots X_{j_q}.
\]

This definition is consistent with the idea of sequences of symbols. In this definition the set \( X \) consists of the symbols, and the free semigroup \langle X \rangle is simply the set of all possible sequences of these symbols.

**Definition 2.10.** The *length* of an element \( x \) in a free semigroup \langle X \rangle, is defined as the length of the concatenated sequence of symbols.

The sequence of symbols that has length \( p \) which only consists of the symbol \( x \) is written as \( x^p \).

Note that \langle X \rangle in fact satisfies the definition of a semigroup, since it does not matter in what groupings the concatenations are evaluated, the only thing that matters is the order of the symbols. This means that concatenation is *associative*, but not *commutative*. 
With free semigroups, it is now possible to define the primary type of algebra studied in this thesis.

**Definition 2.11.** For a set $X = \{X_1, X_2, \ldots, X_n\}$ and a field $k$, the free algebra $k\langle X \rangle$, is the algebra $(k, A, \cdot)$ where $A$ is the vector space with the free semigroup $\langle X \rangle$ as basis and where $x \cdot y$ for $x, y \in k\langle X \rangle$, is defined as the sum of all pairwise semigroup concatenations between $x$ and $y$ such that it distributes over addition.

The set $X$ is called *generators* of $k\langle X \rangle$ and is said to *generate* $k\langle X \rangle$.

The intuition for a free algebra is briefly put just an extension of free semigroups. Informally $k\langle X \rangle$ can be viewed as all possible linear combinations of the sequences in $\langle X \rangle$ with coefficients from the field $k$. An example of this would be $abc + 5ab + 2c + 17\mathbb{1} \in k\langle X \rangle$ where, of course $k = \mathbb{R}$ and $X = \{a, b, c\}$. The binary operator defined for free algebras is best explained using an example. Consider a free algebra $\langle X \rangle$ for $X = \{x, y, z\}$, with coefficients in $\mathbb{R}$. Then the product of the elements $x + 5xy + \mathbb{1}$ and $2z + xzy$ is

$$(x + 5xy + \mathbb{1}) \cdot (2z + xzy) = 2xz + xxzy + 10xyz + 5xyxzy + 2z + xzy$$

It is practical for the purposes of this thesis to view the formal symbols of some free algebra as variables (or indeterminates) and the elements in the algebra itself as polynomials over these variables.

Take the same free algebra $k\langle X \rangle$ as in the example above. Then some elements in said algebra are written as

$$5xy^2z + 12x - 14x^34yz^2 + 17\mathbb{1}$$
$$12x + 7xy$$
$$x^2 + xy + yx + y^2.$$

These elements are similar to ordinary polynomials, but not quite the same. Ordinary polynomials have indeterminates that commute while indeterminates in a free algebra do not. Therefore it is natural to call elements in the free algebra *noncommutative polynomials* and elements in the corresponding free semigroup as *noncommutative monomials*. From here on all *polynomials* and *monomials* are assumed to be noncommutative, unless otherwise stated.
Chapter 3

Reduction systems and the Diamond lemma

When considering commutative free algebras there is the extra property that $xy = yx$. This is the same as a free (noncommutative) algebra with the additional relation $xy = yx$. This extra relation will introduce equivalence classes over the algebra. Each equivalence class is the set of all elements that are considered to be equal. For example $\{xy^2, yxy, y^2x\}$ is an equivalence class that represents all elements equal to (the arbitrarily chosen) element $xy^2$.

If one of the elements from the equivalence class is chosen as “the most simplified” element then it could be used as a representative for the entire equivalence class.

**Definition 3.1.** Given a free algebra $A$, a congruence $E$, and an injective mapping $\varphi : A/E \to A$ the representative of $[a]_E$ is given by $\varphi([a]_E)$ for $a \in A$.

For instance $xy^2$ could be chosen as the representative of the equivalence class specified in the example above. That is, under the relation $xy = yx$, the element $xy^2$ represent one equivalence class in the algebra.

If even more relations are introduced then the equivalence classes will change. For instance consider the relations $y^2 = y$ and $x^2 = x$. These relations will introduce infinite equivalence classes since $y^n$ are all in the same equivalence class for all $n$ (and likewise for $x^n$). Also worth noting is that, for these relations,
there are only four equivalence classes. Those represented by 1, x, y and xy, since for a given monomial

\[ x^{p_1}y^{p_2} \cdots x^{p_{n-1}}y^{p_n} = x^{p_1+p_{n-1}}y^{p_2+\cdots+p_n} = x^p y^q, \]

where \( p, q \in \{0, 1\} \).

Under these relations, it can be useful to think of the relations as some sort of reduction of elements. That is if some element \( X \) is the representative of its corresponding equivalence class, then the relations can be used to reduce every other element in the equivalence class to \( X \).

Of course it is possible to introduce relations that are “inconsistent”. To demonstrate what this means, take \( k = \mathbb{R} \) and \( X = \{a, b, c\} \). Introduce the relations \( a - b = a, a = b \) and \( b - a = b \) over \( k\langle X \rangle \).

Then the element \( a \) can be reduced in several ways. For example \( a = a - b = a - a = b - a = b \). This means that the algebra can only be “consistent” if \( a \) and \( b \) are both zero (that is, the algebra is trivial) meaning that the only equivalence class over the algebra is the algebra itself.

There are several aims of this chapter;

1. Formalize the introduction of relations (or reductions) on free algebras.
2. Define what it means for a set of relations to be “consistent”.

The chapter begins with several definitions that will build up to an important theorem known as the Diamond lemma.

### 3.1 Reduction systems

The first two definitions in this section aims to formally define reductions and how they are induced on an algebra.

**Definition 3.2.** A homomorphism \( h \) over free algebras \( k\langle X \rangle \) and \( k\langle Y \rangle \) is a mapping \( h : k\langle X \rangle \to k\langle Y \rangle \) such that, for \( x_1, x_2 \in k\langle X \rangle \) and \( c \in k \) it holds that

1. \( h(cx_1) = ch(x_1) \),
2. \( h(x_1 + x_2) = h(x_1) + h(x_2) \),
3. \( h(x_1 \cdot x_2) = h(x_1) \cdot h(x_2) \).

A homomorphism \( h : k\langle X \rangle \to k\langle X \rangle \) is called an endomorphism.
Definition 3.3. For any free semigroup \( \langle X \rangle \), and its corresponding free algebra \( k\langle X \rangle \), define;

(i) A reduction system as a set
\[
S = \{(W, f) : W \in \langle X \rangle, f \in k\langle X \rangle\}
\]
(ii) A reduction as an endomorphism \( r_{A\sigma B} : k\langle X \rangle \to k\langle X \rangle \) with \( A, B \in \langle X \rangle \) and \( \sigma = (W_\sigma, f_\sigma) \in S \), such that
\[
r_{A\sigma B}(C) = \begin{cases} r_{A\sigma B}(A)f_\sigma r_{A\sigma B}(B), & \text{if } C = AW_\sigma B \\ C, & \text{otherwise} \end{cases}
\]
that is, \( r_{A\sigma B} \) fixes all monomials in \( a \in k\langle X \rangle \) except \( AW_\sigma B \).
(iii) A reduction \( r_{A\sigma B} \) as acting trivially on \( a \in k\langle X \rangle \) if the monomial \( AW_\sigma B \) is not present in the polynomial \( a \).
(iv) An element \( a \in k\langle X \rangle \) as irreducible if every reduction \( r_{A\sigma B} \) for all \( A, B \in \langle X \rangle \) and \( \sigma \in S \) acts trivially on \( a \).
(v) \( k\langle X \rangle_{\text{irr}} \) as the vector space of all irreducible elements of \( k\langle X \rangle \).
(vi) A finite sequence of reductions \( r_1, \ldots, r_n \) as final on \( a \in k\langle X \rangle \) if
\[
(r_n \circ \ldots \circ r_1)(a) \in k\langle X \rangle_{\text{irr}}
\]
(vii) An element \( a \in k\langle X \rangle \) as reduction-finite if for every infinite sequence of reductions \( r_1, r_2, \ldots \), there is some \( N \) such that \( r_i \) acts trivially on \( (r_{i-1} \circ \ldots \circ r_1)(a) \), for all \( i \geq N \).

A reduction is as such defined as a pair of a monomial and a polynomial \((W, f)\) where the monomial \( W \) can be “reduced” to the polynomial \( f \), even if the monomial is inside a concatenated sequence of monomials. For example, given the free algebra over the reals with the symbols \( \{X, Y\} \) and the reduction system \( \{(XY, XY + Y), (Y^2, Y)\} \) it is possible to reduce the following element
\[
YX^3 + 3YXY \tag{3.1}
\]
into
\[
X^3Y + 3X^2Y + 6XY + 4Y \tag{3.2}
\]
by repeatedly applying the reductions \( Y^2 = Y \) and \( XY = XY + Y \).
A reduction is acting trivially if it does not change the element. For example, neither the reduction $Y^2 = Y$ nor the reduction $YX = X + Y$ will transform (3.2), since there are no $Y^2$ and no $YX$ present in the element.

Therefore all reductions acts trivially on (3.2) and thus it can cannot be reduced further, so it is irreducible.

Since it was possible to reduce (3.1) into (3.2) using finitely many reductions, (3.1) is reduction-finite.

A natural question that arises are whether or not (3.1) can be reduced into anything besides the (3.2). I.e. is (3.2) the “simplest” representative of (3.1)? And if so, can (3.1) be represented by (3.2) uniquely? Call those elements that answers yes to the second question reduction-unique.

**Definition 3.4.** A reduction-finite element $a$, is called reduction-unique if the images of all final sequences are equal. Denote this image by $r_S(a)$ (called the normal form of $a$).

To begin answering the first question it is first necessary to establish what it means for an element to be the “simplest” representative of an element. The first step is to define an ordering on all monomials that represents the “simplicity” (or how “reduced” they are) of each element.

**Definition 3.5.** A partial order is a relation $\leq$ over some set $S$, such that the following holds for all $a, b, c \in S$

(i) $a \leq a$ (Reflexivity),

(ii) if $a \leq b$ and $b \leq a$ then $a = b$ (Antisymmetry),

(iii) if $a \leq b$ and $b \leq c$ then $a \leq c$ (Transitivity).

**Definition 3.6.** For a given semigroup $\langle X \rangle$, with some elements $A, B, B', C \in \langle X \rangle$ a partial order “$\leq$” defined such that $B < B' \Rightarrow ABC < AB'C$, is called a semigroup partial ordering.

The second step is to ensure that all reductions always terminate. This is done by defining the ordering in such a way that each reduction reduces elements into a more well-ordered element.
Definition 3.7. Take \((W_\sigma, f_\sigma) \in S\) such that \(f_\sigma = \sum k_i X_i\) with \(X_i \in \langle X \rangle\). If a partial ordering has the property that \(X_i < W_\sigma\) for all \(\sigma \in S\) and \(X_i\), then the partial ordering is compatible with \(S\).

The final step is to ensure that there is one and only one simplest representative of each polynomial in the algebra. This is done by ensuring that no polynomial can be reduced into more than one irreducible polynomial.

Definition 3.8. An ambiguity of \(S\) is a 5-tuple \((\sigma, \tau, A, B, C)\), with \(\sigma, \tau \in S\) and \(A, B, C \in \langle X \rangle\), where it either is

(i) an overlap ambiguity: \(W_\sigma = AB, W_\tau = BC\) and \(A, B, C \in \langle X \rangle \setminus \{1\}\)

(ii) or an inclusion ambiguity: \(W_\sigma = B, W_\tau = ABC\), where \(\sigma \neq \tau\).

An ambiguity is a monomial that can be reduced by more than one reduction. To construct a normal form these ambiguities must in the end reduce to the same element. Otherwise there would be an ambiguous choice of normal element depending on which reduction is applied. The following definition formalizes this concept;

Definition 3.9. An ambiguity \((\sigma, \tau, A, B, C)\) of \(S\), is resolvable if there exist reductions \(r = r_1 \circ \cdots \circ r_n\) and \(r' = r'_1 \circ \cdots \circ r'_m\), such that

(i) for an overlap ambiguity \(r(f_\sigma C) = r'(Af_\tau)\),

(ii) or for an inclusion ambiguity \(r(Af_\sigma C) = r'(f_\tau)\),

With these definitions it is now possible to formally reason about whether or not a given reduction system over some algebra does have a unique normal form for each element \(a\). That is whether or not there exists a unique simplest representative of each polynomial in the algebra.

### 3.2 The Diamond lemma

This section presents the Diamond lemma which is a theorem that establishes the conditions necessary for a reduction system \(S\) over some algebra \(k\langle X \rangle\) to induce a consistent normal form for each element. That is, all elements \(a\) have a unique normal form \(r_S(a)\) such that \(r_S(a) \leq a\).

For the Diamond lemma to hold the semigroup partial ordering over the reduction system must have the descending chain condition.
Definition 3.10. A semigroup partial order $\leq$ has the descending chain condition if for every sequence

$$A_0 \geq A_1 \geq A_2 \geq \ldots$$

there exists some $n$, such that $A_i = A_n$ for every $i \geq n$.

This means that all strictly decreasing sequences (with respect to the ordering) of monomials must be finite for the descending chain condition to hold.

The proof of the Diamond lemma is omitted in this thesis. For readers who wishes to read the proof or simply wishes to read more, see [3].

Theorem 3.11 (Diamond lemma [3]). If, for a reduction system $S$ over some $k\langle X\rangle$, there exists a semigroup partial ordering $\leq$ on $\langle X\rangle$ that is compatible with $S$ having the descending chain condition, then the following statements are equivalent:

(i) All ambiguities in $S$ are resolvable.

(ii) $x$ is reduction-unique for all $x \in k\langle X\rangle$.

(iii) Take the two-sided ideal $I$ of $k\langle X\rangle$ generated by the elements $W_\sigma - f_\sigma$. A set of representative in $k\langle X\rangle$ for the elements of the algebra $R = k\langle X\rangle/I$ determined by the generators $X$ and the relations $W_\sigma = f_\sigma (\sigma \in S)$ is given by the vector space $k\langle X\rangle_{irr}$ spanned by the $S$-irreducible monomials of $\langle X\rangle$.

Now all the necessary theory has been presented to begin construction of a noncommutative version of the catenoid. The next step is to (in Chapter 4) construct an algebra and apply the Diamond lemma.
Chapter 4

Construction of $\mathcal{C}_\hbar$

The goal of this thesis is to construct a noncommutative algebra $\mathcal{C}_\hbar$ that mimics the behaviour of a surface known as the catenoid. This algebra will be constructed in several steps.

1. First the catenoid will be defined,

2. then a noncommutative free algebra $\mathcal{R}$ will be introduced,

3. a reduction system $S$ (Definition 4.4) will then be induced on $\mathcal{R}$ by basing the rules on the Weyl algebra $\mathcal{W}_\hbar$ and the properties of the catenoid,

4. lastly the diamond lemma (Theorem 3.11) will be used to construct $\mathcal{C}_\hbar$.

The result of this thesis will, in a sense be a noncommutative version of the surface called the catenoid. The reduction system will be constructed to, as closely as possible, mimic the catenoid. Therefore it is of grave importance that a good understanding of the catenoid is established before moving on.

The catenoid is a minimal surface embedded in $\mathbb{R}^3$. What this means is that given some boundary embedded in $\mathbb{R}^3$ there are no surfaces with said boundary that have a smaller area than the minimal surface. In the case of the catenoid the boundary consists of two circles of the same radii that lie on parallel planes where each center point is the projection of each other on respective planes.

What this means in practice is that the catenoid is the (connected) surface with two circles directly above each other as its boundary with the smallest area possible.
The fact that the catenoid actually is a minimal surface will not be shown in this thesis. For readers who are interested in a proof, see [5].

The catenoid has can be parametrized by

\[(x_1, x_2, x_3) = (\cosh u \cos v, \cosh u \sin v, u)\] (4.1)

where \(u \in \mathbb{R}\) and \(v \in [0, 2\pi)\). All smooth functions expressed in \(x_1, x_2\) and \(x_3\) are called the smooth functions over the catenoid. The easiest example of these kinds of functions are all polynomials with \(x_1, x_2\) and \(x_3\) as indeterminates, but of course there are many more. For simplicity the polynomials will be the only smooth functions over the catenoid considered, though there will be some other ones that will be included without any additional work (more on this later).

### 4.1 The generators of \(\mathcal{R}\)

For the intents of this thesis, Weyl algebras are a class of algebras with a set of symbols \(X = \{X_1, X_2, \ldots, X_n\}\) such that \(X_iX_j - X_jX_i = C_{ij}1\); i.e. that the so called commutor of each pair of symbols are the unit element up to a constant \(C_{ij}\) in the corresponding field. The commutors of two elements \(X\) and \(Y\) is denoted \([X,Y] = XY - YX\).

Consider the free algebra \(\mathbb{C}\langle U, V \rangle\) with \([U, V] = i\hbar 1\) where \(\hbar \in \mathbb{R}\). It is a Weyl algebra, where the symbols \(U, V\) are interpreted as noncommutative variants of the parameters \(u, v\) from (4.1).
**Definition 4.1.** Let $\mathcal{W}_\hbar$ denote the quotient algebra $\mathbb{C}\langle U, V \rangle/I$ where $I$ is the ideal generated by $UV - VU - i\hbar 1$.

To generate an algebra from $\mathcal{W}_\hbar$ such that it corresponds to the catenoid, symbols and reductions need to be chosen. The naive approach would be to base these symbols on $x_1, x_2, x_3$ and the reductions on the parametrization. But the reductions will be quite complicated. Therefore another algebra will be generated from a simpler set of indeterminates.

Note that $x_1 = \cosh u \cos v$ and by the definition of $\cosh u$ and by Euler’s formula it can be expressed as

$$x_1 = \frac{1}{4} (e^{u} + e^{-u}) (e^{iv} + e^{-iv}), \quad (4.2)$$

and by the same reasoning

$$x_2 = \frac{1}{4i} (e^{u} + e^{-u}) (e^{iv} - e^{-iv}). \quad (4.3)$$

Therefore all polynomials expressed in $u, e^u, e^{-u}, e^{iv}, e^{-iv}$ will cover all polynomials expressed in $x_1, x_2, x_3$ as well as some other smooth functions over the catenoid. These smooth functions are irrelevant, but as a demonstration, note that

$$e^u = e^{x_3},$$
$$e^{iv} = (x_1 + ix_2)(x_1^2 + x_2^2)^{-\frac{1}{2}},$$
$$e^{-iv} = (x_1 - ix_2)(x_1^2 + x_2^2)^{-\frac{1}{2}}$$

are smooth functions, so all polynomials expressed in $u, e^{\pm u}, e^{\pm iv}$ are smooth functions over $x_1, x_2, x_3$. These are the additional smooth functions mentioned in the beginning of the chapter.

The commutative free algebra generated by

$$u, \quad e^u, \quad e^{-u}, \quad e^{iv}, \quad e^{-iv} \quad (4.4)$$

will therefore contain all the polynomials generated by (4.1), as well some other smooth functions.

It would be beneficial if a corresponding noncommutative free algebra could be generated from noncommutative versions of (4.4).

Introduce the symbols

$$U, \quad R, \quad \tilde{R}, \quad W, \quad \tilde{W} \quad (4.5)$$
where $R, \tilde{R}, W$ and $\tilde{W}$ correspond to $e^U, e^{-U}, e^{iV}, e^{-iV}$ respectively and where $U, V$ are the generators for $\mathcal{W}_h$. It is not meaningful to explicitly state that for example $R = e^U$ since $R$ is a formal symbol and $e^U$ holds no meaning, but it is a useful frame of reference.

To justify this frame of reference, relations between the symbols need to be constructed in such a way that they follow their commutative analogous as closely as possible. To do so a result from Lie algebra called the **Baker-Campbell-Hausdorff formula** will be used.

### 4.2 The Baker-Campbell-Hausdorff formula

In the theory of Lie algebras there exists a formula known as the **Baker-Campbell-Hausdorff formula** (abbreviated BCH) [6]. This formula has the explicit form (one of many possible)

\[
e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{72}[Y,[X,Y]] + ...}
\]

But, if $[X,[X,Y]] = [Y,[X,Y]] = 0$ then the formula reduces to

\[
e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y]} \tag{4.6}
\]

This formula will be used to construct reductions that will simulate the behaviour of exponentials. It is not proven in this thesis, but $\mathcal{W}_h$ is a Lie algebra, and therefore it is to some degree justifiable to apply the BCH formula to $\mathcal{R}$. Note that the use of the BCH formula is strictly formal and is not really well-defined in the context of this thesis, but it will serve as a tool to justify the choices of reductions introduced.

### 4.3 The reduction system $\mathcal{S}$ over $\mathcal{R}$

The free algebra $\mathcal{W}_h$ with symbols $U, V$ where $[U,V] = i\hbar \mathbb{1}$ is used to guide the construction of a new free algebra $\mathcal{R}$ by acting as a reference for the formal symbols $U$ and $V$.

**Definition 4.2.** Let $\mathcal{R}$ be the free algebra constructed from the field $\mathbb{C}$ with the symbols $U, R, \tilde{R}, W$ and $\tilde{W}$. 

4.3. The reduction system $S$ over $\mathcal{R}$

$R$ and $W$ are supposed to correspond to noncommutative versions of $e^u$ and $e^{iv}$. $\tilde{R}$ and $\tilde{W}$ are supposed to represent the *multiplicative inverses* of $R$ and $W$ respectively. Of course $R$ and $W$ are simply formal symbols, so far they have no properties that justify calling them noncommutative analogous of $e^u$ and $e^{iv}$. Likewise for $\tilde{R}$ and $\tilde{W}$ being multiplicative inverses. These properties will be added through the construction of a reduction system $S$.

Recall that an element in a free algebra can be viewed as noncommutative polynomials with the generators as indeterminates. What this means is that all elements in $\mathcal{R}$ can be expressed as multiplications and additions of the generators (4.5); i.e. the only objects that will interact are the generators and some coefficients from $\mathbb{C}$. Therefore, in order to achieve interesting properties, the relationship between all symbols are constructed.

For $\tilde{R}$ and $\tilde{W}$ to correspond to the multiplicative inverses of $R$ and $W$, the following reductions are introduced

$$
R\tilde{R} = 1, \quad \tilde{R}R = 1 \quad (4.7)
$$

$$
W\tilde{W} = 1, \quad \tilde{W}W = 1. \quad (4.8)
$$

Note that $\tilde{R}$ and $\tilde{W}$ might be trivial under the reduction system. Therefore, in order to not cause unnecessary confusion they are not called the inverses until further justification can be supplied.

To impose properties onto $R$ and $W$ such that they mimic the behaviours of exponentials, the BCH formula is used. Note that $[U, [U, V]] = [V, [U, V]] = 0$ so (4.6) should be used. Thus

$$
RW = e^U e^{iv} = e^{U+iV+\frac{1}{2}[U,iV]} = e^{iV+U-\frac{1}{2}[iV,U]} = e^{iV+U+\frac{1}{2}[iV,U]-[iV,U]}
$$

Note that $[iV, U] = -i[U, V] = \hbar \mathbb{1}$, therefore it is reasonable to write

$$
RW = e^{iV+U+\frac{1}{2}[iV,U]-\hbar \mathbb{1}} = e^{-\hbar} e^{iV+U+\frac{1}{2}[iV,U]} = e^{-\hbar} WR
$$

since $[iV + U + \frac{1}{2}[iV,U], -\hbar \mathbb{1}] = 0$. So the following reduction is introduced

$$
RW = e^{-\hbar} WR. \quad (4.9)
$$

By applying (4.7) and (4.8) it is possible to derive

$$
RW\tilde{W} = R = e^{-\hbar} WR\tilde{W} \Leftrightarrow \tilde{W}R = e^{-\hbar} R\tilde{W} \Leftrightarrow R\tilde{W} = e^{\hbar} \tilde{W}R,
$$
so it is plausible to add

\[ R\tilde{W} = e^{\hbar \tilde{W}}R. \]  

(4.10)

Following similar reasoning, these reductions are added

\[ \tilde{R}W = e^{\hbar \tilde{W}}\tilde{R} \]  

(4.11)

\[ \tilde{R}\tilde{W} = e^{-\hbar \tilde{W}}\tilde{R} \]  

(4.12)

The final relations, those that involve \( U \) are a bit more involved. This is because the BCH formula cannot (with ease) be applied to say \( U e^{i\tilde{V}} \).

Instead the formal sum based on the the Taylor expansion of \( e^x \)

\[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \]  

(4.13)

is introduced.

Note that convergence of indefinite sums are outside the scope of this thesis. Therefore it is important to note that this formal sum does not exist, but rather it is introduced as a formal object for which the rules of a definite sum applies.

By substituting in (4.13) for \( e^U \) it follows that

\[ UR = Ue^U = U \left( \sum_{n=0}^{\infty} \frac{1}{n!} U^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} U^{n+1} = \]

\[ = \left( \sum_{n=0}^{\infty} \frac{1}{n!} U^n \right) U = e^U U = RU, \]

namely that \( U \) and \( R \) commutes. Note that

\[ UR\tilde{R} = U = RU\tilde{R} \iff \tilde{R}U = U\tilde{R} \]

so \( R \) and \( \tilde{R} \) should commute with respect to \( U \). Therefore these reductions are added

\[ UR = RU \]  

(4.14)

\[ U\tilde{R} = \tilde{R}U. \]  

(4.15)

The relationship between \( U \) and \( W \) can be derived with the help of the following lemma.
Lemma 4.3. For $U, V$ such that $[U, V] = i\hbar\mathbb{1}$ and $n \geq 0$, it holds that
\[ UV^{n+1} = V^{n+1}U + (n+1)i\hbar V^n. \]

Proof. This lemma is shown by inductive reasoning.

Note that $[U, V] = UV - VU = i\hbar\mathbb{1}$ so $UV = VU + i\hbar\mathbb{1}$, so the lemma holds for $n = 0$. Suppose the lemma holds for all $k \leq n$, i.e.
\[ UV^{k+1} = V^{k+1}U + (k+1)\hbar V^k. \]

If $n = k + 1$, then
\[
UV^{k+2} = UV^{k+1}V = (V^{k+1}U + (k+1)\hbar V^k) V = V^{k+1}UV + (k+1)\hbar V^k = V^{k+2}U + (k+2)\hbar V^{k+1}.
\]

Thus by the principle of induction, the lemma holds for all $n \geq 0$.

Using (4.13) and Lemma 4.3 it is clear that
\[
UV = U e^{iV} = U \left( \sum_{n=0}^{\infty} \frac{1}{n!} (iV)^n \right) = U + \sum_{n=0}^{\infty} \frac{i^{n+1}}{(n+1)!} UV^{n+1} = U + \sum_{n=0}^{\infty} \frac{i^{n+1}}{(n+1)!} (V^n + i(n+1)\hbar V^n) = \left( \sum_{n=0}^{\infty} \frac{1}{n!} V^n \right) U - \hbar \sum_{n=0}^{\infty} \frac{i^n}{n!} V^n = UW - \hbar W.
\]

Also note that
\[
UW\bar{W} = U = WU\bar{W} - \hbar \mathbb{1} \Leftrightarrow \bar{W}U = U\bar{W} - h\bar{W} \Leftrightarrow U\bar{W} = \bar{W}U + h\bar{W}
\]

thus these two reductions are added
\[
UW = UW - \hbar W \quad (4.16)
\]
\[
U\bar{W} = \bar{W}U + h\bar{W}. \quad (4.17)
\]
All relations between $U, R, \tilde{R}, W$ and $\tilde{W}$ have been determined. They are

\begin{align*}
R\tilde{R} &= 1 \\
W\tilde{W} &= 1 \\
WR &= e^hRW \\
\tilde{W}R &= e^{-h}\tilde{R}W \\
RU &= UR \\
WU &= UW + hW
\end{align*}

\begin{align*}
\tilde{R}R &= 1 \\
\tilde{W}W &= 1 \\
W\tilde{R} &= e^{-h}\tilde{R}W \\
\tilde{W}\tilde{R} &= e^h\tilde{R}\tilde{W} \\
\tilde{W}U &= U\tilde{W} - h\tilde{W}.
\end{align*}

Recall that a reduction system is a set of pairs where the first element is a monomial and the second element is a polynomial that the first element reduces into. In order to be able to define a sensible semigroup partial order in the next section, the reduction system will be defined as

**Definition 4.4.** Let $S$ be the reduction system given by

\begin{align*}
(R\tilde{R}, 1), & \quad (\tilde{R}R, 1), \quad (W\tilde{W}, 1), \quad (\tilde{W}W, 1) \\
(WR, e^hRW), & \quad (W\tilde{R}, e^{-h}\tilde{R}W), \quad (\tilde{W}R, e^{-h}\tilde{R}\tilde{W}), \quad (\tilde{W}R, e^h\tilde{R}W) \\
(RU, UR), & \quad (\tilde{R}U, U\tilde{R}), \quad (WU, UW + hW), \quad (\tilde{W}U, U\tilde{W} - h\tilde{W})
\end{align*}

### 4.4 A partial order over $\mathcal{R}$

A reduction system $S$ (see Definition 4.4) over the free algebra $\mathcal{R}$ (see Definition 4.2) has been constructed. In this section it is proven that $\mathcal{R}$ under $S$ is non-trivial.

But first it is important that a semigroup partial order is defined over the indeterminates of $\mathcal{R}$.

**Definition 4.5.** Let $\prec$ be a order over the indeterminates of $\mathcal{R}$, such that

$$1 \prec U \prec R \prec \tilde{R} \prec W \prec \tilde{W}. \quad (4.18)$$

The partial order that will be induced on the monomials of $\mathcal{R}$ will order them by length and how “misordered” the symbols in the monomial are.

Consider for example the element $UWRUR$, which is a permutation of the symbols $U^2R^2W$. The second permutation, $U^2R^2W$ is more “ordered” since the symbols appear in lexicographic order and has been grouped together.
4.4. A partial order over \( \mathcal{R} \)

The “misorder” of a monomial is given by its misordering index.

**Definition 4.6.** Let \( X = X_1X_2 \cdots X_n \) be a monomial of \( \mathcal{R} \) where \( X_i \) is an indeterminate from \( \mathcal{R} \) for all \( i = 1, \ldots, n \). Given \( \prec \) from Definition [4.5] the **misordering index** of \( X \) is the number of pairs \((i, j)\) such that \( i < j \) and \( X_i \succ X_j \). The misordering index is denoted \( m(X) \).

The following lemma demonstrates how to divide the calculation of the misordering index into smaller calculations. The idea is that, given \( X = X_1X_2 \cdots X_nX_{n+1} \cdots X_{n+m} \), the calculation of the misordering index can be broken down into calculating the misordering index of \( X_1X_2 \cdots X_n \) and \( X_{n+1} \cdots X_{n+m} \) as well as how many pairs between these two subsequences of \( X \) that are misordered.

**Lemma 4.7.** Given \( X = X_1X_2 \cdots X_n \) and \( Y = X_{n+1}X_{n+2} \cdots X_{n+m} \), let \( k(X, Y) \) denote the number of pairs \((X_i, X_j)\) for \( i \leq n < j \leq n + m \) such that \( X_i \succ X_j \). Then the following identities hold for all monomials \( X, Y, Z \in \mathcal{R} \):

\[
m(XY) = m(X) + m(Y) + k(X, Y),
\]

\[
k(XY, Z) = k(X, Z) + k(Y, Z)
\]

**Proof.** Let \( p = \{(X_i, X_j) : i < j < n + m\} \). Define \( m' \) such that

\[
m'(A, B) = \begin{cases} 
1, & \text{if } A \succ B \\
0, & \text{otherwise}
\end{cases}
\]

Then, by the definition of the misordering index it holds that

\[
m(XY) = \sum_{(x, y) \in p} m'(x, y).
\]

(4.19)

Note that \( p \) can be partitioned into

\[
p_0 = \{(X_{i_0}, X_{j_0} : i_0 < j_0 \leq n)\}
\]

\[
p_1 = \{(X_{i_1}, X_{j_1} : n < i_1 < j_1 \leq n + m)\}
\]

\[
p_2 = \{(X_{i_2}, X_{j_2} : i_2 \leq n < j_2 < n + m)\},
\]

Therefore, by (4.19) it holds that

\[
m(XY) = \sum_{(x, y) \in p_0} m'(x, y) + \sum_{(x, y) \in p_1} m'(x, y) + \sum_{(x, y) \in p_2} m'(x, y),
\]
which, by definition is equivalent to
\[ m(XY) = m(X) + m(Y) + k(X, Y). \]

Let \( Z = X_{n+m+1}X_{n+m+2} \cdots X_{n+m+k} \). For
\[ p_3 = \{(X_{i_3}, X_{j_3}) : i_3 \leq n + m < j_3 \leq n + m + k\} \]
it holds that
\[ k(XY, Z) = \sum_{(x,y) \in p_3} m'(x, y). \]

But \( p_3 \) can be partitioned into
\[ p_4 = \{(X_{i_4}, X_{j_4}) : i_4 \leq n \text{ and } n + m < j_4 \leq n + m + k\} \]
\[ p_5 = \{(X_{i_5}, X_{j_5}) : n < i_5 \leq n + m < j_5 \leq n + m + k\} \]
so
\[ k(XY, Z) = \sum_{(x,y) \in p_4} m'(x, y) + \sum_{(x,y) \in p_5} m'(x, y) \]
which is equivalent to
\[ k(XY, Z) = k(X, Z) + k(Y, Z). \]

Now all tools are available to construct a semigroup partial order over \( \mathcal{R} \) that has the descending chain condition.

**Theorem 4.8.** Let \( \leq \) be a partial order over the monomials of \( \mathcal{R} \) such that for monomials \( A \) and \( B \) with \( A < B \) it holds that

(i) if \( A \) has a shorter length than \( B \),

(ii) else if \( A \) is a permutation of \( B \) but has a lower misordering index than \( B \),

then \( \leq \) is a semigroup partial order over \( \mathcal{R} \) compatible to \( S \) (Definition 4.4) having the descending chain condition.

**Proof of Theorem 4.8.** Take monomials \( A, B, B' \) and \( C \) such that \( B < B' \). If the length of \( B \) is less than the length of \( B' \) then \( ABC < AB'C \), since the length of \( ABC \) is less than the length of \( AB'C \).
4.5. Ambiguities in $S$

If $m(B) < m(B')$ then, using Lemma 4.7

$$m(ABC) = m(AB) + m(C) + k(AB, C) =$$

$$= m(A) + m(B) + m(C) + k(A, B) + k(A, C) + k(B, C),$$

and likewise

$$m(AB'C) = m(A) + m(B') + m(C) + k(A, B') + k(A, C) + k(B', C).$$

It follows that $m(ABC) < m(AB'C)$, so $ABC < AB'C$. Also $k(A, B) = k(A, B')$ since $k$ only operates on the order between $A$ and $B$ which means that the internal order of $B$ or $B'$ is irrelevant. Thus $\leq$ is a semigroup partial order over $\mathcal{R}$.

To prove that $\leq$ is compatible with $S$ all that has to be done is to check if every monomial in $f_\sigma$ is less than $W_\sigma$ for all $\sigma$. Only one $(W_\sigma, f_\sigma)$ is checked, but the procedure is identical for each case.

Take $\sigma = (WU, UW + hW)$. Note that $f_\sigma = UW + hW$. It is easy to see that $UW < WU$ (by misordering index) and $W < WU$ (by length).

Note that $\leq$ is induced onto $\mathcal{R}$ by assigning two quantities, length and misordering index to each element and then comparing those. Both length and misordering index are positive integer values. Any strictly descending chain of positive integers have finite length, therefore any strictly descending chain (w.r.t. $\leq$) of elements from $\mathcal{R}$ must have finite length. So $\leq$ must have descending chain condition.

Intuitively this order corresponds to how “simple” commutative monomials are perceived and in what order they are written. For example, $x$ is often considered “simpler” than $xyz$, and its often preferred to write $xyz$ rather than $yzx$. Length and the misordering index is a way to capture these notions and imposing them in a noncommutative setting.

4.5 Ambiguities in $S$

By showing that all ambiguities in $S$ are resolvable, the diamond lemma results in the construction of an algebra $C_\hbar$, that in fact is the algebra which captures $S$ over $\mathcal{R}$. Before proceeding a notion of equivalence is needed.
**Definition 4.9.** Let \( \sim \) be a congruence such that \( X \sim Y \) if there exists reductions \( r_1, r_2, \ldots, r_n \) and \( r'_1, r'_2, \ldots, r'_m \) in \( S \) such that

\[(r_1 \circ r_2 \circ \cdots \circ r_n)(X) = (r'_1 \circ r'_2 \circ \cdots \circ r'_m)(Y)\]

All overlap ambiguities are given by \( \sigma, \tau \in S \) such that \( W_\sigma = AB \) and \( W_\tau = BC \), for monomials \( A, B \) and \( C \) not equal to \( 1 \). Since all \( W_\sigma \) and \( W_\tau \) consists of monomials of length 2 it is easy to conclude that all ambiguities are given by all pairs \((W_\sigma, W_\tau)\) such that the first symbol in \( W_\tau \) is equal to the last symbol in \( W_\sigma \). That is, all ambiguities in \( S \) are given by the following monomials:

\[
\begin{align*}
R\tilde{R}U & \quad R\tilde{R}R \quad \tilde{R}RU \quad \tilde{R}\tilde{R}\tilde{R} \\
W\tilde{W}U & \quad W\tilde{W}R \quad W\tilde{W}R \quad W\tilde{W}W \\
\tilde{W}WU & \quad \tilde{W}WR \quad \tilde{W}WR \quad \tilde{W}WW \\
WRU & \quad W\tilde{R}\tilde{R} \quad W\tilde{R}U \quad W\tilde{R}\tilde{R} \\
\tilde{W}RU & \quad \tilde{W}\tilde{R}R \quad \tilde{W}\tilde{R}U \quad \tilde{W}\tilde{R}\tilde{R}.
\end{align*}
\]

Note that there cannot be any *inclusive ambiguities* since there are no \( W_\sigma \) of length 1.

Recall that the definition of *resolvable ambiguities* states, for an overlap ambiguity \((\sigma, \tau, A, B, C)\), that it is resolvable if

\[(r_1 \circ r_2 \circ \cdots \circ r_n)(f_\sigma C) = (r'_1 \circ r'_2 \cdots r'_m)(Af_\tau),\]

for some sequences of reductions \( r_1, r_2, \ldots, r_n \) and \( r'_1, r'_2, \ldots, r'_m \). What this means in practice is that \( Af_\sigma \) and \( f_\tau C \) can, under a finite sequence of reductions, be reduced to the same element. Which can be stated as

\[r_{\|\sigma C}(W_\sigma C) - r_{\|A\tau \|}(AW_\tau) \sim 0.\]

Now it is just a matter of showing that this holds for all ambiguities. Begin with \( R\tilde{R}U \). Applying the reductions given by \( \sigma_1 = (R\tilde{R}, 1) \) and \( \sigma_2 = (\tilde{R}U, U\tilde{R}) \) results in

\[r_{\|\sigma_1 U}(R\tilde{R}U) - r_{R\|\sigma_2 \|}(R\tilde{R}U) = U - RU\tilde{R}.\]

Lastly, apply \( \sigma_3 = (RU, UR) \) and \( \sigma_4 = (R\tilde{R}, 1) \) to get

\[U - r_{U\|\sigma_4 \|}(r_{\|\sigma_3 \|}(RU \tilde{R})) = U - r_{U\|\sigma_4 \|}(UR\tilde{R}) = U - U = 0.\]
Thus $R\tilde{R}U$ is resolvable. This process is quite notation rich, which might hinder the intuition. Therefore a shorthand notation is introduced. Let

$$W_\sigma C - A W_\tau$$

denote

$$r_{\sigma C}W_\sigma - r_{A\tau}A W_\tau.$$ Using this notation the proof that $R\tilde{R}U$ is resolvable becomes

$$R\tilde{R}U - R\tilde{R}U \sim U - \tilde{R}U R \sim U - U \tilde{R}R \sim 0.$$ The proof that $\tilde{R}RU$ is resolvable is almost identical.

Now take $R\tilde{R}R$, then

$$R\tilde{R}R - R\tilde{R}R \sim R - R = 0.$$ Likewise for $\tilde{R}RR, WWW$ and $\tilde{W}WW$.

For most ambiguities the proof follows similar patterns, but to show how the reductions interact, the ambiguity with the most steps will be demonstrated.

$$WR U - WR \sim e^h WR - e^h WR \sim e^h WR - e^h WR = 0$$

Using the same type of reasoning for the remaining ambiguities, it is easy to see that all ambiguities in $S$ are resolvable.

It has now been proven that for the reduction system $S$ over $R$ there exists a semigroup partial order $\leq$ that is compatible with $S$ having the descending chain condition and that all ambiguities in $S$ are resolvable. Therefore it follows from the diamond lemma that all elements in $R$ are reduction-unique under $S$. But most importantly, it follows that for the two-sided ideal $I$ generated by all $W_\sigma - f_\sigma$ in $S$, the quotient algebra $C_h = R/I$ is given by the vector space of all irreducible elements in $R$.

What this means is that the quotient algebra $C_h$ is an algebra that contains the structure of $S$. This differs from $R$ since $S$ was something external that where
imposed on $\mathcal{R}$, but for $\mathcal{C}_h$ it is a part of the algebra itself. This algebra forces all elements into their respective irreducible form which means that all operations in $\mathcal{C}_h$ results in irreducible elements by default. Note that $\tilde{R}R \sim R\tilde{R} \sim \tilde{W}W \sim W\tilde{W} \sim 1$. Therefore it is natural to write $W^{-1} = \tilde{W}$ and $R^{-1} = \tilde{R}$ under $\mathcal{C}_h$.

In this chapter $\mathcal{C}_h$ has been constructed such that the following holds

1. $RR^{-1} = R^{-1}R = WW^{-1} = W^{-1}W = 1$,
2. $WR = e^hRW$,
3. $RU = UR$,
4. $WU = UW + hW$,
5. All elements of $\mathcal{C}_h$ are linear combinations of monomials with the form $U^{k_1}R^{k_2}W^{k_3}$ where $k_1 \in \mathbb{Z}_+$ and $k_2, k_3 \in \mathbb{Z}$ with coefficients in $\mathbb{C}$.

$R$ and $W$ corresponds to the commutative generators $e^u$ and $e^{iv}$ respectively. What has been achieved is a normal form for each element in $\mathcal{R}$. The constructed algebra $\mathcal{C}_h$ is the algebra of all normal forms over $\mathcal{R}$. This normal form guarantees that all elements $A \in \mathcal{C}_h$ has the form $A = \sum U^{k_1}R^{k_2}W^{k_3}$ for $k_1 \in \mathbb{Z}_+$ and $k_1, k_2 \in \mathbb{Z}$. This is a consequence of the semigroup partial order defined in Theorem 4.8, since the minimized misordering index guarantees that all generators(symbols) will be in lexicographic order.
Chapter 5

Geometry of $\mathcal{C}_\hbar$

The concept of geometry can be generalized beyond the classical Euclidean geometry which studies shapes and distances in $\mathbb{R}^3$. Consider vector spaces. They are in essence geometric objects, but their elements do not necessarily correspond to points in Euclidean space. Take for example the vector space of all $n$-degree polynomials. In this vector space each element corresponds to a polynomial. Therefore, if viewed as a geometric space, polynomials are points within this vector space. The vector space of all $n$-degree polynomials is also a subspace to the vector space of all polynomials. So it is possible for “smaller” vector spaces to be embedded within other vector spaces. In this chapter the concept of geometry is generalized for noncommutative algebras.

Recall that algebras are constructed from vector spaces. Therefore the transition into geometric interpretation of an algebra is quite natural. But what does it mean for an algebra to have a geometry? And how is geometry imposed on an algebra? These questions have many different answers depending on the algebra and what behaviours are studied. Recall that the catenoid is a minimal surface. The concept of minimal surfaces can be generalized into the non-commutative case in different ways. One way to introduce it is given by [2]. The exact statement of the definition is irrelevant for this thesis, but it is important to note that this specific definition uses a concept called derivations.

The aim of this thesis to introduce an interesting algebra that in a sense corresponds to the commutative catenoid. To this extent it would be beneficial if derivations were introduced to $\mathcal{C}_\hbar$. 

Holm, 2017.
5.1 Derivations

The generalized partial derivatives over algebras are called derivations. Partial derivatives are central concepts in differential calculus and can be used to define several geometric properties. Examples include area, tangent planes, curvature and many more. To introduce a geometry for $\mathcal{C}_h$ a translation of these concepts into the noncommutative case will be considered. Since in the commutative case these concepts can be defined using partial derivatives, derivations will be used for the noncommutative case.

In this section derivations $D_U$ and $D_V$ will be introduced and proven to exist such that they closely mimic the behaviour of the corresponding partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. First the concept of derivations is formally introduced.

**Definition 5.1.** Let $A$ be an algebra over the field $k$. A map $D : A \rightarrow A$ is a derivation over $A$ if it holds that

$$D(XY) = D(X)Y + XD(Y)$$

and

$$D(X + Y) = D(X) + D(Y).$$

To introduce derivations over $\mathcal{C}_h$ that correspond to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, the derivations $d_U$ and $d_V$ will be defined over $\mathcal{R}$ such that they exactly mimic their commutative counterparts. Then $d_u$ and $d_v$ will be used to define a set of derivations $D_U$ and $D_V$ over $\mathcal{C}_h$.

Since every element in $\mathcal{C}_h$ correspond to an equivalence class in $\mathcal{R}$ under the reduction system $S$, $D_U$ and $D_V$ can be defined in terms of $d_U$ and $d_U$ as such

$$D_i([X]) = [d_i(X)]$$

where $i = U, V$, if $d_i(x) \in [d_i(X)]$ for all $x \in [X]$. More on this in the proof of Theorem 5.2. Note that the derivations $D_U$ and $D_V$ defined in the following theorem will serve as “the” derivations over $\mathcal{C}_h$ for the rest of the thesis, but it is important to keep in mind that they are just one possible set of derivations that can be defined over $\mathcal{C}_h$.

**Theorem 5.2.** There exists non-trivial derivations $D_U : \mathcal{C}_h \rightarrow \mathcal{C}_h$ and $D_V : \mathcal{C}_h \rightarrow \mathcal{C}_h$.
\( \mathcal{C}_h \to \mathcal{C}_h \) such that

\[
\begin{align*}
D_U(U) &= 1 & D_V(U) &= 0 \\
D_U(R) &= R & D_V(R) &= 0 \\
D_U(R^{-1}) &= -R^{-1} & D_V(R^{-1}) &= 0 \\
D_U(W) &= 0 & D_V(W) &= iW \\
D_U(W^{-1}) &= 0 & D_V(W^{-1}) &= -iW^{-1}
\end{align*}
\]

The following lemma deals with derivations over reduction systems and is necessary for the proof of Theorem 5.2.

**Lemma 5.3.** Given a free (associative) algebra \( k\langle X \rangle \) with a reduction system \( S \), the congruence \( \sim \) defined in Definition 4.9 and a derivation \( D : k\langle X \rangle \to k\langle X \rangle \) such that \( D(W_\sigma - f_\sigma) \sim 0 \), then it holds that \( D(X - Y) \sim 0 \) under \( S \) for all \( X, Y \in k\langle X \rangle \) such that \( X \sim Y \).

**Proof.** Note that \( X \sim Y \Rightarrow X - Y \in I \). Therefore \( X - Y = \sum_{A,B} A(W_\sigma - f_\sigma)B \) where \( A, B \in k\langle X \rangle \). By Leibniz rule and \( W_\sigma - f_\sigma \sim 0 \) it holds that

\[
D(X - Y) \sim \sum_{A,B} \left( D(A) \underbrace{(W_\sigma - f_\sigma)B}_{\sim 0} + A D(W_\sigma - f_\sigma)B + A \underbrace{(W_\sigma - f_\sigma)D(B)}_{\sim 0} \right) \sim \sum_{A,B} AD(W_\sigma - f_\sigma)B \sim 0
\]

\[\square\]

**Proof of Theorem 5.2** Let \( d_U : \mathcal{R} \to \mathcal{R} \) and \( d_v : \mathcal{R} \to \mathcal{R} \) be derivations over \( \mathcal{R} \) defined such that

\[
\begin{align*}
d_U(U) &= 1 & d_V(U) &= 0 \\
d_U(R) &= R & d_V(R) &= 0 \\
d_U(\tilde{R}) &= -\tilde{R} & d_V(\tilde{R}) &= 0 \\
d_U(W) &= 0 & d_V(W) &= iW \\
d_U(\bar{W}) &= 0 & d_V(\bar{W}) &= -i\bar{W}
\end{align*}
\]
Then, for the reduction system $S$ defined in Definition 4.4 it holds that
\[ d_i(W_\sigma - f_\sigma) \sim 0 \]
for all $\sigma \in S$ and $i = U, V$. To show this it is just a matter of checking all reductions given in Definition 4.4. For brevity, it is only shown for $(R\tilde{R}, 1), (WR, e^hRW), (RU, UR)$ and lastly $(WU, UW + hW)$. These cases captures all techniques necessary to show the remaining reductions. Note that $d_V(R\tilde{R} - 1) = 0$ by the definition of $d_V$ since it does not involve $W$ or $\tilde{W}$, which are the only indeterminates that are non-zero under $d_V$.

For $(WR, e^hRW)$ neither $d_U$ nor $d_V$ is trivial, so both are shown. Note that $d_u(WR - e^hRW)$ reduces 0, but is not “equal” to 0 in the normal sense.

\[ d_U(WR - e^hRW) = d_U(WR)R + Wd_U(R) - e^h(d_U(R)W + Rd_U(W)) = WR - e^hRW \sim 0, \]

\[ d_V(WR - e^hRW) = d_V(WR)R + Wd_V(R) - e^h(d_V(R)W + Rd_V(W)) = \]

\[ = i(WR - e^hRW) \sim 0. \]

Note that $d_V(RU - UR) \sim 0$ by definition since it does not involve any indeterminates that is non-zero for $d_V$, and

\[ d_U(RU - UR) = d_U(RU)U + Rd_U(U) - d_U(U)R - Ud_U(R) = RU - UR \sim 0. \]

Lastly, see that

\[ d_U(WU - UW - hW) = W - W = 0 \]

\[ d_V(WU - UW - hW) = i(WU - (UW + hW)) \sim 0. \]

By similar arguments all $d_i(W_\sigma - f_\sigma)$ can be reduced to 0. Therefore by Lemma 5.3 it holds that $d_i(X - Y) \sim 0$ for all $X, Y \in \mathcal{R}$ such that $X \sim Y$. Thus for $i = U, V$, $D_i: \mathcal{C}_h \rightarrow \mathcal{C}_h$ defined as

\[ D_i([X]) = [d_i(X)] \]

where $X \in \mathcal{R}$ and $[X]$ denotes the representative of $X$ in $\mathcal{C}_h$ is well defined. For $X, Y \in \mathcal{R}$, note that

\[ D_i([X + Y]) = [d_i(X + Y)] = [d_i(X) + d_i(Y)]. \]
The equivalence classes are induced on $\mathcal{R}$ by the congruence $\sim$, so it holds that $[X + Y] = [X] + [Y]$. Therefore

$$D_i([X + Y]) = [d_i(X)] + [d_i(Y)] = D_i([X]) + D_i([Y]).$$

So both $D_U$ and $D_V$ are linear. Further, by again using that $\sim$ is a congruence, note that $[XY] = [X][Y]$. Therefore

$$D_i([XY]) = [d_i(XY)] = [d_i(X)Y + Xd_i(Y)] = [d_i(X)][Y] + [X] + [d_i(Y)] = D_i([X]][Y] + [X]D_i([Y]),$$

which means $D_U$ and $D_V$ satisfies Leibniz rule. Thus $D_U$ and $D_V$ are derivations over $\mathcal{C}_h$.

\[ \square \]

5.2 A localization $\hat{\mathcal{C}}_h$

Recall that $\mathcal{C}_h$ mimics a class of smooth functions over the generators $x_1, x_2$ and $x_3$ of the commutative catenoid. For instance, $R$ and $R^{-1}$ both correspond to the quotient functions $(x_1 + ix_2)(x_1^2 + x_2^2)^{-\frac{1}{2}}$ and $(x_1 - ix_2)(x_1^2 + x_2^2)^{-\frac{1}{2}}$ respectively over the generators $x_1, x_2$ and $x_3$ for the commutative catenoid.

In this section an extension $\hat{\mathcal{C}}_h$ to $\mathcal{C}_h$ will be constructed such that it contains representations of a larger class of functions.

Recall that the algebra $\mathcal{C}_h$ contains polynomials in the variables $U, R$ and $W$. But not all of these elements are invertible. For instance, there exists an equivalent element to the commutative element $1+u^2$ in $\mathcal{C}_h$, but there are no equivalent elements to $\frac{1}{1+u^2}$. The purpose of localization is to extend an algebra in such a way that equivalent elements to functions of the form $\frac{1}{f}$ is included.

These new functions introduced in $\hat{\mathcal{C}}_h$ will have the form $xs^{-1}$ or $s^{-1}x$, for $x \in \mathcal{C}_h$ and $s \in S$, where $S$ is a subset of $\mathcal{C}_h$ that is closed under multiplication. That is, a set of elements $S \subseteq \mathcal{C}_h$ is constructed such that all elements in $S$ have, either a right- or a left-inverse in $\hat{\mathcal{C}}_h$.

$\hat{\mathcal{C}}_h$ will be constructed using localizations (Theorem 5.5) and the Ore condition (Theorem 5.6). But before that, it is important to define what it means for all elements in $S$ to have an inverse in $\hat{\mathcal{C}}_h$.

**Definition 5.4.** Let $\mathcal{A}$ be an algebra and $S \subseteq \mathcal{A}$ a subset. A homomorphism $f : \mathcal{A} \to \hat{\mathcal{A}}$ to another algebra $\hat{\mathcal{A}}$ such that $f(ab) = f(a)f(b)$ for $a, b \in \mathcal{A}$ is
called \textit{S-inverting} if for each \( s \in S \) it holds that \( f(s) \) has a two-sided inverse in \( \hat{A} \).

So all elements in \( S \subseteq \mathcal{C}_h \) are invertible in some other algebra \( \hat{\mathcal{C}}_h \) if there is an \( S \)-inverting map between them. The following result guarantees that such an \( \hat{\mathcal{C}}_h \) will always exist.

\textbf{Theorem 5.5} (The universal property \cite{4}). Let \( \mathcal{A} \) be an algebra and \( S \) a subset to \( \mathcal{A} \). There exists an algebra \( \mathcal{A}_S \) with an \( S \)-inverting map \( \lambda : \mathcal{A} \rightarrow \mathcal{A}_S \) such that for each \( S \)-inverting map \( f : \mathcal{A} \rightarrow \hat{\mathcal{A}} \) there is a unique map \( f' : \mathcal{A}_S \rightarrow \hat{\mathcal{A}} \) such that \( f = \lambda \circ f' \) for an algebra \( \hat{\mathcal{A}} \).

For a proof of this theorem, see \cite{4}. This theorem is known as the \textit{the universal property} since it deals with the most general form of all algebras. The algebra \( \mathcal{A}_S \) is called a \textit{localization} of \( \mathcal{A} \).

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$\mathcal{A}$};
  \node (AS) at (2,0) {$\mathcal{A}_S$};
  \node (hatA) at (2,-1) {$\hat{\mathcal{A}}$};
  \draw[->] (A) -- (AS) node[midway,above] {$\lambda$};
  \draw[->] (A) -- (hatA) node[midway,left] {$f$};
  \draw[->] (AS) -- (hatA) node[midway,right] {$f'$};
\end{tikzpicture}
\end{center}

\textbf{Figure 5.1: The universal property}

This result is called \textit{universal} because it always holds for all algebras. But it does not say anything about the structure of \( \mathcal{A}_S \), and more specifically it does not say anything about the \textit{kernel} of the \( S \)-inverting map \( \lambda \). That is, it does not determine anything about which elements are mapped to 0 by \( \lambda \). \( \lambda \) could for example map the entire algebra to 0 and the universal property would still hold. Of course then nothing meaningful can be stated about the algebra using the universal property.

The \textit{universal property} will be used to construct a localization \( \hat{\mathcal{C}}_h \) such that it maintains all properties of \( \mathcal{C}_h \) while simultaneously making a subset \( S \) of elements invertible. For the elements in \( S \) to be invertible some constraints needs to be introduced on the kernel of \( \lambda \), since the elements should have their own inverse (i.e. one element should be the inverse of at most one element).

The following theorem, called the \textit{Ore condition} is the main result pertaining to \textit{localizations} in this thesis. The theorem achieves two things, firstly it gives the
conditions which the kernel $\lambda$ must satisfy in order to construct inverses, and secondly it gives a method of constructing said inverses.

**Theorem 5.6 (Ore condition [4]).** Given an algebra $A$ and a subset $S$ such that

(i) $1 \in S$ and $x,y \in S \Rightarrow xy \in S$,

(ii) for any $a \in A$ and $s \in S$ there exists some $b \in A$ and $t \in S$ such that $sb = at$,

(iii) for any $a \in R$ and $s \in S$ if it holds that $sa = 0$ then there must exist some $t \in S$ such that $at = 0$.

Then the kernel of $\lambda$ is given by $\{a \in A : at = 0$ for some $t \in S\}$. For the equivalence relation $\sim$ over $A \times S$ defined as

$$(a, s) \sim (a', s') \Leftrightarrow au = a'u' \in S \text{ and } su = s'u' \in S \text{ for some } u, u' \in A,$$

a localization $A_S$ can be constructed such that $A_S = A \times S/\sim$. Any element in $A_S$ can be written on the form $as^{-1}$ where $a \in A$ and $s \in S$.

To construct a localization for $\mathcal{C}_h$, the subset $S$ will be chosen as $S = \mathcal{C}_h \setminus \{0\}$. Meaning that all elements of $\mathcal{C}_h$ will be invertible in the localization. The universal property proves the existence of such an algebra, call it $\hat{\mathcal{C}}_h$, but it does not provide any hints of how this $\hat{\mathcal{C}}_h$ is constructed. But, if the Ore condition is applicable, an explicit method of construction will be given which is guaranteed to work.

Note that for $\mathcal{C}_h$ (iii) is an empty condition, since there are no zero-divisors. That is, there are no element $0 \neq x \in \mathcal{C}_h$ such that $ax = 0$ or $xa = 0$ for any $a \in \mathcal{C}_h$.

**Lemma 5.7.** There are no $0 \neq q \in \mathcal{C}_h$ such that $pq = 0$ or $qp = 0$ for any $0 \neq p \in \mathcal{C}_h$.

**Proof.** This proof is based on the same argument given in [7]. Assume that $p, q \in \mathcal{C}_h$ such that $pq = 0$ (or $qp = 0$, the argument is identical). Take the highest degree terms with respect to $U$, and out of these elements take those with highest degree with respect to $R$ and lastly take the term with the highest degree with respect to $W$. This term is called the highest ordered term. It follows from $p \neq 0$ that this term exists and it has the form $p_{i_1j_1k_1}U^{i_1}R^{j_1}W^{k_1}$. 

Let \( r = p_{i_1 j_1 k_1} \cdot q_{i_2 j_2 k_2} \), and note that
\[
W^m U^n = U^n W^m + \text{lower degree terms}
\]
\[
R^m U^n = U^n R^n
\]
\[
W^m R^n = e^{\pm n\hbar} R^n W^m,
\]
therefore
\[
p_{i_1 j_1 k_1} U^{i_1} R^{j_1} W^{k_1} \cdot q_{i_2 j_2 k_2} U^{i_2} R^{j_2} W^{k_2} = r U^{i_1} R^{j_1} W^{k_1} U^{i_2} R^{j_2} W^{k_2} =
\]
\[
= r U^{i_1+i_2} R^{j_1} W^{k_1} R^{j_2} W^{k_2} + \text{lower degree terms} =
\]
\[
= r e^{\pm j_2 \hbar} U^{i_1+i_2} R^{j_1} W^{k_1} R^{j_2} W^{k_1+k_2} + \text{lower degree terms},
\]
is a term in \( pq \) which is non-vanishing, which is a contradiction to the assumption that there exists non-zero \( p, q \) such that \( pq = 0 \).

**Proposition 5.8.** The localization \( \hat{C}_h \) can be constructed as \( C_h \times C_h / \sim \), where \( \sim \) is defined as
\[
(a_1, a_2) \sim (a_1', a_2') \iff a_1 b = a_1' b' \text{ and } a_2 b = a_2' b' \text{ where } b, b' \in C_h,
\]
such that all elements \( (a_1, a_2) \) can be written as \( a_1 a_2^{-1} \), where specifically \( (a, \mathbb{1}) \) is written as \( a \).

To prove this proposition, it is sufficient to show that the Ore condition holds for \( C_h \) and \( S = C_h \setminus \{0\} \).

\[ i \] Trivial, since \( S \) is the algebra itself (excluding 0).

\[ ii \] To prove that for given \( p, s \in C_h \) there exists \( q, t \in C_h \) such that \( sq = pt \), a similar argument to that given in [7] will be applied.

Study the equation \( sq - pt = 0 \). Note that \( q \) and \( p \) have undetermined coefficients, and therefore their coefficients will be the variables in the equation. Now take \( n \) such that the degree of \( p \) and \( s \) is less than \( n \), that is
\[
0 \leq \deg_U(p) \leq n \quad 0 \leq \deg_U(s) \leq n
\]
\[
-n \leq \deg_R(p) \leq n \quad -n \leq \deg_R(s) \leq n
\]
\[
-n \leq \deg_W(p) \leq n \quad -n \leq \deg_W(s) \leq n.
\]
Choose \( q \) and \( t \) such that their degree is \( 4n \). Then there are exactly \( 2(4n+1)(8n+1)^2 \) variables in \( sq - pt = 0 \). This is obtained by observing that there are \( 4n+1 \) different powers of \( U \), and \( 8n+1 \) different powers of \( R \) and \( W \) respectively. Therefore there are \( (4n+1)(8n+1)^2 \) different combinations
5.2. A localization \( \hat{\mathcal{C}}_h \)

of these powers. But there are two undetermined polynomials, \( q \) and \( t \) in the equation, therefore there will be two sets of coefficients. So there are \( 2(4n+1)(8n+1)^2 \) coefficients.

The polynomial of \( sq - pt \) will have degree \( 5n \) since \( s \) and \( p \) have degree \( n \) while \( q \) and \( t \) have degree \( 4n \). Therefore there are \( (5n+1)(10n+1)^2 \) equations with \( 2(4n+1)(8n+1)^2 \) variables in the equation system given by \( sq - pt = 0 \).

Note that \( (5n+1)(10n+1)^2 < 2(4n+1)(8n+1)^2 \), so there are more variables than equations, which means that it is an under determined linear system of equations (more variables than equations), so there must exist at least one non-trivial solution to \( sq - pt = 0 \) such that not both \( q \) and \( t \) are the zero element.

Due to Proposition 5.8 there is an algebra \( \hat{\mathcal{C}}_h \) such that it retains all the properties of \( \mathcal{C}_h \) but extends it to include fraction-like elements. Further it has been shown that \( \ker \lambda = \{0\} \), from which it follows that all elements in \( \mathcal{C}_h \) is uniquely mapped to an element in \( \hat{\mathcal{C}}_h \). Consider \( x, y \in \mathcal{C}_h \) such that \( x \neq y \), then \( x - y \neq 0 \). Now assume that \( \lambda(x) = \lambda(y) \) then, since \( \lambda \) is \( S \)-inverting it holds that \( \lambda(x - y) = \lambda(x) - \lambda(y) = 0 \). But \( (x - y) \notin \ker \lambda \) which is a contradiction. Therefore all elements in \( \mathcal{C}_h \) must be uniquely mapped to an element in \( \hat{\mathcal{C}}_h \).

Now it is shown that the derivations constructed for \( \mathcal{C}_h \) also translates to the inverse elements introduced in \( \hat{\mathcal{C}}_h \).

**Proposition 5.9.** Let \( D \) be a derivation over some algebra \( A \). Then given a localization \( \hat{A} \) of \( A \), \( D \) has a uniquely defined extension such that it is defined over \( \hat{A} \).

**Proof.** Since \( D \) is a derivation over \( A \), therefore it must hold that \( D(1) = 0 \). Take \( a \in A \subseteq \hat{A} \) such that there exists a \( a^{-1} \in \hat{A} \). Then \( aa^{-1} = 1 \), so

\[
D(1) = D(aa^{-1}).
\]

Using Leibniz rule this becomes

\[
D(aa^{-1}) = D(a)a^{-1} + aD(a^{-1}) = 0
\]

Which can be rewritten as

\[
D(a^{-1}) = -a^{-1}D(a)a^{-1}, \tag{5.2}
\]

thus showing that there is only one way to introduce a derivation over \( a^{-1} \) independent on the localization and derivation. \( \Box \)
Using Proposition 5.9 it can be concluded that all elements in $\hat{C}_h$ have a derivation.

Even though it has been shown that all elements in $C_h$ have an inverse in $\hat{C}_h$, for most applications it is not desirable for all elements to be invertible. For example when studying the curvature of the surface the corresponding concept of tangent planes is needed. This requires a non-degenerate metric, which is given by the elements

$$1 + \frac{1}{2} e^{-\hbar} (R^2 + R^{-2})$$
$$1 + \frac{1}{2} e^{-\hbar} (R^2 + R^{-2}).$$

For this metric to be non-degenerate $S$ and $T$ needs to have inverses. Note that not all inverses has a natural geometric interpretation, so those elements should be excluded in order to keep the algebra as an geometric object. In this fashion, whenever an element needs to be inverted it is straightforward to pull it from $\hat{C}_h$ into the algebra.
Chapter 6

Conclusion

The noncommutative surface $C_h$ has been constructed such that it closely resembles the catenoid surface. It has been shown that there exists two derivations, $D_u$ and $D_V$ over $C_h$ which are very similar to their commutative counterparts $\partial_u$ and $\partial_v$ respectively. Because the properties of $C_h$ are very close to the properties of the commutative catenoid it is fair to call $C_h$ a noncommutative catenoid.

6.1 Further work

To aid in further work, it has been shown that there exists a localization $\hat{C}_h$ where all elements in $C_h$ has an inverse. This will allow for further investigation of other geometric properties such as

- **Tangent spaces**, which are a generalization of tangent vectors and tangent planes to surfaces. It might for example be interesting to see if the tangent space of $C_h$ retains a general resemblance to the commutative catenoid. If the tangent space is similar to the tangent vectors of the commutative catenoid then it provides even more justification of calling $C_h$ a noncommutative catenoid.

- **Curvature of $C_h$**. The curvature of a surface is, informally speaking a measure of how “curved” a surface is by measuring how much a small subset of the surface deviates from a plane.
• Whether or not $C_h$ actually is a noncommutative minimal surface according to the definition given in [2]. This is probably one of the more appealing subjects of investigation since one of the fundamental properties of the commutative catenoid is precisely that it is a minimal surface.
Bibliography

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