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To our professor and friend María Teresa Lozano

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Abstract We prove that the maximal number $ag + b$ of automorphisms of equisymmetric and complex-uniparametric families of Riemann surfaces appearing in all genera is $4g + 4$. For each integer $g \geq 2$ we find an equisymmetric complex-uniparametric family $\mathcal{A}_g$ of Riemann surfaces of genus g having automorphism group of order $4g + 4$. For $g \equiv -1 \text{mod} 4$ we present another uniparametric family $\mathcal{K}_g$ with automorphism group of order $4g + 4$. The family $\mathcal{A}_g$ contains the Accola-Maclachlan surface and the family $\mathcal{K}_g$ contains the Kulkarni surface.

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1 Introduction

Kulkarni [10] showed that, for any genus $g \equiv 0, 1, 2 \text{mod} 4$, there is a unique surface of genus g with full automorphism group of order $8(g + 1)$ (the surface of Accola-Maclachan [1] and [14]), and for $g \equiv -1 \text{mod} 4$, there is just another surface of genus g (the Kulkarni surface [10]). In [12] Kulkarni shows that, if $g \neq 3$ there is a unique Riemann surface of genus g admitting an automorphism

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of order $4g$, while for $g = 3$ there are two such surfaces (see also [8] and [11]). The surfaces in this last family have exactly $8g$ automorphisms, except for $g = 2$, where the surface has 48 automorphisms.

An equisymmetric family in the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is the subset of $\mathcal{M}_g$ having, as a group of automorphism, a fixed group G acting in a given topological way (see [4]). These families are complex suborbifolds of $\mathcal{M}_g$, the simplest ones are of dimension 0, i.e. points, and after those the simpler ones are complex-dimension one orbifolds, these are the (complex)-uniparametric equisymmetric families of Riemann surfaces that are (complex)-dimension one manifolds, i.e. (non-compact) Riemann surfaces. We prove that the maximum number $ag+b$ of automorphisms of generic Riemann surfaces in equisymmetric and (complex)-uniparametric families of Riemann surfaces appearing in all genera is $4g+4$ (see Theorem 1). The second possible largest number of automorphisms where this fact is verified is $4g$ and this case is studied in [7].

For each integer $g \geq 2$ we find an equisymmetric (complex)-uniparametric family $\mathcal{A}_g$ of Riemann surfaces of genus g where the Riemann surfaces in the family have automorphism group of order $4g + 4$ (see Theorem 2).

The automorphism group of the Riemann surfaces in $\mathcal{A}_g$ is $D_{g+1} \times C_2$, where D_{g+1} denotes the dihedral group of order $2(g + 1)$ and C_2 denotes the cyclic group of order 2. The quotient $X/\mathrm{Aut}(X)$ is the Riemann sphere $\widehat{\mathbb{C}}$ and the meromorphic function $X \to X/\mathrm{Aut}(X) = \widehat{\mathbb{C}}$ have four singular values of orders $2, 2, 2, g + 1$.

For surfaces of genus g with automorphism group of order $4g$ and $g > 30$ the automorphism group is isomorphic to D_{2g} ([7]). For families of surfaces of genus g with $4g + 4$ automorphisms there are surfaces of infinitely many genera with automorphism group non-isomorphic to the automorphism group of the family $\mathcal{A}_g$. In fact, we construct for $g \equiv -1 \mathrm{mod} 4$ another uniparametric family $\mathcal{K}_g$ with automorphism group of order $4g + 4$ (Theorem 4).

Finally we announce that the Riemann surfaces $\mathcal{A}_g$ and $\mathcal{K}_g$ have an anticonformal involutions whose fixed point sets consist of three arcs each, denoted a_1, a_2, b in both cases, corresponding the real Riemann surfaces in the families (i.e. surfaces admitting anticonformal involutions). Hence the Riemann surfaces $\mathcal{A}_g$ and $\mathcal{K}_g$ are, in fact, (non-compact) real Riemann surfaces. Let $\widehat{\mathcal{M}_g}$ be the compactification of $\mathcal{M}_g$ using Riemann surfaces with nodes. The topological closure of $\mathcal{A}_g$ ($\mathcal{K}_g$ resp.) in $\widehat{\mathcal{M}_g}$ has an anticonformal involution with fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) which is a closed Jordan curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of three points, two nodal surfaces and the Accola-Maclahan surface of genus g for $\mathcal{A}_g$, and two points, one nodal surface and the Kulkarni surface for $\mathcal{K}_g$ respectively.

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2 Preliminaries

2.1 Fuchsian groups, Riemann surfaces, automorphisms and uniformization groups

A *Fuchsian group* Γ is a discrete group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}$. We shall consider only Fuchsian groups with compact orbit space. If Γ is such a group then its algebraic structure is determined by its signature

$$(h; [m_1, \ldots, m_r]) \tag{1}$$

The orbit space $\mathbb{H}/\Gamma$ is an orientable surface. The number h is called the *genus* of Γ and equals the topological genus of $\mathbb{H}/\Gamma$. The integers $m_i \geq 2$, called the *periods* , are the branch indices over interior points of $\mathbb{H}/\Gamma$ in the natural projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$.

Associated with each signature there exists a *canonical presentation* for the group Γ with generators:

$x_1, \ldots, x_r$ (elliptic elements),

$a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements);

these generators satisfy the defining relations

$x_i^{m_i} = 1$ (for $1 \leq i \leq r$),

$x_1 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} = 1$ (long relation).

The hyperbolic area of an arbitrary fundamental region of a Fuchsian group Γ with signature (2.1) is given by

$$2\pi\mu(\Gamma) = 2\pi \left(2h - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right) \tag{2}$$

Furthermore, any discrete group Λ of orientation preserving isometries of $\mathbb{H}$ containing Γ as a subgroup of finite index is also a Fuchsian group, and the hyperbolic area of a fundamental region for Λ is given by the Riemann-Hurwitz formula:

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \tag{3}$$

A Riemann surface is a surface endowed with a complex analytical structure. Let X be a compact Riemann surface of genus $g > 1$. Then there is a surface Fuchsian group Γ (that is, a Fuchsian group with signature $(g; [-])$) such that $X = \mathbb{H}/\Gamma$, and if G is a group of automorphisms of X there is a Fuchsian group Δ, containing Γ, and an epimorphism $\theta : \Delta \to G$ such that $\ker \theta = \Gamma$.

2.2 Teichmüller and moduli spaces

Here we follow the reference [13] on moduli spaces of Riemann.

Let s be a signature of Fuchsian groups and let $\mathcal{G}$ be an abstract group isomorphic to Fuchsian groups with signature s. We denote by $\mathbf{R}(s)$ the set of monomorphisms $r : \mathcal{G} \to Aut^{+}(\mathbb{H})$ such that $r(\mathcal{G})$ is a Fuchsian group with signature s. The set $\mathbf{R}(s)$ has a natural topology given by the topology of $Aut^{+}(\mathbb{H})$. Two elements r_1 and $r_2 \in \mathbf{R}(s)$ are said to be equivalent, $r_1 \sim r_2$, if there exists $g \in Aut^{+}(\mathbb{H})$ such that for each $\gamma \in \mathcal{G}$, $r_1(\gamma) = gr_2(\gamma)g^{-1}$. The space of classes $\mathbf{T}(s) = \mathbf{R}(s)/\sim$ is called the *Teichmüller space* of Fuchsian groups with signature s. If the signature s is given in section 2.1, the Teichmüller space $\mathbf{T}(s)$ is homeomorphic to $\mathbb{R}^{d(s)}$, where $d(s) = 6h - 6 + 2r$.

We denote by $\mathbf{T}((g;[-])) = \mathbf{T}_g$ the Teichmüller space of genus g.

Let $\mathcal{G}$ and $\mathcal{G}'$ be abstract groups isomorphic to Fuchsian groups with signatures s and s' respectively. Given an inclusion mapping $\alpha : \mathcal{G} \to \mathcal{G}'$ there is an induced embedding $\mathbf{T}(\alpha) : \mathbf{T}(s') \to \mathbf{T}(s)$ defined by $[r] \mapsto [r \circ \alpha]$.

If a finite group G is isomorphic to a group of automorphisms of Riemann surfaces of genus g, then the action of G is determined by an epimorphism $\theta : \mathcal{D} \to G$, where $\mathcal{D}$ is an abstract group isomorphic to Fuchsian groups with a given signature s and $\ker(\theta) = \mathcal{G}$ is a group isomorphic to surface Fuchsian group of genus g. Then there is an inclusion $\alpha : \mathcal{G} \to \mathcal{D}$ and an embedding $\mathbf{T}(\alpha) : \mathbf{T}(s) \to \mathbf{T}_g$.

3 The maximal order of the automorphism group of an equisymmetric uniparametric family appearing in all genera.

Definition 1 Let a, b be two integers with $a > 0$. We say (a, b) is admissible if for all $g \geq 2$ there is an equisymmetric and (complex)-uniparametric family of Riemann surfaces of genus g and with an automorphism group of order $ag + b$.

We shall denote $\mathcal{A}$ the family of admissible pairs (a, b).

Let (a, b) be an admissible pair and let $\mathcal{F}$ be an equisymmetric and (complex)-uniparametric family of Riemann surfaces of genus g and automorphism group of order $ag + b$. Then there is a signature $s_{(a,b)}$, such that the dimension of the Teichmüller space $\dim \mathbf{T}(s_{(a,b)}) = 1$ and for each $X \in \mathcal{F}$ there is a Fuchsian group Δ with signature $s_{(a,b)}$ such that there is $G \leq \mathrm{Aut}(X)$ with $\mathbb{H}^2/\Delta = X/G$.

Definition 2 Let (a', b'), $(a'', b'') \in \mathcal{A}$, we say $(a', b') \leq (a'', b'')$ if there is g_0 such that $a'g + b' \leq a''g + b''$, for all $g \geq g_0$. In other words $\leq$ is the lexicograpic order in $\mathcal{A}$.

The aim of this paper is to establish that $\max \mathcal{A} = (4, 4)$.

Open problem. Compute $\min \mathcal{A}$

Remark 1 By [7] we have that $\max \mathcal{A} \geq (4, 0)$.

Theorem 1 $\max \mathcal{A} = (4, 4)$.

Lemma 1 *The signatures $s_{(a,b)}$ such that (a, b) may be maximal belong to the set:*

$$\{(0;[2,2,2,3]),(0;[2,2,2,4]),(0;[2,2,2,6]),(0;[2,2,3,3]),(0,[2,2,2,g+1])\}$$

Proof Since $\mathbb{H}^2/\Delta = X/G$, Riemann Hurwitz formula tells us:

$$(ag+b)\mu(s_{(a,b)}) = |G|\,\mu(s_{(a,b)}) = 2(g-1)$$

Since $\dim \mathbf{T}(s_{(a,b)}) = 1$, then $s_{(a,b)} = (0;[m_1,m_2,m_3,m_4])$ or $(1;[m])$. By the Remark 1 if (a,b) is maximal $(a,b) \geq (4,0)$, then by [8], [10], [12] the signature $s_{(a,b)}$ must be of the form either $(0;[2,2,2,n])$, $n \geq 3$ or $(0;[2,2,3,n])$ with $3 \leq n \leq 5$.

Riemann-Hurwitz formula tells us: $(ag+b)\mu(s_{(a,b)}) = 2g-2$

In order to maximize (a,b) we must consider the signatures $s_{(a,b)}$ with small $\mu(s_{(a,b)})$.

1. For $s_{(a,b)} = (0;[2,2,2,3])$ we obtain the smallest $\mu(s_{(a,b)})$ which is $\frac{1}{6}$, and by Riemann-Hurwitz $(a,b) = (12,-12)$.

2. For $s_{(a,b)} = (0;[2,2,2,4])$ we have $\mu(s_{(a,b)}) = \frac{1}{4}$ and $(a,b) = (8,-8)$.

3. For $s_{(a,b)} = (0;[2,2,2,5])$ we have $\mu(s_{(a,b)}) = \frac{3}{10}$, but then the pair is not admissible, since there if $g \not\equiv 1 \bmod 3$ the formula $(ag+b)\frac{3}{10} = 2g-2$ is not possible.

4. For $s_{(a,b)} = (0;[2,2,2,6])$ and $(2,2,3,3)$ we have $\mu(s_{(a,b)}) = \frac{1}{3}$ and $(a,b) = (6,-6)$.

5. For $s_{(a,b)} = (0;[2,2,2,m])$ with $m = 7,8,9$ we have respectively $\mu(s_{(a,b)}) = \frac{5}{14}, \frac{3}{8}, \frac{7}{18}$, and as before correspond to non-admissible pairs. If $s_{(a,b)} = (2,2,3,n)$ with $n = 4,5$ we have respectively $\mu(s_{(a,b)}) = \frac{5}{12}, \frac{7}{15}$ and we have non-admissible pairs.

6. For $s_{(a,b)} = (0;[2,2,2,10])$ we have $(a,b) = (5,-5), ag+b = 5(g-1)$. The signature $(0;[2,2,2,10])$ tells us that there is an element of order 10 in the group G and this fact implies $g \equiv 1 \bmod 2$, then (a,b) is non-admissible.

7. If $s_{(a,b)} = (0;[2,2,2,m])$ with $m > 10$, then $\mu(s_{(a,b)}) \geq \frac{9}{22}$ and:

$$2g-2 = (ag+b)\mu(s_{(a,b)}) \geq (ag+b)\frac{9}{22}$$

Hence $(a,b) \leq \frac{44}{9}(g-1)$, so $(a,b) < (5,b')$ for any integer b'.

8. If we have a pair $(4,b)$, $b > 0$, with $s_{(4,b)} = (0;[2,2,2,m])$, there is a cyclic group in G of order m then m divides $4g+b$. Let $4g+b = km$:

$$2g-2 = (4g+b)\mu(s_{(a,b)}) = (4g+b)(\frac{1}{2}-\frac{1}{m}) = 2g+\frac{b}{2}-k$$

hence $k = \frac{b}{2}+2$ and b is an even integer.

We have $4g+b = (\frac{b}{2}+2)m$. For the case $g = 2$ we have $\frac{b}{2}+2$ divides $b+8$ then $\frac{b}{2}+2$ divides 4 and $b = 4$. Thus a pair $(4,b), b > 0$, is admissible only if $b = 4$. In Theorem 2 of the next section we shall prove that $(4,4)$ is an admissible pair with $s_{(4,4)} = (0;[2,2,2,g+1])$.

By Lemma 1, the only cases that could be greater than $(4,4)$ are $(12,-12)$ with signature $(0;[2,2,2,3])$, $(8,-8)$ with signature $(0;[2,2,2,4])$, and $(6,-6)$ with signatures $(0;[2,2,2,6])$ and $(0;[2,2,3,3])$. In the following Lemma we prove that the pairs $(12,-12)$, $(8,-8)$ and $(6,-6)$ are not admissible.

Lemma 2 *The pairs* $(12,-12)$ *(with* $s_{(12,-12)} = (0;[2,2,2,3])$*),* $(8,-8)$ *(with* $s_{(8,-8)} = (0;[2,2,2,4])$*) and* $(6,-6)$ *(with* $s_{(6,-6)} = (0;[2,2,2,6])$ *or* $(0;[2,2,3,3])$*) are not admissible.*

Proof We show in detail that the case $(12,-12)$, with signature $(0;[2,2,2,3])$, is not admissible. All other cases are discarded in a similar way.

Assume that the pair $(12,-12)$ is admissible, then for all $g \geq 2$ there is a family $\mathcal{F}_g$ of surfaces of genus g such that for each $X = \mathbb{H}/\Gamma \in \mathcal{F}_g$, where Γ is a surface Fuchsian group, there is a group $G_g \leq Aut(X)$ such that X/G_g is uniformized by a Fuchsian group Δ of signature $(0;[2,2,2,3])$, such that $\Gamma \leq \Delta$. By Riemann-Hurwitz formula we know that the order of G_g is $|G_g| = 12(g-1)$.

Consider genera $g \equiv 0 \bmod 3$ such that $g-1$ is a prime integer, $g \geq 12$. In this case there is a unique cyclic group C_{g-1} in G_g. Since the signature of Δ is $(0;[2,2,2,3])$ the group C_{g-1} acts freely of X. Then there is a surface Fuchsian group Λ such that:

$$X = \mathbb{H}/\Gamma \to X/C_{g-1} = \mathbb{H}/\Lambda \to X/G_g = \mathbb{H}/\Delta$$

The surface X/C_{g-1} has genus 2 and the covering $X/C_{g-1} = \mathbb{H}/\Lambda \to X/G_g = \mathbb{H}/\Delta$ is a normal covering with group of automorphisms $D_6 = G_2$. If

$$\left\langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^3 = x_1x_2x_3x_4 = 1 \right\rangle$$

is a canonical presentation of Δ, the monodromy epimorphism $\theta : \Delta \to D_6 = \left\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1s_2)^6 = 1 \right\rangle$ is (up to automorphisms):

$$\theta(x_1) = (s_1s_2)^3, \theta(x_2) = s_1, \theta(x_3) = s_2, \theta(x_1) = (s_1s_2)^2$$

The covering $X \to X/G_g$ is a regular covering with group of automorphisms G_g, that is an extension of C_{g-1} by D_6. Using cohomology of groups (see for instance [2]) the possible extensions are:

$$G_g = C_{g-1} \times D_6, G_g = C_{g-1} \rtimes_2 D_6 = D_{6(g-1)}, G_g = C_{g-1} \rtimes_1 D_6,$$

If $C_{g-1} = \left\langle c : c^{g-1} = 1 \right\rangle$ and $D_6 = \left\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1s_2)^6 = 1 \right\rangle$, the group $C_{g-1} \rtimes_1 D_6$ satisfies $s_1cs_1 = c^{-1}$ and $s_2cs_2 = c$.

The covering $X = \mathbb{H}/\Gamma \to X/G_g$ has a monodromy $\omega : \Delta \to G_g$ that is an epimorphism with $\ker\omega = \Gamma$. We shall analyze the three possibilities for G_g and show that it is not possible the existence of such a monodromy ω.

Case 1. $G_g = C_{g-1} \times D_6$: This group is not generated by elements of order two, but, by the long relation of Δ, the image $\omega(\Delta)$ is generated by $\omega(x_1)$, $\omega(x_2), \omega(x_3)$ which have order two.

For $G_g = C_{g-1} \rtimes_i D_6$, $i = 1,2$ we have that ω is as follows (up to automorphisms):

$\omega(x_1) = c^{j_1}(s_1s_2)^3, \omega(x_2) = c^{j_2}s_1, \omega(x_3) = c^{j_3}s_2$.

Case 2. $G_g = D_{6(g-1)}$. Since $\omega(x_1)$ has order two $j_1 = 1$. Now $\omega(x_2)\omega(x_3)$ must generate C_{g-1} but then $\omega(x_4) = (\omega(x_1)\omega(x_2)\omega(x_3))^{-1}$ have order $3(g-1) \neq 3$.

Case 3. $G_g = C_{g-1} \rtimes_1 D_6$. In this case $s_1cs_1 = c^{-1}$ and $s_2cs_2 = c$, then $j_3 = 1$. And again if $\omega(x_2)\omega(x_3)$ generate C_{g-1} then $\omega(x_4)$ does not have order 3.

The other pairs and signatures may be discarded in a similar way.

4 Equisymmetric uniparametric families with automorphism group of order 4g+4.

Theorem 2 *For every $g \geq 2$ there is an equisymmetric and uniparametric family $\mathcal{A}_g$ of Riemann surfaces of genus g such that if $X \in \mathcal{A}_g$, $D_{g+1} \times C_2 \leq \mathrm{Aut}(X)$, the regular covering $X \to X/D_{g+1} \times C_2$ has four branched points of order $2, 2, 2$ and $g+1$ and $X/(D_{g+1} \times C_2)$ is the Riemann sphere.*

Proof Let Δ be a Fuchsian group with signature $(0; [2, 2, 2, g + 1])$. Let

$$\left\langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^{g+1} = x_1x_2x_3x_4 = 1 \right\rangle$$

be a canonical presentation of Δ. Then we construct an epimorphism

$$\omega : \Delta \to D_{g+1} \times C_2 = \left\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1s_2)^{g+1} = 1 \right\rangle \times \left\langle t : t^2 = 1 \right\rangle$$

defined by:

$$\omega(x_1) = t, \omega(x_2) = ts_1, \omega(x_3) = s_2, \omega(x_4) = s_2s_1$$

The surfaces uniformized by $\ker\omega$ where Δ runs through the Teichmüller space $\mathbf{T}_{(0;[2,2,2,g+1])}$ give us the family $\mathcal{A}_g$

Proof of Theorem 1.

By the Lemmae of the preceding section we have that $\max \mathcal{A} \leq (4, 4)$. Now, by Theorem 2, $(4, 4)$ with the signature $(0; [2, 2, 2, g + 1])$ is admissible, then $\max \mathcal{A} = (4, 4)$.

Remark 2 The surfaces in $\mathcal{A}_g$ are hyperelliptic. The hyperelliptic involution corresponds the generator t of $D_{g+1} \times C_2$ in the proof of Theorem 2.

Following the technics in [6] and [7] and studying the surfaces admitting real forms in the family, we may announce the following result:

Theorem 3 *The real Riemann surface $\mathcal{A}_g$ has an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b, corresponding the real Riemann surfaces in the family. The topological closure of $\mathcal{A}_g$ in $\widehat{\mathcal{M}_g}$ has an anticonformal involution with fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) which is a closed Jordan curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of three points, two nodal surfaces and the Accola-Maclachlan surface of genus g.*

Remark 3 The Accola-Maclachlan surface ($w^2 = z^{2g+2} - 1$) has $8g+8$ automorphisms and appears in all genera, for infinitely many g the number $8g+8$ is the largest order of the automorphism group that any surface of genus g can admit ([1], [14]). The Accola-Maclachlan surface is on the Jordan curve corresponding to the closure of the set of real curves in $\mathcal{A}_g$, this fact explains that there are several real forms for such surface. The real forms of Accola-Maclachlan surfaces are studied in [5]. The topological types of the nodal surfaces may be determined using [9]: One of them consists of two components that are spheres and $g+1$ nodes joining the two components and the other one has also two components of genus $g/2$ and one node joining the two components, if g is even, and two components of genus $(g-1)/2$ and two nodes joining the two components, if g is odd.

The details of proof will appear elsewhere.

For surfaces of genus g with automorphism group of order $4g$, if g is greater than 30, all of them are in an equisymmetric uniparametric family (see [7]). This phenomena does not happen for surfaces with $4g+4$ automorphism. In fact there are equisymmetric uniparametric families different from $\mathcal{A}_g$ and appearing on arbitrary prescribed large genus:

Theorem 4 *Assume* $g \equiv -1 \bmod 4$. *There is an equisymmetric uniparametric family* $\mathcal{K}_g$ *of Riemann surfaces such that there is* G *of order* $4g+4$*, such that* $G \leq \mathrm{Aut}(X)$ *for all* $X \in \mathcal{K}_g$ *and* G *is isomorphic to:*

$$H_g = \left\langle \begin{matrix} t, b, s : t^{g+1} = b^4 = s^2 = 1;\ (bs)^2 = (bt)^2 = 1; \\ b^2 = t^{\frac{g+1}{2}}; sts = t^{\frac{g-1}{2}} \end{matrix} \right\rangle$$

and $X/G = \mathbb{H}/\Gamma'$ *with signature of* Γ'*:* $(0; +; [2, 2, 2, g+1])$

Proof The group H_g is a subgroup of index two of the group:

$$K_g = \langle x, y : x^{2g+2} = y^4 = (xy)^2 = 1;\ y^2 x y^2 = x^{g+2} = 1 \rangle$$

that has $8g+8$ elements, then H_g has $4g+4$ elements. The proof is similar to that of Theorem 2 using now the epimorphism $\omega : \Delta \to H$ defined by:

$$\omega(x_1) = bt, \omega(x_2) = bs, \omega(x_3) = s, \omega(x_4) = t$$

Remark 4 The surfaces in $\mathcal{K}_g$ are non-hyperelliptic. Their automorphism group are isomorphic to the central product of D_4 with the C_{g+1}. The common center of the group is $C_2 = \langle b^2 \rangle$, with no fixed points.

Example 1 As examples of the distinct equisymmetric uniparametric families of Riemann surfaces with $4g+4$ automorphisms we consider families in genera 3, 5 and 7. In genus 3 there are three different such families: the family $\mathcal{A}_3$ of hyperelliptic Riemann surfaces with automorphism group $G = D_4 \times C_2$, the family $\mathcal{K}_3$ formed by non-hyperelliptic Riemann surfaces with automorphism group H_3, the central product of D_4 with C_4, and a third family of non-hyperelliptic Riemann surfaces whith automorphism group $< 4,4|2,2 >= (C_4 \times C_2) \rtimes C_2$ in Coxeter's notation.

In genus five there are two such families: the family $\mathcal{A}_5$ of hyperelliptic Riemann surfaces with automorphism group $G = D_6 \times C_2$, and the family of non-hyperelliptic Riemann surfaces whith automorphism group $< 4,6|2,2 >= (C_6 \times C_2) \rtimes C_2$ in Coxeter's notation. See [3]

In genus seven there are three different such families: the family $\mathcal{A}_7$ of hyperelliptic Riemann surfaces with automorphism group $G = D_8 \times C_2$, the family $\mathcal{K}_7$ formed by non-hyperelliptic Riemann surfaces with automorphism group H_7, the central product of D_4 with C_8, and a third family of non-hyperelliptic Riemann surfaces whith automorphism group $< 4,8|2,2 >= (C_8 \times C_2) \rtimes C_2$ in Coxeter's notation. See [15]

As before we may announce the following result:

Theorem 5 *The real Riemann surface $\mathcal{K}_g$ has an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b, corresponding the real Riemann surfaces in the family. The topological closure $\widehat{\mathcal{K}_g}$ of $\mathcal{K}_g$ in $\widehat{\mathcal{M}_g}$ has an anticonformal involution with fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) which is a closed Jordan curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of two points, one nodal surface and the Kulkarni surface of genus g (note that the nodal surface corresponds to a node of $\widehat{\mathcal{K}_g}$ in $\widehat{\mathcal{M}}_g$.*

Remark 5 The Kulkarni surfaces ($w^{2g+2} = z(z-1)^{g-1}(z+1)^{g+2}$) have $8g + 8$ automorphisms and they are different from the Accola-Maclahan surfaces appearing for $g \equiv -1 \bmod 4$ ([10]). Note that the groups K_g are the groups of automorphisms of the Kulkarni surfaces and these groups contain subgroups of index two isomorphic to H_g, the groups defining the family $\mathcal{K}_g$. The real forms of Kulkarni surfaces are studied in [5]. Using [9] we may deduce that the nodal surface has the topological type of two spheres joined by $g + 1$ nodes.

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