Topology Optimization for Additive Manufacturing
Considering Stress and Anisotropy

Henrik Alm Grundström

LIU-IEI-TEK-A--17/02790—SE

Examensarbete
Institutionen för Ekonomisk och Industriell Utveckling
Linköpings Universitet, Sverige

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Abstract

Additive manufacturing (AM) is a particularly useful manufacturing method for components designed using topology optimization (TO) since it allows for a greater part complexity than any traditional manufacturing method. However, the AM process potentially leads to anisotropic material properties due to the layer-by-layer buildup of parts and the fast and directional cooling. For Ti6Al4V tensile specimens built using electron beam melting (EBM), it has been observed that flat built specimens show superior strength and elastic moduli compared to top built specimens. Designs with the loading direction parallel to the build layers are therefore expected to show greater reliability.

In this thesis a procedure is developed to optimize the AM build orientation considering anisotropic elastic material properties. A transversely isotropic material model is used to represent the in-plane and out-of-plane characteristics of AM produced parts. Two additional design variables are added to the TO formulation in order to control the orientation of the material using a coordinate transformation. Sensitivity analysis for the material direction variables is conducted for compliance as well as maximum von-Mises stress using a $P$-norm stress aggregation function.

The procedures for the AM build orientation optimization and stress constraints are implemented in the finite element software TRINITAS and evaluated using a number of examples in 2D and 3D. It is found that the procedure works well for compliance as well as stress but that a combination of these may lead to convergence issues due to contradicting optimal material orientations. An evaluation of the $P$-norm stress aggregation function showed that a single global stress measure in combination with a stress correction procedure works well for most problems given that the mesh is refined enough to resolve the stresses accurately.
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Nomenclature

Vectors and matrices are written in **bold face letters**.

\( K \) = Global stiffness matrix  
\( K_e = \) Expanded element stiffness matrix  
\( k_e \) = Element stiffness matrix  
\( B_e \) = Global element strain-displacement matrix  
\( b_e \) = Local element strain-displacement matrix  
\( \mathbf{u} \) = Global displacement vector  
\( \mathbf{u}_e \) = Element displacement vector  
\( F \) = Global external force vector  
\( C \) = Constitutive stiffness matrix  
\( S \) = Constitutive compliance matrix  
\( R \) = Rotation matrix  
\( L \) = Transformation matrix  
\( A \) = Boolean expansion matrix  
\( H \) = Filter matrix  
\( \lambda \) = Vector of adjoint variables  
\( \sigma \) = Stress matrix  
\( \epsilon \) = Strain matrix  
\( x \) = Design variables  
\( \rho \) = Physical design variables  
\( \Omega_e \) = Element domain  
\( \Omega \) = Design space  
\( E \) = Young’s modulus  
\( G \) = Shear modulus  
\( C \) = Compliance  
\( R \) = Filter radius  
\( \nu \) = Poisson’s ratio  
\( \sigma \) = Normal stress  
\( \tau \) = Shear stress  
\( \epsilon \) = Normal strain  
\( \gamma \) = Shear strain  
\( \bar{\sigma} \) = Stress normalization factor  
\( \sigma^G \) = Global stress measure  
\( \sigma^{vM}_{e,a} \) = Equivalent von-Mises stress in stress evaluation point \( a \) of element \( e \)  
\( e \) = Element number  
\( p \) = SIMP penalization exponent  
\( q \) = Stress penalization exponent  
\( P \) = P-Norm exponent  
\( n_e \) = Number of finite elements  
\( n_x \) = Number of density design variables  
\( n_a \) = Number of stress evaluation points  
\( n_{a,e} \) = Number of stress evaluation points in element \( e \)
\( \alpha = \) Material orientation angle, damping factor for stress correction
\( \beta = \) Material orientation angle

SO = Structural optimization
TO = Topology Optimization
AM = Additive manufacturing
DED = Direct Energy Deposition
SLM = Selective laser melting
EBM = Electron beam melting
PBF = Powder bed fusion
CAD = Computer aided design
CCW = Counter clockwise
CW = Clockwise
1. Introduction

Additive manufacturing (AM) offers new possibilities and design freedoms when it comes to designing lightweight and highly functional components. Unlike traditional subtractive or formative manufacturing methods, AM is an additive process characterized by a layer by layer build-up of parts. This process allows for a high degree of geometrical complexity which provides new opportunities and challenges for the designers of mechanical components. The possibility to build complex geometries makes additive manufacturing a particularly useful manufacturing method for components designed using topology optimization (TO). Traditionally, the geometries obtained from the TO process have had to be simplified in order to suit a particular manufacturing method, this often requires several design iterations and manual adjustments in order to reach a final design. Alternatively, the TO solution space is restricted to only allow geometries which can be manufactured, which often leads to less optimal designs. In theory, the combination of TO and AM makes it possible to eliminate the need for manual adjustments or overly restrictive design constraints since many of the limitations associated with traditional manufacturing methods are eliminated.

Even though AM allows for higher part complexity than traditional manufacturing methods there are limitations which have to be considered in the design process, ideally many of these should be included in a TO formulation for AM. Many of the limitations on the geometry of AM parts stem from the layer by layer build-up. An AM produced part is in its weakest state during the manufacturing process and all the intermediate stages of the structure have to be self-supported for a successful build. This often requires additional support structures which have to be removed after the build. Removing the support structures is a costly procedure and research is therefore ongoing to include the self-supporting property in TO formulations [1] [2]. For metal AM parts, the supports also act as pathways for heat conduction and are needed to counteract the build-up of thermal residual stresses and heat related build failures [3]. A TO formulation for AM should therefore ideally also include thermal effects. The layer by layer build-up also leads to the so-called staircase effect which in essence means that the surface roughness of a structural member is affected by its orientation with respect to the build direction [4].

Another difficulty is the potentially anisotropic material properties due to the layer-by-layer buildup of parts in combination with the fast and directional cooling. For Ti6Al4V tensile specimens built using electron beam melting (EBM), it has been observed that flat built specimens show superior strength and elastic moduli compared to top built specimens [5]. Designs with the main loading direction parallel to the build layers are therefore expected to show greater reliability. It has also been reported that the fatigue properties of AM components depend on the build orientation [6].

The geometrical freedom associated to the design of AM components, especially in combination with TO, provides a great challenge for design engineers since most commercial CAD programs have been developed with traditional manufacturing processes in mind [3]. It is therefore desirable to automate the transformation of the TO results into solid models which can be used to evaluate and enhance the generated designs. This means that the concepts generated from the TO procedure have to be as mature as possible in order to reduce the amount of manual work necessary in order to reach a final design.
In many industrial applications, especially in the aerospace industry, the goal of the design process is to achieve designs which are as lightweight as possible while still meeting the structural requirements. TO is an efficient tool for producing lightweight conceptual designs with efficient load paths. Traditionally, however, the TO formulations in common use have been based on compliance, which is related to the stiffness of the structure. This severely limits the usability of TO since most components in industrial use are designed based on fatigue or stress requirements rather than stiffness. In order to achieve more mature designs, the stress state in the structure therefore has to be included in the TO formulation. Stress constrained TO have been extensively researched [7] [8] [9] [10], however, most researchers use simple 2D examples and a comparison of the results with a detailed model is often absent.

The goal of this thesis is to develop and implement a TO formulation for AM which considers the build orientation and stress state under the hypotheses of anisotropic elastic properties. The aim is to optimize general 3D structures with anisotropic elastic properties subject to any combination of stress, mass and compliance constraints using the AM build orientation as an additional design variable.
2. Theory & Literature Review

This chapter provides a short review of some basic concepts of solid mechanics and structural optimization which are used throughout this thesis. Current literature is also reviewed for critical subjects such as stress constrained topology optimization and additive manufacturing of metals.

2.1 Linear Elasticity

In linear elasticity it is assumed that any plastic behaviour or time dependence in the response of a structure when subjected to external loading are negligibly small. This assumption is valid for most engineering metals at room temperature when the stress is below the yield limit \( \sigma_Y \). The state of stress in a point of a body is described by 9 stress components collected in the second order stress tensor \( \sigma_{ij} \) (unless otherwise specified, tensors are written in index notation) which in matrix notation can be written as

\[
\sigma = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\]

where \( \sigma_x, \sigma_y \) and \( \sigma_z \) are the normal stresses in the \( x(1), y(2) \) and \( z(3) \) directions respectively and \( \tau_{ij}, i = x, y, z, j = x, y, z, i \neq j \) are the shear stresses acting in a plane with a normal parallel to the \( i \)-axis and direction parallel to the \( j \)-axis. The state of stress in a small volume of material is illustrated in Figure 1.

Equilibrium of the small volume in Figure 1 requires that \( \tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx} \) and \( \tau_{yz} = \tau_{zy} \) which means that the stress tensor is symmetric, i.e. \( \sigma_{ij} = \sigma_{ji} \) with 6 independent stress components [11]. The assumption of linear elasticity permits superposition of stresses and strains from different loads as well as linear scaling of the response from different magnitudes of the load, e.g. if the load \( F_1 \) results in the stress state \( \sigma_{ij}^1 \) and the load \( F_2 \) results in the stress state \( \sigma_{ij}^2 \), then the load \( F = F_1 + F_2 \) results in the stress state \( \sigma_{ij} = \sigma_{ij}^1 + \sigma_{ij}^2 \).
Three sets of fundamental equations are necessary in order to fully describe the elastic response of a solid body when subjected to external forces. A loaded body at rest must be in static equilibrium with respect to each of the coordinate axis, this results in the static equilibrium equations which in tensor notation can be stated as [12]

$$\sigma_{ij,j} + b_i = 0, \tag{2.1}$$

where $b_i$ is the component of volume force acting in the direction of the $i$-axis, the comma sign “,” indicates differentiation with respect to the variable that follows, i.e.

$$\sigma_{xy,x} = \frac{\partial \sigma_{xy}}{\partial x},$$

and the Einstein summation convention is used. The kinematic relations relate the different strain components $\epsilon_{ij}$ to the displacements $u_i$. Assuming small deformations, the kinematic relations can be written

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \tag{2.2}$$

The strain measure used in (2.2) is related to the engineering strain measure $\gamma_{ij}$ ($i \neq j$) through

$$\epsilon_{ij} = \frac{1}{2} \gamma_{ij}, \quad (i \neq j).$$

The third set of equations are the constitutive relations (or material equations). The most commonly used constitutive relation for linear elasticity is Hooke’s Law which relates the stresses to the strains thought the forth order stiffness tensor $C_{ijkl}$. This relation is written as

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \tag{2.3}$$

or equivalently

$$\epsilon_{ij} = S_{ijkl} \sigma_{kl},$$

where $S_{ijkl}$ is the compliance tensor. The components of $C_{ijkl}$ and $S_{ijkl}$ depend on material symmetries which is discussed in Section 2.2.

Combining equation (2.1), (2.2) and (2.3) results in a system of partial differential equations which in combination with given boundary conditions fully describe the linear response of a structure. In Voigt notation the system of equations can be written as

$$\nabla^T C \nabla u(x) + b(x) = 0, \tag{2.4}$$

where $C$ is the stiffness matrix in Voigt notation (see Section 3.1), $u(x) = u(x,y,z)$ is the unknown displacement field and the operator matrix $\nabla$ is defined as
In a linear elastic stress analysis the boundary conditions consist of prescribed displacements \( g(x) \), known as **essential** boundary conditions, and prescribed surface tractions \( h(x) \), known as **natural** boundary conditions. The boundary of the solid is divided into two regions: \( \Gamma_g \) containing the prescribed displacements and \( \Gamma_h \) containing the prescribed surface tractions, see Figure 2.

$$\nabla^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}.$$ 

A boundary value problem can be formulated using (2.4) in combination with the prescribed tractions, displacements and body forces [13]: *Given \( g(x) \), \( h(x) \) and \( b(x) \). Find \( u(x) \) such that*

$$\begin{cases} \nabla^T C \nabla u(x) + b(x) = 0, & \forall x \in \Omega \\ u(x) = g(x), & \forall x \in \Gamma_g \\ N \nabla u(x) = h(x), & \forall x \in \Gamma_h, \end{cases}$$

where, for a 3D problem,

$$N = N(x) = \begin{bmatrix} n_x & 0 & 0 & n_y & 0 & n_z \\ 0 & n_y & 0 & n_x & n_z & 0 \\ 0 & 0 & n_z & 0 & n_y & n_x \end{bmatrix},$$

where \( n_x, n_y \) and \( n_z \) are the components of the outward normal in the \( x, y \) and \( z \) directions respectively. In addition, a compatibility requirement is imposed on the solution \( u(x) \) which express conditions such that material does not penetrate other material and that voids do not form within the body [12]. The boundary value problem (\( S \)) generally lacks closed form solutions but can be efficiently solved for arbitrary geometries and boundary conditions using the **finite element method** which is described in Section 2.3.
2.2 Anisotropic Elasticity

Many engineering materials show a negligible orientation dependence on their elastic properties. These materials are called *isotropic* and their elastic response can be fully described using two material parameters: the Young’s (or Elasticity) modulus $E$ and the Poisson’s ratio $\nu$ [11]. If, however, the response is orientation dependent, as is the case for e.g. wood, composites, monocrystalline metals etc., then these materials are said to be *anisotropic*.

The stiffness tensor $C_{ijkl}$ contains $3^4 = 81$ coefficients which have to be specified in order to describe the response of the material. The number of independent material parameters necessary in order to describe an anisotropic material depends on the number of symmetries that can be identified in the material. A first reduction of the number of parameters comes from the existence of a strain energy potential which requires that the stiffness tensor is symmetric, i.e. $C_{ijkl} = C_{klij}$, this is termed *major symmetry* and a proof of this can be found in e.g. [11]. Another reduction can be done due to *minor symmetry* which comes from the fact that $\sigma_{ij}$ and $\varepsilon_{ij}$ are both individually symmetric. This means that an elastic anisotropic material can have at most 21 independent material parameters.

Further reductions can be made by considering transformations of the material orientation. If the stiffness tensor is invariant to such a transformation, then the material is symmetric with respect to that change in orientation. The transformation of the stiffness tensor is performed as follows

$$C'_{pqmn} = \alpha_{qi}\alpha_{pj}C_{ijkl}\alpha_{km}\alpha_{ln}, \quad (2.5)$$

where $C'_{pqmn}$ contains the material parameters for the new orientation of the material and $\alpha_{ij}$ is a transformation tensor.

If $C'_{ijkl} = C_{ijkl}$ after the transformation in (2.5) when $\alpha_{ij}$ represents a rotation by $180^\circ$ around either the $x$-, $y$- or $z$-axis then the material is said to have *monoclinic* symmetry and the number of independent material parameters are reduced to 13. If the stiffness matrix is invariant to all such $180^\circ$ rotations the material is *orthotropic* and 9 independent parameters are needed in order to describe the response [11].

If, in addition to the orthotropic symmetry, the material is invariant to a single rotation by $90^\circ$ around one of the coordinate axis it shows *tetragonal* symmetry and the number of independent parameters is reduced to 6. A material with *cubic* symmetry is invariant to all such $90^\circ$ rotations and has only 3 independent parameters.

If an orthotropic material is invariant to an arbitrary rotation around one of the coordinate axis the material is *transversely* isotropic and has 5 independent material parameters. Transversely isotropic materials have a plane with homogeneous properties in all directions within the plane and different properties in the normal direction to the plane. An example of a transversely isotropic material is a unidirectional fibre composite where the stiffness varies considerably in the fibre direction compared to the plane of isotropy, see Figure 3.
2.3 The Finite Element Method

The finite element method is a numerical method used to solve general systems of coupled partial differential equations. These appear in a wide range of engineering disciplines including mechanical, thermal, electrical and fluid flow problems. This section provides a brief account of the method based on finding a solution to the boundary value problem (S) rather than a general mathematical approach. A more complete mathematical description can be found in textbooks such as [14].

Instead of solving the boundary value problem (S) directly it is convenient to reformulate the problem using a variational principle. For any kinematically admissible virtual displacement field \( w(x) \), i.e. any displacement field satisfying the essential boundary conditions, superimposed on the unknown displacement field \( u(x) \), the principle of virtual work states that the potential of the external loading must be equal to the elastic strain energy stored in the body. This results in the weak formulation of the boundary value problem: Given \( g(x) \), \( h(x) \) and \( b(x) \). Find \( u(x) \) such that

\[
(W) \quad \left\{ \begin{array}{l}
\int_{\Omega} (\nabla w(x))^T \mathbf{C} \nabla u(x) \, d\Omega = \int_{\Gamma_h} w(x)^T h(x) \, d\Gamma + \int_{\Omega} w(x)^T b(x) \, d\Omega \\
u(x) = g(x), \quad \forall x \in \Gamma_g \\
\forall w(x) \in V = \{w(x) : w(x) = 0, \forall x \in \Gamma_g\}.
\end{array} \right.
\]

The virtual displacements \( w(x) \) are commonly referred to as weight functions since they do not have physical interpretations for other types of problems. In order to find an approximate numerical solution to (W) the unknown displacement field \( u(x) \) is discretized and separated into a known part \( g^h(x) \) and an unknown part \( v^h(x) \)

\[ u(x) \approx u^h(x) = v^h(x) + g^h(x). \]

The domain \( \Omega \) is discretized into a number of subdomains \( \Omega_{e}, e = 1, ..., n_e \) called finite elements. Each finite element has a number of nodes associated to it, see Figure 4.
The displacement field in the domain is approximated using the displacements of the nodes $u_i$ and a displacement assumption which is defined by the shape functions $N_i$ such that

$$
\mathbf{u}^h(x) = \sum_{i=1}^{n_n} N_i(x) u_i + \mathbf{g}^h(x) = \begin{bmatrix}
N_i & 0 & 0 \\
\vdots & N_i & 0 \\
0 & 0 & N_i \\
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i \\
w_i \\
\end{bmatrix} + \mathbf{g}^h(x) = \mathbf{N}u + \mathbf{g}^h(x),
$$

where $u_i, v_i$, and $w_i$ are the displacements of node $i$ in the $x, y$ and $z$ directions respectively. Note that the same notation is used for the continuous displacement field $\mathbf{u}(x)$ and the nodal displacement vector $\mathbf{u}$, the meaning should however be clear from context. In the Galerkin formulation the same shape functions are used to discretize the virtual displacement field $\mathbf{w}(x)$ [13].

$$
\mathbf{w}^h(x) = \mathbf{N}c,
$$

where $c$ are the virtual displacements at the nodes. For simplicity it is here assumed that $\mathbf{g}(x) = \mathbf{0}$, which is the case for all numerical examples in this thesis. The principle of virtual work (also known as the weak form) can now be written in a discrete form as

$$
\int_{\Omega} \mathbf{c}^T \mathbf{N}^T \nabla \mathbf{C} \mathbf{N} \mathbf{u} \, d\Omega = \int_{\Gamma_h} \mathbf{c}^T \mathbf{N}^T \mathbf{h}(x) \, d\Gamma + \int_{\Omega} \mathbf{c}^T \mathbf{N}^T \mathbf{b}(x) \, d\Omega
\implies
\mathbf{c}^T \left( \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega - \int_{\Gamma_h} \mathbf{N}^T \mathbf{h}(x) \, dA - \int_{\Omega} \mathbf{N}^T \mathbf{b}(x) \, d\Omega \right) = 0,
$$

(2.6)

where $\mathbf{B} = \nabla \mathbf{N}$ is the global strain-displacement matrix. Since (2.6) is valid for all kinematically admissible $\mathbf{c}$ it follows that the expression inside the parentheses must vanish. An approximate solution to the boundary value problem can now be obtained by solving a linear system of equations known as the state problem

$$
\mathbf{Ku} = \mathbf{F},
$$

(2.7)

where
\[ K = \int_\Omega B^T C B \, d\Omega \]

is the \textit{global stiffness matrix} and

\[ F = \int_{\Gamma_h} N^T h(x) \, d\Gamma + \int_\Omega N^T b(x) \, d\Omega \]

is the \textit{global load vector}. The Galerkin formulation results in a symmetric global stiffness matrix, i.e. \( K = K^T \), which is an important property for computational efficiency. The global stiffness matrix is assembled from the element stiffness matrices \( k_e \) using an assembly operation represented here by the boolean matrices \( A_e \)

\[
K = \sum_{e=1}^{n_e} A_e^T \left( \int_{\Omega_e} b_e^T C b_e \, d\Omega \right) A_e = \sum_{e=1}^{n_e} A_e^T k_e A_e = \sum_{e=1}^{n_e} K_e
\]

where \( b_e = \nabla N_e \) are the \textit{element strain-displacement matrices} and \( K_e \) are the \textit{expanded element stiffness matrices}.

The shape functions are determined from the type of elements used in the discretized domain. The \textit{order} of the element refers to the polynomial degree of the associated shape functions, e.g. a linear element has linear shape functions. Generally higher order elements give a better approximation of the displacement field but at a higher computational cost [13]. Figure 5 shows all 2D elements used in this thesis. All of these elements have 3D counterparts.

![2D Elements](image)

**Figure 5** – a) Bilinear 4-node quadrilateral b) Quadratic 8-node quadrilateral c) Quadratic 9-node quadrilateral d) 3-node constant-strain triangle e) Quadratic 6-node triangle

The constant-strain triangle (CST) element (Figure 5 d)) gives a very crude approximation of the displacement field and is mainly used for educational purposes [13]. It is included in this thesis for comparison only and should never be used in real life applications. All other elements are widely used in commercial finite element software.
The integration of the element stiffness matrix is performed in a local element coordinate system using Gauss quadrature, e.g. for a 2D element

\[ k_e = \int_{\Omega_e} b_e^T C b_e \, d\Omega \approx \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j b_e(\zeta_i, \eta_j)^T C b_e(\zeta_i, \eta_j) |J(\zeta_i, \eta_j)| t, \]

where \((\zeta_i, \eta_j)\) are the coordinates of the Gauss points in the element coordinate system, \(n\) is the order of the Gauss rule, \(J\) is the Jacobian matrix for the change of coordinates, \(W_i\) are weights associated to the Gauss points and \(t\) is the thickness of the element. A polynomial of degree \(2n - 1\) is integrated exactly using an \(n\)-point Gauss rule [15].

Once the state problem (2.7) is solved, the stress state is evaluated in each element using the local coordinate system according to

\[ \sigma(\tilde{\zeta}_i, \tilde{\eta}_j) = C b_e(\tilde{\zeta}_i, \tilde{\eta}_j) u_e, \]

where the set of points \((\tilde{\zeta}_i, \tilde{\eta}_j)\) correspond to the so-called super convergent points which are the Gauss points of order \(n - 1\), where \(n\) is the order used for the fully integrated element stiffness matrix.

### 2.4 Structural Optimization

There are three main categories of structural optimization problems: size, shape and topology optimization. Size and shape optimization is often used at a late stage in the design process where a number of design features are parameterized and adjusted in order to optimize the final design. This could mean finding the optimal thickness of a structural member or the optimal shape of a cross section. Topology optimization (TO) is the most general form of structural optimization and is mainly used as a conceptual design tool. Given a design space, boundary conditions and relevant constraints a conceptual design is generated which is refined in order to achieve an efficient structure meeting all the given requirements.

A general structural optimization problem can be stated as follows:

\[ (S\Omega) \begin{align*} \min_{x, u} & \quad f_0(x, u) \\ \text{s. t.} & \quad \text{behavioral constraints on } u \\ & \quad \text{design constraints on } x \\ & \quad \text{equilibrium constraints,} \end{align*} \]

where \(\min\) should be read minimize and \(\text{s. t.}\) should be read subject to. The design variables \(x\) describe the geometry of the structure, in more or less detail depending on the type of optimization problem, which may change during the optimization process. The state variables \(u\) describe the response of the structure for a given design, for mechanical problems this often means displacement, stress, strain or force. The equilibrium constraints relate the response of the structure to the design variables through a state equation, e.g. (S) or its discrete counterpart (2.7). The objective function \(f_0\) is used to classify the design and could be a measure of e.g. weight, stiffness, production cost, maximum stress etc. [16].
The formulation of the optimization problem as stated in \((\mathcal{S0})\) is called a **simultaneous formulation** since it includes the equilibrium equations as explicit constraints. For naturally discrete or discretized continuous problems the equilibrium constraints are defined by the state problem (2.7) which for large scale problems can include on the order of \(10^5\) or more equality constraints. Equality constraints can be difficult to handle since many of the popular solution algorithms, including MMA (see Section 2.4.3), do not support them directly. This problem is circumvented by writing the state variables as an implicit function of the design variables: \(u = u(x) = K(x)^{-1}F\), where it is assumed that the load vector \(F\) is independent of the design. The problem can then be rewritten using a **nested formulation** [16]:

\[
(SO)_{nf} \begin{cases} 
\min_x & f_0(x, u(x)) \\
\text{s.t.} & f_i(x, u(x)) \leq 0, \quad i = 1, \ldots, n_c \\
& x \leq x_e \leq \bar{x}, \quad j = 1, \ldots, n_e,
\end{cases}
\]

where \(f_i, i = 1, \ldots, n_c\) are the constraints, \(n_c\) is the number constraints, \(n_e\) is the number of design variables and the bounds on the design variables are defined as a box constraints with the lower bound \(x\) and upper bound \(\bar{x}\).

### 2.4.1 Topology Optimization

In density based TO problems, the finite element method is used to discretize the design space. Each finite element is associated to a design variable \(x_e\) which takes on the value 1 to indicate material or 0 to indicate void in the domain \(\Omega_e\) of the corresponding element, see Figure 6.

![Figure 6](image)

Figure 6 – Design variable associated to finite element.

The design variables are multiplied with the stiffness matrices of the corresponding finite elements in order to decrease the stiffness of the structure in areas of void

\[
k_e = x_e \int_{\Omega_e} B_e^T C_e B_e d\Omega = x_e k_e^0.
\]

For 2D problems the design variables can be viewed as representing material thickness. The global stiffness matrix for the entire design space reads

\[
K(x) = \sum_{e=1}^{n_e} K_e^0 x_e.
\]
Thus, the design variables determine the connectivity of the discretized structure. Since a unique solution to the state problem requires a positive definite global stiffness matrix the design variables cannot take the value 0, since this could render the global stiffness matrix singular, instead a small positive value $\epsilon$ is used as the lower bound on the design variables. Another difficulty arises from the discrete nature of the problem, ideally the design variables should only take values of 1 or $\epsilon$. However, solving the problem as a discrete programming problem is far to computationally expensive for the number of design variables used in TO problems to be of any practical use [7]. A common approach to solve this issue is to use Solid Isotropic Material with Penalization (SIMP) which lowers the effective stiffness for elements with intermediate design variables and hence make them unfavourable in the optimized design. The global stiffness matrix is assembled using the penalized element stiffness matrices:

$$K(x) = \sum_{e=1}^{n_e} K_e^0 x_e^p,$$  

(2.8)

where $p > 1$ is the penalization exponent, commonly set to 3 [16]. The SIMP approach should be viewed as a penalization on the stiffness of the material, i.e. $C_e(x_e) = C_0 x_e^p$, where $C_0$ is the constitutive stiffness matrix of the material [16]. Several difficulties arise from the use of the SIMP penalization scheme. The problem becomes increasingly non-linear and number of local optima may be increased. Computed solutions are in general local optima for this kind of problems.

If low order elements are used checkerboard patterns often appear in the solution because of artificial stiffness attributed to these patterns. Figure 7 shows a cantilever beam optimized for maximum stiffness with a constraint on the allowed volume using 1600 4-noded plane stress elements. As seen in the figure, large areas of checkerboard patterns appear in the solution.

![Figure 7](image)

Figure 7 – Checkerboard patterns appearing due to artificial stiffness when low order elements are used.

Another problem is mesh dependence which comes from the non-existence of a continuous solution to the problem. As the mesh is refined finer and finer structures appear in the solution, this process goes on indefinitely and never converges to a particular solution, see Figure 8.
Figure 8 – Mesh dependence in the solution of TO problems. From the left: 1600, 5625 and 15625 8-noded plane stress elements.

As seen in Figure 8 the formation of checkerboard patterns is not as pronounced when higher order elements are used. The mesh dependence and checkerboarding can be cured by using so-called density filters which restrict the minimum size of the structural members. The filtered design variables $\rho_i(x)$ are defined as a weighted average of the design variables in the surrounding region

$$\rho_i = \sum_{e=1}^{n_e} \frac{\omega_{ie}}{\phi_i} x_e$$

where $\omega_{ie}$ is the weight associated to design variable $e$ in the filter kernel of design variable $i$ and

$$\phi_i = \sum_{e=1}^{n_e} \omega_{ie}.$$  

The weights in the kernel are determined by the type of filter used, e.g.

$$\omega_{ie} = \max(R - \|c_e - c_i\|, 0)$$

for a conical filter, where $R$ is the filter radius, $c_j = (x_j, y_j), j = i, e$ are the coordinates of the centroid of element $j$ and $\|\cdot\|$ denotes the Euclidian norm [17].

Since the kernels of the linear filter only depends on the coordinates of the elements, which do not change during the solution process, they can be calculated once the domain has been discretized and stored in a matrix $H \in \mathbb{R}^{n_e \times n_e}$ such that

$$\rho(x) = Hx$$  \hspace{1cm} (2.9)

where

$$H = \begin{bmatrix}
\omega_{11} & \omega_{12} & \cdots \\
\phi_1 & \phi_1 & \cdots \\
\omega_{21} & \phi_1 & \phi_2 & \cdots \\
\phi_2 & \phi_2 & & \\
\vdots & \vdots & & & \\
\end{bmatrix}.$$  

The filtered design variables $\rho_i(x)$ are referred to as the physical variables since they are used to calculate the properties of the structure. Substituting the physical variables for the design variables in (2.8) one receives
When the SIMP penalization is used with a linear density filter, low order elements can be used without the mesh dependence and checkerboard patterns appearing, see Figure 9.

![Figure 9 – Mesh independency and non-checkerboarding when a linear density filter is used. From the left: 1600, 5625 and 15625 4-noded plane stress elements.](image)

Using the SIMP penalization and a linear density filter a general nested TO problem can be stated as follows:

\[
\begin{align*}
\min_{\mathbf{x}} & \quad f_0(\rho(\mathbf{x}), \mathbf{u}(\rho(\mathbf{x}))) \\
\text{s. t.} & \quad f_i(\rho(\mathbf{x}), \mathbf{u}(\rho(\mathbf{x}))) \leq 0, \quad i = 1, \ldots, n_c \\
& \quad \epsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_e.
\end{align*}
\]

### 2.4.1.1 Studied Objective and Constraint Functions

In this thesis, three different types of objective and constraint functions are considered: mass, stress and compliance. The compliance \( C \) can be viewed as a weighted sum of the deformations of a structure and hence can be used as a measure of stiffness, i.e. minimizing the compliance gives a stiff structure [16]. The compliance is here defined as

\[ C(\mathbf{x}) \equiv \mathbf{F}^T \mathbf{u}(\rho(\mathbf{x})). \]

The mass is simply the sum the masses of the individual elements weighted by the physical design variables

\[ M = \sum_{i=1}^{n_e} \rho_i(\mathbf{x}) m_i, \]

where \( M \) is the total mass of the structure and \( m_i \) is the mass of element \( i \). Compliance and mass are both global measures and can therefore easily be implemented in the (TO) formulation. Stress measures are more challenging since the stress state is calculated locally in each element and therefore results in a large number of constraints in the optimization problem. Furthermore, areas of void may have large stresses since the low stiffness in these areas may result in large strains. There are several methods to circumvent these issues in the literature. Some of these are presented in more detail in Section 2.4.2.
2.4.2 Stress Constrained Topology Optimization

There are three main challenges which have to be overcome in order to efficiently solve stress based topology optimization problems. The first is the so-called singularity phenomenon which arises since the optimal solution often belongs to a degenerate design space of dimension lower than the original problem. The second is the local nature of the stress evaluation which results in a large number of constraints if the stress in each element is considered separately. The third is the highly non-linear behaviour of the stress state as a function of the design which can cause problems unless a numerically stable solution algorithm is used [18].

2.4.2.1 Stress Relaxation

The singularity issue was first observed in stress-constrained truss optimization problems where it could be shown that in an $n$-dimensional problem the admissible region contains degenerate subspaces of dimension $k < n$. Furthermore, the global optimum is often contained in one of these degenerate subspaces [18]. In truss optimization problems a small lower bound $\epsilon > 0$ must be imposed on the cross-sectional areas $x_i$ of the bars in order to avoid singularity of the stiffness matrix. If a bar has reached its lower bound on the cross-sectional area, i.e. $x_i = \epsilon$, in the final iteration of the solution algorithm, it may be manually removed from the final design after the optimization. However, bars with a cross-sectional area corresponding to the lower bound can still experience large stresses potentially restricting the numerical search algorithm from reaching global optima where one or more bars have zero cross-sectional area and hence should have zero stress [19]. Figure 10 shows a schematic representation of a truss optimization problem where the global optimum $G$ is contained in a 1D subspace of the 2D solution space, which due to the non-zero lower bound on the cross-sectional areas is unobtainable using a numerical search algorithm.

![Figure 10 – Example of truss optimization problem with singular global optima $G$. A numerical search algorithm would stop at the local optima $L$.](image)

The same problem arises for discretized continuous structures. When the density variable reaches its lower bound the element associated to it may still experience large stresses due to its low stiffness and hence non-zero strains. In order to be able to find the global optimal solution using numerical algorithms the stress constraints have to be relaxed. It should be pointed out here that the global optimal solution is generally not found since stress-constrained continua problems are highly non-linear and non-convex. However, relaxing the stress constraints enables the numerical algorithm to find better local solutions.
Several relaxation techniques have been proposed in order to circumvent the difficulties associated to the singularity problem. The $\epsilon$-relaxation approach, originally developed for stress constrained truss optimization problems, is based on the idea of converting the bar stress constraints into force constraints and subtracting a small number $\epsilon > 0$ [19]. The method was later adapted to density based topology optimization problems [8]. Consider a discretized continua problem using the SIMP formulation where the objective is to minimize the weight subject to stress constraints:

$$\begin{equation}
(P)_0 \left\{ \begin{array}{l}
\min_{x} \sum_{e=1}^{n_e} \rho_e m_e \\
\text{s. t.} \left\{ \begin{array}{l}
\sigma^\text{VM}_i \leq \sigma_Y, \\
\epsilon \leq x_j \leq 1, \quad i = 1, \ldots, n_a, \\
\epsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_e,
\end{array} \right.
\end{array} \right.
\end{equation}$$

where $\sigma^\text{VM}_i = \sigma^\text{VM}_i (\rho(x))$ is the equivalent von-Mises stress in stress evaluation point $i$ and $\sigma_Y$ is the yield limit of the material. If the physical design variable $\rho_i$ is interpreted as material porosity, the stress constraint should be modified in order to be consistent with the material physics. One simple way of achieving this is to reformulate the stress constraint in $(P)_0$ as [8]

$$\sigma^\text{VM}_i \leq \rho^0 \sigma_Y, \quad i = 1, \ldots, n_a,$$

(2.10)

where $\rho_e$ is associated to the element containing stress evaluation point $i$ and $q = p$ (the SIMP exponent). Since the goal in this thesis is to achieve so called 0-1 or black and white designs, no physical interpretation is made for intermediate values of $\rho_i$.

The stress constraints in $(P)_0$ should only be active for $\rho_e > 0$ since $\rho_e = 0$ represents void in the structure. In order to eliminate this condition the stress constraints are rewritten as

$$\rho_e (\sigma^\text{VM}_i - \rho^p \sigma_Y) \leq 0, \quad i = 1, \ldots, n_a.$$ 

This condition does however not eliminate the singularity problem for the discretized problem since $\rho_e \geq \epsilon$ always holds. In order to avoid the singularity the stress constraints are instead written

$$\rho_e (\sigma^\text{VM}_i - \rho^p \sigma_Y) \leq \epsilon, \quad i = 1, \ldots, n_a,$$

with the additional condition

$$\rho_e \geq \rho_{min} = \epsilon^*$

which guarantees that the stress constraints are never violated for a stress evaluation point in an element with $\rho_e = \rho_{min}$ [19]. Convergence is facilitated with large values of $\epsilon$ and the original problem is attained for $\epsilon = 0$, therefore the authors of [19] suggest a continuation approach where the value of $\epsilon$ is incrementally decreased during the numerical solution of the optimization problem.
Figure 11 shows the schematic representation of the truss optimization problem in Figure 10 when $\varepsilon$-relaxation is used.

The $\varepsilon$-relaxed form of the problem $(\mathcal{P}) \_O$ reads

\[
(\mathcal{P}) \_\varepsilon \begin{cases}
\min_x & \sum_{e=1}^{n_e} \rho_e m_e \\
\text{s.t.} & \begin{cases}
\left( \sigma^M_i - \rho_e \sigma_Y \right) \rho_i \leq \varepsilon, & i = 1, \ldots, n_a \\
\varepsilon^2 \leq x_j \leq 1, & j = 1, \ldots, n_e.
\end{cases}
\end{cases}
\]

Another approach, developed for density-based topology optimization problems, is the $qp$-relaxation [10]. The basic idea is to approximate the consistency condition $q = p$ in (2.10) by choosing $q < p$, thereby avoiding the stress singularity as the $\rho_e \to 0$. Rewriting (2.10) gives

\[
\frac{\sigma^M_i}{\rho_e} \leq \sigma_Y.
\]

In matrix notation the apparent stress state in stress evaluation point $i$ can be written

\[
\sigma = \frac{1}{\rho_e} C \varepsilon = \frac{\rho^p_e}{\rho_e} C_0 \varepsilon,
\]

which means that for $q < p$

\[
\lim_{\rho_e \to 0} \sigma = 0.
\]

It should be noted that the choice $q < p$ is a purely mathematical manipulation and makes the stress state unphysical for intermediate values of $\rho_e$ [10]. In the evaluation of the stress state according to (2.11) the term $p - q$ is often referred to simply as $q$ which gives

\[
\sigma = \rho_e^q C_0 \varepsilon.
\]
This imposes an unfortunate ambiguity as to what is meant by $q$ in the literature but hopefully the meaning is clear from context. In the present thesis the term stress penalization exponent refers to $q$ as it is defined in (2.12). In the $qp$-relaxed form of the problem $(P)_O$ the only modification is that the stresses are evaluated according to (2.12) with $q > 0$. The $qp$-relaxation does not require any additional conditions on the minimum value of the physical design variable $\rho_{\text{min}}$ [10].

### 2.4.2.2 Stress Aggregation

In finite element discretized continuous structures, the stress is evaluated in the super-convergent points of each element. The number of stress evaluation points varies with the type of elements used, but it always holds that $n_a \geq n_e$ (assuming no elements are excluded from the stress calculation). Imposing a stress constraint for each stress evaluation point therefore results in a large number of constraints. The computational cost associated to solving problems with such a large number of constraints makes this approach impractical. If the objective is to minimize the maximum stress for a given volume of material the local nature of the stresses is even more problematic.

The most common way to circumvent these issues is to use an aggregation function and a suitable equivalent stress measure in order to formulate a global stress measure $\sigma^G$. The global stress measure is used to approximate the maximum equivalent stress in the entire structure or a number of subregions. Using this approach the number of constraints can be significantly reduced and problems where the maximum stress is minimized are easy to formulate.

There are several aggregation functions used in the literature; e.g. the Kreissel-Steinhauser (KS) functions, the $P$-norm and the $P$-mean (sometimes referred to as “the modified $P$-norm”) [7]. All of these functions are differentiable approximations of the max operator where the accuracy of the approximation is governed by the parameter $P$. Using the $P$-norm and the von-Mises equivalent stress, the global stress measure is given by

$$\sigma^G_{Pn} = \left( \frac{1}{n_a} \sum_{i=1}^{n_a} (\sigma_{i}^{vM})^P \right)^{\frac{1}{P}} \leq \sigma_{\max}^{vM}, \quad (2.13)$$

where $\sigma_{\max}^{vM} = \max(\sigma_1^{vM}, \sigma_2^{vM}, ..., \sigma_{n_a}^{vM})$. As indicated by (2.13) the $P$-norm overestimates the largest stress in the structure for all finite values of $P > 0$, $\sigma^G_{Pn}$ is therefore an upper bound on the largest von-Mises stress in the structure [20]. The $P$-mean approximation is obtained by dividing by the number of stress evaluation points

$$\sigma^G_{Pm} = \left( \frac{1}{n_a} \sum_{i=1}^{n_a} (\sigma_{i}^{vM})^P \right)^{\frac{1}{P}} \leq \sigma_{\max}^{vM}, \quad (2.14)$$

The $P$-mean approximation underestimates the maximum von-Mises stress for all finite values of $P > 0$. 


Similar to the $P$-norm and the $P$-mean, the KS functions gives an upper $\sigma^U_G$ or lower $\sigma^L_G$ bound of the maximum stress, where

$$\sigma^G_i = \frac{1}{P} \ln \left( \frac{1}{n_a} \sum_{i=1}^{n_a} \exp \left( P \sigma_i^M \right) \right) \leq \sigma^M_{\text{max}} \leq \frac{1}{P} \ln \left( \sum_{i=1}^{n_a} \exp \left( P \sigma_i^M \right) \right) = \sigma^G_i.$$

All of these aggregation functions approximate the maximum stress exactly in the limit as $P \to \infty$. However, due to the highly nonlinear behaviour of the aggregation functions for large $P$, moderate values have to be used in practise. The choice of $P$ is a compromise between the accuracy of the approximation and the smoothness of the aggregate stress function and hence the performance of the numerical solution algorithm [18]. Low values of $P$ may give stress concentrations in the optimized structure while high values can cause convergence issues or increase the number of iterations needed to obtain a solution. Most authors suggest $P$ values in the range $6 \leq P \leq 30$ if a single global stress constraint is used [21] [20] [18] [9].

Using a single global stress measure for the entire structure is the simplest and computationally least expensive way of handling the large amount of stress evaluation points. However, several authors express concerns that a single stress measure is not sufficient to handle local stress concentrations unless excessively large values of $P$ are used [18] [20]. Therefore clustered or regional stress measures are often suggested where the global stress measure is defined for a number of subregions. Using regional stress measures increases the computational cost but generally provides better control over the stress distribution [18]. The authors of [18] used $P = 4$ in combination with a regional division of the stress constraints based on the stress levels and found that 8 regions was sufficient to achieve satisfactory results with the discretization they used. The increase in computational cost from the increase in the number of constraints is somewhat mitigated by the increased smoothness of the stress aggregation function which decreases the number of iterations in the optimization solver. As shown in [20] clustering the stresses based on the stress levels increases the accuracy of the aggregation function when the $P$-mean (or lower bound KS function) is used since

$$\left( \frac{1}{n_c} \sum_{i=1}^{n_a} \left( \sigma_{c,i}^M \right)^P \right)^{1/P} = \sigma_{c,i}^{\text{M}} \quad \text{if} \quad \sigma_{c,i}^M = \sigma_{c,j}^M \forall i, j,$$

where $n_c$ is the number of stress evaluation points in cluster $c$, i.e. if all stresses are the same, the $P$-mean is exact for any value of $P > 0$. In order to increase the accuracy in the approximation of the maximum stress the authors of [18] also used a correction factor $c^{(k)}$ such that

$$\sigma^G = c^{(k)} \left( \sum_{i=1}^{n_a} \left( \sigma_i^M \right)^P \right)^{1/P}.$$
where $c^{(k)}$ is updated in each iteration according to

$$c^{(k)} = \alpha^{(k)} \frac{\sigma_{\text{max}}^{(k-1)}}{\sigma_{G}^{(k-1)}} + (1 - \alpha^{(k)})c^{(k-1)},$$

(2.15)

where $k$ is the current iteration and $\alpha \in (0,1)$ is used to avoid oscillations in the correction factor. A difficulty with the clustering approach is that generally the clusters have to be updated in each iteration of the solution algorithm. Hence the problem is modified as it is solved which may cause convergence issues. The same goes for correction factors.

An alternative to the clustering approach is to use an aggregation function containing global stress measures with several different values of $P$ [9]. This approach can however only be used if the objective is to minimize the stress subject to other constraints. Using the $P$-norm, the stress measures are combined into a single function $\Sigma$ defined as

$$\Sigma = \prod_{P \in \mathcal{P}} \left( \sum_{i=1}^{n_a} \left( \sigma_{i}^{\text{vM}} \right)^{P} \right)^{\frac{1}{P}}, \quad \mathcal{P} = \{P_1, P_2, ..., P_n\}.$$

Since several values of $P$ can be used at once, the difficulty in choosing a suitable value of $P$ is circumvented. Furthermore, the use of correction factors for the stress measure is unnecessary since the stress is used as the objective function. The authors claim that the combination of low and high values in the set $\mathcal{P}$ gives a smooth stress aggregation function in combination with good control over the stress field [9].

In order to avoid numerical issues it is customary to introduce a normalization factor in the stress aggregation function, e.g. the $P$-norm is implemented as

$$\hat{\sigma}_{Pn}^{\hat{\sigma}} = \left( \sum_{i=1}^{n_a} \left( \sigma_{i}^{\text{vM}} \right)^{P} \right)^{\frac{1}{P}} / \sigma \gg 0,$$

where $\sigma \gg 0$ is the normalization factor. This is used to avoid the otherwise excessively large values of the sum [20].
2.4.3 The Method of Moving Asymptotes

The method of moving asymptotes (MMA) is a numerical algorithm originally developed in order to solve large scale structural optimization problems [22]. The general idea is to generate a series of strictly convex approximations of the original problem and solve these instead. Since the approximations are separable and convex they can be efficiently solved using a dual method [16]. There are a number of other methods which use the same concept, however, MMA has the advantage of being able to adjust the level of conservatism in the approximation. The efficiency of MMA and the ability to solve a wide range of problems has made it one of the most popular methods of solving structural optimization problems.

In order to describe the method in a general setting we consider a structural optimization problem \( \mathcal{P} \) which has the same structure as \( \mathcal{T} \), i.e. a nested formulation:

\[
\begin{align*}
(\mathcal{P}) : \min_{x \in \mathbb{R}^n} & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, n_c \\
& \quad x_j \leq x_j \leq \bar{x}_j, \quad j = 1, \ldots, n_e
\end{align*}
\]

As mentioned earlier MMA is not capable of handling equality constraints directly [22].

The general solution procedure for the convex approximation methods can be described as follows:

1. Choose a starting point \( x^{(0)} \) and set \( k = 0 \).
2. Given an iteration point \( x^{(k)} \), calculate \( f_i(x^{(k)}) \) and \( \nabla f_i(x^{(k)}) \), \( i = 0, \ldots, n_c \).
3. Generate a subproblem \((\mathcal{P}^{(k)})\) by replacing the functions \( f_i \) with the approximated function \( f_i^{(k)} \) in \( \mathcal{P} \).
4. Solve \((\mathcal{P}^{(k)})\) and let the solution be the input to the next iteration step, i.e. \( x^{(k+1)} \) is the solution to \((\mathcal{P}^{(k)})\). Set \( k = k + 1 \) and go to step 2.

In general, a suitable stopping criterion is also introduced. In MMA the approximated objective and constraint functions are defined as:

\[
f_i^{(k)} = r_i^{(k)} + \sum_{j=1}^{n} \left( \frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{ij}^{(k)}}{x_j - L_j^{(k)}} \right), \quad i = 0, 1, \ldots, n_c
\]  

(2.16)

where \( U_j^{(k)} \) and \( L_j^{(k)} \) are the *moving asymptotes* which change in each iteration but must always satisfy

\[
L_j^{(k)} < x_j^{(k)} < U_j^{(k)}.
\]

Furthermore,

\[
p_{ij}^{(k)} = \begin{cases} 
(u_j^{(k)} - x_j^{(k)})^2 \frac{\partial f_i}{\partial x_j}, & \text{if } \frac{\partial f_i}{\partial x_j} > 0 \\
0, & \text{if } \frac{\partial f_i}{\partial x_j} \leq 0,
\end{cases}
\]
\[
q_{ij}^{(k)} = \begin{cases} 
0, & \text{if } \frac{\partial f_i}{\partial x_j} \geq 0 \\
-(x_j^{(k)} - L_j^{(k)})^2 \frac{\partial f_i}{\partial x_j}, & \text{if } \frac{\partial f_i}{\partial x_j} < 0
\end{cases}
\]

and

\[
r_i^{(k)} = f_i(x^{(k)}) - \sum_{j=1}^{n} \left( \frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j^{(k)}} + \frac{q_{ij}^{(k)}}{x_j^{(k)} - L_j^{(k)}} \right)
\]

It is easily shown that \( f_i^{(k)} \) is a first order approximation to \( f_i \) at \( x^{(k)} \) given that all the derivatives \( \frac{\partial f_i}{\partial x_j} \) are evaluated at \( x = x^{(k)} \). Since \( f_i^{(k)} \) is a separable function in the variables \( x_j \) the second derivatives are given by

\[
\frac{\partial^2 f_i}{\partial x_j \partial x_l} = \begin{cases} 
\frac{2p_{ij}^{(k)}}{(U_j^{(k)} - x_j^{(k)})^3} + \frac{2q_{ij}^{(k)}}{(x_j^{(k)} - L_j^{(k)})^3}, & \text{if } j = l \\
0, & \text{otherwise}
\end{cases}
\]

which means that its Hessian matrix is diagonal. Furthermore, since \( p_{ij} \geq 0, q_{ij} \geq 0 \) and \( L_j^{(k)} \leq x_j^{(k)} \leq U_j^{(k)} \), \( f_i^{(k)} \) is a convex function. As indicated by (2.16) the curvature of the approximation functions is strongly influenced by the choice of \( U_j^{(k)} \) and \( L_j^{(k)} \). The closer the asymptotes are chosen to \( x_j^{(k)} \) the larger the curvature of the approximation functions and hence the more conservative the approximation of the original problem.

The approximation of \((P)\) in iteration \( k \) can now be expressed as:

\[
\left\{ \begin{array}{l}
\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} \left( \frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j^{(k)}} + \frac{q_{ij}^{(k)}}{x_j^{(k)} - L_j^{(k)}} \right) + r_0^{(k)} \\
\text{s.t. } \sum_{j=1}^{n} \left( \frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j^{(k)}} + \frac{q_{ij}^{(k)}}{x_j^{(k)} - L_j^{(k)}} \right) + r_i^{(k)} - \hat{f}_i \leq 0, \quad i = 1, ..., m \\
\max\{x_j, \alpha_j^{(k)}\} \leq x_j \leq \min\{x_j, \beta_j^{(k)}\}, \quad j = 1, ..., n
\end{array} \right.
\]

where the parameters \( \alpha_j^{(k)} \) and \( \beta_j^{(k)} \) are so called move limits. Since the approximation functions \( f_i^{(k)} \) are not defined for \( x_j = U_j^{(k)} \) or \( x_j = L_j^{(k)} \) the move limits have to be chosen such that:

\[
L_j^{(k)} < \alpha_j^{(k)} < x_j^{(k)} < \beta_j^{(k)} < U_j^{(k)}.
\]

The approximated problem \((\mathbb{P}^{(k)})\) is separable and strictly convex and can hence be solved efficiently using a dual method, for details the reader is referred to [22]. The efficiency of MMA depends on the choice and update procedure for the moving asymptotes \( L_j^{(k)} \) and \( U_j^{(k)} \). Svanberg suggested the following heuristic approach [16]:

22
Given $s_{\text{slower}} \in \mathbb{R}_+ < 1$ and $s_{\text{faster}} \in \mathbb{R}_+ > 1$. For $k < 2$ let

$$L_j^{(k)} = x_j^{(k)} - (\bar{x}_j - x_j) \quad \text{and} \quad U_j^{(k)} = x_j^{(k)} + (\bar{x}_j - x_j).$$

For $k \geq 2$

a) If the problem oscillates in the variable $x_j$, then let

$$L_j^{(k)} = x_j^{(k)} - s_{\text{slower}}(x_j^{(k-1)} - L_j^{(k-1)})$$

$$U_j^{k} = x_j^{(k)} + s_{\text{slower}}(U_j^{(k-1)} - x_j^{(k-1)})$$

b) otherwise let

$$L_j^{(k)} = x_j^{(k)} - s_{\text{faster}}(x_j^{(k-1)} - L_j^{(k-1)})$$

$$U_j^{k} = x_j^{(k)} + s_{\text{faster}}(U_j^{(k-1)} - x_j^{(k-1)})$$

The choice of $s_{\text{faster}}$ and $s_{\text{slower}}$ depends on the type of problem considered. In general the choice is a compromise between efficiency and stability of the algorithm.

### 2.5 Additive Manufacturing

Additive manufacturing (AM) is a collective name for manufacturing methods which, unlike traditional subtractive or formative methods, build up components by adding material in layers. There are two main categories of metal AM processes; direct energy deposition (DED) and powder bed fusion (PBF). The main focus of this thesis is on powder bed fusion processes but a short description of the DED process is included for comparison.

#### 2.5.1 Direct Energy Deposition

DED is a power- or wire-fed process where the material is fed through a nozzle and injected into a melt pool created by a laser on the surface of the workpiece. The melt pool and material feed is protected by an inert gas. A clad bead is created from the relative motion of the workpiece and the processing head. A complete layer of material is produced by overlapping the beads by 50-60% of their width. By adding successive layers it is possible to produce 3D structures with a dimensional accuracy of about 0.1 mm [4]. Feeding the material directly into the melt pool gives the possibility to mix different powders and hence produce functionally graded materials. The small localized fusion and the strong mixing in the melt pool means that materials can be tailored at a microstructure level.

The part complexity achievable by DED is limited compared to PBF processes but the restrictions on part dimensions are less severe. The DED process is illustrated in Figure 12.
2.5.2 Powder Bed Fusion

Powder bed fusion processes can be divided into two main categories: selective laser melting (SLM) and electron beam melting (EBM). This section briefly describes the two processes and gives a short account of the resulting microstructure and mechanical properties.

2.5.2.1 Selective Laser Melting

SLM is a laser based PBF process where a laser scans selected locations of a bed of powder and fuses it to the solid material below by melting it. If the powder is only partially melted the process is called selective laser sintering (SLS). The part is built by successively lowering the powder bed by a specified layer thickness, recoating the bed with powder and melting the powder at the locations corresponding to the material distribution in the current layer. The build process is restricted to a build chamber where a homogeneous flow of inert gas (e.g. Argon or Nitrogen) protects the powder and melt pool from oxidation, the gas also removes the vapour from the melted powder [4]. The PBF process is illustrated in Figure 13.
PBF processes can achieve higher part complexities than DED but part dimensions are restricted by the build chamber, at present time the largest commercial laser PBF machines have a build chamber volume of 800x400x500 mm³ [4].

SLM is a highly complex manufacturing process with many variables which have to be fine-tuned in order to achieve fully dense, metallurgically sound parts with minimum residual stress. These variables include: laser power, scan speed, hatch spacing (distance between laser scan tracks), powder particle size and morphology, powder distribution, layer thickness and scan strategy [4]. In theory almost all metals can be used in the SLM process, however, due to the complexity of the process and its relatively short lifetime, good results are only achieved with a handful of metals. At present time, Aluminium, Titanium and Steel are the most commonly used metals for SLM.

2.5.2.2 Electron Beam Melting

Like SLM, EBM is a powder bed process where a powder is fed from a hopper (Figure 14, 4) and distributed across the build plate (Figure 14, 7) by a rake (Figure 14, 5). The heat source is an electron beam which is generated from an electron gun (Figure 14, 1) and then accelerated using a voltage of about 60 kV, focused by electromagnetic lenses (Figure 14, 2), and directed towards the correct location of the build plane using a magnetic scan coil (Figure 14, 3). The powder is first preheated by a defocused electron beam using a high scan speed achieving temperatures in the powder bed >700° for Ti6Al4V. This leads to sintering of particles in the powder, in order to achieve full melting of the powder the beam is focused and the scan speed is reduced. The powder bed is then lowered and the process is iterated until the part is finished [24]. The scan speed is significantly higher in EBM compared to SLM which means that the process is faster at the expense of surface finish.

Figure 14 – Schematic of an EBM machine. Figure from [25].
2.5.2.3 Microstructure and Mechanical Properties

The microstructure of parts produced using AM is significantly different from those of conventionally produced alloys. The main reason for this is the fast and directional cooling achieved in the AM process. Since the conductivity of the solid metal is higher than that of the powder, heat is primarily conducted away through the previously build layers of the part and support structures [24]. In many materials the cooling process results in columnar structures with elongated grains that grow across the layers in a direction roughly parallel to the build direction [24]. The authors of [26] showed that for SLM of Ti6Al4V the orientation of the columnar grains follow the temperature gradient and that their orientation is dependent on the choice of scan strategy.

The authors of [27] compared the properties of Ti6Al4V produced using EBM and SLM and found significant differences in both the microstructure and mechanical properties. The differences is mainly attributed to the difference in cooling rate between EBM and SLM. The faster cooling rate of the SLM process results in a completely martensitic ($\alpha'$) microstructure while the EBM produced samples contain mainly the $\alpha$ phase which a small amount of $\beta$ phase within the columnar grains oriented in the building direction. The crystal lattice of the $\alpha$ and $\alpha'$ phases form a hexagonal close packed (hcp) structure while the $\beta$ phase is body centred cubic (bcc) [28]. The SLM built samples showed higher tensile and ultimate strength but significantly lowered ductility then the EBM samples. This difference is attributed to the martensitic $\alpha'$ phase in the SLM samples which gives higher strength but lower ductility compared to the $\alpha$ phase in the EBM produced parts. For both processes, the strength is higher in the horizontally built samples compared to the vertically built. There are also differences in the fatigue limits. The SLM samples showed a fatigue limit of 550 MPa compared to 340 MPa for the EBM counterparts.

There does not seem be any consensus in the literature regarding the amount of anisotropy in the mechanical properties of AM produced components. The authors of [5] reports a large difference in the elastic moduli and strength of EBM produced Ti6Al4V specimen for different build orientations. Their experiments indicate that the Young’s modulus may be 40 % lower for vertically build samples as compared to horizontal ones. The authors of [29] compared the tensile properties of SLM produced Ti6Al4V in different build orientations and found no significant differences in strength or elastic moduli. They did however find differences in the failure strain which indicate a difference in ductility between different build orientations. In [6] the authors found a significant difference in the fatigue limit of SLM produced Ti-6Al-4V test specimen built in different orientations. The specimen built in the vertical direction showing the lowest fatigue limit.
3. Method

This section presents the material model and the coordinate transformation used in order to include the AM build orientation in the TO formulation. The sensitivities for stress and compliance are derived and the numerical implementation is discussed.

3.1 Material Model

The material model is based on the hypotheses that the elastic properties of AM produced parts are homogeneous in the build plane with diverging properties in the build direction. This assumption is not completely accurate since several authors have found differences within the build plane as well. It is probable that these differences are due to the choice of scan strategy [26]. A deeper investigation into the influence of process parameters such as scan strategy is however beyond the scope of this work.

These assumptions lead to a transversely isotropic material model. The stiffness $C$ and compliance $S$ matrices for a transversely isotropic material, in Voigt notation, can be written as [11]

$$ C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{31} & C_{31} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11} - C_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} $$

where $C_{ij}$ are constants calculated from 5 independent material parameters and

$$ S = C^{-1} = \begin{bmatrix} \frac{1}{E_1} & \frac{-v_{12}}{E_1} & \frac{-v_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-v_{12}}{E_1} & \frac{1}{E_1} & \frac{-v_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-v_{13}}{E_1} & \frac{-v_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} $$

It follows from major symmetry that

$$ \frac{v_{13}}{E_1} = \frac{v_{31}}{E_3} $$

and in the plane of isotropy
\[ G_{12} = \frac{E_1}{2(1 + \nu_{12})}. \]

The 5 independent material parameters are \( E_1, E_3, \nu_{12}, \nu_{13} \) and \( G_{13} \). The constitutive matrices are defined so that the material properties are isotropic in the \( xy \)-plane (12-plane). The Voigt notation used in this thesis is defined as

\[
\sigma = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \sigma_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \Rightarrow \sigma_V = [\sigma_{11} \sigma_{22} \sigma_{33} \tau_{12} \tau_{23} \tau_{13}]^T,
\]

\[
e = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \varepsilon_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \varepsilon_{33} \end{bmatrix} \Rightarrow \varepsilon_V = [\varepsilon_{11} \varepsilon_{22} \varepsilon_{33} \gamma_{12} \gamma_{23} \gamma_{13}]^T.
\]

### 3.2 Coordinate Transformation

Due to cyclic symmetry in the isotropic plane, only two rotations are necessary in order to uniquely define the orientation of a transversely isotropic material. The transformation of the coordinates is performed using two rotations:

1) **CCW rotation around the \( x \)-axis by the angle \( \alpha \)**

\[
\hat{z}^* = \begin{bmatrix} z_1^* \\
z_2^* \\
z_3^* \end{bmatrix} = R_1 \hat{z} = \begin{bmatrix} 1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} z_1 \\
z_2 \\
z_3 \end{bmatrix}
\]

2) **CCW rotation around the \( y \)-axis by the angle \( \beta \)**

\[
\hat{z}' = \begin{bmatrix} z_1' \\
z_2' \\
z_3' \end{bmatrix} = R_2 \hat{z}^* = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} z_1^* \\
z_2^* \\
z_3^* \end{bmatrix} = R_2 R_1 \hat{z} = R \hat{z}
\]

where \( \hat{z}' \) is normal to the build plane (isotropic plane) in the transformed coordinate system and \( \hat{z} \) is normal to the global \( xy \)-plane (reference plane). The coordinate transformation is illustrated in Figure 15.

![Figure 15 – Coordinate transformation.](image-url)
Hence the rotation matrix for the coordinate transformation is given by

\[
R = R_2 R_1 = \begin{bmatrix}
\cos \beta & \sin \alpha \sin \beta & \cos \alpha \sin \beta \\
0 & \cos \alpha & -\sin \alpha \\
-\sin \beta & \sin \alpha \cos \beta & \cos \alpha \cos \beta
\end{bmatrix}.
\]

In order to represent every possible build direction the unit vector \( \hat{z}'(\alpha, \beta) = [x \ y \ z]^T \) has to be able to reach every point on the unit sphere \( s = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \} \). Since \( R \) is an orthogonal transformation matrix (\( R^T = R^{-1} \)), the unit vector \( \hat{z}'(\alpha, \beta) \) is given by the third column of \( R \), hence

\[
\hat{z}'(\alpha, \beta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \alpha \sin \beta \\ -\sin \alpha \\ \cos \alpha \cos \beta \end{bmatrix}, \quad \|\hat{z}'(\alpha, \beta)\|_2 = 1 \ \forall \alpha, \beta \in \mathbb{R},
\]

where \( \|\cdot\|_2 \) is the Euclidian norm. The point \((x, y, z)\) describing the tip of \( \hat{z}'(\alpha, \beta) \) origin is illustrated in Figure 16.

---

A change of variables from \((\alpha, \beta)\) to \((\theta, \phi)\) such that \( \theta = \alpha - \pi/2 \) and \( \phi = \beta \) gives

\[
\hat{z}'(\theta, \phi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin \theta \sin \phi \\ \cos \theta \\ \sin \theta \cos \phi \end{bmatrix}.
\]

Hence, \( \hat{z}'(\theta, \phi) \) represents a spherical coordinate system with \( r = 1 \), see Figure 17.
Figure 17 - Orientation of the unit vector $\hat{z}'(\alpha, \beta)$ described by a spherical coordinate system.

It is now trivial to see that $\hat{z}'(\theta, \phi)$ can reach any point on the unit sphere $s$ with $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$, hence $\hat{z}'(\alpha, \beta)$ can reach any point on the unit sphere $s$ with $\beta \in [0, 2\pi]$ and $\alpha \in [\pi/2, 3\pi/2]$ (or equally, $\alpha \in [-\pi/2, \pi/2]$) and an arbitrary build orientation can be described.

The coordinate transformation is tested using a simple MATLAB script, Figure 18 a) shows the transformation for $\alpha \in [-\pi/2, \pi/2], \beta \in [0, 2\pi]$. The red dots in Figure 18 show the trace of the tip of the unit vector $\hat{z}'(\alpha, \beta)$, i.e. the point $(x, y, z)$ for different values of $\alpha$ and $\beta$. As seen in the figure, any orientation of the build plane can be described. Due to the reflection symmetry with respect to the build plane the interval for $\beta$ can be reduced to $[-\pi/2, \pi/2]$. Figure 18 b) shows the range of build orientations for $\alpha, \beta \in \Theta = [-\pi/2, \pi/2]$.

Figure 18 – a) Build orientations for $\alpha \in [-\pi/2, \pi/2], \beta \in [0, 2\pi]$, b) Build orientations for $\alpha, \beta \in \Theta$. 
In the utilized Voigt notation, the coordinate transformation matrix is given by

\[
L = \begin{bmatrix}
    a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\
    a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\
    a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\
    a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & (a_{11}a_{22} + a_{12}a_{21}) & (a_{12}a_{23} + a_{13}a_{22}) & (a_{11}a_{23} + a_{13}a_{21}) \\
    a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & (a_{21}a_{32} + a_{22}a_{31}) & (a_{22}a_{33} + a_{23}a_{32}) & (a_{21}a_{33} + a_{23}a_{31}) \\
    a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12} + a_{32}a_{11}) & (a_{32}a_{13} + a_{33}a_{12}) & (a_{31}a_{13} + a_{33}a_{11}) \\
\end{bmatrix}
\]

where \(a_{ij}, i = 1,2,3, j = 1,2,3\) are the components of \(R\) [11]. Note that \(L\) differs from the one in [11] since a different Voigt notation is used. This gives

\[
L = \begin{bmatrix}
    \cos^2 \beta & \sin^2 \alpha \sin^2 \beta & \cos^2 \alpha \sin^2 \beta & \sin \alpha \sin 2\beta & \sin 2 \alpha \sin^2 \beta & \cos \alpha \sin 2\beta \\
    \sin^2 \beta & \sin^2 \alpha & \cos^2 \alpha & \sin 2\alpha & 0 & \cos \alpha \sin 2\beta \\
    0 & 0.5 \sin 2\alpha \sin \beta & -0.5 \sin 2\alpha \sin \beta & \cos \alpha \cos \beta & \cos 2\alpha \cos \beta & -\sin \alpha \cos \beta \\
    0 & 0.5 \sin 2\alpha \cos \beta & -0.5 \sin 2\alpha \cos \beta & -\cos \alpha \sin \beta & \cos 2\alpha \cos \beta & \sin \alpha \sin \beta \\
    -0.5 \sin 2\beta & 0.5 \sin^2 \alpha & 0.5 \cos^2 \alpha & \sin \alpha \cos 2\beta & 0.5 \sin 2\alpha \sin 2\beta & \cos \alpha \cos 2\beta \\
\end{bmatrix}
\]

The transformation of the stiffness matrix (material properties) hence reads

\[
C = L C_0 L^T.
\]

### 3.3 Sensitivity Analysis

MMA is a first order gradient based optimization solver and therefore requires the derivatives of the objective and constraint functions w.r.t. all the design variables. The process of obtaining these derivatives is referred to as sensitivity analysis. In this section the necessary sensitivities are derived. The sensitivities of the compliance and mass functions with respect to the physical design variables \(\rho\) are commonly known and therefore not included here, these can be found in e.g. [16].

#### 3.3.1 Sensitivity of Compliance w.r.t. Material Orientation

The global stiffness matrix \(K\) is given by

\[
K = \sum_{e=1}^{n_{el}} \rho_e^p(x_e) A_e \left( \int_{\Omega_e} b_e^T C(\alpha, \beta) b_e d\Omega \right) A_e^T = \sum_{e=1}^{n_{el}} \rho_e^p(x_e) K_e(\alpha, \beta)
\]

where \(\rho_e\) are the physical design variables, \(x_e\) are the design variables, \(p\) is the SIMP penalty coefficient, \(A_e\) is a boolean expansion matrix, \(b_e\) is the strain-displacement matrix, \(K_e\) is the expanded element stiffness matrix and \(C(\alpha, \beta)\) is the constitutive stiffness matrix for the current material direction, which can be expressed as

\[
C(\alpha, \beta) = L(\alpha, \beta) C_0 L^T(\alpha, \beta).
\]

The compliance \(C\) is given by

\[
C = F^T u(\rho(x), \alpha, \beta)
\]

where \(u\) solves the state problem.
The derivatives of the compliance with respect to the material angles $\alpha$ and $\beta$ are obtained as

$$\frac{\partial C}{\partial \alpha} = F^T \frac{\partial u}{\partial \alpha}$$

Differentiating the state equation w.r.t. $\alpha$ gives

$$\frac{\partial K}{\partial \alpha} \mathbf{u} + K \frac{\partial \mathbf{u}}{\partial \alpha} = 0 \iff \frac{\partial \mathbf{u}}{\partial \alpha} = -K^{-1} \frac{\partial K}{\partial \alpha} \mathbf{u}$$

$$\Rightarrow \frac{\partial C}{\partial \alpha} = -F^T K^{-1} \frac{\partial K}{\partial \alpha} \mathbf{u}.$$ 

Due to the symmetry of the global stiffness matrix ($K^{-1} = K^{-T}$) we have

$$\frac{\partial C}{\partial \alpha} = -(K^{-1}F)^T \frac{\partial K}{\partial \alpha} \mathbf{u} = -u^T \frac{\partial K}{\partial \alpha} u,$$

where

$$\frac{\partial K}{\partial \alpha} = \sum_{e=1}^{n_{el}} \rho_e^p(x) A_e \left( \int_{\Omega_e} b_e^T \frac{\partial C(\alpha, \beta)}{\partial \alpha} b_e \, d\Omega \right) A_e^T.$$

The derivative of the constitutive stiffness matrix $C$ w.r.t. $\alpha$ reads

$$\frac{\partial C(\alpha, \beta)}{\partial \alpha} = \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T,$$

hence,

$$\frac{\partial C}{\partial \alpha} = -u^T \left( \sum_{e=1}^{n_{el}} \rho_e^p(x) A_e \left( \int_{\Omega_e} b_e^T \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T b_e \, d\Omega \right) A_e^T \right) u.$$

Rewriting the above expression in terms of local entities yields

$$\frac{\partial C}{\partial \alpha} = -\sum_{e=1}^{n_{el}} \rho_e^p(x) u_e^T \left( \int_{\Omega_e} b_e^T \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T b_e \, d\Omega \right) u_e,$$

where

$$\frac{\partial L}{\partial \alpha} = \begin{bmatrix}
0 & \sin 2\alpha & \sin 2\alpha \\
0 & \sin 2\alpha & \cos 2\alpha \\
0 & \cos 2\alpha & -\sin 2\alpha \\
0 & \sin 2\alpha & \sin 2\alpha \\
0 & \cos 2\alpha & \cos 2\alpha \\
0 & \sin 2\alpha & \sin 2\alpha \\
0 & \cos 2\alpha & \cos 2\alpha \\
0 & \sin 2\alpha & \cos 2\alpha \\
0.5\sin 2\alpha & -0.5\sin 2\alpha & \cos 2\alpha \\
\end{bmatrix}$$

and

$$\begin{bmatrix}
\sin 2\alpha & \cos 2\alpha & -\sin 2\alpha & \\
\cos 2\alpha & \cos 2\alpha & \sin 2\alpha & \\
-\sin 2\alpha & -\sin 2\alpha & \cos 2\alpha & \\
\cos 2\alpha & \cos 2\alpha & \sin 2\alpha & \\
\sin 2\alpha & \sin 2\alpha & \cos 2\alpha & \\
\cos 2\alpha & \cos 2\alpha & \sin 2\alpha & \\
\sin 2\alpha & \cos 2\alpha & \sin 2\alpha & \\
\cos 2\alpha & \cos 2\alpha & \sin 2\alpha & \\
-0.5\sin 2\alpha & 0.5\sin 2\alpha & \cos 2\alpha & \\
\end{bmatrix}.$$
\[
\frac{dL}{a\beta} = \begin{bmatrix}
-\sin 2\beta & \sin^2 \alpha \sin 2\beta & \cos^2 \alpha \sin 2\beta & 2 \sin \alpha \cos 2\beta & \sin 2\alpha \sin 2\beta & 2 \cos \alpha \cos 2\beta \\
\sin 2\beta & -\sin^2 \alpha \sin 2\beta & -\cos^2 \alpha \sin 2\beta & -2 \sin \alpha \cos 2\beta & 0 & 0 \\
0 & 0.5 \sin 2\alpha \cos \beta & -0.5 \sin 2\alpha \cos \beta & -\cos \alpha \sin \beta & 2 \cos \alpha \cos \beta & \sin \alpha \sin \beta \\
0 & -0.5 \sin 2\alpha \sin \beta & 0.5 \sin 2\alpha \sin \beta & -\cos \alpha \cos \beta & \sin \alpha \cos \beta & \sin \alpha \cos \beta \\
-\cos 2\beta & \sin^2 \alpha \cos 2\beta & \cos^2 \alpha \cos 2\beta & -2 \sin \alpha \sin 2\beta & -2 \cos \alpha \sin 2\beta & -2 \cos \alpha \sin 2\beta
\end{bmatrix}
\]

The integrals
\[
\int_{\Omega_e} b_e^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) b_e d\Omega = \frac{\partial k_e}{\partial \alpha}, \quad e = 1, ..., n_e
\]
are calculated numerically using gauss quadrature of the same order as the element stiffness matrix, using the same local strain displacement matrices.

3.3.2 Sensitivity of Global Stress Measure w.r.t. Design Variables

The local equivalent von Mises stresses \( \sigma_{VM} \) are combined into a single global stress measure \( \sigma^G \) using the \( P \)-norm
\[
\sigma^G = \left( \sum_{i=1}^{n_\alpha} \left( \frac{\sigma_{VM}}{\bar{\sigma}} \right)^p \right)^{\frac{1}{p}},
\]
where \( n_\alpha \) is the number of stress evaluation points and \( \bar{\sigma} \) is a stress reduction factor used to avoid numerical issues due to the large sum. The equivalent von Mises stress in element \( e \) at stress evaluation point \( \alpha \) is given by
\[
\sigma_{VM} = \left( \sigma_{ax}^2 + \sigma_{ay}^2 + \sigma_{az}^2 - \sigma_{ax} \sigma_{ay} - \sigma_{ax} \sigma_{az} - \sigma_{ay} \sigma_{az} + 3 \tau_{axy}^2 + 3 \tau_{axy}^2 + 3 \tau_{axx}^2 \right)^{\frac{1}{2}},
\]
and the stress state in stress evaluation point \( \alpha \) is obtained from
\[
\sigma_{e,\alpha} = \begin{bmatrix}
\sigma_{ax} & \sigma_{ay} & \sigma_{ax} & \tau_{axy} & \tau_{axy} & \tau_{axx}
\end{bmatrix}^T = CB_a u(\rho(x))
\]
where \( B_a \) is the expanded strain-displacement matrix corresponding to stress evaluation point \( \alpha \), \( u(\rho(x)) \) is the global displacement vector and \( C \) is the constitutive stiffness matrix. In order to avoid intermediate design variable values the stresses are penalized according to the \( qp \)-relaxation approach
\[
\sigma_{\alpha} = CB_a u(x) \rho_{e}^q(x)
\]
where \( \rho_e(x) \) is the physical design variable corresponding to the element containing \( \alpha \) and \( q \) is the stress penalization exponent (see Section 2.4.2.1). Hence, using the Einstein summation convention
\[
\frac{\partial \sigma_{\alpha}}{\partial x_j} = \left( \frac{\partial \sigma_{\alpha}}{\partial u} \right)^T \frac{\partial u}{\partial \rho_{i}} \frac{\partial \rho_{i}}{\partial x_j} + \frac{\partial \sigma_{\alpha}}{\partial \rho_{e}} \frac{\partial \rho_{e}}{\partial x_j}, \quad (no \ sum \ over \ e)
\]
The derivative of the displacement vector is obtained from the state equation...
\[ K(\rho(x))u(\rho(x)) = F \Rightarrow \frac{\partial u(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} = -K^{-1}(\rho(x)) \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x)). \]

This gives
\[
\frac{\partial \sigma_a}{\partial x_j} = CB_\alpha u(\rho(x))q\rho^{-1}_e(x)\frac{\partial \rho_e}{\partial x_j} - CB_\alpha \rho^{-1}_e(x)K^{-1}(\rho(x)) \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x)).
\]

The derivatives of the von Mises stress for stress evaluation point \(a\) w.r.t. the stress components is given by
\[
\frac{\partial \sigma^v_a}{\partial \sigma_a} = \frac{1}{2\sigma^v_a}
\begin{bmatrix}
2\sigma_{ax} - \sigma_{ay} - \sigma_{az} \\
2\sigma_{ay} - \sigma_{ax} - \sigma_{az} \\
2\sigma_{az} - \sigma_{ax} - \sigma_{ay} \\
6\tau_{axy} \\
6\tau_{ayz} \\
6\tau_{azx}
\end{bmatrix}.
\]

Hence
\[
\frac{\partial \sigma^v_a}{\partial x_j} = \left(\frac{\partial \sigma^v_a}{\partial \sigma_a}\right)^T \frac{\partial \sigma_a}{\partial x_j} =
\]
\[
= \left(\frac{\partial \sigma^v_a}{\partial \sigma_a}\right)^T \left(CB_\alpha u(\rho(x))q\rho^{-1}_e(x)\frac{\partial \rho_e}{\partial x_j} - CB_\alpha \rho^{-1}_e(x)K^{-1}(\rho(x)) \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x))\right)
\]

The derivative of the global stress measure \(\sigma^G\) w.r.t. the von Mises stress in stress evaluation point \(a\) reads
\[
\frac{\partial \sigma^G}{\partial \sigma^v_a} = \frac{1}{\sigma} \left(\sum_{i=1}^{n_a} \left(\frac{\sigma^v_i}{\sigma}\right)^p\right)^{\frac{1}{p}-1} \left(\frac{\sigma^v_a}{\sigma}\right)^{\frac{1}{p}-1}.
\]

The derivative of the global stress measure with respect to design variable \(x_j\) reads
\[
\frac{\partial \sigma^G}{\partial x_j} = \sum_{a=1}^{n_a} \frac{\partial \sigma^G}{\partial \sigma^v_a} \left(\frac{\partial \sigma^v_a}{\partial \sigma_a}\right)^T \frac{\partial \sigma_a}{\partial x_j} =
\]
\[
= \sum_{a=1}^{n_a} \frac{\partial \sigma^G}{\partial \sigma^v_a} \left(\frac{\partial \sigma^v_a}{\partial \sigma_a}\right)^T \left(CB_\alpha u(\rho(x))q\rho^{-1}_e(x)\frac{\partial \rho_e}{\partial x_j} - CB_\alpha \rho^{-1}_e(x)K^{-1}(\rho(x)) \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x))\right) =
\]
The adjoint variable $\lambda$ is defined as

$$
\lambda^T = \sum_{a=1}^{n_a} \frac{\partial \sigma^G}{\partial \sigma_a^M} \left( \frac{\partial \sigma_a^M}{\partial \sigma_a} \right)^T C B_a u(\rho(x)) q \rho_e^{q-1} \frac{\partial \rho_e}{\partial x_j} \\
- \sum_{a=1}^{n_a} \frac{\partial \sigma^G}{\partial \sigma_a^M} \left( \frac{\partial \sigma_a^M}{\partial \sigma_a} \right)^T C B_a \rho_e^q(x) K^{-1}(\rho(x)) \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x))
$$

The only explicit dependence on $x_j$ in (3.1) comes from the derivative of the physical variables w.r.t. the design variables $\partial \rho_e / \partial x_j$. The physical variables $\rho$ are related to the design variables $x$ through the filter matrix $H$ as

$$
\rho(x) = H x
$$

hence, the derivatives are

$$
\frac{\partial \rho_e}{\partial x_j} = H_{e_j}.
$$

The derivative of the stiffness matrix $K$ w.r.t. design variable $x_j$ now reads

$$
\frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} = \sum_{l=1}^{n_{el}} \mathbf{A}_i (p \rho_i^{p-1} \mathbf{k}_l^p) \mathbf{a}_l^T H_{ij}
$$

and the second term in (3.1) becomes

$$
\lambda^T \frac{\partial K(\rho(x))}{\partial \rho_i} \frac{\partial \rho_i}{\partial x_j} u(\rho(x)) = \lambda^T \left( \sum_{l=1}^{n_{el}} \mathbf{A}_i (p \rho_i^{p-1} \mathbf{k}_l^p) \mathbf{a}_l^T \right) u(\rho(x)) H_{ij}.
$$

The above expression can be rewritten in terms of the a local adjoint variable $\lambda_i$ containing the elements of $\lambda$ associated to the global degrees of freedom of element $i$ and the local displacement vectors $\mathbf{u}_i$, $i = 1, \ldots, n_{el}$ as
\[ \lambda^T \left( \sum_{i=1}^{n_{el}} A_i (p \rho_i^{p-1} k_i^0) A_i^T \right) u(\rho(x)) = \sum_{i=1}^{n_{el}} \lambda_i^T (p \rho_i^{p-1} k_i^0) u_i(\rho(x)). \]

The column vector \( \psi \in \mathbb{R}^{n_{el}} \) is defined such that

\[ \psi_i = \lambda_i^T (p \rho_i^{p-1} k_i^0) u_i(\rho(x)). \]

Hence, the second term of (3.1) becomes

\[ \lambda^T \frac{\partial K(\rho(x))}{\partial \rho_i} u(\rho(x)) = \psi^T H \]

where \( H \) is column \( j \) of \( H \). Similarly, the first term in (3.1) can be written as

\[ \sum_{a=1}^{n_a} \left( \frac{\partial \sigma_G}{\partial \sigma_a} \right)^T C B_a u_i(\rho(x)) q \rho_e^{q-1} \frac{\partial \rho_e}{\partial x_j} = \sum_{a=1}^{n_a} \left( \frac{\partial \sigma_G}{\partial \sigma_a} \right)^T C B_a u_i(\rho(x)) q \rho_e^{q-1} H_{ej} \]

where \( b_a \) is the local strain displacement matrix for stress evaluation point \( a \). The column vector \( \phi \in \mathbb{R}^{n_{el}} \) is defined such that

\[ \phi_i = \sum_{a=1}^{n_{a,i}} \left( \frac{\partial \sigma_G}{\partial \sigma_a} \right)^T C B_a u_i(\rho(x)) q \rho_e^{q-1} \]

where \( n_{a,i} \) is the number of stress evaluation points in element \( i \). The first term in (3.1) now becomes

\[ \sum_{a=1}^{n_a} \left( \frac{\partial \sigma_G}{\partial \sigma_a} \right)^T C B_a u_i(\rho(x)) q \rho_e^{q-1} \frac{\partial \rho_e}{\partial x_j} = \phi^T H_j. \]

The derivative of the global stress measure w.r.t. design variable \( x_j \) can hence be obtained as

\[ \frac{\partial \sigma_G}{\partial x_j} = \phi^T H_j - \psi^T H_j = (\phi - \psi)^T H_j. \]

### 3.3.3 Sensitivity of Global Stress Measure w.r.t. Material Orientation

If the build direction is included in the optimization, the stress state in the structure depends on the design variables as well as the orientation of the material. Using the stress penalization factor \( q \), we have

\[ \sigma_a(x, \alpha, \beta) = C(\alpha, \beta) B_a u(x, \alpha, \beta) \rho_e^q(x), \]

differentiation w.r.t. \( \alpha \) gives
\[
\frac{\partial \sigma_a}{\partial \alpha} = \frac{\partial C(\alpha, \beta)}{\partial \alpha} B_a u(x, \alpha, \beta) \rho^q_e(x) + C(\alpha, \beta) B_a \frac{\partial u(x, \alpha, \beta)}{\partial \alpha} \rho^q_e(x),
\]

where
\[
\frac{\partial C(\alpha, \beta)}{\partial \alpha} = \frac{\partial L}{\partial \alpha} C_0 L^T + LC_0 \left( \frac{\partial L}{\partial \alpha} \right)^T.
\]

From the state equation (2.7) we get
\[
\frac{\partial u(x, \alpha, \beta)}{\partial \alpha} = -K(x, \alpha, \beta)^{-1} \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta),
\]
where
\[
\frac{\partial K(x, \alpha, \beta)}{\partial \alpha} = \sum_{e=1}^{n_{el}} \rho^p_e(x) A_e \left( \int_{\Omega_e} b_e^T \frac{\partial C(\alpha, \beta)}{\partial \alpha} b_e \, d\Omega \right) A_e^T.
\]

This gives
\[
\frac{\partial \sigma_a}{\partial \alpha} = \rho^q_e(x) \left( \frac{\partial L}{\partial \alpha} C_0 L^T + LC_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) B_a u(x, \alpha, \beta)
\]
\[
- C(\alpha, \beta) B_a K(x, \alpha, \beta)^{-1} \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta).
\]

The derivative of the von Mises stress in stress evaluation point \( \alpha \) w.r.t. the orientation angle \( \alpha \) is
\[
\frac{\partial \sigma_{aM}^v}{\partial \alpha} = \left( \frac{\partial \sigma_{aM}^v}{\partial \sigma_a} \right)^T \frac{\partial \sigma_a}{\partial \alpha} =
\]
\[
= \sum_{a=1}^{n_a} \rho^q_e(x) \left( \frac{\partial \sigma_{aM}^v}{\partial \sigma_a} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + LC_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) B_a u(x, \alpha, \beta)
\]
\[
- C(\alpha, \beta) B_a K(x, \alpha, \beta)^{-1} \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta).
\]

The sensitivity of the global stress measure \( \sigma^G \) w.r.t. the orientation angle \( \alpha \) becomes
\[
\frac{\partial \sigma^G}{\partial \alpha} = \sum_{a=1}^{n_a} \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \frac{\partial \sigma_a}{\partial \alpha} \\
= \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) \mathbf{B}_a u(x, \alpha, \beta) \\
- C(\alpha, \beta) \mathbf{B}_a K(x, \alpha, \beta)^{-1} \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta) = \\
= \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) \mathbf{B}_a u_e(x, \alpha, \beta) \\
- \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( C(\alpha, \beta) \mathbf{B}_a K(x, \alpha, \beta)^{-1} \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta) \right).
\]

We define an adjoint variable \( \lambda_\alpha \) as

\[
\lambda_\alpha^T = \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T C(\alpha, \beta) \mathbf{B}_a K(x, \alpha, \beta)^{-1}
\]

\[
\leftrightarrow
\]

\[
K(x, \alpha, \beta) \lambda_\alpha = \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( C(\alpha, \beta) \mathbf{B}_a \right)^T \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha}.
\]

Solving for \( \lambda_\alpha \) and reinserting gives

\[
\frac{\partial \sigma^G}{\partial \alpha} = \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) \mathbf{B}_a u_e(x, \alpha, \beta) \\
- \lambda_\alpha^T \frac{\partial K(x, \alpha, \beta)}{\partial \alpha} u(x, \alpha, \beta).
\]

Rewriting this expression in terms of the only local entities gives

\[
\frac{\partial \sigma^G}{\partial \alpha} = \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) \mathbf{B}_a u_e(x, \alpha, \beta) \\
- \lambda_\alpha^T \left( \sum_{e=1}^{n_e} \rho_e^G(x) \int_{\Omega_e} b_e^T \frac{\partial C(\alpha, \beta)}{\partial \alpha} b_e d\Omega \right) A_e^T \mathbf{u}(x, \alpha, \beta) = \\
= \sum_{a=1}^{n_a} \rho_q^G(x) \frac{\partial^2 \sigma^{\text{VM}}}{\partial \alpha_a^2} \left( \frac{\partial \sigma_a^{\text{VM}}}{\partial \alpha} \right)^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + L C_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) \mathbf{B}_a u_e(x, \alpha, \beta) \\
- \sum_{e=1}^{n_e} \lambda_\alpha^T \rho_e^G(x) \left( \int_{\Omega_e} b_e^T \frac{\partial C(\alpha, \beta)}{\partial \alpha} b_e d\Omega \right) u_e(x, \alpha, \beta).
\]

38
where $\lambda_{a,e}$ contains the elements of $\lambda_a$ corresponding to the global degrees of freedom of element $e$. The final expression for the sensitivity hence becomes

$$\frac{\partial \sigma^G}{\partial \alpha} = \sum_{a=1}^{n_e} \rho_e^a(x) \frac{\partial \sigma^G}{\partial \sigma^a} \left( \frac{\partial L}{\partial \alpha} C_0 L^T + LC_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) b_a \mathbf{u}_e(x, \alpha, \beta)$$

$$- \sum_{e=1}^{n_e} \rho_e^p(x) \lambda_{a,e} \left( \int_{\Omega_e} \mathbf{b}_e^T \left( \frac{\partial L}{\partial \alpha} C_0 L^T + LC_0 \left( \frac{\partial L}{\partial \alpha} \right)^T \right) b_e d\Omega \right) \mathbf{u}_e(x, \alpha, \beta)$$

The sensitivity of $\sigma^G$ w.r.t. $\beta$ is obtained in exactly the same way, simply exchanging $\beta$ for $\alpha$ in the expressions above.
3.4 Numerical Implementation

The TO procedures are implemented in TRINITAS, which is a general purpose finite element program developed at Linköping University by the supervisor of this thesis [30]. The pre-processing, including geometry and mesh generation, as well as post-processing and visualization is performed using existing procedures in TRINITAS. The sensitivity analysis for the stress, mass and compliance constraints and objective functions are implemented by the author of this theses using subroutines which are called from the main program. The implementation of the sensitivities are tested using central difference calculations. The code is implemented in the FORTRAN language. In order to increase the efficiency of the implementation the state problem is solved using Intel’s parallel direct sparse solver included in the Math Kernel Library and the sensitivity analysis is partially parallelized using Open MP.

The optimization problem is solved using MMA, courtesy of Krister Svanberg [22]. Some minor changes are made to the original implementation concerning the asymptote updates for the material direction variables which oscillate when the original implementation of the MMA algorithm is used. This is cured by decreasing the lower limit on the asymptote closeness for these variables.

The implementation is made as general as possible in the sense that any element from the TRINITAS library can be used in 2D and 3D analysis. However, different elements cannot be mixed in the same analysis. The reason for this limitation is to keep the computational efficiency high, a generalization to mixed elements should however be relatively simple, but this is however beyond the scope of this work. The material direction optimization is implemented in 2D and 3D, where the 2D implementation is a simplification of the method derived in Section 3.2 using a single rotation.
4. **Numerical Examples**

In this section a number of numerical examples are evaluated in order to test the implementation. Three TO formulations are mainly considered:

1. \((\mathcal{P}_1)\) : Minimizing the maximum von-Mises stress subject to a mass constraint

\[
\begin{align*}
\min_x & \quad \sigma^G(x) \\
\text{s.t.} & \quad \sum_{e=1}^{n_e} m_e \rho_e(x) \leq M_{\text{max}} \\
& \quad \varepsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_x.
\end{align*}
\]

2. \((\mathcal{P}_2)\) : Minimizing the mass subject to a constraint on the maximum von-Mises stress

\[
\begin{align*}
\min_x & \quad \sum_{e=1}^{n_e} m_e \rho_e(x) \\
\text{s.t.} & \quad \sigma^G(x) \leq \sigma_{\text{max}}^{vM} \\
& \quad \varepsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_x.
\end{align*}
\]

3. \((\mathcal{P}_3)\) : Minimizing the compliance subject to a constraint on the mass and the maximum von-Mises stress

\[
\begin{align*}
\min_x & \quad F^T \mathbf{u}(x) \\
\text{s.t.} & \quad \sigma^G(x) \leq \sigma_{\text{max}}^{vM} \\
& \quad \sum_{e=1}^{n_e} m_e \rho_e(x) \leq M_{\text{max}} \\
& \quad \varepsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_x.
\end{align*}
\]

Slightly different formulations are used in the evaluations of the material orientation examples, these are presented as they appear.
4.1 2D L-Beam

The so-called L-Beam is used extensively in the literature when stress constrained topology optimization is studied [21], [9], [8], [18]. The reason for this is the internal corner which gives large stress concentrations unless avoided, see Figure 19.

Classical compliance based formulations are unable to avoid the stress concentration and hence produces designs which are unusable without major modifications, see Figure 20. With the use of stress constraints it is possible to avoid the internal corner and produce conceptual designs which require less modification.

When stress constraints are used, the dark grey areas in Figure 19 are excluded from the optimization process and their density variables are fixed to 1. The reason for this is to exclude the large stress concentrations in areas of load application and supports so that they do not contribute to the global stress measure.
In order to evaluate the implementation of the constraint and objective functions all three formulations are tested using the L-Beam in Figure 19. The material is a steel with Young’s modulus $E = 205 \, \text{GPa}$, Poisson’s ratio $\nu = 0.3$ and density $\rho = 7790 \, \text{kg/m}^3$, the elastic properties are assumed to be isotropic and a plane stress formulation is used for the constitutive matrix. The side length of the beam is $L = 0.5 \, \text{m}$, the thickness is $t = 75.0 \, \text{mm}$, the filter radius is $R = 6.0 \, \text{mm}$, the stress penalization exponent is $q = 0.5$, the $P$-norm stress aggregation function is used with an exponent of $P = 16$, the SIMP parameter is $p = 3$ and the stress normalization factor is $\bar{\sigma} = 10^6$. A load of $50,000 \, \text{N}$ is applied to a single node at the tip of the beam, see Figure 19. A maximum allowed mass of $30 \, \text{kg}$ is used for (P1) and (P3) which is $30\%$ of the total mass of the design domain and a stress limit of $100 \, \text{MPa}$ is used for (P2) and (P3). The domain is discretized using 8345 4-Node bilinear elements. Unless otherwise specified, the MMA algorithm is run for 500 iterations with no other stopping criteria in all the following examples.

Figure 21 (top) shows the topologies produced from the different formulations. As seen in the figure, all three formulations manage to avoid the stress singularity at the internal corner. Figure 21 (bottom) shows the von Mises stress in the structures, all designs are well below the stress limit of $100 \, \text{MPa}$ (indicated as red in the lower stress plot).

In the final iteration of (P2) the stress aggregation function indicates a largest stress of $100 \, \text{MPa}$ while the “actual” largest stress is $75 \, \text{MPa}$ (as calculated according to (2.12)), i.e. the $P$-norm overestimates the stress by $25\%$. In previous studies, e.g. in [18], correction factors are used in order to circumvent this discrepancy (see Section 2.4.2). Using a slightly modified version of the stress correction in [18], a factor $c$ is calculated in each iteration such that
\[ c^k = \alpha \left( \frac{\sigma^M_{\text{max}}}{\sigma^G} \right)^k + (1 - \alpha)^{k-1}, \]

where \( \sigma^M_{\text{max}} = \max\{\sigma^M_1, \sigma^M_2, \ldots, \sigma^M_n\} \), \( \alpha \in (0,1] \) and \( k \) is the current iteration. The difference from the stress correction in [18] is that the correction factor is updated using the stress in the current iteration instead of the previous (compare with (2.15)) and that \( \alpha \) is fixed. All of the following examples use \( \alpha = 0.5 \). The stress constraint function \( f_1(x) \) is then calculated as

\[ f_1(x) = c \sigma^G(x). \]

Figure 22 shows the results from formulation \((P_2)\) with and without the stress correction. The domain is discretized using 21,530 3-node triangular elements, the filter radius is \( R = 7.0 \text{ mm} \), the stress penalization exponent is \( q = 0.5 \), the \( P \)-norm exponent is \( P = 16 \) and the maximum allowed stress is \( \sigma_{\text{max}} = 120 \text{ MPa} \).

Due to the significant error in the approximated maximum stress the resulting structure is heavier than it needs to be in order to satisfy the constraint when the stress correction is not used (see
Figure 22, left). When the stress correction is used the maximum stress coincides with the approximation and the resulting structure is therefore lighter (see Figure 22, right). The correction factor means that the constraint function is modified in each iteration which may cause convergence issues, no such problems were however experienced during the calculation of these examples. Obviously, if the stress is used as the objective function, this modification is not necessary.

Another approach to decreasing the error in the stress aggregation function is to increase the stress aggregation exponent $P$. Figure 23 shows the results for formulation ($\mathbb{P}_2$) for $P = 12, 18$ and $24$ using the same parameters as for the example in Figure 22. The design domain is discretized using 21,530 6-node triangular elements. The stress correction is not used.

![Image](image_url)

**Figure 23 – Results from formulation ($\mathbb{P}_2$) using different values for the $P$-norm exponent, without stress correction.**

As seen in Figure 23 increasing the $P$-norm exponent has a marginal effect on the stress approximation error. However, the problem gets increasingly difficult to solve for large values of $P$ and there is a significant difference in the resulting topology as $P$ is increased. The examples in Figure 23 are rerun using the stress correction. The results are shown in Figure 24.
The best result (lowest mass) is obtained for $P = 12$ using the stress correction (Figure 24, left). No convergence issues were induced by the stress correction in the examples above. However, if $P$ is chosen too low, convergence problems do occur. Figure 25 shows the results for $P = 8$. The convergence issues are probably due to the failure of the stress aggregation function to discern the locality of the largest stress. Note that this problem does not converge without the stress correction either.

In order to investigate the difference between different element formulations, $(P_2)$ is run using three different element types: 22,236 linear strain triangles, 22,236 6-node serendipity triangles and 21,985 8-node serendipity quadrilaterals. The stress correction is used in these examples. The results are shown in Figure 26. Red in the lower figure indicates areas where the stress is equal to or larger than the limit of 100 MPa.
The higher order elements give designs with slightly larger masses which is probably a result of the better resolution of the stress calculations. The difference in the topology is however not significant.

4.2 3D L-Beam

The 3D L-Beam in this example has the same dimensions and material properties as the 2D variant in Section 4.1. The domain is discretized using 81,120 8-Noded solid brick elements and boundary conditions are applied according to Figure 19. A point load of $F = 50,000 \, N$ is applied at the middle of the beam in the z-direction and the upper plane is fixed in all directions, see Figure 28.
the SIMP parameter is \( p = 3 \) and the stress normalization factor is \( \bar{\sigma} = 10^6 \), the stress correction is not used. Figure 28 shows the resulting topology and stress distribution in TRINITAS. Red in the left figure indicates areas where the stress is equal to or larger than the limit of 100 MPa. The areas of load application and support are excluded from the stress aggregation function.

![Figure 28 – Topology and von Mises stress of the 3D L-Beam using formulation (P_3).](image)

In order to examine the results more closely the topology is exported, converted into a solid body and analysed using the commercial finite element software ANSYS. The conversion from the results shown in Figure 28 to a solid body is performed as follows. The results are exported from TRINITAS as an STL file, see Figure 29. The triangular based STL is then imported into Autodesk Remake where the mesh is analysed and repaired if necessary. The geometry is then exported as a quad based mesh and imported into Autodesk Fusion 360 where the quad based mesh is first converted into a T-spline surface and subsequently a solid body. The solid is exported as a STEP file which is imported into SolidWorks, see Figure 30.
Figure 29 – 3D L-Beam topology exported from TRINITAS as an STL file.

Figure 30 – Solid model of 3D L-Beam.

All the steps from the results in TRINITAS to the analysis in ANSYS are summarized in Figure 31. No smoothing of the geometry is done on purpose, however, some smoothing inevitably occurs during the conversion steps. This is clearly seen when comparing the geometries in Figure 28, Figure 29 and Figure 30.
Figure 31 – Summary of the conversion steps from the TRINITAS result into a solid body which can be analysed in ANSYS.

The beam is meshed with 556,473 10-Node tetrahedral elements in ANSYS, see Figure 32 (left), and the same boundary conditions as in the TO procedure are applied. The right figure shows the resulting von-Mises stress distribution. Red indicates areas where the stress is equal to or larger than the limit of 100 MPa.

Figure 32 – 3D L-Beam discretisation and von-Mises stress distribution in ANSYS. Red indicates stresses over the stress limit of 100 MPa.
Figure 33 shows the von Mises stress distribution at internal corner in the ANSYS model compared to the original topology optimization results in TRINITAS.

The stress at the internal corner of the ANSYS model agrees well with the penalized stress in the same area of the TO results in TRINITAS. The most critical area in the solid model is however just beneath the tip of the beam at the area of the load application, see Figure 32 (right). However, since the elements in this area are excluded from the stress aggregation function in the TO process, comparing the stress in this area is not meaningful.

4.3 3D Bracket

A 3D bracket is optimized for maximum stiffness using formulation \((P_3)\). The dimensions of the design space and the load application are illustrated in Figure 34. The inner surfaces of the three holes on the lower triangular plate are fixed in all directions and a surface traction is applied to the inner surface of the upper hole resulting in a total force of \(F = 283\, kN\).
The material is assumed isotropic with Young’s modulus $E = 115 \text{ GPa}$ and Poisson’s ratio $\nu = 0.33$. The mass density is $\rho = 4430 \text{ kg/m}^3$. The domain is discretized using 133,488 8-node solid brick elements of varying size and shape, see Figure 35.

![Figure 35 – Mesh of 3D bracket design space in TRINITAS.](image)

The mass limit is set to $M_{\text{max}} = 8 \text{ kg}$ which is 27% of the total mass of the design space, the stress limit is set to $\sigma_{\text{max}} = 650 \text{ MPa}$, with $P = 16$ and $q = 0.5$, and the filter radius is set to $R = 8.0 \text{ mm}$. The stress correction is not used. The MMA algorithm is run for 500 iterations with no other stopping criteria. Figure 36 shows the resulting topology converted into a solid model using the procedure described in Section 4.2.

![Figure 36 – Topology of 3D Bracket optimized using formulation (P$_3$).](image)

The global stress measure does not drop below the stress limit of 650 MPa after 500 iterations, the final value is 803 MPa. However, the actual maximum stress does meet the constraint, it is therefore decided not to run any more iterations. The solid model is imported into ANSYS and discretized using 698,806 10-node tetrahedral elements, see Figure 37 (left). The same boundary conditions as in the topology optimization are applied and a linear static stress analysis is performed. The stress distribution from the ANSYS analysis is shown in Figure 37 (right). Red indicates areas where the von-Mises stress is above 570 MPa, which is the maximum stress in the structure according to the TO results in TRINITAS.
As seen in Figure 37 (right), areas of large stress concentrations remain in the structure. The stresses in these areas are significantly higher in the ANSYS model compared to what was indicated by the TO results in TRINITAS. The reason for this discrepancy is most likely that the mesh used for the TO analysis is not accurate enough to resolve the stresses in these areas. Figure 38 show a comparison of the von-Mises stress distribution between the ANSYS model and the TO results in TRINITAS.

Figure 37 – Mesh and von Mises stress distribution for 3D Bracket ANSYS model.

Figure 38 – Comparison of stress distribution between ANSYS and TRINITAS model of bracket. Red indicates areas were the stress is equal to or larger than 570 MPa.
4.4 Simple Material Direction Optimization

An evaluation of the proposed method of material direction optimization is performed using a series of simple numerical examples. A rectangular brick is oriented in different directions w.r.t. the global coordinate system. The stiffness is then maximized using the material direction angles $\alpha$ (rotation w.r.t. global $x$-axis) and $\beta$ (rotation w.r.t. global $y$-axis) as the only active variables. The optimization problem is formulated as follows

$$\min_{\alpha, \beta \in \Phi} F^T u(\alpha, \beta),$$

where $\Phi = [-\pi, \pi]$. The brick is fixed at the lower surface, but allowed to expand and contract in order to achieve a uni-directional stress state. A surface traction of 1 MPa is applied to the upper surface. The boundary conditions are illustrated in Figure 39.

![Boundary Conditions](image)

Figure 39 – Illustration of boundary conditions for material direction optimization.

The material model is transversely isotropic with material parameters according to Table 1. These material parameters are chosen arbitrarily and do not represent any physical material.

| $E_x$, $E_y$ | 120.0 GPa |
| $E_z$ | 30.0 GPa |
| $\nu_{xy}$ | 0.33 |
| $\nu_{xz}$, $\nu_{yz}$ | 0.28 |
| $G_{xy}$ | 45.0 GPa |
| $G_{xz}$, $G_{yz}$ | 20.0 GPa |

Table 1 – Material parameters for material direction optimization examples.

Figure 40 shows the results from the material optimization with the brick oriented in the upright position (see illustration in Figure 40 a)). The material is oriented such that the stiffer isotropic plane is oriented in parallel to the stress state. Figure 40 b) shows the orientation of the isotropic plane (build plane) w.r.t. the global $xy$-plane (reference plane) and Figure 40 c) shows the evolution of the design variables during the iterations of the MMA algorithm as well as the normalized compliance.
Figure 40 – Material direction optimization of brick in upright position.

Figure 41 shows the results when the brick is rotated 45° w.r.t. the global x-axis. The material is again oriented such that the isotropic plane is parallel to the stress.

Figure 41 - Material direction optimization of brick rotated 45° around global x-axis.
Figure 42 shows the results when the brick is rotated \(-45^\circ\) w.r.t. the global \(y\)-axis. This example required more iterations to converge compared to the previous. The results are however similar, the isotropic plane is rotated by \(90^\circ\) w.r.t. the global \(x\)-axis which means that it’s parallel to the stress state. For this value of \(\alpha\) the normal to the isotropic plane is parallel to the global \(y\)-axis which makes \(\beta\) ambiguous. This is due to the rotational symmetry within the isotropic plane and is why \(\beta\) does not seem to converge to any particular value, see Figure 42 c).

The previous results can be obtained using a single rotation, in order to achieve a more complicated orientation the brick in Figure 42 a) is rotated \(45^\circ\) w.r.t. the global \(z\)-axis, the result of this is illustrated in Figure 43 a). The material is once again oriented such that the isotropic plane is parallel to the stress, in order to see this more clearly Figure 43 b) shows the orientation of the isotropic plane in the same viewpoint as the illustration in Figure 43 a). This orientation cannot be said to be optimal in terms of the build plane for AM since the part would be tilted at an angle w.r.t. the build plate, it is however an optimal orientation in terms of stiffness in the load direction.

Figure 42 - Material direction optimization of brick rotated \(-45^\circ\) around global \(y\)-axis.

The previous results can be obtained using a single rotation, in order to achieve a more complicated orientation the brick in Figure 42 a) is rotated \(45^\circ\) w.r.t. the global \(z\)-axis, the result of this is illustrated in Figure 43 a). The material is once again oriented such that the isotropic plane is parallel to the stress, in order to see this more clearly Figure 43 b) shows the orientation of the isotropic plane in the same viewpoint as the illustration in Figure 43 a). This orientation cannot be said to be optimal in terms of the build plane for AM since the part would be tilted at an angle w.r.t. the build plate, it is however an optimal orientation in terms of stiffness in the load direction.
The above examples indicate that the proposed coordinate transformation is capable of finding the best material orientation with regards to stiffness. It can however not be expected that the build orientations achieved from this method are practical since the outer dimensions of the part are not considered.
4.5 3D Bracket with Anisotropic Material Properties

The 3D bracket in Figure 34 is optimized using three different TO formulations, each with and without the material orientation optimization active. The same boundary conditions as in Section 4.3 are used but the material model is now transversely isotropic with material parameters according to Table 2. The mass density is $\rho = 4430 \, kg/m^3$.

Table 2 – Material properties for bracket.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x, E_y$</td>
<td>115 GPa</td>
</tr>
<tr>
<td>$E_z$</td>
<td>40 GPa</td>
</tr>
<tr>
<td>$\nu_{xy}$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\nu_{xz}, \nu_{yz}$</td>
<td>0.28</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>43.0 GPa</td>
</tr>
<tr>
<td>$G_{xz}, G_{yz}$</td>
<td>20.0 GPa</td>
</tr>
</tbody>
</table>

The domain is discretized using 50,750 8-node solid brick elements of varying size and shape, see Figure 44. The first two optimizations are performed using formulation (B$_1$):

$$\left\{ \begin{array}{l}
\min_{\mathbf{x}, \mathbf{u}, \alpha, \beta} F^T \mathbf{u}(\mathbf{x}, \alpha, \beta) \\
\text{s.t.} \quad \sum_{e=1}^{n_e} m_e \rho_e(\mathbf{x}) \leq M_{\max} \\
\quad \epsilon \leq x_j \leq 1, \quad j = 1, ..., n_x \\
\quad \alpha, \beta \in [-\pi, \pi],
\end{array} \right.$$  

with $\alpha$ and $\beta$ are fixed to 0 when the material orientation optimization is inactive. The mass limit is set to $M_{\max} = 12 \, kg$ which is 40% of the total mass of the design space and the filter radius is set to $R = 8.0 \, mm$.

Figure 44 – Discretisation of design space for the 3D bracket.
Figure 45 shows the resulting topology from formulation \((B_1)\) with and without the material orientation optimization active. The procedure described in Section 4.2 is used to obtain the solid bodies shown in the figure.

As seen in Figure 45 there is a clear difference in the topology when the material direction optimization is used, the resulting material orientation is illustrated in the upper figures. This also results in a lower compliance (higher stiffness) in the structure, see Figure 46 (left). Figure 46 (right) shows the evolution of the material angles during the optimization.
In formulation \((\mathbb{B}_2)\) a global stress constraint is included:

\[
\begin{align*}
\min_{x, \alpha, \beta} & \quad F^T u(x, \alpha, \beta) \\
\text{s.t.} & \quad \sum_{e=1}^{n_e} m(e) p_e(x) \leq M_{\text{max}} \\
& \quad \sigma^G(x, \alpha, \beta) \leq \sigma_{\text{max}}^{\text{vM}} \\
& \quad \varepsilon \leq x_j \leq 1, \quad j = 1, \ldots, n_x \\
& \quad \alpha, \beta \in [-\pi, \pi],
\end{align*}
\]

where \(\sigma_{\text{max}}^{\text{vM}} = 500\) MPa, \(P = 18\) and \(q = 0.4\). The stress correction is not used. Figure 47 shows the resulting topology from formulation \((\mathbb{B}_2)\) with and without the material orientation optimization active. There is still a difference in the resulting topologies, however, the material orientation angles do not seem to converge to any particular value, see Figure 48 (right).

Figure 47 - Comparison of topologies obtained from \((\mathbb{B}_2)\).
Left: Without material orientation optimization.
Right: With material orientation optimization.

Figure 48 (left) shows the evolution of the compliance and global stress measure during the optimization process. The dent in the compliance when the material orientation optimization is active is when the stress constraint is fulfilled.
In order to investigate the behaviour of the global stress measure as a function of the material angles the global stress for the topology in Figure 47 (right) is evaluated for all values of $\alpha$ in the range $[-\pi/2, \pi/2]$ keeping all other variables fixed. The results can be seen in Figure 49.

The value of $\alpha$ seems to be in the region of the global minimum of $\sigma^G(\alpha)$ for the current topology. It is however probable, looking at the results when the stress constraint is not used, that the best material orientation for maximum stiffness is another one. In order to investigate this, the compliance is not used in the final example.
In the final example the mass is minimized subject to a global stress constraint using formulation \((B_3)\):

\[
\begin{aligned}
&B_3 \quad \min_x \sum_{e=1}^{n_e} m_e \rho_e(x) \\
&\text{s.t.} \quad \sigma^G(x, \alpha, \beta) \leq \sigma_{\text{max}}^\nu M \\
&\qquad \epsilon \leq x_j \leq 1, \\
&\qquad \alpha, \beta \in [-\pi, \pi], \\
\end{aligned}
\]

where \(\sigma_{\text{max}}^\nu M = 450 \text{ MPa}\) with \(P = 18\) and \(q = 0.4\). The stress correction is used in this example. Figure 50 shows the resulting topologies with and without the material direction optimization.

As seen in Figure 50, using the material orientation optimization did not result in any significant difference in the topology. The mass is however slightly lower when it is active. Figure 51 shows the evolution of the mass and the global stress measure during the optimization process.
Figure 51 – Mass (left) and global stress measure (right) evolution with and without material direction optimization for formulation (𝔹₃).

Figure 52 shows the evolution of the material angles during the optimization process. The optimal material orientation seems to be a very different one for the global stress measure compared to the stiffness which is probably the reason why the material orientation angles did not converge in formulation (𝔹₂).

Figure 52 – Evolution of material orientation angles for 3D Bracket using formulation (𝔹₃).
5. Discussion & Conclusions

The implementation of the stress constraint using a single global stress measure seems to be adequate to achieve good results for a wide variety of problem sizes. However, the discrepancy between the global stress measure and the actual largest stress in the structure is very hard to predict, therefore the use of a stress correction factor seems to be necessary in order to give a good approximation of the maximum stress. The literature study also indicated that using a separation of the stress evaluation points into clusters may allow for lower $P$ values to be used, thereby relieving some of the tendency to local optima experienced during the course of this work. The clustering approach was however not implemented due to the limited time available.

A general problem with the use of the stress aggregation function and stress penalization is appropriate choices of the parameters $P$ and $q$ which are not obvious and may differ for different problems. Although no clear recommendations can be given, using $P = 16$ and $q = 0.5$ seems to be a good initial choice for most problems. Another difficulty is the choice of starting guess for the density variables. It is recommended that the user test several different values for the parameters and starting guess in order achieve good results.

The evaluation of the TO results using more detailed models revealed large discrepancies in the maximum stress in some cases (see Section 4.2 and 4.3). This is probably due to the elements and mesh densities used in the TO models which were not accurate enough to resolve the stresses and therefore stress concentrations remained in the optimized structures. The use of higher order elements is probably necessary in order to achieve better results. The comparison of the 2D L-Beam showed that the results may differ depending on the choice of element formulation, see Figure 26. This was however not tested for 3D structures due to the excessively large computational cost. Using 10-node tetrahedral elements is probably a good choice for stress constrained TO due to their comparatively low cost compared to the 20-Node brick elements.

Conclusive evidence that the elastic properties of AM produced Ti6Al4V parts are anisotropic could not be found in the literature review. However, if this turns out to be the case, the proposed method of combining the TO process with finding the optimal build orientation seems to work in most of the tested examples. The examples in Section 4.5 indicate that there can be a contradiction between the best material orientations for minimum stress compared to maximum stiffness. This is probably the reason for the non-convergence of formulation ($𝔹_2$).
6. Future Work

More refined meshes have to be used in order to evaluate the global stress constraint for more complicated 3D geometries. The structured meshes used for the 3D examples in this work were not fine enough to resolve all the areas of stress concentration in the bracket in Section 4.3 and 4.5. An unstructured mesh with 10-node tetrahedral elements is probably the best compromise between computational cost, accuracy in stress calculation and element size. In order to use these elements effectively a 3D unstructured mesh generator has to be implemented in TRINITAS.

The global stress measure used in this work is based on the equivalent von-Mises stress since it is simple and easy to implement. However, the von-Mises stress is not appropriate for fatigue loadings since it does not differ between tensile and compressive stresses. In order to use fatigue constraints other stress measures must be implemented, e.g. the maximum principal stress or the largest surface tangential stress.

All of the examples presented in this work use a single loadcase. This makes it easier to interpret and evaluate the results but often lead to non-robust structures in the sense that even a small difference in the loading conditions may lead to catastrophic failure, this non-robustness is clearly seen in the 3D bracket examples in Section 4.3 and 4.5. TRINITAS already contains procedures for time-stepping and multiple load application so a generalization to an arbitrary number of load cases should be quite simple.

AM applications place high demands on the robustness of optimized structures due to the large uncertainties associated with the process. Beyond multiple loadcases, a TO formulation for AM should be able to account for all these uncertainties and produce robust conceptual designs which are not too sensitive to variations in e.g. geometry, material properties and residual stresses. An effective TO formulation for AM also requires a deeper investigation into the material properties and associated uncertainties expected from the process.

The material orientation optimization presented in this work is limited to elastic anisotropic material properties. There are of course a number of other factors which have to be considered in the optimization of the build direction for AM parts, such as support structures, build height, orientation dependent surface roughness, orientation dependent fatigue properties, efficient cooling, residual stresses etc. If the build direction is used as a parameter in the TO formulation, as many as possible of these factors should be considered.
References


[14] T. J. R. Hughes, The Finite Element Method: Linear Static and Dynamic Analysis, New York:


