COUPLING REQUIREMENTS FOR MULTIPHYSICS PROBLEMS POSED ON TWO DOMAINS*

FATEMEH GHASEMI[†] AND JAN NORDSTRÖM[†]

Abstract. We consider two hyperbolic systems in first order form of different size posed on two domains. Our ambition is to derive general conditions for when the two systems can and cannot be coupled. The adjoint equations are derived and well-posedness of the primal and dual problems is discussed. By applying the energy method, interface conditions for the primal and dual problems are derived such that the continuous problems are well posed. The equations are discretized using a high order finite difference method in summation-by-parts form and the interface conditions are imposed weakly in a stable way, using penalty formulations. It is shown that one specific choice of penalty matrices leads to a dual consistent scheme. By considering an example, it is shown that the correct physical coupling conditions are contained in the set of well posed coupling conditions. It is also shown that dual consistency leads to superconverging functionals and reduced stiffness.

Key words. well posed problems, high order finite differences, stability, summation-by-parts, weak interface conditions, dual consistency, stiffness, superconvergence

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1. Introduction. Roughly speaking, a well posed initial boundary value problem requires that a unique solution estimated in terms of data exists. The most common procedure for showing well-posedness is the so-called energy method. In this method, one multiplies the governing partial differential equations (PDEs) with the solution, integrates by parts, and imposes a minimal number of boundary conditions, such that an energy estimate is obtained [9, 15, 25]. This procedure leads to a well posed problem. However, for the coupling of multiphysics problems [8, 17, 22, 23, 33, 27] at an interface, this procedure is less well known. The reason for that is the somewhat more unclear nature of coupling conditions compared to boundary conditions [12, 17, 22, 23, 26], even though there are similarities. Well-posedness, especially related to the coupling of multiphysics problems, is discussed in [19, 20, 23, 9, 15].

First, accuracy relations must exist such that a combination of variables for one set of PDEs at the interface is equal to another combination of variables for the other set. These sets of variables must also have the same physical properties. For an illustrating example, see [23]. Second, the number of accuracy relations must fit both problems. Too many conditions ruin existence and too few ruin uniqueness. If the number of accuracy relations is too low, additional conditions requiring external data must be added. If the number of accuracy relations is too high, only a subset can be used. Third, the accuracy relations must be such that no artificial growth or decay is generated.

Sensitivity analysis for systems of PDEs is often used in optimization, parameter estimation, model simplification, data assimilation, optimal control, uncertainty analysis, and experimental design [13, 14]. For problems involving a large number of sensitivity parameters, the adjoint method is the most efficient [16]. In this paper,

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[†]Computational Mathematics, Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden (fatemeh.ghasemi@liu.se, jan.nordstrom@liu.se).

we derive the adjoint equations and interface conditions for these types of problems, such that the adjoint (or dual) problem is also well posed and stable.

It has been shown that dual consistent summation-by-parts (SBP) discretizations lead to superconvergent functionals [10]. Additionally, in [2, 3, 4], it was shown that the dual boundary conditions depend on the primal boundary conditions and vice versa. The fact that the dual and primal boundary conditions are mutually dependent leads to new types of dual consistent boundary conditions. We will use the same technique and derive dual consistent interface conditions. The main goal of this paper is to derive well posed dual interface conditions and a stable dual consistent scheme.

The rest of the paper proceeds as follows. In section 2, the interface conditions, well-posedness, and stability of the primal problem are derived. The dual problem, its interface conditions, well-posedness, and stability are presented in section 3. We analyze dual consistency for the discrete problems in section 4. Section 5 contains an example of the theory developed here and the numerical results for both the primal and dual problems. Finally, in section 6 we summarize and draw conclusions.

2. The primal problem. A typical example of problems we consider in this paper is the coupling of the linearized symmetrized Euler equations for fluid flow

$$(1) \quad u_t + Au_x = 0, \quad -1 < x < 0,$$

to the wave equation for the deformation of a solid,

$$(2) \quad v_{tt} - c_s^2 v_{xx} = 0, \quad 0 < x < 1.$$

The matrix and vector in (1) are

$$(3) \quad A = \begin{bmatrix} \bar{w} & c_f/\sqrt{\gamma} & 0 \\ c_f/\sqrt{\gamma} & \bar{w} & c_f\sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & c_f\sqrt{\frac{\gamma-1}{\gamma}} & \bar{w} \end{bmatrix} \text{ and } u = \left[\frac{c_f\rho}{\sqrt{\gamma}\bar{\rho}}, w, \frac{-c_f\rho}{\bar{\rho}\sqrt{\gamma(\gamma-1)}} + \frac{p\sqrt{\gamma/(\gamma-1)}}{\bar{\rho}c_f} \right]^T,$$

respectively. In (3), ρ, w, p , and c_f are respectively the density, the velocity, the pressure perturbation, and the speed of sound. The solution we have linearized around is denoted by overbars and γ is the ratio of specific heats [1]. In (2), v is the displacement and $c_s = \sqrt{E/\rho_s}$ is wave speed, where E is the elastic modulus and ρ_s is the density of solid. The subscripts f and s refer to the values from the fluid and the solid, respectively.

The second order equation (2) can be rewritten as a first order system given by

$$(4) \quad U_t + BU_x = 0, \quad B = \begin{bmatrix} 0 & c_s \\ c_s & 0 \end{bmatrix},$$

where $U = [v_t, -c_s v_x]^T$ and v_t and v_x are the velocity and the stress, respectively.

The possibility to couple problems like (1) and (2) (or (4)) is the main topic in this paper.

2.1. The general formulation. We will consider interface conditions for the hyperbolic systems

$$(5) \quad \begin{aligned} u_t + Au_x &= 0, & -1 < x < 0, & \quad t > 0, \\ u(x, 0) &= f(x), & -1 < x < 0, & \quad t = 0, \end{aligned}$$

and

$$(6) \quad \begin{aligned} v_t + Bv_x &= 0, & 0 < x < 1, \quad t > 0, \\ v(x, 0) &= g(x), & 0 < x < 1, \quad t = 0. \end{aligned}$$

In (5)–(6), A, B are $m \times m$ and $n \times n$ symmetric constant matrices, respectively, and u, v are unknown vectors of sizes m and n . In general we have $m \neq n$. For clarity we often ignore the boundary conditions at $x = \pm 1$ and focus on the interface conditions at $x = 0$.

2.2. The interface conditions and well-posedness. The following definition of well-posedness is used.

DEFINITION 2.1. *Consider the coupled problem (5)–(6) with homogeneous boundary conditions and a minimal number of boundary and coupling conditions. The coupled problem is well posed if*

$$\frac{d}{dt}(\|u\|_2^2 + \alpha_c^p \|v\|_2^2) \leq 0,$$

where α_c^p is a positive free weight, $\|u\|_2^2 = \int_{-1}^0 u^T u \, dx$, and $\|v\|_2^2 = \int_0^1 v^T v \, dx$.

We apply the energy method by multiplying (5) and (6) with u^T and v^T , respectively, integrating in space and adding them together. The result is

$$(7) \quad \frac{d}{dt}(\|u\|_2^2 + \alpha_c^p \|v\|_2^2) = w^T E w|_{x=0}.$$

In (7), $w = [u, v]^T$ and $E = \text{diag}(-A, \alpha_c^p B)$. In the following, all terms are evaluated at $x = 0$, unless stated otherwise.

We denote $k = k_A^- + k_B^+$ as the number of positive eigenvalues of E , where k_A^-, k_B^+ are the number of positive eigenvalues of $-A$ and B , respectively. This means that exactly k interface (or accuracy) conditions are required for a well posed problem. Let the interface conditions be described by

$$(8) \quad C^p u = D^p v,$$

where the matrices C^p and D^p have k linearly independent rows. We will seek matrices C^p and D^p such that the coupled problem (5)–(6) is well posed according to Definition 2.1. We will use the same technique as in [9, 15] to find these matrices.

Since A and B are symmetric, we have

$$(9) \quad \begin{aligned} A &= X \Lambda_A X^T, \quad X = [X_+, X_-], \quad \Lambda_A = \begin{bmatrix} \Lambda_A^+ & 0 \\ 0 & \Lambda_A^- \end{bmatrix}, \\ B &= Y \Lambda_B Y^T, \quad Y = [Y_+, Y_-], \quad \Lambda_B = \begin{bmatrix} \Lambda_B^+ & 0 \\ 0 & \Lambda_B^- \end{bmatrix}. \end{aligned}$$

In (9), X_+, X_- and Y_+, Y_- are column matrices containing the eigenvectors related to the positive and negative eigenvalues of A and B , respectively. The diagonal block matrices Λ_A^+, Λ_A^- and Λ_B^+, Λ_B^- contain the positive and negative eigenvalues of A and B , respectively. For simplicity (and without restriction) we assume that there are no zero eigenvalues.

By using (9), the relation (7) can be reformulated as

$$\begin{aligned} \frac{d}{dt}(\|u\|_2^2 + \alpha_c^p \|v\|_2^2) &= -(X^T u)^T \Lambda_A (X^T u) + \alpha_c^p (Y^T v)^T \Lambda_B (Y^T v) \\ &= -[(X_+^T u)^T \Lambda_A^+ (X_+^T u) + (X_-^T u)^T \Lambda_A^- (X_-^T u)] \\ &\quad + \alpha_c^p [(Y_+^T v)^T \Lambda_B^+ (Y_+^T v) + (Y_-^T v)^T \Lambda_B^- (Y_-^T v)]. \end{aligned} \quad (10)$$

We consider interface conditions of the general forms

$$\begin{aligned} (\Lambda_A^- X_-^T - R_A X_+^T) u &= (T_A Y_-^T + S_A Y_+^T) v, \\ (\Lambda_B^+ Y_+^T - R_B Y_-^T) v &= (T_B X_+^T + S_B X_-^T) u. \end{aligned} \quad (11)$$

The matrices R_A, T_A, S_A, R_B, T_B , and S_B are of sizes $k_A^- \times k_A^+, k_A^- \times k_B^-, k_A^- \times k_B^+, k_B^+ \times k_B^-, k_B^+ \times k_A^+$, and $k_B^+ \times k_A^-$, respectively, where k_A^+ and k_B^- are the number of negative eigenvalues of $-A$ and B , respectively.

2.2.1. Strongly imposed interface conditions. By inserting (11) into (10) we obtain

$$\frac{d}{dt}(\|u\|_2^2 + \alpha_c^p \|v\|_2^2) = - \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}^T \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ (m_{12})^T & m_{22} & m_{23} & m_{24} \\ (m_{13})^T & (m_{23})^T & m_{33} & m_{34} \\ (m_{14})^T & (m_{24})^T & (m_{34})^T & m_{44} \end{bmatrix}}_M \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}, \quad (12)$$

where

$$\begin{aligned} m_{11} &= \Lambda_A^+ + R_A^T (\Lambda_A^-)^{-1} R_A - \alpha_c^p T_B^T (\Lambda_B^+)^{-1} T_B, & m_{23} &= 0, \\ m_{12} &= \alpha_c^p T_B^T (\Lambda_B^-)^{-1} S_B, & m_{24} &= \alpha_c^p S_B^T (\Lambda_B^+) R_B, \\ m_{13} &= R_A^T (\Lambda_A^-)^{-1} S_A, & m_{33} &= S_A^T (\Lambda_A^-)^{-1} S_A, \\ m_{14} &= R_A^T (\Lambda_A^-)^{-1} T_A - \alpha_c^p T_B^T (\Lambda_B^+)^{-1} R_B, & m_{34} &= S_A^T (\Lambda_A^-)^{-1} T_A, \\ m_{22} &= -\alpha_c^p S_B^T (\Lambda_B^+)^{-1} S_B, & m_{44} &= -\alpha_c^p (\Lambda_B^- + R_B^T (\Lambda_B^+)^{-1} R_B) \\ & & & \quad + T_A^T (\Lambda_A^-)^{-1} T_A. \end{aligned}$$

To obtain a well posed problem, the matrices $R_A, T_A, S_A, R_B, T_B, S_B$, and α_c^p must be chosen such that M in (12) is positive semidefinite.

PROPOSITION 2.2. *If $S_A \neq 0$ or $S_B \neq 0$, then the coupled problem (5)–(6) is not well posed.*

Proof. If $S_A \neq 0$ or $S_B \neq 0$, then the diagonal elements m_{22} or m_{33} are negative and consequently the matrix M in (12) cannot be positive semidefinite. $\square$

From now on we consider $S_A = S_B = 0$, which leads to the interface conditions

$$(\Lambda_A^- X_-^T - R_A X_+^T) u = T_A Y_-^T v, \quad (\Lambda_B^+ Y_+^T - R_B Y_-^T) v = T_B X_+^T u, \quad (13)$$

which can be rewritten as

$$\begin{bmatrix} -\Lambda_A^- & 0 \\ 0 & \Lambda_B^+ \end{bmatrix} \begin{bmatrix} -X_-^T & 0 \\ 0 & Y_+^T \end{bmatrix} - \begin{bmatrix} -R_A & T_A \\ -T_B & R_B \end{bmatrix} \begin{bmatrix} -X_+^T & 0 \\ 0 & Y_-^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \quad (14)$$

In (13) (or (14)), we specify the ingoing characteristic variables in terms of the outgoing ones and incoming data.

Remark 2.3. The form (13) automatically gives the correct number $k = k_A^- + k_B^+$ of interface conditions.

For future reference we write the final coupled problem as

$$(15) \quad \begin{aligned} u_t + Au_x &= 0, & -1 < x < 0, \quad t > 0, \\ u(x, 0) &= f(x), & -1 < x < 0, \quad t = 0, \\ v_t + Bv_x &= 0, & 0 < x < 1, \quad t > 0, \\ v(x, 0) &= g(x), & 0 < x < 1, \quad t = 0, \\ C^p u(0, t) &= D^p v(0, t), & x = 0, \quad t > 0, \end{aligned}$$

where

$$C^p = \begin{bmatrix} T_B X_+^T \\ \Lambda_A^- X_-^T - R_A X_+^T \end{bmatrix}, \quad D^p = \begin{bmatrix} \Lambda_B^+ Y_+^T - R_B Y_-^T \\ T_A Y_-^T \end{bmatrix}.$$

Now, (12) can be rewritten as

$$(16) \quad \frac{d}{dt} (\|u\|_2^2 + \alpha_c^p \|v\|_2^2) = - \begin{bmatrix} X_+^T u \\ Y_-^T v \end{bmatrix}^T \underbrace{\begin{bmatrix} m_{11}^p & m_{12}^p \\ (m_{12}^p)^T & m_{22}^p \end{bmatrix}}_{M^p} \begin{bmatrix} X_+^T u \\ Y_-^T v \end{bmatrix},$$

where

$$(17) \quad m_{11}^p = m_{11}, \quad m_{12}^p = m_{14}, \quad m_{22}^p = m_{44}.$$

In the following proposition, we will find matrices R_A, R_B, T_A , and T_B such that the coupled problem (15) leads to an energy estimate.

PROPOSITION 2.4. *If the matrices R_A, R_B, T_A, T_B and the parameter α_c^p are chosen such that*

$$(18) \quad \begin{aligned} \Lambda_A^+ + R_A^T (\Lambda_A^-)^{-1} R_A &> 0, & T_B^T (\Lambda_B^+)^{-1} T_B &\leq (\Lambda_A^+ + R_A^T (\Lambda_A^-)^{-1} R_A) / \alpha_c^p, \\ \Lambda_B^- + R_B^T (\Lambda_B^+)^{-1} R_B &< 0, & T_A^T (\Lambda_A^-)^{-1} T_A &\geq \alpha_c^p (\Lambda_B^- + R_B^T (\Lambda_B^+)^{-1} R_B), \end{aligned}$$

and

$$(19) \quad R_A^T (\Lambda_A^-)^{-1} T_A = \alpha_c^p T_B^T (\Lambda_B^+)^{-1} R_B,$$

then, the matrix M^p in (16) is positive semidefinite and an energy estimate is obtained for (15).

Proof. Since (19) is satisfied, $m_{12}^p = 0$ and the matrix M^p has the form

$$\begin{bmatrix} m_{11}^p & 0 \\ 0 & m_{22}^p \end{bmatrix},$$

which is positive semidefinite due to the conditions in (18). $\square$

Remark 2.5. Note that if $m_{11}^p = 0$ or $m_{22}^p = 0$ and $m_{12}^p \neq 0$, the coupling cannot be done.

Remark 2.6. By replacing x with $-x$ for $-1 \leq x \leq 0$, the coupled problem (5)–(6) can be considered as the initial boundary value problem

$$(20) \quad \begin{aligned} w_t + Gw_x &= 0, \quad 0 \leq x \leq 1, \\ w(x, 0) &= [f(x), g(x)]^T, \end{aligned}$$

where $w = [u, v]^T$ and $G = \text{diag}(-A, B)$. Boundary conditions for the system (20) at $x = 0$ will be the interface conditions for the coupled problem (5)–(6). By applying the same technique as in [9, 15], we can derive the boundary conditions

$$(21) \quad (\Lambda_G^+ Z_+^T - R Z_-^T) w = 0, \quad Z_+ = \begin{bmatrix} -X_- & 0 \\ 0 & Y_+ \end{bmatrix}, \quad Z_- = \begin{bmatrix} -X_+ & 0 \\ 0 & Y_- \end{bmatrix},$$

at $x = 0$. In (21), $\Lambda_G^+ = \text{diag}(-\Lambda_A^-, \Lambda_B^+)$ and R is a 2×2 block matrix. If we choose the matrix R as

$$(22) \quad R = \begin{bmatrix} -R_A & T_A \\ -T_B & R_B \end{bmatrix},$$

then the conditions (14) and (21) are equivalent.

2.2.2. Weakly imposed interface conditions. In order to prepare for the numerical approximation, we impose the interface condition (8) weakly. The result is

$$(23) \quad \begin{aligned} \frac{d}{dt} (\|u\|_2^2 + \alpha_c^p \|v\|_2^2) &= -u^T A u + \alpha_c^p v^T B v \\ &+ u^T \Sigma_L (C^p u - D^p v) + (u^T \Sigma_L (C^p u - D^p v))^T \\ &+ \alpha_c^p \left[v^T \Sigma_R (D^p v - C^p u) + (v^T \Sigma_R (D^p v - C^p u))^T \right], \end{aligned}$$

where Σ_L and Σ_R are penalty matrices of sizes $m \times k$ and $n \times k$, respectively. The relation (23) can be rewritten in matrix form as

$$(24) \quad \frac{d}{dt} (\|u\|_2^2 + \alpha_c^p \|v\|_2^2) = - \begin{bmatrix} u \\ v \end{bmatrix}^T \underbrace{\begin{bmatrix} n_{11}^p & n_{12}^p \\ (n_{12}^p)^T & n_{22}^p \end{bmatrix}}_{N^p} \begin{bmatrix} u \\ v \end{bmatrix},$$

where

$$(25) \quad \begin{aligned} n_{11}^p &= A - \Sigma_L C^p - (\Sigma_L C^p)^T, \quad n_{12}^p = \Sigma_L D^p + \alpha_c^p (\Sigma_R C^p)^T, \\ n_{22}^p &= -\alpha_c^p (B + \Sigma_R D^p + (\Sigma_R D^p)^T). \end{aligned}$$

Next, we must find the matrices Σ_L and Σ_R such that the matrix N^p in (24) is positive semidefinite. Let

$$\tilde{N}^p = \begin{bmatrix} I & E^p \\ 0 & I \end{bmatrix}^T N^p \begin{bmatrix} I & E^p \\ 0 & I \end{bmatrix},$$

and note that if $\tilde{N}^p$ is positive semidefinite, so is N^p . The choice $E^p = -(n_{11}^p)^{-1} n_{12}^p$ gives

$$\tilde{N}^p = \begin{bmatrix} n_{11}^p & 0 \\ 0 & -(n_{12}^p)^T (n_{11}^p)^{-1} n_{12}^p + n_{22}^p \end{bmatrix}.$$

By choosing Σ_L and Σ_R such that

$$(26) \quad n_{11}^p > 0, \quad -(n_{12}^p)^T (n_{11}^p)^{-1} n_{12}^p + n_{22}^p \geq 0,$$

then the right-hand side of (24) is bounded.

The above procedure is formalized as follows.

PROPOSITION 2.7. *By choosing the penalty matrices Σ_L and Σ_R such that (26) is satisfied, the coupled problem (15) is well posed.*

In the following proposition, we specify special choices of matrices Σ_L and Σ_R , which lead to well-posedness. In section 4, it will be shown that the discrete approximation of (15) is dual consistent for these specific choices.

PROPOSITION 2.8. *If the matrices R_A, T_A, R_B, T_B , and α_c^p satisfy (18) and (19), then the matrices*

$$(27) \quad \Sigma_L = X \begin{bmatrix} 0 & 0 \\ 0 & I_A^- \end{bmatrix}, \quad \Sigma_R = -Y \begin{bmatrix} I_B^+ & 0 \\ 0 & 0 \end{bmatrix}$$

guarantee that (26) is satisfied. In (27), I_A^- and I_B^+ are identity matrices of size k_A^- and k_B^+ , respectively.

Proof. By inserting the matrices Σ_L and Σ_R into (25) we obtain

$$N^p = \begin{bmatrix} X_+^T \\ X_-^T \\ Y_+^T \\ Y_-^T \end{bmatrix}^T \bar{N}^p \begin{bmatrix} X_+^T \\ X_-^T \\ Y_+^T \\ Y_-^T \end{bmatrix}, \quad \text{where } \bar{N}^p = \begin{bmatrix} \Lambda_A^+ & R_A^T & -\alpha_c^p T_B^T & 0 \\ R_A & -\Lambda_A^- & 0 & T_A \\ -\alpha_c^p T_B & 0 & \alpha_c^p \Lambda_B^+ & -\alpha_c^p R_B \\ 0 & T_A^T & -\alpha_c^p R_B^T & -\alpha_c^p \Lambda_B^- \end{bmatrix}.$$

Note that the interface terms in (16) can be rewritten as

$$- \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}^T \tilde{M}^p \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}, \quad \text{where } \tilde{M}^p = \begin{bmatrix} m_{11}^p & 0 & 0 & m_{12}^p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (m_{12}^p)^T & 0 & 0 & m_{22}^p \end{bmatrix}.$$

The matrix $\tilde{M}^p$ is positive semidefinite due to the result obtained for the strong interface conditions in Proposition 2.4. To take advantage of this, we rewrite the matrix $\bar{N}^p$ as $\bar{N}^p = \hat{M}^p + \hat{N}^p$, where

$$\hat{N}^p = \begin{bmatrix} -R_A^T (\Lambda_A^-)^{-1} R_A & R_A^T & 0 & -R_A^T (\Lambda_A^-)^{-1} T_A \\ R_A & -\Lambda_A^- & 0 & T_A \\ 0 & 0 & 0 & 0 \\ -T_A^T (\Lambda_A^-)^{-1} R_A & T_A^T & 0 & -T_A^T (\Lambda_A^-)^{-1} T_A \end{bmatrix} \\ + \alpha_c^p \begin{bmatrix} +T_B^T (\Lambda_B^+)^{-1} T_B & 0 & -T_B^T & +T_B^T (\Lambda_B^+)^{-1} R_B \\ 0 & 0 & 0 & 0 \\ -T_B & 0 & \Lambda_B^+ & -R_B \\ +R_B^T (\Lambda_B^+)^{-1} T_B & 0 & -R_B^T & +R_B^T (\Lambda_B^+)^{-1} R_B \end{bmatrix}.$$

The matrix $\hat{N}^p$ which is due to the use of weak interface conditions can be rewritten as

$$\hat{N}^p = L_A (C_A \otimes (\Lambda_A^-)^{-1}) L_A^T + \alpha_c^p L_B (C_B \otimes (\Lambda_B^+)^{-1}) L_B^T,$$

where

$$L_A = \begin{bmatrix} R_A & 0 & 0 & 0 \\ 0 & \Lambda_A^- & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T_A \end{bmatrix}, \quad C_A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \end{bmatrix},$$

$$L_B = \begin{bmatrix} T_B & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \Lambda_B^+ & 0 \\ 0 & 0 & 0 & R_B \end{bmatrix}, \quad C_B = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 \end{bmatrix},$$

and $\otimes$ denotes the Kronecker product [11]. The eigenvalues of the matrices C_A and C_B are $\{-3, 0, 0, 0\}$, which implies that the matrix $\hat{N}^p$ is positive semidefinite. $\square$

The difference between the right-hand side in the strong estimate (16) and the right-hand side in the weak estimate (24) is

$$R = - \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}^T \hat{N}^p \begin{bmatrix} X_+^T u \\ X_-^T u \\ Y_+^T v \\ Y_-^T v \end{bmatrix}.$$

We can expand the term R by using

$$C_A = X_A \Lambda_C^A X_A^T, \quad X_A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \Lambda_C^A = \text{diag}([-3, 0, 0, 0]),$$

$$C_B = X_B \Lambda_C^B X_B^T, \quad X_B = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \Lambda_C^B = \Lambda_C^A,$$

and we find that

$$R = - \begin{bmatrix} R_A X_+^T u \\ \Lambda_A^- X_-^T u \\ Y_+^T v \\ T_A Y_-^T v \end{bmatrix}^T (X_A \Lambda_C^A X_A^T \otimes (\Lambda_A^-)^{-1}) \begin{bmatrix} R_A X_+^T u \\ \Lambda_A^- X_-^T u \\ Y_+^T v \\ T_A Y_-^T v \end{bmatrix}$$

$$- \alpha_c^p \begin{bmatrix} T_B X_+^T u \\ X_-^T u \\ \Lambda_B^+ Y_+^T v \\ R_B Y_-^T v \end{bmatrix}^T (X_B \Lambda_C^B X_B^T \otimes (\Lambda_B^+)^{-1}) \begin{bmatrix} T_B X_+^T u \\ X_-^T u \\ \Lambda_B^+ Y_+^T v \\ R_B Y_-^T v \end{bmatrix}$$

$$= (C^p u - D^p v)^T \Lambda^{-1} (C^p u - D^p v), \quad \Lambda = \text{diag}(\Lambda_A^-, -\alpha_c^p \Lambda_B^+).$$

Remark 2.9. The additional seemingly dissipative term R in the weak energy rate is proportional to the interface condition (13) squared and is obviously zero. A non-zero truly dissipative term of the same form will appear in the discrete approximation.

2.3. The semidiscrete primal problem. We now consider finite difference approximations of (15) in SBP form [6, 18, 28]. The interface conditions are implemented using simultaneous approximation terms (SAT) as described in [7, 24, 29, 31].

The semidiscrete SBP-SAT formulation of (15) is

$$(28) \quad \begin{aligned} \mathbf{u}_t + (D_u \otimes A)\mathbf{u} &= P_u^{-1} e_N \otimes \Sigma_L (C^p u_N - D^p v_0), \\ \mathbf{v}_t + (D_v \otimes B)\mathbf{v} &= P_v^{-1} e_0 \otimes \Sigma_R (D^p v_0 - C^p u_N), \end{aligned}$$

where the outer boundary conditions are ignored as in the continuous case. The discrete solutions are arranged as

$$\mathbf{u} = (u_{01}, \dots, u_{0m}, \dots, u_{N1}, \dots, u_{Nm})^T, \quad \mathbf{v} = (v_{01}, \dots, v_{0n}, \dots, v_{M1}, \dots, v_{Mn})^T,$$

where $u_N = [u_{N1}, \dots, u_{Nm}]^T$, $v_0 = [v_{01}, \dots, v_{0n}]^T$. In (28), $D_{u,v} = P_{u,v}^{-1} Q_{u,v}$ are difference operators which approximate the first derivative, and the subscripts u, v denote the direction they operate along. The difference operators satisfy $P_{u,v} = P_{u,v}^T > 0$ and $Q_{u,v} + Q_{u,v}^T = \text{diag}([-1, 0, \dots, 0, 1])$. $e_N = (0, \dots, 1)^T$ and $e_0 = (1, \dots, 0)^T$ are $N+1$ and $M+1$ unit vectors, respectively. The penalty matrices Σ_L and Σ_R have the same dimensions as in the continuous case and will be chosen for stability. For a comprehensive review of the SBP-SAT technique, see [32].

2.3.1. Stability conditions at the interface. The discrete energy method is applied to (28) by multiplying the two equations with $\mathbf{u}^T(P_u \otimes I)$, $\mathbf{v}^T(P_v \otimes I)$, respectively, and adding the result. By defining the discrete norms $\|\mathbf{u}\|_{P_u \otimes I}^2 = \mathbf{u}^T(P_u \otimes I)\mathbf{u}$, $\|\mathbf{v}\|_{P_v \otimes I}^2 = \mathbf{v}^T(P_v \otimes I)\mathbf{v}$, using the symmetry properties of A, B and the SBP property of $Q_{u,v}$, we obtain

$$(29) \quad \begin{aligned} \frac{d}{dt} (\|\mathbf{u}\|_{P_u \otimes I}^2 + \alpha_d^p \|\mathbf{v}\|_{P_v \otimes I}^2) &= -u_N^T A u_N + \alpha_d^p v_0^T B v_0 \\ &\quad + u_N^T \Sigma_L (C^p u_N - D^p v_0) + (u_N^T \Sigma_L (C^p u_N - D^p v_0))^T \\ &\quad + \alpha_d^p \left[v_0^T \Sigma_R (D^p v_0 - C^p u_N) + (v_0^T \Sigma_R (D^p v_0 - C^p u_N))^T \right], \end{aligned}$$

which is similar to the continuous (weak) energy rate in (23).

The relation (29) can be rewritten in the matrix form

$$(30) \quad \frac{d}{dt} (\|\mathbf{u}\|_{P_u \otimes I}^2 + \alpha_d^p \|\mathbf{v}\|_{P_v \otimes I}^2) = - \begin{bmatrix} u_N \\ v_0 \end{bmatrix}^T \mathbf{N}^p \begin{bmatrix} u_N \\ v_0 \end{bmatrix},$$

where we find that $\mathbf{N}^p = N^p$ in (24) by letting $\alpha_d^p = \alpha_c^p$.

We immediately arrive at the following proposition.

PROPOSITION 2.10. *By choosing the penalty matrices Σ_L and Σ_R such that (26) is satisfied, the semidiscrete approximation (28) of the coupled problem (15) is stable.*

Similar to the continuous case, we will specify a set of penalty matrices which leads to stability.

PROPOSITION 2.11. *If the matrices R_A, T_A, R_B, T_B , and α_d^p satisfy (18), and (19), then the matrices*

$$(31) \quad \Sigma_L = X \begin{bmatrix} 0 & 0 \\ 0 & I_A^- \end{bmatrix}, \quad \Sigma_R = -Y \begin{bmatrix} I_B^+ & 0 \\ 0 & 0 \end{bmatrix}$$

satisfy (26).

Proof. See the proof of Proposition 2.8. □

Remark 2.12. The derivation in the discrete case is analogous to the continuous one above due to the mimicking properties of the SBP-SAT technique. In fact, the interface conditions and penalty matrices are already derived in the continuous setting; see [21] for more details on this technique.

The discrete energy rate (30) is similar to the continuous one with the additional term

$$\mathbf{R} = (C^p u_N - D^p v_0)^T (\Lambda)^{-1} (C^p u_N - D^p v_0), \quad \Lambda = \text{diag}(\Lambda_A^-, -\alpha_d^p \Lambda_B^+),$$

added onto the right-hand side. This term was identically zero in the continuous case but now adds a small amount of dissipation, since u_N and v_0 are approximations of $u(0, t)$ and $v(0, t)$. As we refine the mesh, the term $C^p u_N - D^p v_0$ goes to zero and the additional dissipation vanishes.

3. The dual problem. Consider the linear functional

$$J(w) = (u, h)_L + (v, l)_R,$$

where h and l are weight functions and $(u, h)_L = \int_{-1}^0 u^T h \, dx$, $(v, l)_R = \int_0^1 v^T l \, dx$. To derive the dual problem, we add forcing functions f_L, f_R to the right-hand sides of (5)–(6) and seek functions ϕ and ψ such that

$$\int_0^T J(w) dt = \int_0^T (\phi, f_L)_L + (\psi, f_R)_R \, dt.$$

As an initial step, we observe that

$$\begin{aligned} \int_0^T J(w) \, dt &= \int_0^T J(w) \, dt \\ &\quad - \int_0^T (\phi, u_t + Au_x - f_L)_L \, dt - \int_0^T (\psi, v_t + Bv_x - f_R)_R \, dt. \end{aligned}$$

Next, we use integration by parts to get

$$\begin{aligned} \int_0^T J(w) dt &= \int_0^T (\phi, f_L)_L + (\psi, f_R)_R \, dt \\ (32) \quad &+ \int_0^T (u, \phi_t + A\phi_x + h)_L \, dt + \int_0^T (v, \psi_t + B\psi_x + l)_R \, dt \\ &- \int_{-1}^0 [\phi^T u]_0^T \, dx - \int_0^1 [\psi^T v]_0^T \, dx - \int_0^T \left([\phi^T Au]_{-1}^0 + [\psi^T Bv]_0^1 \right) dt. \end{aligned}$$

The dual boundary and interface conditions are the minimal number of conditions such that

$$(33) \quad \int_0^T \left([\phi^T Au]_{-1}^0 + [\psi^T Bv]_0^1 \right) dt = 0.$$

In the next subsection, explicit interface conditions will be derived. By choosing homogeneous initial and final conditions

$$u(x, 0) = v(x, 0) = \phi(x, T) = \psi(x, T) = 0$$

for the primal and dual problem, the terms $\int_{-1}^0 [\phi^T u]_0^T \, dx$ and $\int_0^1 [\psi^T v]_0^T \, dx$ in (32) vanish, and the following dual equations are obtained:

$$(34) \quad \begin{aligned} -\phi_t - A\phi_x &= h, & -1 < x < 0, \\ -\psi_t - B\psi_x &= l, & 0 < x < 1. \end{aligned}$$

3.1. The dual boundary and interface conditions. The interface conditions (13) grouped together with the interface terms in (33) yield

$$0 = (\phi^T A u - \psi^T B v) = (\phi^T X_+ \Lambda_A^+ + \phi^T X_- R_A - \psi^T Y_+ T_B) X_+^T u \\ + (\phi^T X_- T_A - \psi^T Y_+ R_B - \psi^T Y_- \Lambda_B^-) Y_-^T v$$

and arrive at the dual interface conditions

$$(35) \quad (\Lambda_A^+ X_+^T + R_A^T X_-^T) \phi = T_B^T Y_+^T \psi, \quad (\Lambda_B^- Y_-^T + R_B^T Y_+^T) \psi = T_A^T X_-^T \phi.$$

Remark 3.1. Note that the dual interface conditions are given by the primal ones. See [2, 3, 4] for similar effects regarding boundary conditions.

The dual interface condition (35) can be written more compactly as

$$(36) \quad C^a \phi = D^a \psi,$$

where

$$C^a = \begin{bmatrix} \Lambda_A^+ X_+^T + R_A^T X_-^T \\ T_A^T X_-^T \end{bmatrix}, \quad D^a = \begin{bmatrix} T_B^T Y_+^T \\ \Lambda_B^- Y_-^T + R_B^T Y_+^T \end{bmatrix}.$$

For completeness, the dual boundary conditions must also be determined such that

$$(37) \quad \phi^T A u|_{x=-1} = 0, \quad \psi^T B v|_{x=+1} = 0.$$

To choose dual boundary conditions, the boundary conditions for the primal problem must exist. We consider the following general homogeneous boundary conditions for the primal problem:

$$(\Lambda_A^+ X_+^T - R_l X_-^T) u(-1, t) = 0, \quad (\Lambda_B^- Y_-^T - R_r Y_+^T) v(+1, t) = 0,$$

where the matrices R_l and R_r are such that a well posed primal problem is obtained. This yields

$$\phi^T A u|_{x=-1} = (\phi^T X_+ R_l + \phi^T X_- \Lambda_A^-) X_-^T u|_{x=-1}, \\ \psi^T B v|_{x=+1} = (\psi^T Y_+ \Lambda_B^+ + \psi^T Y_- R_r) Y_+^T v|_{x=+1},$$

and we choose the dual boundary conditions

$$(\Lambda_A^- X_-^T + R_l^T X_+^T) \phi(-1, t) = 0, \quad (\Lambda_B^+ Y_+^T + R_r^T Y_-^T) \psi(+1, t) = 0,$$

such that all boundary terms vanish and (37) is satisfied.

The initial conditions for the dual problem are given at time $t = T$. The time transformation $\tau = t - T$ inserted into (34) results in

$$(38) \quad \begin{aligned} \phi_t - A \phi_x &= h, & -1 < x < 0, \quad \tau > 0, \\ \phi(x, 0) &= 0, & -1 < x < 0, \quad \tau = 0, \\ \psi_t - B \psi_x &= l, & 0 < x < 1, \quad \tau > 0, \\ \psi(x, 0) &= 0, & 0 < x < 1, \quad \tau = 0, \\ C^a \phi(0, \tau) &= D^a \psi(0, \tau), & x = 0, \quad \tau > 0, \end{aligned}$$

which is the final form of the coupled dual problem. Note that the dual interface conditions $C^a \phi = D^a \psi$, given in (36), specify the ingoing characteristic variables in terms of outgoing ones and incoming data, just as in the primal problem.

3.1.1. Strongly imposed dual interface conditions. We apply the energy method to (38) and ignore the boundary terms as for the primal problem. By using the dual interface conditions (35), we get

$$(39) \quad \begin{aligned} \frac{d}{d\tau}(\|\phi\|_2^2 + \alpha_c^a \|\psi\|_2^2) &= \phi^T A \phi|_{x=0} - \alpha_c^a \psi^T B \psi|_{x=0} \\ &= \begin{bmatrix} X_-^T \phi \\ Y_+^T \psi \end{bmatrix}^T \underbrace{\begin{bmatrix} m_{11}^a & m_{12}^a \\ (m_{12}^a)^T & m_{22}^a \end{bmatrix}}_{M^a} \begin{bmatrix} X_-^T \phi \\ Y_+^T \psi \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} m_{11}^a &= \Lambda_A^- + R_A(\Lambda_A^+)^{-1} R_A^T - \alpha_c^a T_A(\Lambda_B^-)^{-1} T_A^T, \\ m_{12}^a &= -R_A(\Lambda_A^+)^{-1} T_B^T + \alpha_c^a T_A(\Lambda_B^-)^{-1} R_B^T, \\ m_{22}^a &= -\alpha_c^a \Lambda_B^+ - \alpha_c^a R_B(\Lambda_B^-)^{-1} R_B^T + T_B(\Lambda_A^+)^{-1} T_B^T, \end{aligned}$$

and α_c^a is a positive weight.

In the following proposition, we will find the matrices R_A, R_B, T_A , and T_B such that the coupled problem (38) leads to an energy estimate.

PROPOSITION 3.2. *If the matrices R_A, R_B, T_A, T_B and the parameter α_c^a are chosen such that*

$$(40) \quad \begin{aligned} \Lambda_A^- + R_A(\Lambda_A^+)^{-1} R_A^T &< 0, & T_A(\Lambda_B^-)^{-1} T_A^T &\geq (\Lambda_A^- + R_A(\Lambda_A^+)^{-1} R_A^T) / \alpha_c^a, \\ \Lambda_B^+ + R_B(\Lambda_B^-)^{-1} R_B^T &> 0, & T_B(\Lambda_A^+)^{-1} T_B^T &\leq \alpha_c^a (\Lambda_B^+ + R_B(\Lambda_B^-)^{-1} R_B^T), \end{aligned}$$

and

$$(41) \quad R_A(\Lambda_A^+)^{-1} T_B^T = \alpha_c^a T_A(\Lambda_B^-)^{-1} R_B^T,$$

then, the matrix M^a in (39) is negative semidefinite and an energy estimate is satisfied for (38).

Proof. The proof proceeds as in Proposition 2.4. $\square$

3.1.2. Weakly imposed dual interface conditions. As in the primal problem, we impose the interface condition (36) weakly. The result is

$$(42) \quad \begin{aligned} \frac{d}{d\tau}(\|\phi\|_2^2 + \alpha_c^a \|\psi\|_2^2) &= \phi^T A \phi - \alpha_c^a \psi^T B \psi \\ &\quad + \phi^T \Sigma_L (C^a \phi - D^a \psi) + (\phi^T \Sigma_L (C^a \phi - D^a \psi))^T \\ &\quad + \alpha_c^a \left[\psi^T \Sigma_R (D^a \psi - C^a \phi) + (\psi^T \Sigma_R (D^a \psi - C^a \phi))^T \right], \end{aligned}$$

where the penalty matrices Σ_L and Σ_R are of size $m \times (k_A^+ + k_B^-)$ and $n \times (k_A^+ + k_B^-)$, respectively.

Remark 3.3. The derivation procedure below for the dual problem is the same as for the primal problem.

The relation (42) can be rewritten in the matrix form

$$(43) \quad \frac{d}{d\tau}(\|\phi\|_2^2 + \alpha_c^a \|\psi\|_2^2) = \begin{bmatrix} \phi \\ \psi \end{bmatrix}^T \underbrace{\begin{bmatrix} n_{11}^a & n_{12}^a \\ (n_{12}^a)^T & n_{22}^a \end{bmatrix}}_{N^a} \begin{bmatrix} \phi \\ \psi \end{bmatrix},$$

where

$$\begin{aligned} n_{11}^a &= A + \Sigma_L C^a + (\Sigma_L C^a)^T, \quad n_{12}^a = -\Sigma_L D^a - \alpha_c^a (\Sigma_R C^a)^T, \\ n_{22}^a &= -\alpha_c^a \left(B - \Sigma_R D^a - (\Sigma_R D^a)^T \right). \end{aligned}$$

By using the rotation technique, as in the primal problem, we arrive at the following.

PROPOSITION 3.4. *The choice of penalty matrices Σ_L and Σ_R such that*

$$(44) \quad n_{11}^a < 0, \quad -(n_{12}^a)^T (n_{11}^a)^{-1} n_{12}^a + n_{22}^a \leq 0,$$

holds leads to well-posedness of the coupled problem (38).

As in the primal problem, we specify special choices of matrices Σ_L and Σ_R , which lead to well-posedness.

PROPOSITION 3.5. *If the matrices R_A, T_A, R_B, T_B , and α_c^a satisfy (40) and (41), then the matrices*

$$(45) \quad \Sigma_L = -X \begin{bmatrix} I_A^+ & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_R = Y \begin{bmatrix} 0 & 0 \\ 0 & I_B^- \end{bmatrix}$$

guarantee that (44) holds. In (45), I_A^+ and I_B^- are identity matrices of size k_A^+ and k_B^- , respectively.

Proof. See the proof of Proposition 2.7. $\square$

3.2. The semidiscrete approximation of the dual problem. The corresponding semidiscrete SBP-SAT formulation of (38) is

$$(46) \quad \begin{aligned} \phi_\tau - (D_u \otimes A)\phi &= P_u^{-1} e_N \otimes \Sigma_L (C^a \phi_N - D^a \psi_0), \\ \psi_\tau - (D_v \otimes B)\psi &= P_v^{-1} e_0 \otimes \Sigma_R (D^a \psi_0 - C^a \phi_N), \end{aligned}$$

where the outer boundary conditions are ignored as in the primal case. The vectors ϕ_N and ψ_0 are arranged as $\phi_N = [\phi_{N1}, \dots, \phi_{Nm}]^T$, $\psi_0 = [\psi_{01}, \dots, \psi_{0n}]^T$. The penalty matrices Σ_L and Σ_R have the same dimensions as in the continuous case.

3.2.1. Stability conditions at the interface. Applying the discrete energy method to (46) leads to

$$(47) \quad \begin{aligned} \frac{d}{d\tau} (\|\phi\|_{P_u \otimes I}^2 + \alpha_d^a \|\psi\|_{P_v \otimes I}^2) &= \phi_N^T A \phi_N - \alpha_d^a \psi_0^T B \psi_0 \\ &+ \phi_N^T \Sigma_L (C^a \phi_N - D^a \psi_0) + (\phi_N^T \Sigma_L (C^a \phi_N - D^a \psi_0))^T \\ &+ \alpha_d^a \left[\psi_0^T \Sigma_R (D^a \psi_0 - C^a \phi_N) + (\psi_0^T \Sigma_R (D^a \psi_0 - C^a \phi_N))^T \right], \end{aligned}$$

which is similar to the continuous case (42). The relation (47) can be rewritten in the matrix form

$$\frac{d}{dt} (\|\mathbf{u}\|_{P_u \otimes I}^2 + \alpha_d^p \|\mathbf{v}\|_{P_v \otimes I}^2) = \begin{bmatrix} \phi_N \\ \psi_0 \end{bmatrix}^T \mathbf{N}^a \begin{bmatrix} \phi_N \\ \psi_0 \end{bmatrix},$$

where we find that $\mathbf{N}^a = N^a$ in (43) by letting $\alpha_a^p = \alpha_c^a$.

We immediately arrive at the following proposition.

PROPOSITION 3.6. *By choosing the penalty matrices Σ_L and Σ_R such that (44) is satisfied, the semidiscrete approximation (46) of the coupled problem (38) is stable.*

Similar to the continuous case, we will specify a set of penalty matrices which leads to stability.

PROPOSITION 3.7. *If the matrices R_A, T_A, R_B, T_B , and α_d^a satisfy (40) and (41), then the matrices*

$$(48) \quad \Sigma_L = -X \begin{bmatrix} I_A^+ & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_R = Y \begin{bmatrix} 0 & 0 \\ 0 & I_B^- \end{bmatrix}$$

guarantee that (44) holds and that (46) is stable.

Proof. See the proof of Proposition 2.8. $\square$

Remark 3.8. Just as in the primal problem, the penalty matrices in the discrete case are the same as in the continuous one, due to the similarity in analysis of the continuous and discrete problem.

4. Dual consistency. To investigate dual consistency, we rewrite (28) and (46) into the following form:

$$(49) \quad P\mathbf{w}_t + L^p \mathbf{w} = PF, \quad P\boldsymbol{\theta}_\tau + L^a \boldsymbol{\theta} = PH,$$

where

$$L^p = + \begin{bmatrix} Q_u \otimes A & 0 \\ 0 & Q_v \otimes B \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & -\Sigma_L C^p & \Sigma_L D^p & \\ & & \Sigma_R C^p & -\Sigma_R D^p & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix},$$

$$L^a = - \begin{bmatrix} Q_u \otimes A & 0 \\ 0 & Q_v \otimes B \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & -\Sigma_L C^a & \Sigma_L D^a & \\ & & \Sigma_R C^a & -\Sigma_R D^a & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix},$$

and

$$P^{-1} = \begin{bmatrix} P_u^{-1} \otimes I_m & 0 \\ 0 & P_v^{-1} \otimes I_n \end{bmatrix}.$$

In (49), $\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T$, $\boldsymbol{\theta} = [\boldsymbol{\phi}, \boldsymbol{\psi}]^T$, and $F = [F_L, F_R]$, $H = [H_L, H_R]$, where F_L, F_R, H_L and H_R are the discrete forms of f_L, f_R, h_L , and h_R , respectively.

Remark 4.1. The similarity between the second matrix in L^p and L^a is due to the similarity of the right-hand sides in (28) and (46).

The schemes in (49) are dual consistent [2, 10] if

$$(50) \quad (L^p)^T = L^a.$$

Since $Q_{u,v} + Q_{u,v}^T = E_N - E_0$, the requirement (50) leads to the condition

$$(51) \quad \begin{bmatrix} A - (C^p)^T \Sigma_L^T & (C^p)^T \Sigma_R^T \\ (D^p)^T \Sigma_L^T & -(D^p)^T \Sigma_R^T - B \end{bmatrix} = \begin{bmatrix} -\Sigma_L C^a & \Sigma_L D^a \\ \Sigma_R C^a & -\Sigma_R D^a \end{bmatrix}.$$

We find the following.

PROPOSITION 4.2. *If the penalty matrices $\Sigma_{L,R}$ and $\Sigma_{L,R}$ are chosen as in (31) and (48), respectively, then the SBP-SAT discretization of the coupled problem (15) is dual consistent*

Proof. A direct insertion of (31) and (48) into (51) yields the result. $\square$

Remark 4.3. By choosing the penalty matrices such that (26) is satisfied but (51) is not, the semidiscrete approximation (28) is stable but dual inconsistent.

Remark 4.4. We summarize what has been done so far below:

- Well posed interface conditions for the primal problem have been derived. The interface conditions are imposed and penalty matrices are obtained such that the continuous problem is well posed. The penalty matrices for the continuous problem lead directly to a stable discrete primal problem.
- By using the same strategy as in the primal problem, the interface conditions and penalty matrices for the continuous coupled dual problem are derived. These matrices lead to a stable discrete dual problem. The dual interface condition is determined by the primal interface condition.
- A specific set of penalty matrices for the primal and dual problem can be chosen such that the discrete problems are stable and the primal discrete problem is dual consistent.

5. The physical example. In this section, we consider the physical example in section 2. We will investigate if the mathematical theory derived earlier will provide us with the physical coupling conditions. The physical interface conditions come from mechanical principles: (1) continuity and (2) force balance (Newton's second law). According to the first principle the fluid and solid velocities must match at the interface; otherwise the fluid will be detached from the solid. This means that one of the interface conditions is

$$(52) \quad w = v_t.$$

The fluid pushes on the solid with a traction p and the solid pushes back with an equal and opposite traction, which is called σ , and we get $\sigma = -p$. For a linear elastic material, according to Hooke's law, we have $\sigma = E v_x$, which implies that the second interface condition is

$$(53) \quad p = -E v_x.$$

Now, we apply the mathematical approach. The matrices A and B in (3) and (4) can be written as

$$A = X_+ \Lambda_A^+ X_+^T + X_- \Lambda_A^- X_-^T, \quad B = Y_+ \Lambda_B^+ Y_+^T + Y_- \Lambda_B^- Y_-^T,$$

where

$$X_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\frac{2(\gamma-1)}{\gamma}} & 0 & -\sqrt{2/\gamma} \\ \sqrt{1/\gamma} & 1 & \sqrt{\frac{(\gamma-1)}{\gamma}} \end{bmatrix}^T, \quad X_- = \frac{1}{\sqrt{2}} [\sqrt{1/\gamma}, -1, \sqrt{(\gamma-1)/\gamma}]^T,$$

$$Y_+ = \frac{1}{\sqrt{2}} [1, 1]^T, \quad Y_- = \frac{1}{\sqrt{2}} [1, -1]^T,$$

and

$$\Lambda_A^+ = \begin{bmatrix} \bar{w} & 0 \\ 0 & \bar{w} + c_f \end{bmatrix}, \quad \Lambda_A^- = \bar{w} - c_f, \quad \Lambda_B^+ = c_s, \quad \Lambda_B^- = -c_s.$$

According to Proposition 2.4, the unknown matrices R_A, R_B, T_A , and T_B must be chosen such that the conditions (18) and (19) are satisfied. There are different choices for these unknown matrices which lead to well-posedness.

One of the choices is the set of characteristic interface conditions

$$(54) \quad R_A = [0, 0], \quad T_A = \sqrt{c_s(c_f - \bar{w})}, \quad R_B = 0, \quad T_B = \left[0, \sqrt{c_s(\bar{w} + c_f)}\right],$$

and $\alpha_c^p = 1$. But the choices in (54) do not lead to the physical conditions (52) and (53). Hence, these conditions lack consistency. By inspection we find that the matrices

$$(55) \quad \begin{aligned} R_A &= \left[0, \frac{(c_f - \bar{w})(\bar{\rho}c_f - E/c_s)}{\bar{\rho}c_f + E/c_s}\right], & R_B &= -\frac{c_s(\bar{\rho}c_f - E/c_s)}{\bar{\rho}c_f + E/c_s}, \\ T_A &= \frac{2E/c_s(c_f - \bar{w})}{\bar{\rho}c_f + E/c_s}, & T_B &= \left[0, \frac{2c_s\bar{\rho}c_f}{\bar{\rho}c_f + E/c_s}\right] \end{aligned}$$

lead to

$$w + p = v_t - Ev_x, \quad \bar{\rho}c_fw + p = \bar{\rho}c_fv_t - Ev_x,$$

which is equivalent to the physical conditions (52) and (53) and hence consistent. The choice (55) and $\alpha_c^p = \rho_s(c_f - \bar{w})/\bar{\rho}c_f$ imply that the conditions (18) and (19) are satisfied, which means that the coupled problem (1)–(2) is well posed.

Finally, we also apply boundary conditions to the model problem. Since two eigenvalues of matrix A are positive, two boundary conditions are needed at $x = -1$. The matrix B has one negative eigenvalue, which means that one boundary condition is needed at $x = 1$. We choose the general characteristic boundary conditions

$$(X_+^T - R_l X_-^T) u(-1, t) = g_l(t), \quad (Y_-^T - R_r Y_+^T) v(1, t) = g_r(t),$$

where R_l and R_r are matrices of appropriate size.

5.1. Numerical results. In the numerical calculations, we will use the manufactured solutions

$$\begin{aligned} u_1(x, t) &= \sin(6\pi(x - t)), & u_2(x, t) &= 0, & u_3(x, t) &= \cos(6\pi(x - t)), \\ v_1(x, t) &= v_2(x, t) = e^{t/\lambda} \sin(8\pi(x - t)) \end{aligned}$$

with $\lambda = 0.05$ and the parameters $\bar{w} = 0.5, c_f = 1, \bar{\rho} = 1, \gamma = 1.4, \rho_s = 1$, and $c_s = 1$. The manufactured solution will provide data for the forcing function, boundary conditions, and initial function. The rate of convergence is calculated as

$$q_u = \ln \left(\frac{\|\mathbf{u}^{N_1} - \mathbf{u}\|_{P_u \otimes I}}{\|\mathbf{u}^{N_2} - \mathbf{u}\|_{P_u \otimes I}} \right) / \ln \left(\frac{N_1}{N_2} \right), \quad q_v = \ln \left(\frac{\|\mathbf{v}^{N_1} - \mathbf{v}\|_{P_v \otimes I}}{\|\mathbf{v}^{N_2} - \mathbf{v}\|_{P_v \otimes I}} \right) / \ln \left(\frac{N_1}{N_2} \right),$$

where $\mathbf{u}$ and $\mathbf{v}$ are the analytical solutions and $\mathbf{u}^{N_i}$ and $\mathbf{v}^{N_i}$ are the corresponding numerical solutions with N_i grid points.

TABLE 1
Rate of convergence q_u for $u = (u_1, u_2, u_3)$.

N	$SBP21$	$SBP42$	$SBP63$	$SBP84$
20	-	-	-	-
40	2.593	2.764	3.442	3.823
80	2.102	3.146	4.264	4.956
120	2.033	3.070	4.510	5.148
180	2.014	3.035	4.550	5.160
240	2.007	3.018	4.632	5.128

TABLE 2
Rate of convergence q_v for $v = (v_1, v_2)$.

N	$SBP21$	$SBP42$	$SBP63$	$SBP84$
20	-	-	-	-
40	2.077	2.161	3.121	5.070
80	2.024	2.849	4.197	4.939
120	2.008	2.998	4.241	4.878
180	2.003	3.032	4.301	4.849
240	2.002	3.028	4.411	4.932

The time-integration in this section is done using the classical fourth order explicit Runge–Kutta scheme [5] with $CFL\ number = 0.01$. We choose the final time $T = 1.0$ with time step $\Delta t = CFL \times \Delta x$, which makes the time-error negligible. The semi-discrete scheme was implemented using SBP operators $SBP21$, $SBP42$, $SBP63$, and $SBP84$, which gives a global accuracy of 2, 3, 4, and 5, respectively [28, 30]. The results can be seen in Tables 1 and 2.

5.2. Superconvergence of the functional. In this section, we investigate the impact of dual consistency. A dual consistent scheme is obtained by choosing the penalty matrices as in (31). Small changes in the coefficients of the penalty matrices in (31) lead to a stable but dual inconsistent scheme. In the dual inconsistent case, we choose

$$\Sigma_L = \frac{3}{2}X \begin{bmatrix} 0 & 0 \\ 0 & I_A^- \end{bmatrix}, \quad \Sigma_R = -\frac{3}{2}Y \begin{bmatrix} I_B^+ & 0 \\ 0 & 0 \end{bmatrix},$$

which leads to stability.

In Tables 3 and 4, the rates of convergence of the functional

$$J(w) = \int_{-1}^0 u_1 + u_2 + u_3 \, dx + \int_0^1 v_1 + v_2 \, dx$$

TABLE 3
Rate of functional convergence for the dual inconsistent discretization.

N	$SBP21$	$SBP42$	$SBP63$	$SBP84$
20	-	-	-	-
40	1.887	1.859	3.981	6.232
80	1.963	2.829	4.544	4.567
120	1.987	2.950	4.625	4.681
180	1.994	2.978	4.631	4.777
240	1.997	2.988	4.661	4.965

TABLE 4
Rate of functional convergence for the dual consistent discretization.

N	$SBP21$	$SBP42$	$SBP63$	$SBP84$
20	-	-	-	-
40	2.303	2.464	6.541	5.331
80	2.077	4.101	6.123	7.439
120	2.024	4.269	6.101	8.400
180	2.010	4.256	6.022	8.533
240	2.005	4.218	6.010	8.644

FIG. 1. Spectral radius of the semidiscrete approximation problem (56) for different operators and numbers of grid points.

are calculated and clear superconvergence can be seen. The superconvergent functional is obtained without requiring any knowledge about the solution of the dual equations. In order to have a dual consistent scheme, we only need the dual equation and its interface conditions, which are obtained by knowing the interface conditions of the primal problem. This means that superconverging functionals are obtained at no extra computational cost [2].

5.3. Stiffness. In practice, the wave speeds between fluid (c_f) and solid (c_s) are different, which may cause stiffness. We will check the stiffness of coupled problem (1)–(2) by computing the spectral radius of the resulting semidiscrete formulation

$$(56) \quad \mathbf{w}_t = K\mathbf{w} + F.$$

In (56), $\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T$ and the matrices K and F contain the complete spatial discretization including the coupling terms and the boundary data, respectively. Figure 1 shows the spectral radius of matrix K for dual consistent and dual inconsistent cases. Clearly, the dual consistent scheme leads to smaller spectral radius and less stiffness. One can also see that the stiffness increases as c_s/c_f increases above one.

6. Summary and conclusions. We have considered the coupling of two general hyperbolic systems. The energy method was used to derive general well posed interface conditions. It was shown that the derived interface conditions lead to a well posed problem for both weak and strong imposition.

The equations were discretized using finite differences of SBP-SAT form. The penalty matrices were derived in the analysis of the continuous problem. Almost no additional derivations were necessary.

Next, the dual problem and its well posed interface conditions were derived using the energy method. The interface conditions were imposed weakly and strongly also for the dual problem. The weak interface procedures lead directly to stability of the numerical approximation as in the primal problem. The numerical scheme was shown to be dual consistent for specific choices of the penalty matrices.

By considering a physical example, it was shown that the mathematical interface conditions contain the physically correct interface conditions. The mathematical theory can also narrow the search for well posed and accurate interface conditions.

The rate of convergence was verified by the method of manufactured solution and the result was consistent with the SBP-SAT theory. Superconverging functionals were obtained for dual consistent discretizations. It was also found that dual consistency reduced the stiffness of the discretizations.

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