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On The Number of Partial Steiner Systems

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Abstract: We give a simple proof of the result of Grable on the asymptotics of the number of partial Steiner systems $S(t, k, m)$. © 2000 John Wiley & Sons, Inc. *J Combin Designs* 8:347–352, 2000

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1. INTRODUCTION

A partial Steiner system $S(t, k, m)$ is a collection of k -subsets of an m -element set M such that each t -subset is contained in at most one k -subset from $S(t, k, m)$. When every t -subset of M is contained in exactly one k -subset from $S(t, k, m)$, we have a classical Steiner system on the set M with parameters t and k . Some bounds of the number of such systems for $t = 2, k = 3$ and $t = 3, k = 4$ were obtained in [1], [9], [7] and [6]. Very little is known about the number of classical Steiner systems for large t and k .

The number of distinct partial Steiner systems $S(t, k, m)$ we denote by $s(t, k, m)$. For two sequences f_m and g_m we write $f_m \sim g_m$ if $f_m/g_m \rightarrow 1$ as $m \rightarrow \infty$.

In [5] Grable announced that using the Rödl nibble algorithm [8] and generalizing the result in [3] he proved the following:

Theorem 1. *Let t and k be two fixed positive integers, $t < k$. Then*

$$\ln s(t, k, m) \sim \frac{k-t}{(k)_t} m^t \ln m \text{ as } m \rightarrow \infty,$$

where $(k)_t = k(k-1) \dots (k-t+1)$.

The Rödl nibble algorithm is a very powerful but not an easy technique. In this paper we give a simple proof of Theorem 1. We use the result of Frankl and Rödl [4] concerning the existence of nearly perfect matchings in hypergraphs. A *hypergraph* H is a pair (V, E) , where V is a finite set of *vertices* and E is a finite family of subsets of V , called *edges*. A hypergraph is r -uniform if every edge contains precisely r vertices. The number of edges of a hypergraph H containing a vertex v is called the degree of v and denoted by $d_H(v)$ or simply $d(v)$. For two distinct vertices u and v of a hypergraph H , the number of edges containing both u and v is denoted by $d_H(u, v)$ or simply $d(u, v)$. A *matching* in a hypergraph is a collection of pairwise disjoint edges. We will use the result in [4] in the following slightly weaker form.

Theorem 2. *Let integer $r \geq 3$ and real $\delta > 0$ be fixed, and H be an r -uniform hypergraph on n vertices. There exists $\gamma > 0$ and $n_0 > 0$ such that if for some D and for every pair of distinct vertices u and v of H the following two conditions hold:*

- (1) $(1 - \gamma)D \leq d(v) \leq (1 + \gamma)D$,
- (2) $d(u, v) \leq D/(\ln n)^4$,

then for all $n \geq n_0$ H has a matching containing at least $\lceil (1 - \delta)^{\frac{n}{r}} \rceil$ edges.

2. PROOF OF THEOREM 1

Let $r = \binom{k}{t}$, $n = \binom{m}{t}$, $d = \binom{m-t}{k-t}$, and $q = \binom{m}{k}$. Consider the family $F = \{A_1, \dots, A_q\}$ of all k -subsets of an m -set M . We define the hypergraph $H(t, k, m)$ corresponding to the family F in the following way: The vertex set V of $H(t, k, m)$ is the set of all t -subsets of M and the edge set is $E = \{e_1, \dots, e_q\}$, where each e_i consists of all t -subsets which are contained in a k -subset A_i . For each matching $\{e_{i_1}, \dots, e_{i_l}\}$ of $H(t, k, m)$ the corresponding set $\{A_{i_1}, \dots, A_{i_l}\}$ in F is a partial Steiner system $S(t, k, m)$. Since this correspondence is one-to-one, the number $s(t, k, m)$ of all partial Steiner systems is equal to the number of matchings in the hypergraph $H(t, k, m)$. Note, that each edge of the hypergraph $H(t, k, m)$ contains exactly $r = \binom{k}{t}$ vertices, that is, the hypergraph is r -uniform.

Let δ be real, $0 < \delta < 1/2$. We define a random subfamily $F(p)$ of the family F by choosing independently each k -subset of F with probability $p = d^{-1+\delta}$. Taking into account the one-to-one correspondence between k -subsets of the m -set M and the edges of the hypergraph $H(t, k, m)$ we obtain a random subhypergraph $H_p = H_p(t, k, m)$ corresponding to the random family $F(p)$.

Lemma 1. *Let X be the random variable equal to the number of partial Steiner systems $S(t, k, m)$ in the random family $F(p)$ each containing at least $T = \lceil (1 - \delta)^{\frac{n}{r}} \rceil$ k -subsets. Then, $\mathbf{P}\{X \geq 1\} \geq 1 - \delta$ for sufficiently large m .*

The proof of Lemma 1 will be given in Section 3. It uses the observation that since the vertex degrees and pair degrees of random subhypergraph H_p are sums of independent indicator variables, the Chernoff bounds prove that H_p almost always satisfy the conditions of Theorem 2 and, therefore, contains a matching with at least $T = \lceil (1 - \delta)^{\frac{n}{r}} \rceil$ edges.

Let N be the number of partial Steiner systems $S(t, k, m)$ each containing at least T k -subsets. Let us compare upper and lower bounds of the probability $\mathbf{P}\{X \geq 1\}$.

A lower bound. By Lemma 1

$$\mathbf{P}\{X \geq 1\} \geq 1 - \delta.$$

Markov's inequality shows that

$$\mathbf{P}\{X \geq 1\} \leq \mathbf{E}X \leq Np^T.$$

Thus,

$$N \geq (1 - \delta)p^{-T},$$

which implies the inequality

$$s(t, k, m) \geq N \geq (1 - \delta)d^{(1-\delta)2n/r}. \quad (1)$$

An upper bound. The number of matchings $s(t, k, m)$ satisfies the following trivial upper bound:

$$s(t, k, m) \leq \sum_{j=1}^{\lfloor \frac{n}{r} \rfloor} \binom{q}{j} \leq \frac{n}{r} \binom{q}{\lfloor \frac{n}{r} \rfloor}.$$

Taking into account that $\binom{q}{j} \leq (\frac{eq}{j})^j$, and $q = \frac{nd}{r}$ we obtain

$$s(t, k, m) \leq \frac{n}{r} \left(\frac{\frac{nd}{r}}{\lfloor \frac{n}{r} \rfloor} \right)^{\lfloor \frac{n}{r} \rfloor} \leq \frac{n}{r} (ed)^{n/r} \leq d^{(1+\delta)n/r}. \quad (2)$$

Combining the inequalities (1) and (2) which hold for any fixed $0 < \delta < \frac{1}{2}$ and sufficiently large m , we obtain that

$$\ln s(t, k, m) \sim \frac{n}{r} \ln d \text{ as } m \rightarrow \infty.$$

Taking into account that

$$\frac{n}{r} = \frac{(m)_t}{(k)_t}, \quad (m)_t \sim m^t, \quad \ln d = \ln \binom{m-t}{k-t} \sim (k-t) \ln m,$$

we obtain that

$$\ln s(t, k, m) \sim \frac{k-t}{(k)_t} m^t \ln m \text{ as } m \rightarrow \infty.$$

3. PROOF OF LEMMA 1

Let $D(u)$ be the random variable equal to the degree of vertex u in $H_p(t, k, m)$, and for each pair of distinct vertices u and v let $D(u, v)$ be the random variable equal to the number of edges in $H_p(t, k, m)$ containing both u and v . For all vertices $u, v \in V$ let $A(u)$ denote the event: $|D(u) - \mathbf{E}D(u)| > (\ln d)^{-1} \mathbf{E}D(u)$ and $B(u, v)$ denote the event: $D(u, v) - \mathbf{E}D(u, v) > (\ln n)^{-5} d^\delta$. We need the following technical lemma.

Lemma 2. For sufficiently large m

$$\mathbf{P}\{A(u)\} \leq \exp\{-d^{\delta/2}\}, \quad \mathbf{P}\{B(u, v)\} \leq \exp\{-d^{\delta/2}\}.$$

Proof of Lemma 2. We use the following bounds for probabilities of large deviations of sums $Z = \sum_{i=1}^L z_i$ of independent random variables $z_1, \dots, z_L$ such that z_i takes two values 0 and 1, and $\mathbf{P}\{z_i = 1\} = p$, $\mathbf{P}\{z_i = 0\} = 1 - p$ (see [2]):

$$\mathbf{P}\{|Z - \mathbf{E}Z| > \varepsilon \mathbf{E}Z\} \leq 2\exp\{-(\varepsilon^2/3)\mathbf{E}Z\}, \quad \text{if } 0 \leq \varepsilon \leq 1, \quad (3)$$

$$\mathbf{P}\{Z - \mathbf{E}Z > \varepsilon \mathbf{E}Z\} \leq \exp\{(\varepsilon - (1 + \varepsilon)\ln(1 + \varepsilon))\mathbf{E}Z\}, \quad \text{if } \varepsilon \geq 0. \quad (4)$$

For each vertex u we define independent random variables $y_1, \dots, y_{d(u)}$ such that $y_j = 1$ iff the j th edge containing u is in $H_p(t, k, m)$, and $y_j = 0$, otherwise. It is clear, that

$$D(u) = \sum_{j=1}^{d(u)} y_j, \quad \mathbf{E}D(u) = \sum_{j=1}^{d(u)} \mathbf{E}y_j = pd = d^\delta.$$

Using (3) we obtain

$$\mathbf{P}\{|D(u) - \mathbf{E}D(u)| > (\ln d)^{-1} \mathbf{E}D(u)\} \leq 2\exp\{-(2\ln d)^{-2} d^\delta\} \leq \exp\{-d^{\delta/2}\}.$$

Similarly, for every two distinct vertices u and v we define independent random variables $x_1, \dots, x_{d(u,v)}$ such that $x_j = 1$ iff the j th edge containing both u and v is in $H_p(t, k, m)$, and $x_j = 0$ otherwise. Clearly,

$$D(u, v) = \sum_{j=1}^{d(u,v)} x_j.$$

Since $d = \binom{m-t}{k-t} \leq (m-t)^{k-t}$, we obtain, for sufficiently large m ,

$$\mathbf{E}D(u, v) \leq p \max_{u \neq v} d(u, v) \leq p \binom{m-t-1}{k-t-1} \leq (k-t)d^{\delta-1/(k-t)} \leq d^\delta (\ln n)^{-6}. \quad (5)$$

Let $d^\delta = g\mathbf{ED}(u, v)$, $\theta = (\ln n)^{-5}$. Then (5) implies $\theta g > \ln n$. This and (4) imply

$$\begin{aligned} \mathbf{P}\{D(u, v) - \mathbf{ED}(u, v) > \theta d^\delta\} &= \mathbf{P}\{D(u, v) - \mathbf{ED}(u, v) > \theta g \mathbf{ED}(u, v)\} \\ &\leq \exp\{(\theta g - (1 + \theta g)\ln(1 + \theta g))\mathbf{ED}(u, v)\} \\ &\leq \exp\{-\theta g(\ln(\theta g) - 1)\mathbf{ED}(u, v)\} \leq \exp\{-d^\delta(\ln n)^{-5}\} \leq \exp\{-d^{\delta/2}\}. \end{aligned}$$

The proof of Lemma 2 is complete.

Lemma 2 and inequality (4) imply, for sufficiently large m ,

$$\begin{aligned} \mathbf{P}\left\{\bigcup_{u \in V} A(u)\right\} &\leq n \max_u \mathbf{P}\{A(u)\} \leq m^t \exp\{-m^{\delta/4}\} \leq \delta/2, \\ \mathbf{P}\left\{\bigcup_{u, v \in V, u \neq v} B(u, v)\right\} &\leq n^2 \max_{u \neq v} \mathbf{P}\{B(u, v)\} \leq m^{2t} \exp\{-m^{\delta/4}\} \leq \delta/2. \end{aligned}$$

Note, that the event $\overline{\bigcup_{u \in V} A(u)}$ implies, for every vertex u , the inequality

$$|D(u) - d^\delta| \leq (\ln d)^{-1} d^\delta, \quad (6)$$

and the event $\overline{\bigcup_{u, v \in V, u \neq v} B(u, v)}$ implies for every two distinct vertices u, v the inequality

$$D(u, v) \leq \mathbf{ED}(u, v) + (\ln n)^{-5} d^\delta \leq 2d^\delta (\ln n)^{-5}. \quad (7)$$

Let γ and n_0 be the constants in Theorem 2. It is clear that for sufficiently large m the inequality (6) implies the condition 1 of Theorem 2 with $D = d^\delta$, and the inequality (7) implies the condition 2 of Theorem 2 with $D = d^\delta$. Thus, by Theorem 2

$$\mathbf{P}\{X \geq 1\} \geq 1 - (\mathbf{P}\left\{\bigcup_{u \in V} A(u)\right\} + \mathbf{P}\left\{\bigcup_{u, v \in V, u \neq v} B(u, v)\right\}) \geq 1 - \delta.$$

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