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Some Panconnected and Pancyclic Properties of Graphs with a Local Ore-type Condition

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Abstract. Asratian and Khachatryan proved that a connected graph G of order at least 3 is hamiltonian if $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|$ for any path uvw with $uv \notin E(G)$, where $N(x)$ is the neighborhood of a vertex x .

We prove that a graph G with this condition, which is not complete bipartite, has the following properties:

- a) For each pair of vertices x, y with distance $d(x, y) \geq 3$ and for each integer n , $d(x, y) \leq n \leq |V(G)| - 1$, there is an $x - y$ path of length n .
 - (b) For each edge e which does not lie on a triangle and for each n , $4 \leq n \leq |V(G)|$, there is a cycle of length n containing e .
 - (c) Each vertex of G lies on a cycle of every length from 4 to $|V(G)|$.
- This implies that G is vertex pancyclic if and only if each vertex of G lies on a triangle.

1. Introduction

We use Bondy and Murty [6] for terminology and notation not defined here and consider finite simple graphs only. For each vertex u of a graph G we denote by $N(u)$ the set of all vertices of G adjacent to u . The distance between vertices u and v is denoted by $d(u, v)$. A path with x and y as end vertices is called an $x - y$ path. An $x - y$ path is called a Hamilton path if it contains all the vertices of G . A graph G is Hamilton-connected if every two vertices of G are connected by a Hamilton path.

Let G be a graph of order $p \geq 3$. G is called panconnected if for each pair of distinct vertices x and y and for each l , $d(x, y) \leq l \leq p - 1$, there is an $x - y$ path of length l in G . G is called pancyclic if it contains a cycle of length l for each l satisfying $3 \leq l \leq p$. G is called vertex pancyclic (edge pancyclic) if each vertex (edge) of G lies on a cycle of every length from 3 to p inclusive.

Let $M_1 = \{K_{n,n}, n \geq 2\}$ and $M_2 = \{G: K_{n,n} \subseteq G \subseteq K_n \vee K_n^c, n \geq 3\}$.

The following results are known.

Theorem 1 (Ore [10]). *Let G be a graph of order $p \geq 3$, where $d(u) + d(v) \geq p$ for each pair of nonadjacent vertices u and v . Then G is hamiltonian.*

A graph G satisfying the condition of Theorem 1 is called an Ore graph.

Theorem 2 (Bondy [5]). *An Ore graph G is pancyclic if and only if $G \notin \mathcal{M}_1$.*

Note that Theorem 2 is a corollary from more general theorems of Bondy [5] and Häggkvist, Faudree and Schelp [8].

Theorem 3 [7, 11]. *Let G be an Ore graph of order $p \geq 4$. Then each vertex of G lies on a cycle of every length from 4 to p inclusive, unless $G \in \mathcal{M}_1$.*

Theorem 4 (Asratian¹ and Khachatryan [9]). *Let G be a connected graph of order at least 3 where $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|$ for any path uvw with $uv \notin E(G)$. Then G is hamiltonian.*

A simpler proof of Theorem 4 was suggested in [2]. Clearly, Theorem 4 implies Theorem 1. Moreover, while Theorem 1 only applies to graphs G with diameter 2 and large edge density ($|E(G)| \geq \frac{1}{4} \cdot |V(G)|^2$), Theorem 4 applies to infinite classes of graphs G with small edge density ($|E(G)| \leq \text{constant} \cdot |V(G)|$) and large diameter ($\geq \text{constant} \cdot |V(G)|$).

Denote by L_0 the set of graphs of order at least 5 satisfying the conditions of Theorem 4.

Theorem 5 [4]. *A graph $G \in L_0$ is pancyclic if and only if $G \notin \mathcal{M}_1$.*

Theorem 6 [1]. *A graph $G \in L_0$ is Hamilton-connected if and only if it is 3-connected and $G \notin \mathcal{M}_2$.*

We prove here that a graph $G \in L_0 \setminus \mathcal{M}_1$ has the following properties;

- (a) For each pair of vertices x, y with $d(x, y) \geq 3$ and for each integer n , $d(x, y) \leq n \leq |V(G)| - 1$, there is an $x - y$ path of length n .
- (b) For each edge e which does not lie on a triangle and for each n , $4 \leq n \leq |V(G)|$, there is a cycle of length n containing e .
- (c) Each vertex of G lies on a cycle of every length from 4 to $|V(G)|$.

The last property implies the following: A graph $G \in L_0 \setminus \mathcal{M}_1$ is vertex pancyclic if and only if each vertex of G lies on a triangle.

Theorems 2, 3 and 5 follow from our results.

2. Notations and Preliminary Results

Let P be a path of G . We denote by $\vec{P}$ the path P with a given orientation and by $\overleftarrow{P}$ the path P with the reverse orientation. If $u, v \in V(P)$, then $u\vec{P}v$ denotes the consecutive vertices of P from u to v in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. We use w^+ to denote the successor of w

¹ In [9] the last name of the first author was transcribed as Hasratian.

on $\vec{P}$ and w^- to denote its predecessor. Also we denote by $N(P)$ the set of vertices v outside P with $N(v) \cap V(P) \neq \emptyset$. If $W \subseteq V(P)$ then $W^+ = \{w^+ / w \in W\}$ and $W^- = \{w^- / w \in W\}$. We will say that a path $\vec{P}$ contains a triangle $a_1 a_2 a_3 a_1$ if $a_1, a_2, a_3 \in V(P)$, $a_1 a_3 \in E(G)$ and $a_1^+ = a_2 = a_3^-$. A path $\vec{P}$ containing a triangle Δ is denoted by $\vec{P}^\Delta$. The set of all triangles contained in $\vec{P}^\Delta$ we denote by $T(\vec{P}^\Delta)$.

Analogous terminology is used with respect to cycles as well.

We assume that an $x - y$ path $\vec{P}$ has an orientation from x to y .

A path (cycle) on n vertices will be denoted by P_n (respectively, C_n).

Let A and B be two disjoint subsets of vertices of a graph G . We denote by $\varepsilon(A, B)$ the number of edges in G with one end in A and the other in B .

Proposition 1 [9]. $G \in L_0$ if and only if for any path uvw with $uv \notin E(G)$ $|N(u) \cap N(v)| \geq |N(w) \setminus (N(u) \cup N(v))|$ holds.

Corollary 1. If $G \in L_0$ then G is 2-connected and $|N(u) \cap N(v)| \geq 2$ for each pair of vertices u, v with $d(u, v) = 2$.

Proof. Let $d(u, v) = 2$ and $w \in N(u) \cap N(v)$. Then $u, v \in N(w) \setminus (N(u) \cup N(v))$. Therefore, by Proposition 1, $|N(u) \cap N(v)| \geq 2$. This implies 2-connectedness of G . $\square$

Proposition 2. Let $G \in L_0$ and x, y be two distinct vertices of G with $d(x, y) = l \geq 3$. Then there exists an $x - y$ path P_{l+2}^A .

Proof. Let $P_{l+1} = u_0 u_1 \dots u_l$ be an $x - y$ path of length l where $u_0 = x$ and $u_l = y$. Since P_{l+1} is an $x - y$ path of minimum length then

$$N(u_i) \cap V(P_{l+1}) = \{u_{i-1}, u_{i+1}\} \quad \text{for each } i = 1, \dots, l-1. \quad (1)$$

Suppose that no vertex v outside P_{l+1} is adjacent to two consecutive vertices of P_{l+1} . Then, since $u_1 \in N(u_0) \cap N(u_2)$ and $u_2 \in N(u_1) \cap N(u_3)$ we obtain from (1), using Proposition 1, the following:

$$d(u_2) - 1 \geq |N(u_0) \cap N(u_2)| \geq |N(u_1) \setminus (N(u_0) \cup N(u_2))| = d(u_1), \quad (2)$$

$$d(u_1) - 1 \geq |N(u_1) \cap N(u_3)| \geq |N(u_2) \setminus (N(u_1) \cup N(u_3))| = d(u_2). \quad (3)$$

But (2) contradicts (3). Hence there are vertices u_i and v such that $0 \leq i \leq l-1$, $v \notin P_{l+1}$ and $vu_i, vu_{i+1} \in E(G)$. This implies that there is an $x - y$ path $P_{l+2}^A = u_0 \dots u_i v u_{i+1} \dots u_l$ of length $l+1$ with $\Delta = u_i v u_{i+1} u_i$. $\square$

Proposition 3. Let $G \in L_0 \setminus \mathcal{M}_1$. Then each edge of G lies on a triangle or on a cycle C_5^A .

Proof. Let $G \in L_0$ and assume that there is an edge e of G which lies neither on a triangle nor on a cycle C_5^A . Let $e = v_1 w_1$, $d(w_1) \geq d(v_1)$ and $N(w_1) = \{v_1, \dots, v_n\}$. Clearly, $v_1 v_j \notin E(G)$ for each $j = 2, \dots, n$ because e does not lie on a triangle. Since $d(v_1, v_2) = 2$ there is a vertex $w_2 \in N(v_1) \cap N(v_2)$, $w_2 \neq w_1$. Clearly, $v_2 v_j \notin E(G)$ for each j , $3 \leq j \leq n$. (Otherwise e lies on a cycle C_5^A .) Then, using Proposition 1, we have

$$n = d(w_1) \geq d(v_1) \geq |N(v_1) \cap N(v_2)| \geq |N(w_1) \setminus (N(v_1) \cup N(v_2))| = |N(w_1)| = n.$$

Therefore $n = d(v_1) = |N(v_1) \cap N(v_2)|$. Let $N(v_1) \cap N(v_2) = \{w_1, \dots, w_n\}$. Clearly, $w_i w_j \notin E(G)$ for each pair i, j where $2 \leq i < j \leq n$. (Otherwise there is a cycle C_5^d containing e .) Since $d(w_1, w_i) = 2$ then, using Proposition 1, we have $|N(w_1) \cap N(w_i)| \geq |N(v_1) \setminus (N(w_1) \cup N(w_i))| = |N(v_1)| = n$ for each $i = 2, \dots, n$. Therefore $w_i v_j \in E(G)$ for each pair i, j , where $1 \leq i, j \leq n$.

Let us show that $V(G) = \{v_1, \dots, v_n, w_1, \dots, w_n\}$. Suppose that there is a vertex $u \in V(G) \setminus \{v_1, \dots, v_n, w_1, \dots, w_n\}$ such that $uw_i \in E(G)$ for some i , $2 \leq i \leq n$. Then $uv_j \notin E(G)$ for each $j = 2, \dots, n$. (Otherwise there is a cycle C_5^d containing e .) Since $d(u, v_1) = 2$ then, using Proposition 1, we have

$$|N(u) \cap N(v_1)| \geq |N(w_i) \setminus (N(u) \cup N(v_1))| \geq n + 1$$

because $u, v_1, \dots, v_n \in N(w_i) \setminus (N(u) \cup N(v_1))$. But then $n = d(v_1) \geq |N(v_1) \cap N(u)|$, a contradiction. Hence $N(w_i) = \{v_1, \dots, v_n\}$ for each $i = 1, \dots, n$.

Using the same arguments it is possible to prove that $N(v_j) = \{w_1, \dots, w_n\}$ for each $j = 1, \dots, n$. Consequently $G = K_{n,n}$. $\square$

Corollary 2. *Let $G \in L_0 \setminus \mathcal{M}_1$. Then each vertex of G lies on a triangle or on a cycle C_5^d .*

Corollary 3. *Each graph $G \in L_0 \setminus \mathcal{M}_1$ contains a triangle.*

3. Main Results

Proposition 4. *Let $G \in L_0$ and x, y be two distinct vertices of G . If there is an $x - y$ path P_n such that $n = |V(G)| - 1$ and $|N(v) \cap \{x, y\}| \leq 1$ for the unique vertex v outside P_n then there exists a Hamilton $x - y$ path of G .*

Proof. W.l.o.g. we assume $vy \notin E(G)$. Let W_v denote the set $N(v) \cap V(P_n)$ and $p = |W_v|$. If $vw^+ \in E(G)$ for some $w \in W_v$ then there is a Hamilton $x - y$ path. Suppose that $vw^+ \notin E(G)$ for each $w \in W_v$. Since $n = |V(G)| - 1$ we have that $N(v) \cap N(w^+) \subseteq W_v$ for each $w \in W_v$. Suppose that $w^+ g^+ \notin E(G)$ for each pair $w, g \in W_v$. Then

$$\sum_{w \in W_v} |N(v) \cap N(w^+)| \leq \varepsilon(W_v, W_v^+)$$

and

$$\sum_{w \in W_v} |N(w) \setminus (N(v) \cup N(w^+))| \geq \varepsilon(W_v, W_v^+) + p$$

since $v \in N(w) \setminus (N(v) \cup N(w^+))$ for each $w \in W_v$. By Proposition 1 we have

$$\sum_{w \in W_v} |N(v) \cap N(w^+)| \geq \sum_{w \in W_v} |N(w) \setminus (N(v) \cap N(w^+))|.$$

But the last inequality contradicts the previous two. So $w^+ g^+ \in E(G)$ for some pair of vertices w, g where w occur on P_n before g . Then there exists a Hamilton $x - y$ path $P_{n+1} = x \tilde{P}_n w v g \tilde{P}_n w^+ g^+ \tilde{P}_n y$. $\square$

The next technical lemma plays a key role in the proofs of the subsequent theorems.

Lemma. *Let $G \in L_0$ and x, y be two distinct vertices of G . If there exists an $x - y$ path P_n^d such that $4 \leq n \leq |V(G)| - 2$ and $|N(v) \cap \{x, y\}| \leq 1$ for each $v \in V(G) \setminus V(P_n^d)$ then there exists an $x - y$ path $P_{n+t}^{d_1}$ such that $1 \leq t \leq 2$ and $V(P_n^d) \subset V(P_{n+t}^{d_1})$.*

The proof of the lemma will be given later.

Theorem 7. *Let $G \in L_0$ and $e = xy$ be an edge of G which does not lie on a triangle. Then e lies on a cycle C_n for each n , $4 \leq n \leq |V(G)|$.*

Proof. It is sufficient to prove that there exists an $x - y$ path P_n for each n , $4 \leq n \leq |V(G)|$. Since e does not lie on a triangle then, by Proposition 3, e lies on a cycle C_5^d . Hence there is an $x - y$ path P_4 and an $x - y$ path P_5^d . Suppose that there exist $x - y$ paths $P_4, \dots, P_{n-1}$ and an $x - y$ path P_n^d for some n , $5 \leq n \leq |V(G)| - 1$. If $n = |V(G)| - 1$ then, by Proposition 4, there exists a Hamilton $x - y$ path. If $n \leq |V(G)| - 2$ then, by the lemma, there exists an $x - y$ path $P_{n+t}^{d_1}$, where $1 \leq t \leq 2$. If $t = 2$ and $\Delta_1 = w^-ww^+w^-$ then we can obtain an $x - y$ path P_{n+1} by deleting the vertex w from $P_{n+2}^{d_1}$.

Repetition of our argument shows that there is an $x - y$ path P_n for each n , $4 \leq n \leq |V(G)|$. $\square$

Using Proposition 2 instead of Proposition 3 and the same arguments as in the proof of Theorem 7, we can prove the following.

Theorem 8. *Let $G \in L_0$ and x, y be two distinct vertices of G with $d(x, y) \geq 3$. Then for each n , $d(x, y) + 1 \leq n \leq |V(G)|$, there exists an $x - y$ path P_n .*

Let a cycle $\vec{C}$ contain a triangle $\Delta = a_1a_2a_3a_1$. We call the vertex a_2 a centre of Δ and denote it by $s(\Delta)$.

Proposition 5. *Let $G \in L_0 \setminus \mathcal{M}_1$. Then each vertex g of G lies on a cycle C_4^d or on a cycle C_5^d such that $g \neq s(\Delta)$.*

Proof. If g does not lie on a triangle then, by Corollary 2, g lies on a cycle C_5^d and $g \neq s(\Delta)$. Now suppose that g lies on a triangle $xgyx$.

Case 1. $d(g) = 2$. Since $|V(G)| \geq 5$ and, by Corollary 1, G is 2-connected then must exist a vertex v with $d(g, v) = 2$. Clearly, $|N(g) \cap N(v)| \geq 2$. Since $d(g) = 2$ then $N(g) \cap N(v) = \{x, y\}$. So we have a cycle $C_4^d = xvygx$ with $\Delta = xvyx$ and $s(\Delta) = v$.

Case 2. $d(g) \geq 3$ and g has a common neighbour with x or y outside Δ .

W.l.o.g. suppose $xv, vg \in E(G)$ for some $v \notin \{x, y, g\}$. Then there is a cycle $C_4^d = xvgyx$ with $\Delta = xvgx$ and $s(\Delta) = v$.

Case 3. $d(g) \geq 3$ and g has no common neighbour with x and y outside Δ .

Consider a vertex $v \in N(g) \setminus \{x, y\}$. Since $d(v, x) = 2$ we have that $|N(x) \cap N(v)| \geq 2$ and there exists a vertex $z \in N(v) \cap N(x)$, $z \neq g$. Now we have a cycle $C_5^d = xygvzx$ with $\Delta = xygzx$ and $s(\Delta) = y$. $\square$

Theorem 9. *Let $G \in L_0 \setminus \mathcal{M}_1$. Then for each vertex g of G and for each n , $4 \leq n \leq |V(G)|$, there is a cycle of length n containing g .*

Proof. By Proposition 5, g lies on a cycle C_r^d where $s(\Delta) \neq g$ and $4 \leq r \leq 5$. Hence there is a cycle C_4 containing g .

Suppose there exist cycles $C_4, \dots, C_n$, $4 \leq n \leq |V(G)| - 1$, containing g such that the last cycle C_n contains a triangle $\Delta = a_1 a_2 a_3 a_1$ with $a_2 = s(\Delta) \neq g$ i.e. $C_n = C_n^d$. If there exists a vertex $v \in N(C_n^d)$ which is adjacent to two consecutive vertices w and w^+ of C_n^d then there is a cycle $C_{n+1}^d = w v w^+ \tilde{C}_n^d w$ with $\Delta_1 = w v w^+$ where $s(\Delta_1) = v \neq g$. Now let no vertex $v \in N(C_n^d)$ be adjacent to two consecutive vertices of C_n^d . Consider a vertex $v \in N(C_n^d)$ and the triangle $\Delta = a_1 a_2 a_3 a_1$. Clearly, $g \neq a_2$.

Let us define vertices x and y , a triangle Δ_0 and an $x - y$ path $P_n^{d_0}$ in the following way.

If $g \in W_v$ and $g \neq a_3$ then $x = g$, $y = g^-$, $\Delta_0 = \Delta$ and $P_n^{d_0} = g \tilde{C}_n^d g^-$.

If $g \in W_v$ and $g = a_3$, then $x = g$, $y = g^+$, $\Delta_0 = a_3 a_2 a_1 a_3$ and $P_n^{d_0} = g \tilde{C}_n^d g^+$.

If $g \notin W_v$ and $g \neq a_1$ then $x = g^+$, $y = g$, $\Delta_0 = \Delta$ and $P_n^{d_0} = g^+ \tilde{C}_n^d g$.

If $g \notin W_v$ and $g = a_1$ then $x = g^-$, $y = g$, $\Delta_0 = a_3 a_2 a_1 a_3$ and $P_n^{d_0} = g^- \tilde{C}_n^d g$.

Clearly, $|N(v) \cap \{x, y\}| \leq 1$ for each $v \in N(P_n^{d_0})$. If $n = |V(G)| - 1$ then, by Proposition 4, there exists an Hamilton $x - y$ path. Clearly, since x and y are adjacent, there also exists a Hamilton cycle of G . If $n \leq |V(G)| - 2$ then, by the lemma, there is an $x - y$ path $P_{n+t}^{d_1}$ such that $1 \leq t \leq 2$ and $V(P_n^{d_0}) \subset V(P_{n+t}^{d_1})$. Since $g \in \{x, y\}$ then $g \neq s(\Delta_1)$. The path $P_{n+t}^{d_1}$ define a cycle $C_{n+t}^{d_1}$. If $t = 2$ we can obtain a cycle C_{n+1} containing g by deleting the centre $s(\Delta_1)$ from $C_{n+t}^{d_1}$.

Repetition of our argument shows that there is a cycle C_n for each n , $4 \leq n \leq |V(G)|$. $\square$

Clearly, Theorem 5 follows from Theorem 9 and Corollary 3. Using Theorem 9 we can formulate a criterion for a graph $G \in L_0$ to be vertex pancyclic.

Theorem 10. *A graph $G \in L_0$ is vertex pancyclic if and only if every vertex of G lies on a triangle.*

Corollary 5. *Let G be a connected graph of order at least 3 where $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| + 1$ for any path uvw with $uv \notin E(G)$. Then G is vertex pancyclic if and only if every vertex of G lies on a triangle.*

Let us point out that for each $p \geq 7$ the graph $K_3 \vee (K_1 + K_{p-4})$ satisfies the condition of Corollary 5 but contains a vertex that does not lie on a triangle. The next result immediately follows from Theorem 9.

Corollary 6. *Let $G \in L_0 \setminus \mathcal{M}_1$ be a graph of order $p \geq 5$ where $d(u) + d(v) \geq p$ for each pair of vertices u, v with $d(u, v) = 2$. Then each vertex of G lies on a cycle of every length from 4 to p .*

Finally we give the proof of the lemma.

Proof. Since G is connected and $n < |V(G)|$ then $N(P_n^d) \neq \emptyset$. For each $v \in N(P_n^d)$

we denote by W_v the set $N(v) \cap V(P_n^d)$. Let $U_1 = \{v \in N(P_n^d) / |W_v| = 1\}$ and $U_2 = N(P_n^d) \setminus U_1$.

Suppose that there does not exist any $x - y$ path $P_{t+n}^{d_1}$ such that $1 \leq t \leq 2$ and $V(P_n^d) \subset V(P_{t+n}^{d_1})$. Then the following properties hold.

Property 1. $vw^+ \notin E(G)$ for each $v \in N(P_n^d)$ and each $w \in W_v$.

Property 2. If $v \in U_1$, $W_v = \{w\}$ and $w \notin \{x, y\}$ then the set $T(\vec{P}_n^d)$ contains the unique triangle $w^-ww^+w^-$.

Proof. Let $a_1a_2a_3a_1$ be a triangle from the set $T(\vec{P}_n^d)$. Suppose $a_2 \neq w$. Since $d(v, w^-) = 2 = d(v, w^+)$ then, by Corollary 1, there exist vertices v_1 and v_2 such that $v_1 \in (N(v) \cap N(w^-)) \setminus V(P_n^d)$ and $v_2 \in (N(v) \cap N(w^+)) \setminus V(P_n^d)$. This gives an $x - y$ path

$$P_{n+2}^{d_1} = \begin{cases} x\vec{P}_n^dw^-v_1vw\vec{P}_n^dy & \text{if } a_2 \in w^+\vec{P}_n^dy \\ x\vec{P}_n^dwv_2w^+\vec{P}_n^dy & \text{if } a_2 \in x\vec{P}_n^dw^- \end{cases}$$

with $\Delta_1 = a_1a_2a_3a_1$ such that $V(P_n^d) \subset V(P_{n+2}^{d_1})$, a contradiction. $\square$

Property 3. $U_2 \neq \emptyset$.

Proof. Suppose that $U_2 = \emptyset$. Let $v \in U_1$, $W_v = \{w_1\}$ and $\Delta_1 = a_1a_2a_3a_1$ be a triangle from the set $T(\vec{P}_n^d)$. Three cases are possible: $w_1 = x$, $w_1 = y$ and $w_1 \notin \{x, y\}$.

Let $w_1 = x$. Since $d(v, w_1^+) = 2$ and $|W_v| = 1$ then, by Corollary 1, there is a vertex $v_1 \in (N(v) \cap N(w_1^+)) \setminus V(P_n^d)$. Hence, by Property 2, $a_1 = w_1$, $a_2 = w_1^+$ and $a_1a_2a_3a_1$ is the unique triangle in the set $T(\vec{P}_n^d)$. Since $v_1 \in U_1$, $d(v_1, a_3) = 2$ and $|W_{v_1}| = 1$ then, by Corollary 1, there is a vertex $v_2 \in (N(v_1) \cap N(a_3)) \setminus V(P_n^d)$. Clearly, $v_2 \in U_1$, $W_{v_2} = \{a_3\}$ and $a_2^-a_2^+ \notin E(G)$. This contradicts Property 2.

Using the same arguments we obtain contradictions in the cases $w = y$ and $w \notin \{x, y\}$. Hence, $U_2 \neq \emptyset$. $\square$

Property 4. Let $v \in U_2$, $vy \notin E(G)$ and $Q \subseteq W_v = \{w_1, \dots, w_p\}$. Then

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \geq \sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \quad (4)$$

Furthermore, if $a_1a_2a_3a_1$ is a triangle from $T(\vec{P}_n^d)$ with $\{a_1, a_2\} \cap Q = \emptyset$ then

$$N(v) \cap N(w_i^+) \subseteq W_v \quad \text{for each } w_i \in Q \quad (5)$$

and

$$w_i^+w_j^+ \notin E(G) \quad \text{for each pair of vertices } w_i, w_j \in Q \quad (6)$$

Proof. Since $d(v, w_i^+) = 2$ for each $w_i \in Q$ then (4) follows from Proposition 1. If (5) does not hold then there exist a vertex $v_1 \in (N(v) \cap N(w_i^+)) \setminus W_v$ for some $w_i \in Q$ and an $x - y$ path $P_{n+2}^{d_1} = x\vec{P}_n^dw_1vv_1w_i^+\vec{P}_n^dy$ with $\Delta_1 = a_1a_2a_3a_1$, a contradiction. So, (5) must hold.

If $w_i^+w_j^+ \in E(G)$ for some pair $w_i, w_j \in Q$, $i < j$, then there is an $x - y$ path $P_{n+1}^{d_1} = x\vec{P}_n^dw_1vw_j\vec{P}_n^dw_i^+w_j^+\vec{P}_n^dy$ with

$$\Delta_1 = \begin{cases} a_1 a_2 a_3 a_1 & \text{if } a_1 \notin w_i^+ \vec{P}_n^{\Delta} w_j \\ a_3 a_2 a_1 a_3 & \text{otherwise} \end{cases}$$

a contradiction. So, (6) holds. $\square$

Property 5. Let $v \in U_2$ and $vy \notin E(G)$. If $a_1 a_2 a_3 a_1$ is a triangle from the set $T(\vec{P}_n^{\Delta})$ then $W_v \cap \{a_1, a_2\} \neq \emptyset$.

Proof. Let $w_1, \dots, w_p$ denote the vertices of W_v occurring on $\vec{P}_n^{\Delta}$ in the order of their indices. Suppose $\{a_1, a_2\} \cap W_v = \emptyset$. Set $Q = W_v$. Then, by Property 4, we have (4), (5) and (6). Furthermore, we have

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \leq \varepsilon(Q, Q^+) \quad (7)$$

and

$$\sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq \varepsilon(Q, Q^+) + p \quad (8)$$

because $v \notin Q^+$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \dots, p$. But (7) and (8) contradict (4). So, $\{a_1, a_2\} \cap W_v \neq \emptyset$. $\square$

Property 6. $|W_v| = 2$ for each vertex $v \in U_2$. Moreover $w^- w^+ \in E(G)$ for each $w \in W_v \setminus \{x, y\}$.

Proof. Let $v \in U_2$. Then $|W_v \cap \{x, y\}| \leq 1$ by the assumption (in the lemma). W.l.o.g. we may assume that $vy \notin E(G)$. Let $w_1, \dots, w_p$ be the vertices of W_v occurring on $\vec{P}_n^{\Delta}$ in the order of their indices. By Property 5 we have that $W_v \cap \{a_1, a_2\} \neq \emptyset$ for each triangle $a_1 a_2 a_3 a_1$ from the set $T(\vec{P}_n^{\Delta})$. Let k be the minimum i , $1 \leq i \leq p$, for which $w_i^- w_i^+ \in E(G)$ or $w_i w_i^{++} \in E(G)$.

Case 1. $k < p$.

Let $a_1 a_2 a_3 a_1$ be a triangle with $w_k \in \{a_1, a_2\}$. Set $Q = W_v \setminus \{w_k\}$. Since $Q \cap \{a_1, a_2\} = \emptyset$ then by Property 4 all of (4), (5) and (6) hold. Since the vertices w_k and w_i^+ can be adjacent for each $w_i \in Q$, we have

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \leq \varepsilon(Q, Q^+) + p - 1. \quad (9)$$

If $w_{k+1}^- w_{k+1}^+ \notin E(G)$ then

$$\sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq \varepsilon(Q, Q^+) + p \quad (10)$$

because $w_{k+1}^- \notin Q^+$, $w_{k+1}^- \in N(w_{k+1}) \setminus (N(v) \cup N(w_{k+1}^+))$ and $v \notin Q^+$, $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $w_i \in Q$.

But (9) and (10) contradict (4). Consequently $w_{k+1}^- w_{k+1}^+ \in E(G)$. Moreover note that $w_k^+ \neq w_{k+1}^-$, since otherwise we find an $x - y$ path of length $n + 1$ containing a triangle. Now set $Q = W_v \setminus \{w_{k+1}\}$. Then, by Property 4, all of (4), (5) and (6) also hold for the new Q . Clearly, $w_j^+ w_{k+1} \notin E(G)$ for each $j \neq k$, $1 \leq j \leq p$. (Otherwise there is an $x - y$ path P_{n+1}^{Δ} with $\Delta_1 = a_1 a_2 a_3 a_1$, a contradiction.) Hence, we have

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| \leq \varepsilon(Q, Q^+) + 1. \quad (11)$$

Furthermore,

$$\sum_{w_i \in Q} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq \varepsilon(Q, Q^+) + p - 1 \quad (12)$$

because $v \notin Q^+$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $w_i \in Q$. It follows from (11), (12) and (4) that $p = 2$. So, $k = 1$, $p = 2$ and $w_2^- w_2^+ \in E(G)$.

If $w_1 \neq x$ then we can obtain $w_1^- w_1^+ \in E(G)$ by considering a $y - x$ path $\tilde{P}_n^d$ and using the above arguments.

Case 2. $k = p$.

We assume that $x = w_1$. (Otherwise we can consider a $y - x$ path $\tilde{P}_n^d$ and use the same arguments as in Case 1). Set $Q = \{w_1, \dots, w_{p-1}\}$. Let $a_1 a_2 a_3 a_1$ be a triangle with $w_p \in \{a_1, a_2\}$. By Property 4 all of (4), (5) and (6) hold because $Q \cap \{a_1, a_2\} = \emptyset$. Furthermore, since the vertices w_p and w_i^+ can be adjacent for each $w_i \in Q$, (9) also holds. On the other hand the inequality (12) also holds, because $v \notin Q^+$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $w_i \in Q$. It follows from (9), (12) and (4) that

$$\sum_{w_i \in Q} |N(v) \cap N(w_i^+)| = \varepsilon(Q, Q^+) + p - 1 \quad (13)$$

This implies that $w_i^+ w_p \in E(G)$ for $i = 1, \dots, p - 1$. Now we have that $p \geq |N(v) \cap N(w_i^+)| \geq |N(w_p) \setminus (N(v) \cup N(w_i^+))|$ and $\{v, w_1^+, \dots, w_{p-1}^+\} \subseteq N(w_p) \setminus (N(v) \cup N(w_i^+))$. Therefore, for each $i = 1, \dots, p - 1$ we have

$$N(w_p) \setminus (N(v) \cup N(w_i^+)) = \{v, w_1^+, \dots, w_{p-1}^+\} \quad (14)$$

Since $w_p^+ \in N(w_p) \setminus N(v)$, (14) implies that

$$w_i^+ w_p^+ \in E(G) \quad \text{for each } i = 1, \dots, p - 1 \quad (15)$$

Suppose that $w_p^- w_p^+ \notin E(G)$. Then, by the definition of p , $w_p w_p^{++} \in E(G)$. Since $w_p^{++} \in N(w_p) \setminus N(v)$, (14) implies that $w_1^+ w_p^{++} \in E(G)$. By (15), we also have that $w_1^+ w_p^+ \in E(G)$. Then G has an $x - y$ path $P_{n+1}^{d_1} = x v w_p \tilde{P}_n^d w_1^+ w_p^+ \tilde{P}_n^d y$ with $\Delta_1 = w_1^+ w_p^+ w_p^{++} w_1^+$; a contradiction. So, $w_p^- w_p^+ \in E(G)$ and, therefore, $w_{p-1}^+ \neq w_p^-$. Moreover, (14) implies that $w_1^+ w_p^- \in E(G)$ since $w_p^- \in N(w_p) \setminus N(v)$.

If $p \geq 3$ then G has an $x - y$ path $P_{n+1}^{d_1} = x v w_{p-1} \tilde{P}_n^d w_1^+ w_p^- \tilde{P}_n^d w_{p-1}^+ w_p \tilde{P}_n^d y$ with $\Delta_1 = w_{p-1}^+ w_p w_p^+ w_{p-1}^+$; a contradiction. Therefore, $p = 2$ and $w_2^- w_2^+ \in E(G)$. $\square$

Now using the properties above we will obtain a contradiction. Let $v \in U_2$, $W_v = \{w_1, w_2\}$ and assume that w_1 occur on $\tilde{P}_n^d$ before w_2 . W.l.o.g. we may assume that $vy \notin E(G)$. Then, by Property 6, $w_2^- w_2^+ \in E(G)$. Clearly, $w_1^+ w_2 \in E(G)$ because $|N(v) \cap N(w_1^+)| \geq 2$, $N(v) \cap N(w_1^+) \subseteq W_v$ and $|W_v| = 2$. Since G is 2-connected and $n \leq |V(G)| - 2$ we have that $N(P_n^d) \setminus \{v\} \neq \emptyset$. Let us show

$$uw_2^- \notin E(G) \quad \text{for each } u \in N(P_n^d) \setminus \{v\} \quad (16)$$

Suppose $uw_2^- \in E(G)$ for some $u \in N(P_n^d) \setminus \{v\}$. Then, by Property 2, $u \in U_2$. For the vertex $g = w_2^-$ we have $g^- \neq w_1$ and $g \in W_u \setminus \{x, y\}$. Hence, by Property 6, $g^- g^+ \in E(G)$. But then for the triangle $g^- g g^+$ we have $\{g^-, g\} \cap W_v = \emptyset$. This contradicts Property 5. So, (16) is proved.

Now let us show that

$$uw_2 \notin E(G) \quad \text{for each } u \in N(P_n^d) \setminus \{v\} \quad (17)$$

Suppose that $uw_2 \in E(G)$ for some $u \in N(P_n^d) \setminus \{v\}$. Clearly, $uv \notin E(G)$. (Otherwise there exists an $x - y$ path $P_{n+2}^d = x\tilde{P}_n^d w_1 v u w_2 w_1^+ \tilde{P}_n^d w_2^- w_2^+ \tilde{P}_n^d y$ with $\Delta_1 = v u w_2 v$, a contradiction.)

We have $w_2^-, v, u \in N(w_2) \setminus (N(w_2^-) \cup N(v))$ and $d(w_2^-, v) = 2$. Therefore, by Proposition 1, $|N(w_2^-) \cap N(v)| \geq 3$. Since $|W_v| = 2$ there is a vertex $v_1 \in (N(v) \cap N(w_2^-)) \setminus W_v$ which contradicts (16). So (17) is proved.

Consider a vertex $u \in N(P_n^d) \setminus \{v\}$. By (17) and Property 2 we have $u \in U_2$. Using Property 5 with the triangle $w_2^- w_2 w_2^+$, the vertex u and the path $\tilde{P}_n^d$ we obtain that $uy \in E(G)$. Then $ux \notin E(G)$ because $|N(u) \cap \{x, y\}| \leq 1$. Let $W_u = \{g_1, g_2\}$ where $g_2 = y$. Then, by Property 6, $g_1^- g_1^+ \in E(G)$.

Using Property 5 with the triangle $w_2^+ w_2 w_2^-$, the vertex u and the path $\tilde{P}_n^d$, we obtain $uw_2^+ \in E(G)$. Suppose that $w_2^+ \neq y$. Then $g_1 = w_2^+$ and $g_1^- g_1^+ \in E(G)$. This implies $w_2^- g_1^+ \in E(G)$. (Otherwise $w_2^-, g_1^+, u \in N(g_1) \setminus (N(u) \cup N(g_1^+))$ and, by Proposition 1, $|N(u) \cap N(g_1^+)| \geq 3$. Since $|W_u| = 2$ there exists a vertex $u_1 \in (N(u) \cap N(g_1^+)) \setminus W_u$ and an $x - y$ path $P_{n+2}^d = x\tilde{P}_n^d g_1 u u_1 g_1^+ \tilde{P}_n^d y$ with $\Delta_1 = w_2^- w_2 w_2^+ w_2^-$, a contradiction.) But now we obtain an $x - y$ path $P_{n+1}^d = x\tilde{P}_n^d w_1 v w_2 w_1^+ \tilde{P}_n^d w_2^- g_1 g_1^+ \tilde{P}_n^d y$ with $\Delta_1 = w_2^- g_1 g_1^+ w_2^-$, a contradiction.

So, $w_2^+ = g_2 = y$. Then using Property 5 with the triangle $g_1^- g_1 g_1^+$, the vertex v and the $x - y$ path $\tilde{P}_n^d$, we obtain $w_1 \in \{g_1^-, g_1\}$. If $g_1 = w_1$ then $N(v) \cap N(w_1^+) = \{w_1, w_2\}$, $\{u, v, w_1^+\} \subseteq N(w_1) \setminus N(w_1^+)$ and $v, w_1^+ \in N(w_1) \setminus (N(v) \cap N(w_1^+))$ imply $uv \in E(G)$. But then there is a path $P_{n+1}^d = x\tilde{P}_n^d w_1 u v w_2 w_1^+ \tilde{P}_n^d w_2^- y$ with $\Delta_2 = w_1 u v w_1$, a contradiction. So, $w_1 = g_1^-$.

Now we have: $w_1^+ = g_1, w_2^+ = y = g_2, g_1^- g_1^+ \in E(G), w_2^- w_2^+ \in E(G), w_1^+ w_2 \in E(G), d(w_1, u) = 2$ and $w_1^+ \in N(w_1) \cap N(u)$. Clearly, $N(u) \cap N(w_1) \subseteq W_u$. (Otherwise there exists a vertex $u_1 \in (N(u) \cap N(w_1)) \setminus W_u$ and an $x - y$ path $P_{n+2}^d = x\tilde{P}_n^d w_1 u_1 u w_1^+ \tilde{P}_n^d y$ with $\Delta_1 = w_2^- w_2 w_2^+ w_2^-$, a contradiction.) Hence, using Proposition 1, we obtain

$$2 \geq |N(w_1) \cap N(u)| \geq |N(w_1^+) \setminus (N(w_1) \cup N(u))| \quad (18)$$

On the other hand, we have $w_1 u, u w_2 \notin E(G)$ and $u, w_1, w_2 \in N(w_1^+)$. Then $w_1 w_2 \in E(G)$. (Otherwise $u, w_1, w_2 \in N(w_1^+) \setminus (N(u) \cup N(w_1))$ which contradicts (18).) Now we obtain an $x - y$ path $P_{n+1}^d = x\tilde{P}_n^d w_1 v w_2 w_1^+ \tilde{P}_n^d w_2^- y$ with $\Delta_1 = w_1 v w_2 w_1$, a contradiction. The proof of the lemma is complete. $\square$

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References

1. Asratian, A.S.: A criterion for some hamiltonian graphs to be Hamilton-connected. *Australasian J. Comb.* **10**, 193–198 (1994)
2. Asratian, A.S., Broersma, H.J., van den Heuvel, J., Veldman, H.J.: On graphs satisfying a local Ore-type condition. *J. Graph Theory* **21**, 1–10 (1996)

3. Asratian, A.S., Khachatrian, N.K.: Investigation of graph's hamiltonicity using neighborhoods of vertices (Russian). *Doklady Acad. Nauk Armenian SSR* **81**, 103–106 (1985)
4. Asratian, A.S., Sarkisian, G.V.: On cyclic properties of some hamiltonian graphs (Russian). *Diskretnaja Matematika* **3**, 91–104 (1991)
5. Bondy, J.A.: Pancyclic graphs 1. *J. Comb. Theory Ser. B* **11**, 80–84 (1971)
6. Bondy, J.A., Murty, U.S.R.: *Graph Theory with Applications*, MacMillan, London and Elsevier, New York
7. Xiaota, Cai: On the panconnectivity of Ore graph. *Scientia Sinica*, **27**, 684–694 (1984)
8. Häggkvist, R., Faudree, R.J., Schelp, R.H.: Pancyclic graphs-connected Ramsey number. *Ars Combinatoria* **11**, 37–49 (1981)
9. Asratian, A.S., Khachatrian, N.K.: Some localization theorems on hamiltonian circuits. *J. Comb. Theory Ser. B* **49**, 287–294 (1990)
10. Ore, O.: Note on hamiltonian circuits. *Amer. Math. Monthly* **67** 55 (1960)
11. Zhang Ke Min, Holton, D.A., Bau Sheng: On vertex-pancyclic graphs. Preprint (1989)