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A Regularity Statistic for Images

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Abstract

Measures of statistical regularity or complexity for time series are pervasive in many fields of research and applications, but relatively little effort has been made for image data. This paper presents a method for quantifying the statistical regularity in images. The proposed method formulates the entropy rate of an image in the framework of a stationary Markov chain, which is constructed from a weighted graph derived from the Kullback-Leibler divergence of the image. The model is theoretically equal to the well-known approximate entropy (ApEn) used as a regularity statistic for the complexity analysis of one-dimensional data. The mathematical formulation of the regularity statistic for images is free from estimating critical parameters that are required for ApEn.

Keywords: Image complexity, entropy rate, Markov chain, Kullback-Leibler divergence, regularity statistics.

1 Introduction

The study of complexity has been mainly concerned with time series generated by dynamical systems. Most well-known methods for discovering the nonlinear phenomena of time series are based on the concepts of chaos and nonlinear dynamics, and their applications can be found extensively across many disciplines of sciences, medicine, health, and engineering [1]-[5]. Yet relatively little effort has been made toward the development of methods for quantifying the complexity of image content, which inherently exists in many forms of this kind of data.

Representative and chronological reports on various measures of complexity of images include fractal surface measurement methods (isarithmetic, variogram, triangular prism) for characterizing the complexity of remote-sensing landscape images [6], measuring image complexity with information channel maximizing the mutual information [7], mean information gain for quantifying habitat change in a forest ecosystem [8], image complexity for steganalysis using the shape parameters of the generalized Gaussian distribution of wavelets [9], modeling of visual complexity based on fuzzy entropic distance functions [10], image complexity using independent component analysis (ICA) for content-based image retrieval [11], quantification of image complexity using singular value decomposition (SVD) [12], geostatistics and nonlinear dynamics analysis of biomedical signals [13], visual complexity using multiple parameters related to the mechanisms of visual processing [14], measure of image complexity based on compression quality [15], complexity measures for noise filtering [16], chaos analysis and nonlinear dynamics [17]-[19], fractal analysis for texture classification [20], and qualitative evaluation of visual complexity based on the psychology of artworks [21].

Given wide range of applications in different fields, the definition of complexity of an image is still not well-defined [14]. It is rather problem dependent as different methods introduce different standpoints for quantifying the complexity of images. The Shannon entropy of an image intensity histogram has been considered as a measure of image complexity, but this formulation does not consider the spatial distribution of pixels in images [7]. Other methods address the issue of image complexity in the context of image compression. The association between complexity and compression stems from the concept of the Kolmogorov complexity introduced in algorithmic information theory. The Kolmogorov complexity is known as the descriptive complexity of an object being equivalent to the length of the shortest computer program that produces the object as the output from basic elements [22]. However, in addition to the difficulty of the computation of the Kolmogorov complexity, it has been known that the Kolmogorov complexity is not associated with the underlying nature of visual appearance in images [11, 14].

This paper presents a method for measuring image complexity in the context of a regularity statistic, which is a notion of complexity addressed by ApEn to quantify the pattern of regularity in one-dimensional data [23]-[26]. By using the image histogram, which conveys statistical information about the image intensity, each row or column of an image is then normalized to represent a finite probability distribution. The weights between image rows or columns can then be computed using the Kullback-Leibler divergence (KLD) [22] to construct a weighted graph of the image. By imposing constraints on the graph of the image as a stationary Markov chain, the entropy rate of the Markov chain, which is the weighted transition entropy, can be computed. Such an entropy rate has been proved to be equal to the regularity statistic obtained from ApEn [23]. The idea of mapping an image to a

graph, whose vertices are pixels and edge weights are pairwise similarities, was proposed to construct a graph consisting of a subset of edges [27]. The formulation here derives a weighted graph whose vertices are either rows or columns of an image and edge weights are measures of differences between two probability distributions of the corresponding pairs of image rows or columns, and both separate sources of image information are combined as the entropy sum to represent the image regularity statistic.

The motivation for using the KLD to determine the weights of an image-based graph edges is based on several theoretical aspects. In comparison with other metric functions such as the Euclidean distance, the KLD conveys statistical meaning and is geometrically important, because its asymmetric property exists for a manifold of probability distributions while no distance functions can take on this measure [28]. Moreover, the KLD has three special properties that make them important in information processing [29]: 1) an information-theoretic function that satisfies the data processing inequality, 2) being an exponential rate of optimal classifier performance probabilities, and 3) its Hessian matrix is proportional to the Fisher information matrix. It is also proved that the KLD is the only dissimilarity measure of two probability distributions, which satisfies the characterization of entropy [30]. The rest of this paper is organized as follows. Section 2 presents the formulation of ApEn and its proof showing that ApEn is equal to the entropy rate of a first-order stationary Markov chain. Section 3 describes how an image can be induced to a weighted graph using the KLD and the image histogram. Section 4 shows the computation of the entropy rate of a Markov chain of a KLD-weighted graph of an image, which can be used as a tool for quantifying the regularity or complexity of image data. Experiment results of image quality assessment obtained from the proposed method are presented in Section 5. Finally, Section

6 is the concluding remarks of the research findings.

2 ApEn and Entropy Rate of a Markov Chain

The formulation of ApEn [23] is outlined as follows [19]. Let a time series or sequence $\mathbf{u} = (u_1, u_2, \dots, u_N)$. The whole sequence is first split into a set of vectors, each with a predetermined length m : $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N-m+1})$, where $\mathbf{x}_i = (u_i, u_{i+1}, \dots, u_{i+m-1})$, $i = 1, 2, \dots, N - m + 1$. ApEn further requires a predetermined positive tolerance value r that is used for decision making if a pair of vectors are similar to each other. The probability of vector \mathbf{x}_i that is similar to vector \mathbf{x}_j is computed as

$$C_i(m, r) = \frac{1}{N - m + 1} \sum_{j=1}^{N-m+1} \theta(d(\mathbf{x}_i, \mathbf{x}_j)), \quad (1)$$

where $\theta(d(\mathbf{x}_i, \mathbf{x}_j))$ is the Heaviside step function defined as

$$\theta(d(\mathbf{x}_i, \mathbf{x}_j)) = \begin{cases} 1 & : d(\mathbf{x}_i, \mathbf{x}_j) \leq r \\ 0 & : d(\mathbf{x}_i, \mathbf{x}_j) > r \end{cases} \quad (2)$$

The distance between the two vectors can be obtained by

$$d(\mathbf{x}_i, \mathbf{x}_j) = \max_k (|u_{i+k-1} - u_{j+k-1}|), k = 1, 2, \dots, m. \quad (3)$$

The probabilities of all vectors being similar to one another are computed as

$$C(m, r) = \frac{1}{N - m + 1} \sum_{i=1}^{N-m+1} \log(C_i(m, r)). \quad (4)$$

ApEn is expressed as a family of formulas, with fixed m and r , as

$$ApEn(m, r) = \lim_{N \rightarrow \infty} (C(m, r) - C(m + 1, r)). \quad (5)$$

For the first-order stationary Markov chain of a discrete state space X , with $r < |x - y|$, $x \neq y$, where x and y are state-space values, $\forall m$, it is shown in [23] that ApEn is equal to the entropy rate of the Markov chain, which is

$$ApEn(m, r) = - \sum_{x \in X} \sum_{y \in X} \pi_x p_{xy} \log p_{xy}, \quad (6)$$

where π_x is the density function of x , and p_{xy} is the probability of transition from x to y .

The development of the entropy rate of a stationary Markov chain of a weighted graph of an image is described in the subsequent sections.

3 A Markovian KLD-Weighted Graph

To model an image as a discrete stochastic process, let the sequence of random variables $\{X_i\}$, $i = 1, \dots, N$, represent the sequence of N rows or N columns of the image. Here, using image rows and columns for extracting the spatial information of the image is to allow for the computation of the KLD, which will be presented subsequently. In addition, the use of vertical and horizontal orientations for capturing spatial content in images has been shown to be effective in several studies, such as the semi-variogram [31, 32] and Sobel kernels [33]. Such a sequence of random variables of the image is assumed to be stationary with respect to shifts in the space index, that is $Pr(X_1 = x_1, X_2 = x_2, \dots, X_N = x_N) = Pr(X_{1+s} = x_1, X_{2+s} = x_2, \dots, X_{N+s} = x_N)$, $\forall N$, $\forall s$, and $\forall x_i \in \mathbf{X}$. The assumption of stationarity is particularly found appropriate for texture images that consists of a single

type of texture and natural images [34], where the local statistical properties do not change with spatial locations in the image. The discrete stochastic process of the image is also a Markov chain, in which each X_i is assumed to be dependent only on its immediately preceding random variable and conditionally independent of all other preceding random variables, that is $Pr(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_1 = x_1) = Pr(X_{n+1} = x_{n+1} | X_n = x_n)$. The focus here is on the first-order Markov chain because, firstly, it is possible to convert the Markov chain of any order into the first-order Markov chain by appropriately enlarging the state space, and secondly, it is to avoid a model that can be cumbersome and impractical from both statistical and computational standpoints. Similar Markovian property that assumes the value of a pixel depends directly only on the values of its neighboring pixels and is independent of other pixels has been introduced for the construction of Markov random fields for texture analysis in images [35].

The stationary Markov process of an image can be constructed as a weighted graph G that consists of two finite sets: a set of vertices $V(G)$ and a set of edges $D(G)$, where each edge is associated with a pair of vertices called its endpoints. A weight $w_{ij} \geq 0$ is assigned on the edge joining its endpoints from i to j (for an undirected graph, $w_{ij} = w_{ji}$), which can be obtained as the Kullback-Leibler divergence of the two probability distributions of image rows or columns i and j as follows.

Let $g = (g_1, g_2, \dots, g_M)$ and $h = (h_1, h_2, \dots, h_M)$ be two discrete probability distributions of two corresponding image rows or columns i and j of length M , respectively, obtained from the image histogram. A weight w_{ij} of graph G of an image can be computed by the KLD as a statistical measure of the departure of the candidate distribution h from the reference model g as follows [22]:

$$w_{ij} = D(g||h) + D(h||g), \quad (7)$$

where

$$D(g||h) = \sum_{k=1}^M g_k \log \frac{g_k}{h_k}, \quad (8)$$

$$D(h||g) = \sum_{k=1}^M h_k \log \frac{h_k}{g_k}, \quad (9)$$

where $0 \log(0/0) = 0$, $0 \log(0/q) = 0$, and $p \log(p/0) = \infty$.

Based on the properties of the KLD, $w_{ij} \geq 0$. Figure 1 shows the process of transforming an image into a graph whose edge weights are obtained by the KLD based on the histogram of the image, where the normalization of global probability distribution given by the image histogram is performed to ensure the summation of the probability density function of each image row or column is equal to one (each row or column histogram is normalized to 1) to allow the calculation of the KLD between rows or columns.

4 Entropy Rate of a Markov Chain of an Image

After modeling an image as a KLD-weighted graph, the entropy rate of the Markov chain of a weighted graph can be readily computed as follows. Let $\mathbf{X} = \{X_1, X_2, \dots, X_N\}$ be a sequence of N random variables. The entropy rate for the stochastic process \mathbf{X} , denoted as $H(\mathbf{X})$, which grows with N , is derived in [22], which is known as the average transition entropy in terms of the entropy of the stationary distribution and the total number of edges in the weighted graph. Because the edge weights of the graph are the KLD values, the derivations of the probabilities of connecting the edges and the state probabilities are taken

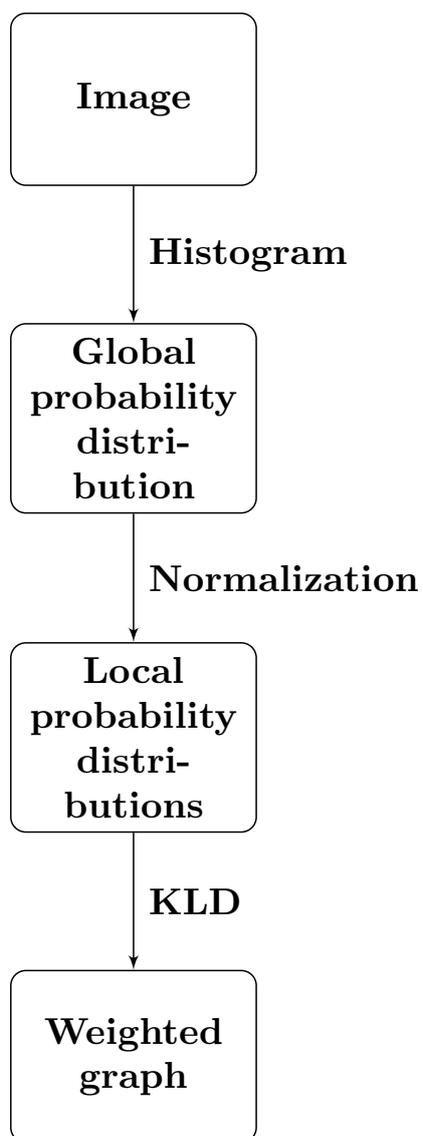


Figure 1: Procedure for constructing a weighted graph of an image using its pixel value histogram information and KLD-based edge weights.

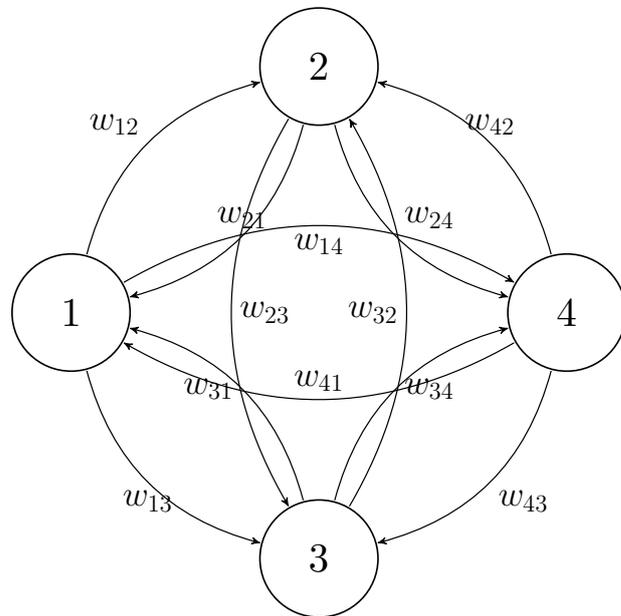


Figure 2: A 4-state Markov chain of a KLD-weighted graph of an image, where states are either rows or columns of the image.

as the complement. The entropy rate for a stochastic process is mathematically expressed as [22]

$$H(\mathbf{X}) = \lim_{N \rightarrow \infty} \frac{1}{N} H(X_1, X_2, \dots, X_N), \quad (10)$$

where the limit exists.

Furthermore, the conditional entropy rate of \mathbf{X} , denoted as $H^*(\mathbf{X})$, is defined as

$$H^*(\mathbf{X}) = \lim_{N \rightarrow \infty} \frac{1}{N} H(X_N | X_{N-1}, X_{N-2}, \dots, X_1), \quad (11)$$

where the limit exists.

For a stationary stochastic process, the limit expressed in Equations (10) and (11) exists and $H(\mathbf{X}) = H^*(\mathbf{X})$. Thus, for a stationary Markov chain, the entropy rate can be calculated as [22]

$$H(\mathbf{X}) = H^*(\mathbf{X}) = H(X_2 | X_1) = - \sum_i \mu_i \sum_j p_{ij} \log p_{ij}, \quad (12)$$

where

$$p_{ij} = 1 - \frac{w_{ij}}{c_i}, \quad (13)$$

which is the probability of the edge connecting node i to node j , and c_i is the total weight of edges emitting from node i :

$$c_i = \sum_j w_{ij}, \quad (14)$$

and μ_i is the stationary probability of state i , which is proportional to the weight of all edges emitting from node i , expressed as follows

$$\mu_i = 1 - \frac{c_i}{2C}, \quad (15)$$

where

$$C = \sum_i \sum_{j > i} w_{ij}. \quad (16)$$

It can be of interest to obtain the entropy rate as an average of the total number of the connected edges in the weighted graph, denoted as $E(\mathbf{X})$:

$$E(\mathbf{X}) = -\frac{1}{M} \sum_i \mu_i \sum_j p_{ij} \log p_{ij}, \quad (17)$$

where M is the total number of the connected edges in the graph.

Figure 2 shows an example of a 4-state Markov chain for the KLD-weighted graph, where the states are either rows or columns of an image. Thus, by this formulation, the complexity analysis takes into account statistical information related to both intensity and structure of an image.

The complexity of an image is obtained as the average of the sum of the complexity measures for the rows and columns of the image. For a purely isotropic image, which does not exist in applications, the complexity measures in the rows and columns of the image are the same. For an image of partially isotropic or anisotropic data, the complexity measures in the rows and columns are expected to be different. As the Markov chain states are the rows or the columns of the image, images will have the same complexity if their rows and columns have

the same respective probability distributions, indicating the same spatial pattern of intensity occurrences in both horizontal and vertical directions in the images.

5 Experiments and Discussion on Image Quality Assessment

The measures of complexity in images provide useful insights into studying the abstract information of image content that can be used as a tool for assessing image quality with respect to the effects of resolution reduction [15] and distortion [36]. It is known that the CSIQ (Categorical Subjective Image Quality) database [36] consists of images that reveal a trend of complexity with changes in the resolution reduction, and an attempt to test methods that can show this trend has been recently reported in literature [15]. There are 5 categories of images in the CSIQ database: *animals*, *landscape*, *people*, *plants*, and *urban*. Each category has 30 color (standard RGB) images, and each image is of size 512×512 pixels. It is of interest to test if the proposed regularity statistic as a measure of image complexity can consistently capture this trend for image resolution changes. For the construction of the Markov chain of an image, the number of bins in the histogram is determined by the image type. For a grayscale image, the histogram has 256 bins. For a color image represented by an indexed image, which consists of a data matrix and a color map matrix, the number of bins in the histogram is equal to the number of entries in the color map. In this study, the images were converted to grayscale in order to avoid the effect of color on compression as suggested in [15]. Figure 3 shows 5 selected grayscale images for the 5 categories and their



(a)



(b)



(c)



(d)



(e)



(f)



(g)



(h)



(i)



(j)

Figure 3: CSIQ images and their 50%-reduced resolution: elk (a) and its reduced resolution (b), lake (c) and its reduced resolution (d), native American (e) and its reduced resolution (f), mushroom (g) and its reduced resolution (h), Boston (i) and its reduced resolution (j).

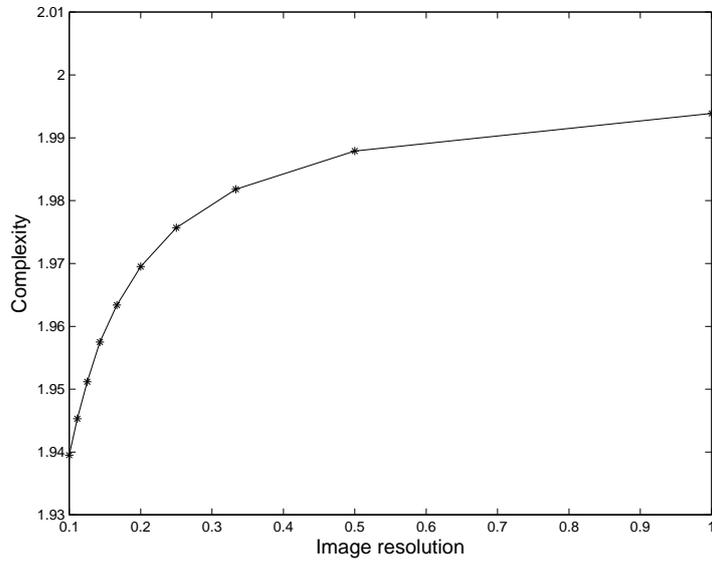


Figure 4: Image complexity vs. image resolution of the CSIQ *animal* category.

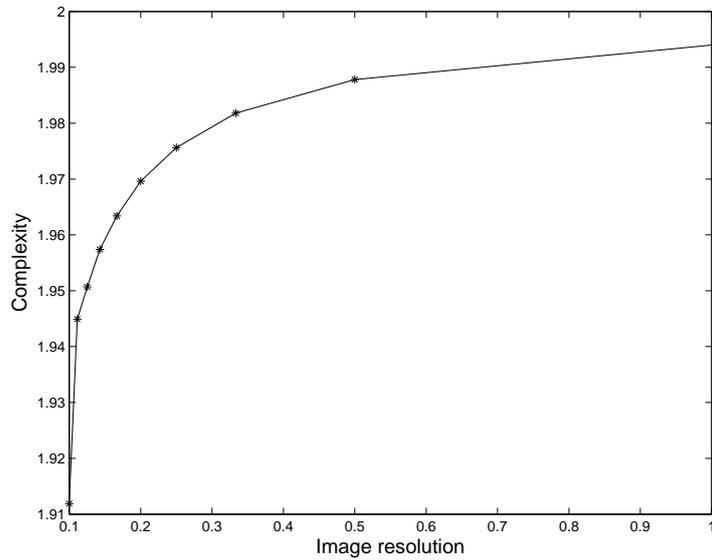


Figure 5: Image complexity vs. image resolution of the CSIQ *landscape* category.

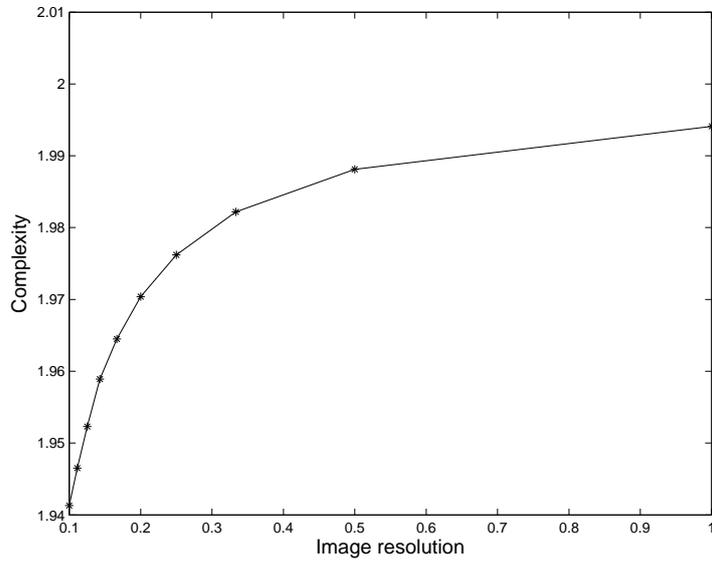


Figure 6: Image complexity vs. image resolution of the CSIQ *people* category.

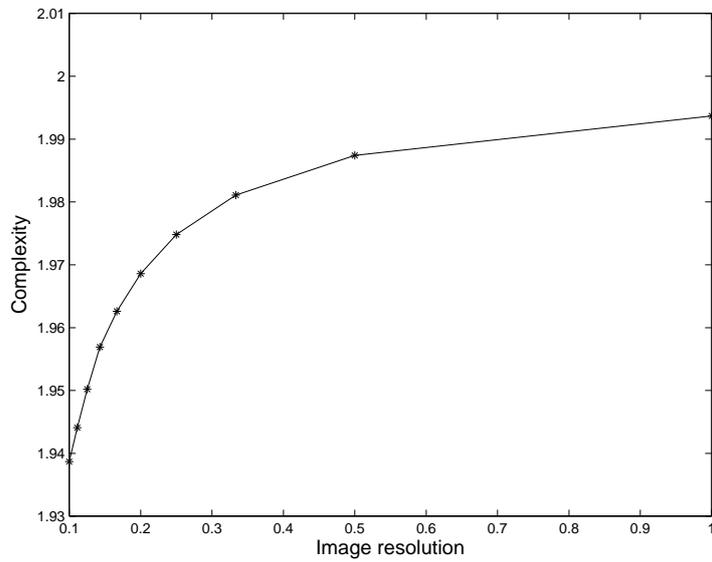


Figure 7: Image complexity vs. image resolution of the CSIQ *plant* category.

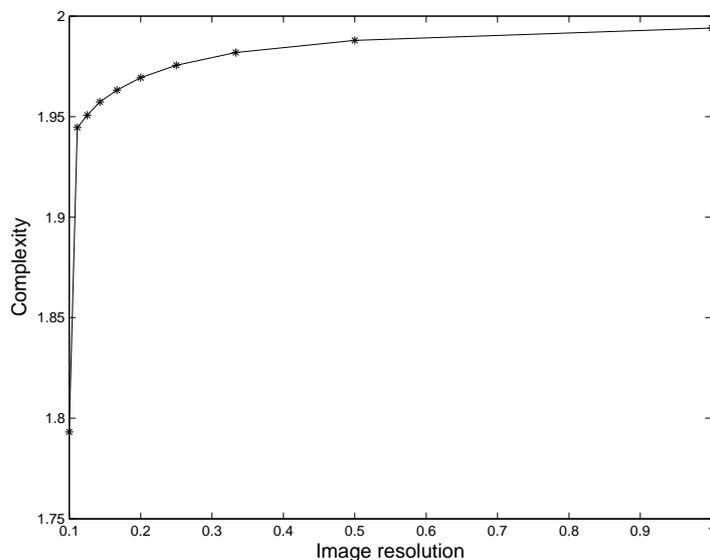


Figure 8: Image complexity vs. image resolution of the CSIQ *urban* category.

images of the reduced resolution of 50%. Figures 4-8 are the plots of complexity vs. image resolution for the 5 categories of *animals*, *landscape*, *people*, *plants*, and *urban*, respectively. All the five plots consistently show a trend of increasing image complexity with increasing image resolutions from 1/10 to 1, where the curves are steeper with reductions from 1/10 to 1/2. This complexity indicator can be useful as a computational perception tool for assessing and designing image quality data.

In Figures 4-8, the image complexity displays a logarithmic growth with respect to the image resolution, in which a steep ascent up to about 40% can be observed in the image resolution and afterwards the slope is much gentler. It has been known that the definition of image complexity is not as simple as it appears to be [15], making it difficult for a direct comparison of various complexity measures. However, in comparison with the image complexity measures as a function of compression levels for JPEG and JPEG2000 compression presented in [15],

the image complexity defined by lossy compression is a monotonically increasing function of the compression level, while the image complexity defined by the root-mean-square error is not a monotonic function of the compression level, where the complexity values drop after a certain compression level after its increasing values. Furthermore, in comparison with the complexity measures against different image resolutions presented in [15], the proposed image complexity defined as the entropy rate exhibits a smoother function of the image resolution, bearing a resemblance to the logarithmic form, of which similar nonlinear behavior has been found in many complex deterministic systems [37, 38].

The presented experiments have focused on the variations of complexity in images with their resolutions, which suggest useful implications in many applications, where it might contribute to providing a ground truth reference or an absolute complexity for assessing among images. Such applications include the quality assessment of image and video databases [39], content-based image retrieval [11], and image and texture classification [40, 41]. Particularly, the measure of image complexity has been utilized to determine the level of image compression and bandwidth, where an image with lower complexity indicates easier compression and less bandwidth requirement [42, 15].

6 Conclusion

The formulation of a new regularity statistic as a measure of complexity in terms of entropy rate in images has been presented. It is theoretically equivalent to the regularity statistic for quantifying complexity in time series known as ApEn. In other words, the entropy rate of the Markov chain of an image is equivalent to $ApEn(m, r)$ for a time series, where the

latter depends on two parameters m and r to quantify the similarity between two variables in the state space. The direct calculation of the entropy rate out of the weighted graph not only provides a procedure for quantifying the irregularity of an image but also circumvents the necessity of defining the two parameters required by the ApEn. This implies, through the expression of Equation (6), that the calculation of the entropy rate selects specific and intrinsic parameters m and r , separating the image Markov chain states. Based on the experimental results, the proposed method is promising as a potential method for quantifying image complexity, with a particular application to image quality assessment. Other appropriate applications of the proposed method for quantifying image complexity are also promising.

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