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Abstract

For many properties $P$, Bondy and Chvátal (1976) have found sufficient conditions such that if a graph $G + uv$ has property $P$ then $G$ itself has property $P$. In this paper we will give a generalization that will improve ten of these conditions.

1. Introduction

Our notation and terminology follows Berge [1] and Harary [7]. We denote the set of all graphs of order $n$ by $R_n$. The distance between vertices $u$ and $v$ in the graph $G = (V(G), E(G))$ is denoted by $d_G(u, v)$. Let $k$ be a positive integer. For each $u \in V(G)$ we denote by $N^k(u)$ and $M^k(u)$ the sets of all $v \in V(G)$ with $d_G(u, v) = k$ and $d_G(u, v) \leq k$, respectively.

The $k$-closure of $G$ is the graph $C_k(G)$ obtained from $G$ by recursively joining pairs of non-adjacent vertices whose degree-sum is at least $k$, until no such pair remains.

For many properties $P$, Bondy and Chvátal [2] have found sufficient conditions such that if a graph $G + uv$ has property $P$, then $G$ itself has property $P$. In particular it is shown (by paraphrasing Ore’s proof [10]) that if $G \in R_n$, $uv \notin E(G)$, $d_G(u) + d_G(v) \geq n$ and $G + uv$ is hamiltonian, then $G$ is hamiltonian. Using this condition Bondy and Chvátal [2] have found the following sufficient condition for a graph to be hamiltonian: If the graph $C_n(G)$ is hamiltonian, then $G$ is hamiltonian. In particular, if $n \geq 3$ and $C_n(G) = K_n$, then $G$ is hamiltonian. It was noted in [2], that many generalizations of Dirac’s condition [6] including those of Chvátal [4] and Las Vernas [9], guarantee that $C_n(G) = K_n$. It was shown in [5], that if $C_n(G) = K_n$ then $|E(G)| \geq \lceil (n + 2)^2 / 8 \rceil$. 


In this paper we will give a generalization that will improve the conditions of Bondy–Chvátal for ten properties considered in [2]. For example, we prove that if \( G + uv \) is hamiltonian, \( d_G(u, v) = 2 \) and
\[
d_G(u) + d_G(v) \geq |M_G^k(u)| + |N_G^k(u) \cap N_G^l(v)| + r
\]
then \( G \) is hamiltonian. Using this condition, we define a new closure of the graph \( G \), which has \( C_n(G) \) as a spanning subgraph, and \( G \) is hamiltonian if and only if this new closure of \( G \) is hamiltonian. It is shown that for every \( n \geq 6 \) there is \( G \in R_n \) such that \( |E(G)| = 2n - 3 \) and the new closure of \( G \) is a complete graph.

These results can be viewed as a step towards a unification of the various known results on the existence of hamiltonian cycles in undirected graphs.

We will use the methods of proof that were used in [2].

2. Stability and closures

Let \( P \) be a property defined on \( R_n \) and \( r \) be an integer.

**Definition 1.** The property \( P \) is \((k, r)\)-stable, \( k \geq 2 \), if whenever \( G + uv \) has property \( P \), \( d_G(u, v) = 2 \) and
\[
d_G(u) + d_G(v) \geq |M_G^k(u)| + |N_G^{k+1}(u) \cap N_G^l(v)| + r
\]
then \( G \) itself has property \( P \).

**Remark 1.** If \( k \geq 3 \) and \( d_G(u, v) = 2 \) then (2.1) is equivalent to
\[
d_G(u) + d_G(v) \geq |M_G^k(u)| + r
\]
because
\[
N_G^{k+1}(u) \cap N_G^l(v) = \emptyset.
\]

**Remark 2.** If \( d_G(u, v) = 2 \) then (2.1) is equivalent to
\[
|N_G^l(u) \cap N_G^l(v)| \geq 1 + \sum_{j=2}^{k} |N_G^j(u) \setminus N_G^l(v)| + r
\]
because
\[
|M_G^k(u)| = 1 + \sum_{j=1}^{k} |N_G^j(u)|, \quad d_G(u) = |N_G^l(u)|, \quad d_G(v) = \sum_{j=1}^{3} |N_G^j(u) \cap N_G^l(v)|
\]
and
\[
N_G(u) \setminus N_G^1(v) = N_G^j(u), \quad N_G^j(u) \cap N_G^l(v) = \emptyset \quad \text{for } j \geq 4.
\]

From Definition 1 we have the following.
Proposition 1. If property $P$ is $(k, r)$-stable and $m > k \geq 2$, $t > r$, then:

(a) $P$ is $(m, r)$-stable,
(b) $P$ is $(k, t)$-stable.

A property $P$ is called $(n + r)$-stable [2] if whenever $G \in R_n$, $G + uv$ has property $P$ and $d_G(u) + d_G(v) \geq n + r$, then $G$ itself has property $P$.

Proposition 2. If property $P$ is $(k, r)$-stable, $k \geq 2$ and $r \geq -1$, then $P$ is $(n + r)$-stable.

Proof. Assume $G \in R_n$, $G + uv$ has property $P$ and $d_G(u) + d_G(v) \geq n + r$. Clearly,

$$d_G(u, v) = 2 \quad \text{and} \quad d_G(u) + d_G(v) \geq |M^k_G(u)| + |N^{k+1}_G(u) \cap N^1_G(v)| + r.$$ 
Hence $G$ has property $P$ which completes the proof. □

In [2], the smallest integer $r(P)$ was found for many properties $P$ such that $P$ is $(n + r(P))$-stable.

In this paper we will find for ten of these properties $P$ the smallest integer $k(P) \geq 2$ such that $P$ is $(k(P), r(P))$-stable.

Definition 2. Let $G \in R_n$, $H \in R_n$ and let $H$ be a supergraph of $G$. We shall say that $H$ is a $(k, r)$-closure of $G$, $k \geq 2$, if

$$d_H(u) + d_H(v) < |M^k_H(u)| + |N^{k+1}_H(u) \cap N^1_H(v)| + r$$

for all $uv \notin E(H)$ with $d_H(u, v) = 2$ and there exists a sequence of graphs $H_1, \ldots, H_m$ such that $H_1 = G$, $H_m = H$ and for $1 \leq i \leq m - 1$ $H_{i+1} = H_i + u_1v_i$, where $d_H(u_i, v_i) = 2$ and

$$d_H(u_i) + d_H(v_i) \geq |M^k_H(u_i)| + |N^{k+1}_H(u_i) \cap N^1_H(v_i)| + r.$$ 

A $(k, r)$-closure of a graph is certainly not unique. For example, the graph $G$ in Fig. 1 has two $(2, 0)$-closures, namely $G + uv$ and $G + uw$.

It is not difficult to see that if $r \geq -1$ then $C_n + r(G)$ is a subgraph of each $(k, r)$-closure of $G$, $k \geq 2$.

From Definition 1 and 2 we have the following.

Proposition 3. If $P$ is $(k, r)$-stable, $k \geq 2$ and some $(k, r)$-closure of $G$ has property $P$, then $G$ itself has property $P$. 

![Fig. 1.](image-url)
3. The Hamiltonian Property

Lemma 1. Let \( G \in R_n \), \( n \geq 3 \). If \( u_1, u_2, \ldots, u_n \) is a Hamiltonian path of \( G \), \( d_G(u_1, u_n) = 2 \), and

\[
d_G(u_1) + d_G(u_n) \geq |M_G^2(u_1)| + |N_G^2(u_1) \cap N_G^1(u_n)|
\]

(3.1)

then there is a \( m \) such that \( 2 \leq m \leq n-2 \), \( u_m u_{m+1} \in E(G) \) and \( u_n u_m \in E(G) \).

Proof. Let \( N^1_G(u_1) = \{u_1, \ldots, u_t\} \). If \( u_n u_{t-1} \notin E(G) \) for every \( j \), \( 1 \leq j \leq t \), then

\[
|N^1_G(u_1) \cap N^1_G(u_n)| + |N^2_G(u_1) \cap N^1_G(u_n)| < |M^2_G(u_1)| - d_G(u_1).
\]

But then

\[
d_G(u_n) < |M^2_G(u_1)| + |N^2_G(u_1) \cap N^1_G(u_n)| - d_G(u_1)
\]

because

\[
d_G(u_n) = \sum_{j=1}^{n} |N^1_G(u_1) \cap N^1_G(u_n)|.
\]

This contradicts (3.1) and completes the proof.

Theorem 1. The property of containing a Hamiltonian cycle is \((2,0)\)-stable.

Proof. Let \( G \in R_n \), \( n \geq 3 \), \( d_G(u, v) = 2 \) and

\[
d_G(u) + d_G(v) \geq |M^2_G(u)| + |N^2_G(u) \cap N^1_G(v)|.
\]

Suppose that \( G + uv \) is Hamiltonian, but \( G \) is not. Then, \( G \) has a Hamiltonian path \( u_1, u_2, \ldots, u_n \) with \( u_1 = u, u_n = v \). From Lemma 1, there is an integer \( m \) such that \( 2 \leq m \leq n-2 \), \( u_m u_{m+1} \in E(G) \) and \( u_1 u_{m+1} \in E(G) \). But then \( G \) has the Hamiltonian cycle \( u_1 u_2 \cdots u_m u_n u_{n-1} \cdots u_{m+1} u_1 \). This contradicts the hypothesis, and completes the proof.

From Theorem 1 and Proposition 1 it follows that the property of containing a Hamiltonian cycle is \((3,0)\)-stable. Hence, from Remark 1 we have the following.

Corollary 1. Let \( G \in R_n \), \( n \geq 3 \). If \( d_G(u, v) = 2 \), \( d_G(u) + d_G(v) \geq |M^2_G(u)| \) and \( G + uv \) is Hamiltonian, then \( G \) is Hamiltonian.

Remark 3. If the \((2,0)\)-closure of \( G \) has the Hamiltonian cycle \( C \), then, by using Lemma 1, one can transform \( C \) into a Hamiltonian cycle in \( G \) in exactly the same way that the Hamiltonian cycle in \( C_n(G) \) was transformed into a Hamiltonian cycle in \( G \) (see [2]).
Corollary 2. Let \( G \in \mathbb{R}_n \), \( n \geq 3 \). If \( K_n \) is the \((2, 0)\)-closure of \( G \), then \( G \) is hamiltonian.

Theorem 2. For every \( n \geq 6 \) there is \( G \in \mathbb{R}_n \) such that \( |E(G)| = 2n - 3 \) and \( K_n \) is the \((2, 0)\)-closure of \( G \).

Proof. Let \( t \) be the integer part of the number \( n/2 \). Consider a sequence of graphs \( G_1, \ldots, G_t \), such that \( G_i = K_n \), \( V(G_i) = \{u_1, u_2, \ldots, u_n\}, i = 1, \ldots, t \) and

\[
E(G_{i-k+1}) = \{u_iu_j \mid 2k - 1 < i < j < n\}
\cup \{u_{2i-1}u_{2i}, u_{2i-1}u_{2i+1}, u_{2i}u_{2i+1}, u_{2i}u_{2i+2} \mid i = 1, \ldots, k - 1\}
\]

for every \( k, 2 \leq k \leq t \). (For \( n = 8 \) the graphs \( G_1, G_2, G_3 \) are shown in Fig. 2.)

Clearly

\[
|E(G_1)| = 2n - 3 \quad \text{and} \quad |E(G_{i-k+2})| - |E(G_{i-k+1})| = 2n - 4k + 1, \ k = 2, \ldots, t.
\]

We shall show that \( G_i \) is a \((2, 0)\)-closure of \( G_1 \). For each \( k, 2 \leq k \leq t \), define \( H_{k,0}, H_{k,1}, \ldots, H_{k,2n-4k+1} \) to be a sequence of graphs such that \( H_{k,0} = G_{t-k+2} \) and

\[
H_{k,i+1} = \begin{cases} 
H_{k,i} + u_{2i-1}u_{2i} & \text{for } i = 0, 1, \ldots, n - 2k - 1, \\
H_{k,i} + u_{2n-2k-i}u_{2k-3} & \text{for } i = n - 2k, \ldots, 2n - 4k.
\end{cases}
\]

It is not difficult to verify that if \( 2 \leq k \leq t \), \( 0 \leq i < 2n - 4k + 1 \) and \( H_{k,i+1} = H_{k,i} + u_{2k}u_{n-3} \), then

\[
d_{H_{k,i}}(u_p, u_r) = 2
\]

and

\[
d_{H_{k,i}}(u_p) + d_{H_{k,i}}(u_r) \geq |M_{H_{k,i}}(u_p)| + |N_{H_{k,i}}(u_p) \cap N_{H_{k,i}}(u_r)|.
\]

Hence \( G_i \) is a \((2, 0)\)-closure of \( G_1 \) and this completes the proof. \( \square \)

4. Other properties

By \( C_s \) and \( P_s \) we mean a cycle and a path on \( s \) vertices, respectively.

Theorem 3. Let \( n, s \) be positive integers with \( 4 \leq s \leq n \). Then the property of containing a \( C_s \) is \((2, n-s)\)-stable.
Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^2(u) \cap N_G^1(v)| + n - s. \quad (4.1)$$

From Remark 2 we have that (4.1) is equivalent to

$$|N_G^2(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)| + n - s. \quad (4.2)$$

If $G + uv$ contains a $C_s$ but $G$ does not, then $G$ contains a path $u_1, u_2, \ldots, u_s$ with $u_1 = v, \ u_s = u$. Let $H$ be the subgraph of $G$ induced by $\{u_1, u_2, \ldots, u_s\}$. Then $H + uv$ is hamiltonian but $H$ is not. Clearly, $v \in N_G^2(u) \setminus N_G^1(v)$ and

$$|N_G^2(u) \setminus N_G^1(v)| \geq |N_H^1(u) \setminus N_H^1(v)| + n - s. \quad (4.3)$$

From (4.2) and (4.3) we have $|N_H^1(u) \cap N_H^1(v)| \geq 1$, and so $d_H(u, v) = 2$. Now from Theorem 1 and Remark 2, it follows that

$$|N_H^1(u) \cap N_H^1(v)| < 1 + |N^2_H(u) \setminus N^1_H(v)|. \quad (4.4)$$

It's clear, that $|N_H^2(u) \setminus N_H^1(v)| \leq |N_G^2(u) \setminus N_G^1(v)|$. From (4.3) and (4.4) we can deduce that

$$|N_G^2(u) \cap N_G^1(v)| \leq |N_H^1(u) \cap N_H^1(v)| + n - s \leq |N_H^2(u) \setminus N_H^1(v)| + n - s \leq |N_G^2(u) \setminus N_G^1(v)| + n - s. \quad (4.5)$$

This contradicts (4.2) and completes the proof. $\Box$

Theorem 4. Let $n, s$ be positive integers such that $s$ is even and $4 \leq s < n$. Then the property of containing a $C_s$ is $(4, n - s - 1)$ stable.

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + n - s - 1. \quad (4.6)$$

From Remark 2 we have that (4.6) is equivalent to

$$|N_G^2(u) \cap N_G^1(v)| \geq n - s + \sum_{j=2}^{4} |N_G^2(u) \setminus N_G^1(v)|. \quad (4.7)$$

If $G + uv$ contains a $C_s$ but $G$ does not, then $G$ contains a path $u_1, u_2, \ldots, u_s$ with $u_1 = v, \ u_s = u$. Let $H$ be the subgraph of $G$ induced by $\{u_1, u_2, \ldots, u_s\}$. As in the proof of Theorem 3, we have (4.5). It's clear, that (4.5) and (4.7) imply

$$|N_G^2(u) \setminus N_G^1(v)| = |N_G^2(u) \setminus N_G^1(v)| = 0,$$

$$|N_H^1(u) \cap N_H^1(v)| = |N_H^2(u) \setminus N_H^1(v)| = |N_G^2(u) \setminus N_G^1(v)|,$$

and

$$|N_G^2(u) \cap N_G^1(v)| = |N_H^1(u) \cap N_H^1(v)| + n - s. \quad (4.8)$$

Since $n > s$, $u$ and $v$ have a common neighbour $w$.

Clearly,

$$\{k \mid 2 \leq k \leq s - 2, u_k u_k+1 \in E(G), u_k u_k+1 \in E(G)\} = \emptyset, \quad (4.9)$$
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because in fact if \( u_n u_k \in E(G) \) and \( u_1 u_{k+1} \in E(G) \) for some \( k \), then
\[ u_1 u_2 \cdots u_k u_{k-1} \cdots u_{k+1} \] is a \( C_s \) in \( G \).

In addition we have \( u_1 u_3 \notin E(G) \), for otherwise \( u_1 u_3 u_4 \cdots u_k u_1 \) is a \( C_s \) in \( G \). Similarly, we have \( u_1 u_{s-2} \notin E(G) \) for otherwise \( u_1 u_2 \cdots u_s w u_1 \) is a \( C_s \) in \( G \).

Let \( \mathcal{N}^1_H(u) \cap \mathcal{N}^1_H(v) = \{ u_{i_0}, \ldots, u_{i_t} \} \), \( i_0 = 0 \) and \( i_1 < \cdots < i_t \) if \( t \geq 2 \). Then (4.9) and \( u_i \in \mathcal{N}^3_H(u) \setminus \mathcal{N}^1_H(v) \) imply that for \( j, 0 \leq j \leq t-1 \), there exist \( r_j \), such that
\[ i_j < r_j < i_{j+1} \] and \( u_{i_j} \in \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \). We can take \( r_0 = 1 \).

We will now show that \( i_t = s - 1 \). Suppose \( i_t < s - 1 \). Then (4.9) and \( u_i u_{s-2} \notin E(G) \) imply that there exists \( r_t \) such that \( i_t < r_t \leq s - 2 \), \( u_{i_t} \notin E(G) \), \( u_{i_t} \notin E(G) \) and \( u_{i_t} \notin E(G) \). But then \( \{ u_i \mid i = 0, 1, \ldots, t \} \subseteq \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \) and \( |\mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v)| \geq t + 1 \), which contradicts (4.8). Therefore \( i_t = s - 1 \).

Next, note that if \( 2 \leq i \leq s - 3 \), then
\[ u_i u_i \in E(G) \Rightarrow u_i u_{i+1} \notin E(G). \] (4.10)

Otherwise \( u_1 \cdots u_i u_{i+1} u_{i+2} \cdots u_{s-1} u_1 \) is a \( C_s \) in \( G \).

We have that
\[ d_H(u_3, u) \leq 4 \] and \( \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) = \mathcal{N}^4_H(u) \setminus \mathcal{N}^3_H(v) = \emptyset \).

Therefore \( d_G(u_3, u) \leq 2 \). If \( d_G(u_3, u) = 1 \), then from (4.9) and (4.10) we have \( u_4 \in \mathcal{N}^3_H(u) \setminus \mathcal{N}^1_H(v) \). This implies \( \{ u_4, u_n, \ldots, u_{n-1} \} \subseteq \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \) and \( |\mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v)| \geq t + 1 \) which contradicts (4.8).

If \( d_G(u_3, u) = 2 \) and \( i_t \geq 4 \) then \( \{ u_3, u_n, \ldots, u_{n-1} \} \subseteq \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \), which contradicts (4.8).

Let \( d_G(u_3, u) = 2 \) and \( i_1 = 2 \). Then \( t \geq 2 \) and \( u_1 u_{i_1} \notin E(G) \), \( j = 1, \ldots, t \), because if \( u_1 u_{i_1} \in E(G) \) for some \( j \), then \( u_1 u_1 \cdots u_i u_2 u_3 \cdots u_{i_1-j} u_1 \) is a \( C_s \) in \( G \).

It follows from (4.10) that \( u_{i_1-j} \in \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \), \( j = 1, \ldots, t \).

Also, \( i_{j+1} - i_j = 2 \) for every \( j = 1, \ldots, t - 1 \), because if \( i_{j+1} - i_j > 2 \) for some \( j \), then
\[ \{ u_{i_1-i_j} \}, \ldots, u_{i_1-i_j} u_{i_1} \} \subseteq \mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v) \] and \( |\mathcal{N}^2_H(u) \setminus \mathcal{N}^1_H(v)| \geq t + 1 \), which contradicts (4.8).

Therefore \( s = 2t + 1 \), which contradicts the hypothesis, that \( s \) is even, and completes the proof. \( \square \)

Fig. 3 (with \( n = 10, s = 8 \)) and its obvious generalization show that the property of containing a \( C_s \) with \( s = 2p < n \) is not \((3, n-s-1)\)-stable for \( s \geq 8 \).
**Theorem 5.** Let $n$, $s$ be positive integers with $4 \leq s \leq n$. Then the property of containing a $P_s$ is $(4, -1)$-stable.

**Proof.** Let $G \in R_n$, $d_G(u, v) = 2$ and
\[ d_G(u) + d_G(v) \geq |M^*_{2s}(u)| - 1. \]

From Remark 2 we have that (4.11) is equivalent to
\[ |N^1_G(u) \cap N^1_G(v)| \geq \sum_{j=2}^{s} |N^1_G(u) \setminus N^1_G(v)|. \]  

Suppose $G + uv$ contains a $P_s$ but $G$ does not. Then $G + uv$ contains a path $u_1, u_2, \ldots, u_s$ with $u_m = u$, $u_{m+1} = v$ for some $m$. Let $N^1_G(u) \cap N^1_G(v) = \{u_i, \ldots, u_j\}$, $i_0 = 1$, $i_{t+1} = s$, $i_0 < i_1 < \cdots < i_{t+1}$ and let $i_k < m < i_{k+1}$. Clearly,
\[ \{ j \mid 1 \leq j \leq s, u_m u_j \in E(G), u_{m+1} u_{j+1} \in E(G) \} = \emptyset \]
because if $u_m u_j \in E(G)$ and $u_{m+1} u_{j+1} \in E(G)$ for some $j$, then $G$ contains a $P_s$ where
\[ P_s = [u_1 u_2 \cdots u_m u_{m-1} \cdots u_j u_{j+1} u_{m+1} u_{m+2} \cdots u_s] \]
in addition we have $u_k u_m \notin E(G)$ and $u_k u_{m+1} \notin E(G)$. Then for each $j$, $j \neq k$, $1 \leq j \leq t$, there is a $u_{r_j}$ such that $i_j < r_j < i_{j+1}, u_{r_j-1} \in E(G), uu_{r_j} \notin E(G)$ and $uu_{r_j} \notin E(G)$. Therefore $u_{r_j} \in N^2_G(u) \setminus N^1_G(v), j \neq k, 1 \leq j \leq t$, and
\[ |N^1_G(u) \cap N^1_G(v)| \leq |N^2_G(u) \setminus N^2_G(v)|. \]

It follows from (4.12) and (4.13) that $N^3_G(u) \setminus N^1_G(v) = N^2_G(u) \setminus N^1_G(v) = \emptyset$ and
\[ t = |N^1_G(u) \cap N^1_G(v)| = |N^2_G(u) \setminus N^1_G(v)|. \]

If $uu_1 \notin E(G)$ then $u_1 \in N^2_G(u) \setminus N^1_G(v)$. Then
\[ \{ u_{r_j} \mid j \neq k, 1 \leq j \leq k \} \cup \{ u_1, v \} \subseteq N^1_G(u) \setminus N^1_G(v) \]
and $|N^2_G(u) \setminus N^1_G(v)| \geq t + 1$. This contradicts (4.14).

If $uu_1 \in E(G)$, then $i_j < m$, for otherwise
\[ u_{i_1+i_2+i_3} \cdots u_{i_k i_{k+1}} \cdots u_1 u_2 \cdots u_{i_k} u_{m+1} u_{m+2} \cdots u_s \]
is a $P_s$ in $G$. Therefore
\[ \{ v, u_1, \ldots, u_s \} \subseteq N^2_G(u) \setminus N^1_G(v) \quad \text{and} \quad |N^2_G(u) \setminus N^1_G(v)| \geq t + 1. \]

This contradicts (4.14) and completes the proof. \[ \square \]

Fig. 4 (with $n = s = 7$) and its obvious generalization show that the property of containing a $P_s$ is not $(3, -1)$-stable for $s \geq 7$. 

Theorem 6. Let \( n, s \) be positive integers with \( 4 \leq s \leq n \). Then the property of containing a \( P_s \) is \((2, 0)\)-stable.

Proof. Let \( G \in R_n, \ d_G(u, v) = 2 \) and
\[
d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)|. \tag{4.15}
\]
From Remark 2 we have that (4.15) is equivalent to
\[
|N_G^3(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)|. \tag{4.16}
\]
Suppose \( G + uv \) contains a \( P_s \) but \( G \) does not. Then \( G + uv \) contains a path \( u_1, u_2, \ldots, u_s \) with \( u_m = u, \ u_{m+1} = v \) for some \( m, 1 \leq m \leq s - 1 \). As in the proof of Theorem 5, we have \( |N_G^1(u) \cap N_G^1(v)| \leq |N_G^2(u) \setminus N_G^1(v)| \). This contradicts (4.16) and completes the proof. \( \Box \)

Corollary 3. Let \( n, s \) be positive integers with \( 4 \leq s \leq n \). Then the property of containing a \( P_s \) is \((3, 0)\)-stable.

Corollary 3 follows from Theorem 6 and Proposition 1. From Theorem 5, Corollary 3 and Remark 1 we have the following.

Corollary 4. If \( d_G(u) + d_G(v) \geq \min\{|M_G^1(u)| - 1, |M_G^1(u)|\}, \ d_G(u, v) = 2 \) and \( G + uv \) contains a \( P_s \), then \( G \) contains a \( P_s \).

Theorem 7. Let \( n, s \) be positive integers with \( s \leq n - 3 \). Then the property of being \( s \)-hamiltonian (see [3]) is \((2, s)\)-stable.

Proof. Let \( G \in R_n, \ d_G(u, v) = 2 \) and
\[
d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)| + s. \tag{4.17}
\]
From Remark 2 we have that (4.17) is equivalent to
\[
|N_G^3(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)| + s. \tag{4.18}
\]
Suppose that for some set \( W \) of at most \( s \) vertices of \( G \), \( (G + uv) - W \) is hamiltonian but \( H = G - W \) is not. We have
\[
|N_G^1(u) \cap N_G^1(v)| \leq |N_H^1(u) \cap N_H^1(v)| + s.
\]
Together with (4.18) this implies that
\[
|N_H^1(u) \cap N_H^1(v)| \geq 1 \quad \text{and} \quad d_H(u, v) = 2.
\]
Then from Theorem 1 and Remark 2 we have
\[ |N_B^1(u) \cap N_B^1(v)| < 1 + |N_B^2(u) \setminus N_B^1(v)|. \]
Hence
\[ |N_B^1(u) \cap N_B^1(v)| \leq |N_B^1(u) \cap N_B^1(v)| + s \leq |N_B^2(u) \setminus N_B^1(v)| + s \]
\[ \leq |N_B^2(u) \setminus N_B^1(v)| + s. \]
This contradicts (4.18) and completes the proof. \(\square\)

The following Theorems 8–12 are obtained by using the same arguments as in [2].

**Theorem 8.** Let \( n, s \) be positive integers with \( s \leq n - 3 \). Then the property of being \( s \)-edge-hamiltonian (see [8]) is \((2, s)\)-stable.

**Theorem 9.** Let \( n, s \) be positive integers with \( s \leq n - 4 \). Then the property of being \( s \)-hamiltonian-connected (see [1]) is \((2, s + 1)\)-stable.

**Theorem 10.** Let \( n, s \) be positive integers with \( s \leq n - 2 \). Then the property of containing \( K_{2,s} \) is \((2, s - 2)\)-stable.

**Theorem 11.** Let \( n, s \) be positive integers with \( s \leq n - 2 \). Then the property of being \( s \)-connected is \((2, s - 2)\)-stable.

**Theorem 12.** Let \( n, s \) be positive integers with \( s \leq n - 2 \). Then the property of being \( s \)-edge-connected is \((2, s - 2)\)-stable.

**References**