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Neutral shielding and cloaking of magnetic fields using isotropic media

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Abstract. A method for designing magnetic shields that do not perturb applied multipole fields in the static regime is developed. Cylindrical core-shell structures with two layers characterized by homogeneous isotropic permeabilities are found to support neutral shielding of multipole fields and unique cloaking solutions of arbitrary multipole order. An extra degree of freedom is provided by every layer added to the structure which may be exploited with an effective design formula for cloaking of additional field terms. The theory is illustrated with numerical simulations.

Keywords: magnetic shielding, neutral inclusion, cloaking, multipole fields

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1. Introduction

Electromagnetic shielding is the practice of reducing the electromagnetic field in a space with barriers made of conductive or magnetic materials. Passive shields can reduce the electromagnetic interference between devices and their environments, and they are necessary for the reliable operation of many electronic devices. Shields are used to either expel fields from an area or to confine fields to a region in, e.g., superconducting quantum interface devices for precision measurements of magnetic fields [1], magnetic resonance imaging machines and tokamaks in magnetic confinement fusion [2, 3]. Further more, electromagnetic shields are commonly used in advanced nanotechnology research facilities, biomedical research laboratories, continuous beam accelerators and various facilities such as transformer vault and switchgear in electrical-power industry [4, 5]. For frequencies above 100 kHz, satisfactory shielding performance can be achieved by a conductive shell (Faraday shield), which makes use of the eddy current to absorb the electromagnetic waves [6]. For a low-frequency electromagnetic wave or static electric or magnetic field, shields made of high magnetic permeability metal alloys can be used, such as sheets of μ -metal, to channel flux lines around the shielded region [7, 8, 9, 10]. The shielding effectiveness depends on the permeability, which generally drops off at both very low magnetic field strengths and at high field strengths where the material becomes saturated. So to achieve low residual fields, magnetic shields often consist of several enclosures one inside the other, each of which successively reduces the inner field [11, 12, 13, 14]. Also, superconducting materials can expel magnetic fields by the Meissner effect. However, a single shield with relative permeability of $0 \leq \mu_r < 1$ (diamagnetic) or $\mu_r > 1$ (paramagnetic, ferromagnetic) disturbs the external magnetic field in free space.

There are many situations in which it may be desirable to hide a sensitive object from a magnetic field, but not to disturb the applied field. In this way, external functions of the field, such as resolving objects in a magnetic resonance imaging scan, are preserved. A direct current magnetic cloak that shields the inner region without affecting the exterior space requires an inhomogeneous and anisotropic medium characterized by a response that is diamagnetic in one direction and paramagnetic in the perpendicular directions [15, 16], as demanded by transformation optics [17] in the limit

of zero frequency. These specifications were met in an experiment for a cylindrical cloak [18] using a novel design for the required diamagnetic metamaterials [19]. Also magnetic cloaking of uniform static fields has been realized using a superconducting-ferromagnetic bilayer [20].

In this paper, we develop a method for designing magnetic shields made of isotropic materials that do not perturb applied multipole fields in the static regime. The idea is inspired by the concept of neutral inclusion in the theory of composites with respect to a uniform field [21, 22, 23, 24, 25]. Another motivation comes from the theoretical proposals and experimental efforts to realize cloaking by manipulating various materials [17, 26, 27, 28, 29, 30]. Perfect cloaking requires that the shield and the object we wish to hide do not disturb the wave field in the exterior domain for an incident wave from any source. This means that the solution to Maxwell's equations remains exactly the same in the exterior domain as if we use the vacuum permittivity and permeability everywhere in space. Such a requirement is rather stringent but can be met by using materials with their physical moduli equal to negative numbers, zero or infinite, which might be realized by metamaterials. An approximate cylindrical cloak for microwaves has been realized from a reduced set of effective anisotropic and inhomogeneous material parameters [31]. Relaxing the requirement of perfect cloaking one can use plasmonic covers to cancel the dominant scattering from an object [32]. In this approach there is no need for anisotropy and several layers can provide a discrete inhomogeneity to obtain multifrequency solutions [33]. In the quasi-static limit the dipole case is nothing but the transparency condition for a uniform field [34].

We present a formula for making cylindrical core-shell structures with concentric layers, characterized by isotropic and homogeneous permeabilities, neutral to applied multipole fields (section 3). In the limit of a perfect diamagnetic (or an infinite μ -metal) response of the innermost layer we find exact cloaking solutions for every multipole order. This is an extensions of the double-layer dipole cloak [20] and every layer added to this structure constitutes an extra degree of freedom to obtain simultaneous cloaking solutions. The theory is illustrated with numerical simulations for a few-layer shields (section 4) and their significance in the problem of cloaking is discussed as we conclude (section 5). Next we describe the methodology (section 2).

Figure 1. Schematic picture of a uniform magnetic field $\mathbf{H}_1$ over concentric cylindrical domains with isotropic relative permeabilities $\mu_1, \mu_2, \dots, \mu_k$ in free space ($\mu_{k+1} = 1$) and radii $R_1, R_2, \dots, R_k$ centered at the origin.

2. Methodology

We consider cylindrical core-shell structures with concentric layers made of materials with homogeneous and isotropic permeabilities. The exterior region is considered to be free space with the vacuum permeability μ_0 . The core can be either diamagnetic, paramagnetic (ferromagnetic) or be free space. If there are no currents present, the magnetic field $\mathbf{H}$ can be written in terms of a magnetic scalar potential ψ as $\mathbf{H} = -\nabla\psi$. Using the consecutive relation for the magnetic flux density $\mathbf{B} = \mu\mathbf{H}$, the potential satisfies

$$\nabla \cdot (\mu \nabla \psi) = 0, \quad (1)$$

where μ is the permeability. In polar coordinates (r, θ) the potential can be expanded as

$$\psi^{(j)} = \sum_{n=1}^{\infty} \psi_n^{(j)}(r, \theta) = \sum_{n=1}^{\infty} \left(A_n^{(j)} r^n + \frac{B_n^{(j)}}{r^n} \right) \cos n\theta \quad (2)$$

in each medium with relative permeability $\mu_j = \mu/\mu_0$ for $j = 1, 2, \dots, k, k+1$, see figure 1. The number of layers in the shell is then $k-1$. The coefficients $A_n^{(j)}$ and $B_n^{(j)}$ are determined by the boundary conditions. By enforcing continuity of the potential (2) and of the normal component of the magnetic flux density $\mathbf{B} = -\mu\nabla\psi$ at the interface $r = R_j$ between μ_j and μ_{j+1} , the boundary conditions can be expressed as

$$\begin{bmatrix} R_j^{2n} & 1 \\ \mu_j & -\mu_j R_j^{-2n} \end{bmatrix} \begin{bmatrix} A_n^{(j)} \\ B_n^{(j)} \end{bmatrix} = \begin{bmatrix} R_j^{2n} & 1 \\ \mu_{j+1} & -\mu_{j+1} R_j^{-2n} \end{bmatrix} \begin{bmatrix} A_n^{(j+1)} \\ B_n^{(j+1)} \end{bmatrix}. \quad (3)$$

For non-zero μ_j the matrix equation can be rewritten as

$$\begin{bmatrix} A_n^{(j)} \\ B_n^{(j)} \end{bmatrix} = T_j \begin{bmatrix} A_n^{(j+1)} \\ B_n^{(j+1)} \end{bmatrix}, \quad j = 1, 2, \dots, k. \quad (4)$$

The coefficients are related by the transfer matrix

$$T_j = \frac{1}{1 + \Gamma_j} \begin{bmatrix} 1 & \Gamma_j R_j^{-2n} \\ \Gamma_j R_j^{2n} & 1 \end{bmatrix}, \quad (5)$$

where we have defined the mismatch factor

$$\Gamma_j = \frac{\mu_j - \mu_{j+1}}{\mu_j + \mu_{j+1}}. \quad (6)$$

This factor takes values in the range $-1 \leq \Gamma_j \leq 1$. The relation between the coefficients interior and exterior to a $(k-1)$ -layer shield is found by iteration of (4) as

$$\begin{bmatrix} A_n^{(1)} \\ B_n^{(1)} \end{bmatrix} = \prod_{j=1}^k T_j \begin{bmatrix} A_n^{(k+1)} \\ B_n^{(k+1)} \end{bmatrix}. \quad (7)$$

We consider $A_n^{(k+1)}$ as given in terms of an applied field strength $H_n = |\nabla\psi_n^{(k+1)}| = nr^{n-1}|A_n^{(k+1)}|$ in the limit $r \rightarrow \infty$, originating from a magnetic multipole of order n at infinity. The monopole ($n = 0$) term is zero implying that $A_0^{(k+1)}$ is an unphysical constant taken to be zero, which is why we consider $n \geq 1$ in the potential (2). The dipole ($n = 1$) case corresponds to the uniform field in figure 1. Due to the fact that the potential must be finite within the core $r < R_1$, we have $B_n^{(1)} = 0$. Neutral shielding of each multipole field $\mathbf{H}_n = -\nabla\psi_n^{(k+1)}$ separately is found by equating (7) to $B_n^{(k+1)} = 0$. This means that $T_1 T_2 \dots T_k$ is an upper triangular matrix. We treat a few-layer shields in some detail and give a formula relating the permeabilities in the general case.

3. Analytical results

3.1. Single-layer shields

For one-layer shields the neutrality condition applied to equation (7) with $k = 2$ ($R_k = R_2$) using (5) is

$$\Gamma_2 = -\Gamma_1 \left(\frac{R_1}{R_2} \right)^{2n}, \quad (8)$$

where the allowed values of the mismatch factors Γ_1 and Γ_2 are determined by requiring that the ratio R_1/R_2 is less than unity. By expanding Γ_1 and Γ_2 using (6), we arrive at the permeability relation

$$\mu_3 = \mu_2 + \frac{2\mu_2(\mu_1 - \mu_2) \left(\frac{R_1}{R_2} \right)^{2n}}{2\mu_2 + (\mu_1 - \mu_2) \left[1 - \left(\frac{R_1}{R_2} \right)^{2n} \right]}. \quad (9)$$

This formula is exactly the same as the transparency condition for Mie scattering on an electrically small coated disc in the quasi-static limit [32]. We note that the exterior permeability μ_3 can be seen as the effective permeability of the coated inclusion with respect to an arbitrary multipolar field of order $n \geq 1$. For a uniform field ($n = 1$) the effective permeability is equivalent to the Maxwell Garnet mixing rule for the composite with the area fraction $(R_1/R_2)^2$.

Figure 2. The mismatch factors Γ_3 and Γ_2 as functions of Γ_1 giving neutral shielding (10) of dipole ($n = 1$) and quadrupole ($n = 2$) fields for $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$.

3.2. Double-layer shields

For two-layer shields the neutrality condition applied to equation (7) with $k = 3$ using (5) reads

$$\Gamma_3 = -\frac{\Gamma_2 + \Gamma_1 \left(\frac{R_1}{R_2}\right)^{2n}}{1 + \Gamma_2 \Gamma_1 \left(\frac{R_1}{R_2}\right)^{2n}} \left(\frac{R_2}{R_3}\right)^{2n}. \quad (10)$$

The allowed values of the mismatch factors Γ_1, Γ_2 and Γ_3 are determined by requiring that the ratios R_1/R_2 and R_2/R_3 are positive and less than unity. For a given geometry the mismatch factors Γ_2 and Γ_3 can be uniquely determined as functions of Γ_1 ; equating the mismatch factor Γ_3 ($n = 1$) to Γ_3 ($n = 2$) in (10) yields a quadratic equation in Γ_2 , which admits one acceptable ($|\Gamma_2| \leq 1$) analytical solution. This solution and the corresponding mismatch factor Γ_3 are plotted in figure 2 as functions of Γ_1 for the thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$. We observe that Γ_2 is a decreasing function of Γ_1 , whilst Γ_3 is an increasing function of Γ_1 , and that they have opposite signs. A magnetic shield satisfying these conditions is neutral to both dipole ($n = 1$) and quadrupole ($n = 2$) fields. The shielding effectiveness is commonly quantified by the shielding factor, S , which is defined by the ratio of the magnetic field without the shield to the magnetic field with the shield at the same point. The shielding factor is unity for complete transparency and turns to infinity in the limit of perfect screening. Here, we consider the inverse of the shielding factor in order to get values in the unit interval. Solving equation (7) for $B_n^{(1)} = B_n^{(4)} = 0$ using (10) we find the ratio

$$\frac{A_n^{(1)}}{A_n^{(4)}} = \frac{(1 - \Gamma_1)(1 - \Gamma_2)}{(1 + \Gamma_3) \left[1 + \Gamma_2 \Gamma_1 \left(\frac{R_1}{R_2}\right)^{2n} \right]} \quad (11)$$

Figure 3. The inverse shielding factor (13) for $n = 1, 2$ as functions of Γ_1 using $R_1/R_2 = 0.75$. The mismatch factors Γ_2 and Γ_3 are those in figure 2 giving neutrality to both dipole and quadrupole fields.

of the core coefficient $A_n^{(1)}$ to the exterior coefficient $A_n^{(4)}$. Removing the shield we get the ratio

$$\frac{\tilde{A}_n^{(1)}}{A_n^{(4)}} = \frac{2\mu_4}{\mu_1 + \mu_4} \quad (12)$$

of the core coefficient $\tilde{A}_n^{(1)}$ to the exterior coefficient $A_n^{(4)}$. (Formally this follows from the coefficient relation (11) by putting $\mu_2 = \mu_3 = \mu_4$ so that $1 - \Gamma_1 = 2\mu_4/(\mu_1 + \mu_4)$ and $\Gamma_2 = \Gamma_3 = 0$.) Without a shield the core scatters the applied field unless $\mu_1 = \mu_4$ in which case (12) is unity. The inverse of the shielding factor can be expressed as

$$S^{-1} = \frac{A_n^{(1)}}{\tilde{A}_n^{(1)}} = \frac{1 + \Gamma_2 \Gamma_1 + \Gamma_3(\Gamma_1 + \Gamma_2)}{(1 - \Gamma_3^2) \left[1 + \Gamma_2 \Gamma_1 \left(\frac{R_1}{R_2}\right)^{2n} \right]}, \quad (13)$$

taking the value 1 for complete transparency and 0 for perfect screening. This expression follows from equations (11) and (12) with repeated use of the mismatch factors (6). In figure 3 we plot S^{-1} for $n = 1, 2$ as functions of Γ_1 using $R_1/R_2 = 0.75$ and the mismatch factors in figure 2. The inverse shielding factor (13) clearly decreases with increasing order n since Γ_1 and Γ_2 have opposite signs and $(R_1/R_2)^n$ decreases with increasing value of n . The larger the absolute value of Γ_1 the better the shielding.

By expanding Γ_2 and Γ_3 in equation (10) using (6), we find the permeability relation for the two-layer shield to be

$$\mu_4 = \mu_3 + \frac{2\mu_3(\mu_2^* - \mu_3) \left(\frac{R_2}{R_3}\right)^{2n}}{2\mu_3 + (\mu_2^* - \mu_3) \left[1 - \left(\frac{R_2}{R_3}\right)^{2n} \right]}, \quad (14)$$

Figure 4. The permeability ratios μ_2/μ_4 and μ_3/μ_4 for neutral shielding of dipole ($n = 1$), quadrupole ($n = 2$), and octupole ($n = 3$) fields as functions of Γ_1 in the case (a) $\mu_1/\mu_4 = 0.3782$, (b) $\mu_1/\mu_4 = 1$ and (c) $\mu_1/\mu_4 = 2.6444$. The thickness ratios are $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$.

where

$$\mu_2^* = \mu_2 + \frac{2\mu_2\Gamma_1 \left(\frac{R_1}{R_2}\right)^{2n}}{1 - \Gamma_1 + \Gamma_1 \left[1 - \left(\frac{R_1}{R_2}\right)^{2n}\right]} \quad (15)$$

denotes the effective permeability (9) of the single-layer shield, expressed in terms of the mismatch factor Γ_1 , the ratio R_1/R_2 , and the relative permeability

$$\mu_2 = \mu_1 \frac{1 - \Gamma_1}{1 + \Gamma_1}. \quad (16)$$

Figure 5. The inverse shielding factor (13) with $\mu_1/\mu_4 = 1$ and $R_1/R_2 = 0.75$ for $n = 1, 2, 3$ as functions of Γ_1 . The permeabilities μ_2 and μ_3 are those in figure 4(b) giving neutral shielding of each multipole field separately.

The permeability relation (14) means that a multipole field in the exterior region ($r > R_3$) is not perturbed by the core-shell structure ($0 < r < R_3$), which forms an inclusion in the background with μ_4 as the effective permeability. Figure 4 shows the permeability ratios μ_3/μ_4 in (14) and μ_2/μ_4 in (16) for neutral shielding of dipole ($n = 1$), quadrupole ($n = 2$) and octupole ($n = 3$) fields as functions of Γ_1 for three characteristic values of μ_1/μ_4 using the thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$. When $\mu_1/\mu_4 < 1$ (as in figure 4(a)) the curves (14) for two different n intersect in the range $-1 < \Gamma_1 < 0$, corresponding to simultaneous solutions at a specific permeability ratio $\mu_3/\mu_4 < 1$. When $\mu_1/\mu_4 > 1$ (as in figure 4(c)) the curves (14) for two different n intersect in the range $0 < \Gamma_1 < 1$, corresponding to simultaneous solutions at a specific permeability ratio $\mu_3/\mu_4 > 1$. When $\mu_1/\mu_4 = 1$ (figure 4(b)) all curves intersect each other at $\Gamma_1 = 0$, corresponding to the trivial solution $\mu_1 = \mu_2 = \mu_3 = \mu_4$. However, separate multipole solutions for neutral shielding of the core exist for nontrivial material parameters. In this case there is no scattering when the shield is removed and consequently the inverse of the corresponding shielding factors become zero in the limits $\Gamma_1 \rightarrow 1$ ($\Gamma_2 \rightarrow -1$) and $\Gamma_1 \rightarrow -1$ ($\Gamma_2 \rightarrow 1$) as shown in figure 5. In these limits we can place any object in the central region without affecting the shielding effectiveness.

In the limit $\mu_2 \rightarrow 0$ ($\mu_2^* \rightarrow 0$), the permeability relation (14) reduces to

$$\frac{\mu_3}{\mu_4} = \frac{1 + \left(\frac{R_2}{R_3}\right)^{2n}}{1 - \left(\frac{R_2}{R_3}\right)^{2n}}. \quad (17)$$

This gives perfect screening of the core and thus defines a cloaking solution for the applied multipole field of order $n \geq 1$, depending on the ratios μ_3/μ_4 and R_2/R_3 .

Figure 6. The permeability ratio μ_3/μ_4 as a function of R_2/R_3 in the case $\mu_2 = 0$ for cloaking of dipole ($n = 1$), quadrupole ($n = 2$) and octupole ($n = 3$) fields, respectively.

Figure 6 shows the ratio (17) as a function of R_2/R_3 for cloaking of dipole ($n = 1$), quadrupole ($n = 2$) and octupole ($n = 3$) fields, respectively. This formula is an extension of the dipole case [20] to multipolar fields. For $R_2/R_3 = 0.8$ the values (17) can also be read at $\Gamma_1 = 1$ in figure 4 and both $\mu_3/\mu_4 = 4.5556$ ($n = 1$) and $\mu_3/\mu_4 = 2.3875$ ($n = 2$) will be used in the numerical simulations. A perfect diamagnetic response of the inner layer ($\mu_2 = 0$) corresponds to $\Gamma_1 = -\Gamma_2 = 1$ in which case equation (10) reduces to $\Gamma_3 = (R_2/R_3)^{2n}$, being identical to the expression (17). In the limit $\mu_2 \rightarrow \infty$ ($\mu_2^* \rightarrow \infty$), we obtain

$$\frac{\mu_3}{\mu_4} = \frac{1 - \left(\frac{R_2}{R_3}\right)^{2n}}{1 + \left(\frac{R_2}{R_3}\right)^{2n}}, \quad (18)$$

which is another cloaking solution of order $n \geq 1$. The permeability ratio (18) is inverted as compared to (17) and the values are read at $\Gamma_1 = -1$ in figure 4 for the thickness ratio $R_2/R_3 = 0.8$. This solution can also be found directly from the neutrality condition (10). The limit $\mu_2 \rightarrow \infty$ corresponds to $\Gamma_1 = -\Gamma_2 = -1$ in which case equation (10) reduces to $\Gamma_3 = -(R_2/R_3)^{2n}$, being identical to the expression (18). Hence double-layer shields support unique cloaking solutions for every multipole field, that is, there are no concurrent solutions since the curves never intersect each other.

3.3. Multi-layer shields

In general, we find that neutral $(k - 1)$ -layer shields can be calculated iteratively by putting the exterior permeability μ_{k+1} equal to the effective permeability (with respect to a multipole field of order $n \geq 1$)

$$\mu_k^* = \mu_k + \frac{2\mu_k(\mu_{k-1}^* - \mu_k) \left(\frac{R_{k-1}}{R_k}\right)^{2n}}{2\mu_k + (\mu_{k-1}^* - \mu_k) \left[1 - \left(\frac{R_{k-1}}{R_k}\right)^{2n}\right]} \quad (19)$$

Figure 7. The permeability ratio μ_4/μ_5 as a function of μ_3/μ_4 for cloaking of dipole ($n = 1$), quadrupole ($n = 2$) and octupole ($n = 3$) fields. The thickness ratios are $R_2/R_3 = R_3/R_4 = 0.8$.

of the core-shell structure for $k \geq 2$. Here μ_k is the permeability of the k th domain, R_{k-1}/R_k is the corresponding thickness ratio and μ_{k-1}^* is the effective permeability of the $(k - 2)$ -layer shield, taking μ_1^* to be the core permeability μ_1 . This is an extension of a neutral cylindrical coated inclusion [23] from uniform to multipolar fields. As an example, we consider the three-layer shield defined by confining μ_4 to the layer $R_3 < r < R_4$ in figure 1 and introducing μ_5 as the exterior permeability. In the case of zero permeability of the innermost layer, $\mu_2 = 0$, we obtain

$$\frac{\mu_5}{\mu_4} = 1 + \frac{2 \left[\frac{\mu_3}{\mu_4} \frac{1 - \left(\frac{R_2}{R_3}\right)^{2n}}{1 + \left(\frac{R_2}{R_3}\right)^{2n}} - 1 \right] \left(\frac{R_3}{R_4}\right)^{2n}}{2 + \left[\frac{\mu_3}{\mu_4} \frac{1 - \left(\frac{R_2}{R_3}\right)^{2n}}{1 + \left(\frac{R_2}{R_3}\right)^{2n}} - 1 \right] \left[1 - \left(\frac{R_3}{R_4}\right)^{2n} \right]}, \quad (20)$$

using $\mu_4^* = \mu_5$ in (19) and $\mu_3^* = \mu_4$ in (17) for the effective permeabilities. The permeability ratio μ_4/μ_5 is plotted versus μ_3/μ_4 in figure 7 for cloaking of dipole ($n = 1$), quadrupole ($n = 2$) and octupole ($n = 3$) fields, respectively, using the geometry $R_2/R_3 = 0.8$, $R_3/R_4 = 0.8$. Curves corresponding to different orders intersect each other once. The intersection points can be found analytically by solving (20) for the order pairs. For example, the dipole and quadrupole curves in figure 7 intersect each other at $(\mu_3/\mu_4, \mu_4/\mu_5) = (10, 8765, 0.5846)$. Taking $\mu_5 = 1$ the layers alternate between being diamagnetic and paramagnetic for the simultaneous solutions. The values of the permeability ratio μ_4/μ_5 in the limits $\mu_3 \rightarrow 0$ and $\mu_3 \rightarrow \infty$ happen to coincide with those of (17) and (18), respectively, due to the equal thickness ratios $R_3/R_4 = R_2/R_3 = 0.8$. The cloaking solution (20) can also be expressed

in terms of mismatch factors as

$$\Gamma_4 = \frac{\Gamma_3 - \left(\frac{R_2}{R_3}\right)^{2n}}{\Gamma_3 \left(\frac{R_2}{R_3}\right)^{2n} - 1} \left(\frac{R_3}{R_4}\right)^{2n}, \quad (21)$$

which is the neutrality condition obtained from (7) with $k = 4$ using (5) in the case $\Gamma_1 = -\Gamma_2 = 1$. For a given geometry the relation (21) defines cloaking solutions of order n with two degrees of freedom, constituted by the mismatch factors Γ_3 and Γ_4 . One extra degree of freedom is provided by every layer (mismatch factor) added to the magnetic shields which may be exploited to cloak additional multipole fields.

4. Numerical simulations

We illustrate the results by numerical simulations.

4.1. Neutral shielding

In figure 8 we illustrate the neutral performance of two-layer shields with thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$ using COMSOL Multiphysics. A uniform magnetic field is applied in the surrounding medium with relative permeability $\mu_4 = 1$. The horizontal lines are field lines and the color tables show the absolute values of the magnetic flux densities for an implementation of a diamagnetic core with $\Gamma_1 = -0.8$ ($\mu_1 = 0.3782$, $\mu_2 = 3.4034$, $\mu_3 = 0.6526$) in figure 8(a) and a paramagnetic core with $\Gamma_1 = 0.8$ ($\mu_1 = 2.6444$, $\mu_2 = 0.2938$, $\mu_3 = 1.5323$) in figure 8(c). The material parameters used come from the analytical expressions (14) and (16). The neutral shielding performance is excellent in the dipole ($n = 1$) case and it should apply also to a quadrupole ($n = 2$) field since the numerical simulations correspond to the intersections of the dipole and quadrupole curves at $\Gamma_1 = -0.8$ and $\Gamma_1 = 0.8$ in figure 4(a) and figure 4(c), respectively. For simultaneous solutions either the core is diamagnetic ($0 < \mu_1 < 1$), the inner layer paramagnetic ($\mu_2 > 1$) and the outer layer diamagnetic ($0 < \mu_3 < 1$) or the types of responses are interchanged so that the core is paramagnetic, the inner layer diamagnetic and the outer layer paramagnetic. The only common solution for a vacuum core ($\mu_1 = 1$) is the trivial one with $\mu_2 = \mu_3 = 1$ at $\Gamma_1 = 0$ in figure 4(b). However, separate multipole solutions for neutral shielding of the core exist for nontrivial material parameters. Simulation results of a vacuum core implementation with $\Gamma_1 = 0.8$ ($\mu_1 = 1$, $\mu_2 = 0.1111$, $\mu_3 = 3.3000$) is shown in figure 8(b) in the dipole case. One may note that the theory relies on the core to have homogeneous permeability. If we were to place an object in the vacuum core, such as a circuit board that needs to be shielded from external magnetic fields, the neutral shielding would

Figure 8. Simulation results of neutral magnetic shields with thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = 0.8$ using COMSOL Multiphysics. The surrounding medium is vacuum with relative permeability $\mu_4 = 1$ where a uniform magnetic field is applied. Field lines are drawn and the color tables show the absolute values of the magnetic flux densities for (a) a diamagnetic core with $\Gamma_1 = -0.8$ ($\mu_1 = 0.3782$, $\mu_2 = 3.4034$, $\mu_3 = 0.6526$), (b) a vacuum core with $\Gamma_1 = 0.8$ ($\mu_1 = 1$, $\mu_2 = 0.1111$, $\mu_3 = 3.3000$) and (c) a paramagnetic core with $\Gamma_1 = 0.8$ ($\mu_1 = 2.6444$, $\mu_2 = 0.2938$, $\mu_3 = 1.5323$).

be destroyed unless the layer permeabilities satisfy the cloaking solution (17) or (18). In order to illustrate neutral shielding and cloaking of multipole fields we invoke a transformation.

4.2. A conformal mapping

A uniform magnetic field with strength H_1 at the origin, which is transverse to the cylindrical core-shell structure and is invariant over the cross section in figure 1 can be thought of as originating from a magnetic

Figure 9. Applying the conformal mapping (22) to the problem in figure 1 gives the complementary problem with a magnetic dipole $\mathbf{m}_1$ at the origin and relative permeabilities $\mu'_j = \mu_{k+2-j}$ for $j = 1, 2, \dots, k, k+1$.

dipole at infinity. Applying the transformation

$$r' = R_2 R_3 / r, \quad \theta' = \theta, \quad (22)$$

turns the geometry inside out in that it maps the circle R_2 to the circle R_3 , the origin to infinity and vice versa. Hence the magnetic dipole at infinity, which created the uniform field, is transformed to a magnetic dipole at the origin, see figure 9. Any potential (2) that is a valid solution to the boundary value problem (1) is also a solution to the transformed configuration due to invariance of the Laplace equation under the transformation (22). This result can be viewed as a basic application of transformation optics [17] at zero frequency ensuring that all material properties are maintained as isotropic. In view of the conformal mapping (22) the layer ordering j is reversed such that $\mu'_j = \mu_{k+2-j}$ and the individual radii R'_j are different, but the permeability values (19) remain invariant since the thickness ratios $R'_j / R'_{j-1} = R_{j-1} / R_j$ are preserved. Hence neutral field expulsion of nonuniform fields (applied at infinity) by magnetic shields and their performance can be visualized by neutral field confinement using magnetic multipoles at the origin.

4.3. Cloaking

Taking the limit $\mu_2 \rightarrow 0$ of the two-layer neutral shield in figure 8(b) with $R_1/R_2 = 0.75$, $R_2/R_3 = 0.8$ and $\mu_1 = \mu_4 = 1$ yields the cloaking solution (17). Application of the conformal mapping (22) results in a cloak with the layer ordering reversed, i.e., $\mu'_2 = \mu_3$ and $\mu'_3 = \mu_2$, and the thickness ratios $R'_3/R'_2 = R_2/R_3$ and $R'_2/R'_1 = R_1/R_2$ preserved, see figure 9. This configuration is implemented in COMSOL Multiphysics for numerical illustration of cloaking of a magnetic dipole in figure 10(a) and a quadrupole (realized by two magnetic dipoles) in figure 10(d) centered at the origin. Field lines are

drawn and the color tables show the absolute values of the magnetic flux densities. The cloak's inner layer is paramagnetic with relative permeability $\mu_3 = 4.5556$ ($n = 1$) in figure 10(a) and $\mu_3 = 2.3875$ ($n = 2$) in figure 10(d), and the outer layer is a perfect diamagnet, $\mu_2 = 0$. For comparison, we calculate the fields without the cloaks by substituting the layers by the vacuum permeability and plot the differences between the magnetic flux densities with and without the cloak for the dipole excitation in figure 10(b) and the quadrupole excitation in figure 10(e). The differences in the cores are not identically zero but the neutral shielding performance is very good indeed. The discrepancy is attributed to the physical realization of the magnetic dipoles. The good performance becomes even more clear if the wrong shield is used with respect to the applied field. The latter is illustrated in figure 10(c) and figure 10(f) where the shields have been interchanged, i.e., $\mu_3(n = 1) \leftrightarrow \mu_3(n = 2)$ resulting in significant variations in the cores. This result confirms the fact that dipole and quadrupole fields cannot be simultaneously cloaked.

We exploit the degrees of freedom of a three-layer shield to illustrate that the structure supports simultaneous cloaking solutions of dipole and quadrupole fields as predicted by (20). In figure 11(a) and figure 11(d) we show simulation results of uniform field expulsion by a magnetic cloak with thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = R_3/R_4 = 0.8$ in vacuum using COMSOL Multiphysics. Field lines are drawn and the color tables show the absolute values of the magnetic flux densities with the cloak (figure 11(a)) and the norm of the differences between the magnetic flux densities with and without the cloak (figure 11(d)). The interior and exterior medium has the vacuum permeability, $\mu_1 = \mu_5 = 1$, and the layer closest to the shielded region is a perfect diamagnet ($\mu_2 = 0$). The middle layer is paramagnetic with relative permeability $\mu_3 = 6.3584$ and the layer closest to the applied field is diamagnetic with relative permeability $\mu_4 = 0.5846$ corresponding to the intersection of the dipole ($n = 1$) and quadrupole ($n = 2$) curves at $(\mu_3/\mu_4, \mu_4/\mu_5) = (10, 8765, 0.5846)$ in figure 7. Applying the conformal mapping (22) to figure 11(a) results in a cloak with the layer ordering reversed but with the thickness ratios $R'_2/R'_1 = 0.75$ and $R'_4/R'_3 = R'_3/R'_2 = 0.8$ preserved. In figure 11(b) and figure 11(c) we show simulation results of neutral field confinement by exciting a dipole and a quadrupole centered at the origin, respectively. In this case no field lines permeate the perfect diamagnetic material of the outer screen. The results for the norm of the differences between the magnetic flux densities with and without the cloaks in figure 11(e) and figure 11(f) show good agreement with theory in that the interior fields are not perturbed by the cloaks.

Figure 10. Simulation results of two-layer magnetic shields with thickness ratios $R'_3/R'_2 = R_2/R_3 = 0.8$ and $R'_2/R'_1 = R_1/R_2 = 0.75$ using COMSOL Multiphysics. The interior medium is vacuum with relative permeability $\mu_4 = 1$ where a magnetic dipole ((a)–(c)) or a quadrupole ((d)–(f)) is centered at the origin. Field lines are drawn in white and the color tables show the absolute values of the magnetic flux densities. The cloak's inner layer is paramagnetic with relative permeability μ_3 ($n = 1$) = 4.5556 ((a)–(b)) or μ_3 ($n = 2$) = 2.3875 ((d)–(e)) and the outer layer is a perfect diamagnet, $\mu_2 = 0$. The exterior medium has the vacuum permeability, $\mu_1 = 1$. The norm of the differences between the magnetic flux densities with and without the shield ((b) and (e)) and the same but for interchanged shields ((c) and (f)).

5. Discussion and conclusion

In conclusion, we designed magnetic shields that do not perturb applied multipole fields in the static regime and evaluated their shielding effectiveness. Specifically, we presented an effective permeability relation for making cylindrical core-shell structures with concentric layers, characterized by isotropic and homogeneous permeabilities, neutral to arbitrary applied multipole fields from which cloaking solutions for every multipole order can be found. The two-layer cloaking formulas are generalizations of the dipole case [20]. By adding

a third layer to the core-shell structure two concurrent multipole fields can be exactly cloaked. By extension, magnetic shields with k layers support simultaneous cloaking solutions of $k - 1$ different multipole fields. The recursive formula can be used to design the cloaks, which is a stationary case of the general cloaking problem for the particular geometry. The solutions are exact and thereby different from approximation methods such as homogenization [35, 36] or numerical optimization [37] of multilayered structures. However, to include every multipole field in the solution we have to let k turn to infinity, resulting in an infinite series of

Figure 11. Simulation results of a three-layer magnetic cloak with thickness ratios $R_1/R_2 = 0.75$ and $R_2/R_3 = R_3/R_4 = 0.8$ for uniform field expulsion ((a) and (d)) and the complementary dipole field confinement cloak with thickness ratios $R'_2/R'_1 = R_1/R_2 = 0.75$ and $R'_4/R'_3 = R_3/R_4 = 0.8 = R_2/R_3 = R'_3/R'_2$ ((b) and (e)), being a cloak also for quadrupole fields ((c) and (f)), using COMSOL Multiphysics. The layers closest to the shielded regions are perfect diamagnets ($\mu_2 = 0$), the middle layers are paramagnetic with relative permeability $\mu_3 = 6.3584$, and the layers closest to the applied fields are diamagnetic with relative permeability $\mu_4 = 0.5846$. The interior and exterior media have the vacuum permeability, $\mu_1 = \mu_5 = 1$. Field lines are drawn and the color tables show the absolute values of the magnetic flux densities with the cloaks ((a)–(c)) and the norm of the differences between the magnetic flux densities with and without the cloaks ((d)–(f)).

alternating diamagnetic and paramagnetic layers. We expect that such a discrete inhomogeneity mimics the continuously varying anisotropic material parameters in the transformation optics recipe for a 2D cloak [31].

The presented analytical results can be adopted to phenomena described by the Laplace's equation. This form occurs in many situations of physics and cloaking, including static thermal cloaks [38, 39], direct current electric cloaks [40] and diffusive light cloaks [41]. We note that it is possible to derive analogous theoretical results for other canonical geometries, e.g., in spherical

symmetry, following a similar methodology.

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