Strain Energy of Bézier Surfaces

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Abstract

Bézier curves and surfaces are used to great success in computer-aided design and finite element modelling, among other things, due to their tendency of being mathematically convenient to use. This thesis explores the different properties that make Bézier surfaces the strong tool that it is. This requires a closer look at Bernstein polynomials and the de Casteljau algorithm. To illustrate some of these properties, the strain energy of a Bézier surface is calculated. This demands an understanding of what a surface is, which is why this thesis also covers some elementary theory regarding parametrized curves and surface geometry, including the first and second fundamental forms.

Keywords:
Surface geometry, Bézier curves, Bézier surfaces.

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Nomenclature

<table>
<thead>
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<tr>
<td>$S$</td>
<td>surface</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>parametrized curve</td>
</tr>
<tr>
<td>$N$</td>
<td>standard unit normal</td>
</tr>
<tr>
<td>$n$</td>
<td>normal of a curve</td>
</tr>
<tr>
<td>$S^2$</td>
<td>the unit sphere</td>
</tr>
<tr>
<td>$T$</td>
<td>tangent plane</td>
</tr>
<tr>
<td>$f_{x_i}$</td>
<td>partial derivative of a function with respect to $x_i$</td>
</tr>
<tr>
<td>$\mathcal{G}$</td>
<td>the Gauss map</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>the Weingarten map</td>
</tr>
<tr>
<td>$\langle \cdot, \cdot \rangle$</td>
<td>symmetric bilinear form</td>
</tr>
<tr>
<td>$I$</td>
<td>the first fundamental form</td>
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<td>$B$</td>
<td>Bézier curve</td>
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<td>$X$</td>
<td>Bézier surface</td>
</tr>
<tr>
<td>$B^n_i$</td>
<td>Bernstein polynomial</td>
</tr>
<tr>
<td>$b$</td>
<td>control point</td>
</tr>
<tr>
<td>$\times$</td>
<td>vector product</td>
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<tr>
<td>$\cdot$</td>
<td>dot product</td>
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<tr>
<td>$| \cdot, | | |}</td>
<td>length of a vector</td>
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Brämä, 2018.
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Chapter 1

Introduction

Bézier curves and surfaces are named after the French engineer Pierre Bézier, who is one of the founders of the field of geometric modelling. He worked at Renault where he used them to design car bodies. There, he developed the CAD/CAM system known as UNISURF. A good account of Bézier’s work can be found in Pierre Bézier: An engineer and a mathematician of Pierre-Jean Laurent and Paul Sablonnière ([Laur-Sabl]).

Bézier surfaces have a wide range of applications in the field of CAD and computer modelling because they possess certain algorithmic properties which enables efficient methods for rendering and analyzing shapes. To explain why, this thesis covers the fundamentals of differential geometry, including the theory of surface geometry. We follow the material in Elementary Differential Geometry by Andrew Pressley and Differential Geometry of Curves and Surfaces by Manfredo P. do Carmo.

In surface geometry we regard every part, or patch, of a surface as being equivalent to an open subset of a plane. One way to look at it is that the surface is the shape the plane takes as it is being deformed by bending and stretching. We say that they are homeomorphic and that the map between them is a homeomorphism. This map is also called the parametrization of the patch. By studying the behavior of curves going through a point of a surface, we can define local metric properties of the surface at this point. This is done by defining a vector space, which we call the tangent plane, which is spanned by tangent vectors of the curves going through this point. The first and second fundamental forms are symmetric bilinear forms defined on the tangent plane. Together, they determine the metric properties of the surface. The inverse function theorem allows the neighborhood of every point of a surface to be parametrized as a Monge patch. This enables the strain energy of the surface
at this point to be expressed in terms of the fundamental forms. Since they are independent of the choice of parametrization, the strain energy can be calculated for any given parametrization.

A Bézier curve is defined by its control polygon and the Bernstein basis polynomials. The control polygon is a set of points and the Bézier curve is contained within the convex hull of these points. Thus, given a control polygon, there is exactly one corresponding Bézier curve. The Bernstein basis polynomials induce certain practical properties, such as the de Casteljau algorithm which enables every point of the Bézier curve to be recursively defined by the points of its control polygon. Not only that, but the derivatives are recursively defined as well. A Bézier surface, which can be seen as the product of two Bézier curves, also has these properties. By extension, this means that the fundamental forms of a Bézier surface are given by the corresponding control polygon. This simplifies the process of calculating the strain energy, among other things, considerably and is one of the reasons why Bézier surfaces are so prevalent in technical applications such as CAD - computer-aided design. Chapter 3 of which the majority is based on the material presented in Applied Geometry for Computer Graphics and CAD by Duncan Marsh, is dedicated to Bézier curves and surfaces. A brief account can also be found in Bonneau’s dissertation.

This thesis covers the calculation of the strain energy of a Bézier surface and uses the results found in the dissertation in Variational Design of Rational Bézier Curves and Surfaces by Georges-Pierre Bonneau. Strain energy is a central concept of the design and production of many technical implementations, such as cars and airplanes. In chapter 4 a comparison of the strain energy of four different Bézier surfaces is presented, mainly for the purpose of giving the reader a glimpse of the vast possibilities this mathematical tool has to offer. A short introduction to future work and possible continuations of this thesis concludes this chapter and can be further studied in the dissertation of Bonneau.
Chapter 2

Surface Geometry

Before introducing the concept of a Bézier surface we must first have a firm understanding of surface geometry. Loosely speaking, a surface is a subset of $\mathbb{R}^3$ where the local environment, which we call a patch, of every point of the surface appears to be a part of a plane. Indeed, we will regard every such patch as being essentially equivalent to a part of a plane, allowing the surface to locally be parametrized by an open subset of $\mathbb{R}^2$. The surface will then be the union of all these patches, in a way resembling a patchwork quilt. A helpful analogy would perhaps be the one where the open subsets of $\mathbb{R}^2$ are referred to as charts of the surface. This collection of charts that together make up the whole surface would then be called the atlas of that surface, much like an atlas of the surface of the Earth, containing charts of different countries and continents.

By studying the properties of curves going through a point on a surface, we can study the local properties of that surface in a small neighborhood of that point. With every point of the surface belonging to at least one patch, curves going through this point can be expressed in terms of the parametrization of the associated patch. This will enable the defining of two important properties of the surface, namely the first and second fundamental forms, which are essential to the process of calculating the strain energy of a surface.

In section 2.1 we will start by looking at some basic properties of parametrized curves after which section 2.2 covers the parametrization of a surface. This will be followed by two sections about the first and second fundamental forms. In sections 2.1 through 2.4 we follow the material presented in [Pressley] and [Do Carmo]. An account of the notion of strain energy at the end of section 2.4 can be found in [Bonneau].
2.1 Curves

In this section we cover only what is required in order for the reader to be able to follow the material in section 2.2. For a more comprehensive account of curves, see the introductory chapters of [Do Carmo] or [Pressley].

Definition 2.1.1 (Parametrized curve). A \textit{parametrized curve} $\gamma$ is a map $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$, for some open interval $(\alpha, \beta)$. We write $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$, $t \in (\alpha, \beta)$, and define the following properties:

- The \textit{tangent vector} of $\gamma$ at the point $\gamma(t)$ is given by
  \[ \dot{\gamma}(t) = (\dot{\gamma}_1(t), \dot{\gamma}_2(t), \ldots, \dot{\gamma}_n(t)) . \]
- $\gamma(t)$ is a \textit{regular point} of $\gamma$ if $\dot{\gamma}(t) \neq 0$. We say that $\gamma$ is a \textit{regular curve} if all its points are regular.
- The \textit{arc-length} of $\gamma$ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by
  \[ s(t) = \int_{t_0}^{t} \| \dot{\gamma}(u) \| \, du . \]
- The \textit{speed} of $\gamma$ at the point $\gamma(t)$ is $\| \dot{\gamma}(t) \|$, and $\gamma$ is said to be a \textit{unit-speed} curve if $\| \dot{\gamma}(t) \| = 1$ for all $t \in (\alpha, \beta)$.
- The \textit{curvature} at a point $p$ of a regular curve is the magnitude of the deviation of $\gamma$ from the tangent line, i.e. the line through $p$ parallel to the tangent vector. Denoted $\kappa$, the curvature of $\gamma$ is given by
  \[ \kappa = \frac{\| \ddot{\gamma} \times \dot{\gamma} \|}{\| \dot{\gamma} \|^3} \quad (2.1) \]
  or, in the case of a unit-speed curve, by $\kappa = \| \ddot{\gamma} \|$.

Example 2.1.2 (Regular and unit-speed parametrizations). The upper half of the unit circle in the $xy$-plane in $\mathbb{R}^3$, with the $x$-axis excluded, is given by $y = \sqrt{1-x^2}$, where $x \in (-1, 1)$ and $z = 0$. Let $\gamma_1$ and $\gamma_2$ be different parametrizations of this curve, such that
\[
\begin{align*}
\gamma_1(\theta) &= (\cos \theta, \sin \theta, 0), \ \theta \in (0, \pi) \\
\gamma_2(t) &= (t, \sqrt{1-t^2}, 0), \ t \in (-1, 1).
\end{align*}
\]

The respective tangent vector is then given by
\[
\begin{align*}
\dot{\gamma}_1(\theta) &= (- \sin \theta, \cos \theta, 0), \ \theta \in (0, \pi) \\
\dot{\gamma}_2(t) &= \left(1, \frac{-t}{\sqrt{1-t^2}}, 0\right), \ t \in (-1, 1) .
\end{align*}
\]
where \( \| \dot{\gamma}_1 \| = 1 \quad \forall \theta \in (0, \pi) \) and \( \| \dot{\gamma}_2 \| = \frac{1}{\sqrt{1-t^2}} \). We conclude that both parametrizations are regular and that \( \gamma_1 \) is unit-speed whereas \( \gamma_2 \) is not. Since the curvature is a property of the curve itself, one might suspect that it is independent of the choice of parametrization so long as it is regular. Indeed,

\[
\begin{align*}
\dot{\gamma}_1(\theta) &= (-\cos \theta, -\sin \theta, 0), \quad \theta \in (0, \pi) \\
\dot{\gamma}_2(t) &= \left(0, \frac{1}{\sqrt{1-t^2}}, 0\right), \quad t \in (-1, 1)
\end{align*}
\]

and by equation (2.1) we have that

\[
\begin{align*}
\kappa_1 &= \| \ddot{\gamma} \| = 1 \\
\kappa_2 &= \frac{\| \dot{\gamma}_2 \times \ddot{\gamma}_2 \|}{\| \dot{\gamma}_2 \|^3} = \frac{\| (0, 0, \frac{1}{(1-t^2)^{3/2}}) \|}{\| (1, \frac{-t}{\sqrt{1-t^2}}, 0) \|^3} = 1,
\end{align*}
\]

and thus \( \kappa_1 = \kappa_2 \).

The dot notation is frequently used in physics to indicate the derivative with respect to time, and has a natural interpretation. If we imagine a curve being the trace of a particle moving through space, then the parametrization would tell us the speed, among other things, of the particle at every point in time. The expression "unit-speed" can then be interpreted as the speed of the particle being constantly 1. It turns out there is a way to always find a unit-speed parametrization of a regular curve.

**Proposition 2.1.3.** A parametrized curve \( \gamma \) has a unit-speed reparametrization if and only if it is regular. If \( \gamma \) has a unit-speed reparametrization \( \tilde{\gamma} \), then it is essentially parametrized by its arc-length \( s \), i.e. \( \tilde{\gamma}(s(t)) = \gamma(t) \).

Thus, proposition 2.1.3 enables the assumption that every regular parametrized curve is unit-speed. Now, consider a point \( p \) and a curve \( \gamma \) going through this point. Since we have assumed that \( \gamma \) is unit-speed, we know that

\[
\dot{\gamma} \cdot \dot{\gamma} = 1. \tag{2.2}
\]

Differentiating (2.2) with respect to \( t \) yields

\[
\dot{\gamma} \cdot \dot{\gamma} + \dot{\gamma} \cdot \ddot{\gamma} = 0 \iff \ddot{\gamma} \cdot \dot{\gamma} = 0.
\]

Hence, \( \dot{\gamma} \) and \( \ddot{\gamma} \) are orthogonal unit vectors. If we let \( t = \dot{\gamma} \), \( n = \frac{\kappa}{\| \ddot{\gamma} \|} \) and \( b = t \times n \), where \( \kappa = \| \ddot{\gamma} \| \), then \( \{t, n, b\} \) form an orthonormal basis of \( \mathbb{R}^3 \) called the Frenet trihedron. Here, \( t \) is the tangent vector; \( n \) is the principal
normal vector and \( b \) is the binormal vector of \( \gamma \) at \( p \). The planes spanned by \( \{t, b\} \), \( \{n, b\} \) and \( \{t, n\} \) are called the rectifying, normal and osculating plane, respectively. Furthermore, the equations

\[
\begin{align*}
\dot{t} &= \kappa n \\
\dot{b} &= \tau n \\
\dot{n} &= -\tau b - \kappa t
\end{align*}
\]

are called the Frenet formulas, where \( \kappa \) is called the curvature and \( \tau \) is called the torsion. Thus, \( \kappa \) and \( \tau \) measure the deviation of \( \gamma \) from its rectifying and osculating plane, respectively, at the point \( p \). In other words, a curve with no curvature is a straight line and a curve which is completely contained within a plane has no torsion. Together, the curvature and torsion completely determines the local properties of a curve up to an isometry of \( \mathbb{R}^3 \).

### 2.2 Parametrization of a Surface

In the introduction of this chapter we used the phrase "essentially equivalent" when referring to the relation between the surface patch and the associated open subset of \( \mathbb{R}^2 \), but we shall need to be more specific. The following definition uses the terms in the introductory analogy and can be found in [Pressley].

**Definition 2.2.1 (Surface).** A subset \( S \) of \( \mathbb{R}^3 \) is a surface if, for every point \( p \in S \), there is an open set \( U \subseteq \mathbb{R}^2 \) and an open set \( W \) in \( \mathbb{R}^3 \) containing \( p \) such that \( U \) is homeomorphic to \( V = W \cap S \). If \( \sigma : U \to V \) is a homeomorphism, we say that \( (U, \sigma) \) is a chart of \( S \) such that \( \sigma(u_0, v_0) = p \), \((u, v) \in U\), and \( \sigma \) is the parametrization of \( V \).

For the remainder of this thesis we will assume that every \( \sigma \) is a smooth map, see [A.0.1]. A definition of homeomorphism can be found in [A.0.2].

**Remark 2.2.2.** In most cases we will only write \( \sigma \) when actually referring to a chart \( (U, \sigma) \) of a surface \( S \). It is then implied that \( \sigma \) is a chart of \( S \) with the domain \( U \) such that \( \sigma \) parametrizes a patch \( V \) of \( S \).

As mentioned earlier, one way to define the local properties of a surface \( S \) is to study the behaviour of curves in \( S \) passing through a point \( p \in S \). Let \( \gamma \) be such a curve. We will define a tangent vector of \( S \) at \( p \) to be the tangent vector of \( \gamma \) at \( p \). Of course, there is an infinite amount of curves passing through \( p \), which leads us to defining the tangent plane of \( S \) at \( p \) to be the set of all tangent vectors of \( S \) at \( p \).
Defintion 2.2.3 (Tangent vector and tangent plane). A tangent vector \( v \) to a surface \( S \) at a point \( p \in S \) is the tangent vector at \( p \) of a curve in \( S \) passing through \( p \). The tangent plane \( T_p S \) of \( S \) at \( p \) is the vector space consisting of all tangent vectors to \( S \) at \( p \). Let \( \sigma : U \to V \) be a parametrization of an open set \( V \) on a surface \( S \) containing a point \( p \in S \), and let \((u,v)\) be coordinates in \( U \). The tangent plane of \( S \) at \( p \) is the vector subspace of \( \mathbb{R}^3 \) spanned by the vectors \( \sigma_u \) and \( \sigma_v \) (the derivatives are evaluated at the point \((u_0,v_0) \in U \) such that \( \sigma(u_0,v_0) = p \)).

If \( \{\sigma_u,\sigma_v\} \) is a basis of the tangent plane \( T_p S \) and \( \gamma \) is a curve through \( p \), then \( \gamma(t) = \sigma(u(t),v(t)) \) and, by the use of the chain rule, the tangent vector \( \dot{\gamma} \) can then be expressed in terms of the basis \( \{\sigma_u,\sigma_v\} \) as

\[
\dot{\gamma}(t) = \dot{u}(t)\sigma_u + \dot{v}(t)\sigma_v.
\]

(2.3)

This result will be used in section 2.3 where we define the first fundamental form of a surface.

Definition 2.2.3 clarifies the need for the charts of \( S \) to be smooth. However, we need to make sure that the charts fit together properly to cover the whole of \( S \) correctly. In other words, we would like \( S \) to be a smooth surface. This requires the charts to be regular and compatible.

Definition 2.2.4 (Regular parametrization). A parametrization \( \sigma : U \to \mathbb{R}^3 \) is called regular if it is smooth and the vectors \( \sigma_u \) and \( \sigma_v \) are linearly independent at all points \((u,v) \in U \). The standard unit normal of a surface \( S \) at a point \( p \) is then defined to be

\[
\mathbf{N}_p = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|},
\]

where \( \mathbf{N}_p \neq 0 \) if \( \sigma \) is regular.

Now, let \( \sigma \) and \( \tilde{\sigma} \) be regular charts of \( S \) such that \( \sigma : U \to V \) and \( \tilde{\sigma} : \tilde{U} \to \tilde{V} \) and let \( \Omega = V \cap \tilde{V} \) be an open set. Thus, \( \Omega \) is parametrized by both \( \sigma \) and \( \tilde{\sigma} \). We say that the charts are compatible if the transition function \( \Phi : U \to \tilde{U} \), where \( \Phi = \tilde{\sigma}^{-1} \circ \sigma \), is a smooth bijective map and its inverse, \( \Phi^{-1} : \tilde{U} \to U \), is smooth. In this case, \( \Phi \) could also be called the reparametrization of \( \Omega \). To summarize:

Definition 2.2.5 (Altas of a smooth surface). A surface \( S \) is smooth if its charts are regular and compatible. A collection of such charts is then called an atlas of \( S \). Furthermore, we say that the maximal atlas of \( S \) is the union of all atlases of \( S \).
Example 2.2.6 (Catenoïd). The resulting surface of revolution, $S$, when revolving the catenary curve $x = \cosh z$ around the $z$-axis is called a catenoïd and is parametrized by

$$\sigma(u, v) = (\cosh u \cos v, \cosh u \sin v, u), \quad u \in \mathbb{R}, \quad v \in [0, 2\pi].$$

However, the fact that $\sigma(u, 0) = \sigma(u, 2\pi)$ means that $\sigma$ is not a bijective map and hence not a homeomorphism. In order for $\sigma$ to be a chart of $S$, we have to restrict $v$ to, for example, the open interval $(0, 2\pi)$. By adding the chart

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = (\cosh \tilde{u} \cos \tilde{v}, \cosh \tilde{u} \sin \tilde{v}, \tilde{u}), \quad \tilde{u} \in \mathbb{R}, \quad \tilde{v} \in (-\pi, \pi)$$

we find that the union of the two charts cover the whole of $S$. Some calculations reveal that both $\sigma$ and $\tilde{\sigma}$ are regular. The reparametrization $\tilde{u} = u$ and $\tilde{v} = v - \pi$ is a linear map and is therefore smooth and bijective with a smooth inverse, which means that the charts are compatible. We can conclude that $\sigma$ and $\tilde{\sigma}$ together make up an atlas of $S$. By changing the intervals of $v$ and $\tilde{v}$ in a suitable way, one can find infinitely many atlases of $S$.

There is one last restriction we shall have to impose on a surface which has to do with its orientation. The standard unit normal of the tangent plane at $p \in \Omega \subseteq S$ should be independent of the choice of parametrization. However, if $\sigma$ and $\tilde{\sigma}$ are compatible charts of $\Omega$ and $\Phi$ is the transition map $\Phi : \tilde{\sigma} \to \sigma$, then

$$\left( \begin{array}{c} \tilde{\sigma}_u \\ \tilde{\sigma}_v \end{array} \right) = \left( \begin{array}{cc} \partial_u \\ \partial_v \end{array} \right) \left( \begin{array}{c} \sigma_u \\ \sigma_v \end{array} \right),$$

where $J(\Phi) = \left( \begin{array}{cc} \partial_u \\ \partial_v \end{array} \right)$ is the Jacobian matrix of the transition map $\Phi$ and both $\{\sigma_u, \sigma_v\}$ and $\{\tilde{\sigma}_u, \tilde{\sigma}_v\}$ are bases of the tangent plane $T_pS$. If $p$ and $\tilde{p}$ are the same point in $\Omega$, expressed in terms of their associated parametrizations, then the standard unit normal at $\tilde{p}$ is given by

$$N_{\tilde{\sigma}} = \frac{\tilde{\sigma}_u \times \tilde{\sigma}_v}{\|\tilde{\sigma}_u \times \tilde{\sigma}_v\|} = \frac{\det(J(\Phi))}{\|\det(J(\Phi))\|} \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \pm N_p,$$

where the sign depends on $\Phi$.

Definition 2.2.7 (Orientable surface). We say that a surface $S$ is orientable if there exists an atlas $A$ such that, if $\Phi$ is a transition map between any two different compatible charts in $A$, then $\det(J(\Phi)) > 0$ where $\Phi$ is defined.

A casual way of putting it is that we need to be able to tell the "inside" and "outside" of a surface apart. This means that by walking on the outside of an oriented surface one can never end up on the inside.
Example 2.2.8 (Möbius strip). The *Möbius strip* is an example of a surface that is not orientable.

Remark 2.2.9. Henceforth we will assume that all surfaces are smooth and orientable. This simplification will not lead to any major restrictions in later chapters.

By studying the change of the standard unit normal one can study the curvature of a surface. We will introduce two new concepts which will be used to define the second fundamental form of a surface.

Let $N_p$ be the standard unit normal of the tangent plane $T_pS$. Since $|N_p| = 1$ we can define a map that takes each point $p \in S$ to a point on the unit sphere, $S^2$.

**Definition 2.2.10 (Gauss map).** The map that takes points on a surface $S$ to a point on the unit sphere $S^2$ is called the *Gauss map* and is given by

$$ G : S \rightarrow S^2 $$

$$ p \mapsto N_p $$

A more detailed motivation regarding the following definition can be found in [Pressley](#).

**Definition 2.2.11 (Derivative of a map).** Let $f : S \rightarrow \hat{S}$ be the smooth map that takes points $p \in S$ to points $f(p) \in \hat{S}$. The *derivative* $D_pf$ of $f$ at $p$ is then the map that takes tangent vectors of $S$ at $p$ to tangent vectors of $\hat{S}$ at $f(p)$, in other words $D_pf : T_pS \rightarrow T_{f(p)}\hat{S}$.

The derivative of $G$ is then the linear map from $T_pS$ to $T_{G(p)}S^2$. Here, $G(p)$ is determined by its tangent plane $T_{G(p)}S^2$, i.e. the plane perpendicular to $G(p)$ going through the origin. But, since $G(p) = N_p$ and $N_p$ is determined by the tangent plane $T_pS$, it follows that $T_{G(p)}S^2 = T_pS$ and thusly $D_pG : T_pS \rightarrow T_pS$.

**Definition 2.2.12 (Weingarten map).** The map

$$ \mathcal{W} : T_pS \rightarrow T_pS $$

$$ \mathcal{W} = -D_pG $$

is called the *Weingarten map*.

Thus, $\mathcal{W}$ describes the way the standard unit normal changes when moving across a surface. The greater the curvature of the surface, the faster the change of the standard unit normal. Note that the Weingarten map is self-adjoint, see [A.0.5](#). This characteristic will be used in section 2.4.

**Remark 2.2.13.** The minus sign in $\mathcal{W} = -D_pG$ is only a convention to simplify future calculations.
2.3 The First Fundamental Form

When measuring the distance between two points in $\mathbb{R}^3$ one simply draws a line between the two points and then measures the length of the line. The distance between two points on a surface is in general not the length of a straight line joining the two points, however, hence we will need to define the metric properties of a regular smooth surface. To do this, we must define a symmetric bilinear form of a surface, see A.0.4.

Example 2.3.1. The dot product on $\mathbb{R}^n$ is perhaps the most well-known example of a symmetric bilinear form.

Let $S$ be a surface and let $\sigma : U \to V$ be a chart of $V \subseteq S$ with $(u, v) \in U \subseteq \mathbb{R}^2$. In section 2.2 we saw that the tangent vector of a curve $\gamma$ moving through a point $p \in S$ can be written as $\dot{\gamma}(t) = \dot{u}(t)\sigma_u + \dot{v}(t)\sigma_v$ in the basis $\{\sigma_u, \sigma_v\}$ of the tangent plane $T_pS$, see equation 2.3. Let $w_1, w_2 \in T_pS$ be two tangent vectors expressed in terms of the basis $\{\sigma_u, \sigma_v\}$ as $w_1 = \mu_1\sigma_u + \nu_1\sigma_v$ and $w_2 = \mu_2\sigma_u + \nu_2\sigma_v$, respectively, where $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{R}$. We can then define an inner product of $T_pS$ to be the dot product restricted to tangent vectors, i.e.

$$\langle w_1, w_2 \rangle = w_1 \cdot w_2 = \mu_1\mu_2(\sigma_u \cdot \sigma_u) + (\mu_1\nu_2 + \mu_2\nu_1)(\sigma_u \cdot \sigma_v) + \nu_1\nu_2(\sigma_v \cdot \sigma_v).$$

This bilinear form is called the first fundamental form, $I$, of $S$ and can be written in matrix form as

$$I(w_1, w_2) = \begin{pmatrix} \mu_1 & \nu_1 \end{pmatrix} \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_u \cdot \sigma_v & \sigma_v \cdot \sigma_v \end{pmatrix} \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}. \quad (2.6)$$

We also make the following denotations:

$$E = \sigma_u \cdot \sigma_u, \quad F = \sigma_u \cdot \sigma_v \quad \text{and} \quad G = \sigma_v \cdot \sigma_v. \quad (2.7)$$

Definition 2.3.2 (First fundamental form). Let $T_pS$ be the tangent plane of $S$ at the point $p$ and let $w_1, w_2 \in T_pS$ be tangent vectors to $S$ at $p$. The symmetric bilinear form $I$, defined as

$$I : T_pS \times T_pS \to \mathbb{R}$$

$$(w_1, w_2) \mapsto \langle w_1, w_2 \rangle,$$

such that $I(w_1, w_2) = w_1 \cdot w_2$, is called the first fundamental form of $S$ at $p$. Given a parametrization $\sigma$, the matrix of the first fundamental form is then defined as $F_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, with $E, F$ and $G$ as in (2.7).
It is important to note that the first fundamental form is, in general, different for each point of the surface. The fact that $I(w_1, w_2) = I(w_2, w_1)$ verifies the statement of the first fundamental form being symmetric.

**Example 2.3.3 (Plane).** Let $\Pi$ be the plane in $\mathbb{R}^3$ parametrized by $\sigma(u, v) = (u, v, k)$, where $(u, v) \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Since $\sigma_u = (1, 0, 0)$ and $\sigma_v = (0, 1, 0)$, the first fundamental form of $\Pi$ is $F_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If we let $w_1 = a\sigma_u + b\sigma_v$ and $w_2 = c\sigma_u + d\sigma_v$ be two tangent vectors in the tangent plane of a point in $\Pi$, then we have that

$$I(w_1, w_2) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd,$$

as expected.

**Example 2.3.4 (Torus).** A torus with axis of revolution the $z$-axis in $\mathbb{R}^3$ is parametrized by

$$\sigma(\theta, \varphi) = ((R + r \cos \theta) \cos \varphi, (R + r \cos \theta) \sin \varphi, r \sin \theta), \quad 0 < r < R,$$

where $r$ is the radius of the tube and $R$ is the distance from the $z$-axis to the center of the tube. We will find the matrix of the first fundamental form of the torus.

$$\sigma_\theta = (-r \sin \theta \cos \varphi, -r \sin \theta \sin \varphi, r \cos \theta)$$

$$\sigma_\varphi = (- (R + r \cos \theta) \sin \varphi, (R + r \cos \theta) \cos \varphi, 0)$$

$$||\sigma_\theta||^2 = r^2 \sin^2 \theta \cos^2 \theta + r^2 \sin^2 \theta \sin^2 \varphi + r^2 \cos^2 \theta = r^2$$

$$||\sigma_\varphi||^2 = (R + r \cos \theta)^2 \sin^2 \varphi + (R + r \cos \theta)^2 \cos^2 \varphi = (R + r \cos \theta)^2$$

$$\sigma_\theta \cdot \sigma_\varphi = r \sin \theta (R + r \cos \theta) \sin \varphi \cos \varphi - r \sin \theta (R + r \cos \theta) \sin \varphi \cos \varphi = 0$$

$$\therefore F_I = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r \cos \theta)^2 \end{pmatrix}$$

It is worth noting that the first fundamental form is independent of $\varphi$, which is to be expected since the torus is symmetric with respect to the $z$-axis.

**Example 2.3.5 (Catenoid).** We will find the first fundamental form of the
Chapter 2. Surface Geometry

catenoid defined in example 2.2.6

\[ \sigma_u = (\sinh u \cos v, \sinh u \sin v, 1) \]
\[ \sigma_v = (-\cosh u \sin v, \cosh u \cos v, 0) \]
\[ E = \|\sigma_u\|^2 = \sinh^2 u + 1 = \cosh^2 u \]
\[ G = \|\sigma_v\|^2 = \cosh^2 u \]
\[ F = \sigma_u \cdot \sigma_v = 0 \]
\[ \therefore F I = \begin{pmatrix} \cosh^2 u & 0 \\ 0 & \cosh^2 u \end{pmatrix} \]

As in the case of the torus, the catenoid is also symmetric with respect to the z-axis and thus the first fundamental form is independent of v.

Proposition 2.3.6 (Length of a Curve). The length of \( \gamma(t) \) is

\[ \int \| \dot{\gamma}(t) \| \, dt = \int \sqrt{I(\dot{\gamma}(t))} \, dt = \int \sqrt{E \ddot{u}(t)^2 + 2F \ddot{u}(t) \ddot{v}(t) + G \ddot{v}(t)^2} \, dt, \quad (2.8) \]

where \( \sigma(u(t), v(t)) \) is the parametrization of \( S \) and \( E, F \) and \( G \) are the coefficients of the first fundamental form defined in (2.7).

Example 2.3.7 (Meridian of a Torus). The meridian of a surface of revolution is the intersection of the surface and a half-plane, with the axis of revolution as its boundary. Let \( \gamma \) be the meridian of a torus where \( \varphi = \varphi_0 \). In this case, the meridian is a circle with radius \( r \), so the length of \( \gamma \) should be \( 2\pi r \), see example 2.3.4. We calculate the length of \( \gamma \) using the results found in the proposition above. The meridian can be expressed in terms of the parametrization of the torus as

\[ \gamma(t) = \sigma(\theta(t), \varphi_0), \]

where \( \theta(t) = 2\pi t, \ t \in [0, 1] \), and \( \varphi = \varphi_0 \) is a constant. In example 2.3.4 we found the first fundamental form of a torus to be

\[ F I = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r \cos \theta(t))^2 \end{pmatrix}. \]

The length of \( \gamma \) is then

\[ \int_0^1 \sqrt{I(\dot{\gamma}(t))} \, dt = \int_0^1 \sqrt{r^2 \ddot{\theta}(t)^2} \, dt = \int_0^1 2\pi r \, dt = 2\pi r. \]

Continuing, we will use the first fundamental form of a surface to calculate the area of a part of that surface. Let \( S, U, V \), and \( \sigma \) be the same as in the
2.4. The Second Fundamental Form

We have seen that the first fundamental form can be used to measure distances and areas on a surface. In order to determine the curvature of a surface we will define another symmetric bilinear form of a surface, namely the second fundamental form.

Recall that the Weingarten map describes the change of the standard unit normal at a point on a surface as one moves away from the tangent plane. From definition [2.2.12] we know that \( W \) is a self-adjoint map, i.e. \( W : T_p S \rightarrow T_p S \). As a consequence of this, \( W \) is adjoint to the first fundamental form \( I \).

Let \( w_1, w_2 \in T_p S \), with \( \{ \sigma_u, \sigma_v \} \) being a basis of the tangent plane \( T_p S \) and let, in this case, \( N \) be the standard unit normal of \( S \) at \( p \). That is, \( w_1 = \mu_1 \sigma_u + v_1 \sigma_v \) and \( w_2 = \mu_2 \sigma_u + v_2 \sigma_v \). The bilinear form

\[
I(w_1, w_2) = I(W(w_1), w_2)
\]

\[
= (\mu_1 W(\sigma_u) + v_1 W(\sigma_v)) \cdot (\mu_2 \sigma_u + v_2 \sigma_v)
\]

\[
= (\mu_1 (-N_u) + v_1 (-N_v)) \cdot (\mu_2 \sigma_u + v_2 \sigma_v)
\]

\[
= \mu_1 \mu_2 (-N_u \cdot \sigma_u) + \mu_1 v_2 (-N_u \cdot \sigma_v) + \mu_2 v_1 (-N_v \cdot \sigma_u)
\]

\[
+ v_1 v_2 (-N_v \cdot \sigma_v)
\]

\[
= \mu_1 \mu_2 (N \cdot \sigma_{uu}) + (\mu_1 v_2 + \mu_2 v_1)(N \cdot \sigma_{uv}) + v_1 v_2 (N \cdot \sigma_{vv}),
\]
is called the second fundamental form of the surface \( S \) at the point \( p \). In the last step used the fact that differentiating \( \mathbf{N} \cdot \sigma_u = 0 \) and \( \mathbf{N} \cdot \sigma_v = 0 \) with respect to \( u \) and \( v \), respectively, yields

\[
\begin{cases}
\sigma_{uu} \cdot \mathbf{N} = -\sigma_u \cdot \mathbf{N}_u \\
\sigma_{uv} \cdot \mathbf{N} = -\sigma_u \cdot \mathbf{N}_v \\
\sigma_{uv} \cdot \mathbf{N} = -\sigma_v \cdot \mathbf{N}_u \\
\sigma_{vv} \cdot \mathbf{N} = -\sigma_v \cdot \mathbf{N}_v
\end{cases}
\]

The second fundamental form can be written in matrix form as

\[
\Pi(w_1, w_2) = \begin{pmatrix} \mu_1 & \nu_1 \end{pmatrix} \begin{pmatrix} \sigma_{uu} \cdot \mathbf{N} & \sigma_{uv} \cdot \mathbf{N} \\ \sigma_{uv} \cdot \mathbf{N} & \sigma_{vv} \cdot \mathbf{N} \end{pmatrix} \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}.
\]

We also make the following denotations:

\[ L = \sigma_{uu} \cdot \mathbf{N}, \quad M = \sigma_{uv} \cdot \mathbf{N} \quad \text{and} \quad N = \sigma_{vv} \cdot \mathbf{N}. \quad (2.10) \]

**Definition 2.4.1 (Second fundamental form).** Let \( T_pS \) be the tangent plane of \( S \) at the point \( p \) and let \( w_1, w_2 \in T_pS \) be tangent vectors to \( S \) at \( p \). Let \( \mathcal{W} \) be the Weingarten map and let \( \mathbf{N} \) be the standard unit normal at \( p \). The symmetric bilinear form \( \Pi \), defined as

\[ \Pi : T_pS \times T_pS \to \mathbb{R} \\
(w_1, w_2) \mapsto \langle \mathcal{W}(w_1), w_2 \rangle, \]

such that \( \Pi(w_1, w_1) = \mathbb{I}(\mathcal{W}(w_1), w_2) \), is called the second fundamental form of \( S \) at \( p \). Given a parametrization \( \sigma \), the matrix of the second fundamental form is then defined as \( \mathcal{F}_\Pi = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \), with \( L, M \) and \( N \) defined as in \( (2.10) \).

Some verifications will reveal that \( \Pi(w_1, w_2) = \Pi(w_2, w_1) \), indicating that the second fundamental form is indeed symmetric.

**Example 2.4.2 (Torus, continued).** Continuing example \( 2.3.4 \) we calculate the
second fundamental form of the Torus.

\[
\sigma_{\theta\theta} = (-r \cos \theta \cos \varphi, -r \cos \theta \sin \varphi, -r \sin \theta)
\]
\[
\sigma_{\varphi\varphi} = -(R + r \cos \theta) \cos \varphi, -(R + r \cos \theta) \sin \varphi, 0)
\]
\[
\sigma_{\theta\varphi} = (r \sin \theta \sin \varphi, -r \sin \theta \cos \varphi, 0)
\]
\[
\sigma_{\varphi\theta} = \sigma_{\theta\varphi}
\]
\[
\sigma_{\theta} \times \sigma_{\varphi} = -r(R + r \cos \theta)(\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta)
\]
\[
N = \frac{\sigma_{\theta} \times \sigma_{\varphi}}{||\sigma_{\theta} \times \sigma_{\varphi}||} = -(\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta)
\]
\[
N \cdot \sigma_{\theta\theta} = r \cos \theta \cos \varphi \cos \theta \cos \varphi + \cos \theta \sin \varphi \sin \theta = b
\]
\[
N \cdot \sigma_{\varphi\varphi} = \cos \theta \cos \varphi \cos \theta \cos \varphi + \cos \theta \sin \varphi \sin \theta = \cos \theta \sin \theta \sin \varphi \sin \varphi - r \sin \theta \cos \theta \sin \varphi \sin \varphi = 0
\]
\[
\therefore \ F_{II} = \begin{pmatrix} r & 0 \\ 0 & \cos \theta (R + r \cos \theta) \end{pmatrix}
\]

Example 2.4.3 (Catenoid, continued). Calculating the second fundamental form of the Catenoid yields some interesting results.

\[
\sigma_{uu} = (\cosh u \cos v, \cosh u \sin v, 0)
\]
\[
\sigma_{vv} = (- \cosh u \cos v, - \cosh u \sin v, 0)
\]
\[
\sigma_{uv} = (\sinh u \sin v, \sinh u \cos v, 0)
\]
\[
\sigma_{u} \times \sigma_{v} = \cosh u(- \cos v, - \sin v, \sinh u)
\]
\[
N = \frac{\sigma_{u} \times \sigma_{v}}{||\sigma_{u} \times \sigma_{v}||} = \frac{1}{\cosh u}(- \cos v, - \sin v, \sinh u)
\]
\[
N \cdot \sigma_{uu} = -1
\]
\[
N \cdot \sigma_{vv} = 1
\]
\[
N \cdot \sigma_{uv} = 0
\]
\[
\therefore \ F_{II} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

The catenoid is a minimal surface. This kind of surfaces has many interesting properties, but is outside the scope of this thesis. More information regarding minimal surfaces can be found in [Do Carmo] and [Pressley].

Proposition 2.4.4. Let $F_1$ and $F_{II}$ be the matrices of the first- and second fundamental forms, respectively, and let $W$ be the matrix of the Weingarten map. Then,

\[
W = F_1^{-1}F_{II}.
\]
Proof. We first note that, since $I$ is a symmetric bilinear form, $\det(I) > 0$ and thus the inverse $F_I^{-1}$ exists. Suppose that $\{\sigma_u, \sigma_v\}$ is a basis of the tangent plane at a point on a surface and let $N$ be the standard unit normal at this point. Then $W(\sigma_u) = -N_u$ and $W(\sigma_v) = -N_v$, where $N_u$ and $N_v$ can be written as
\[ \begin{cases} -N_u = a\sigma_u + b\sigma_v, \\ -N_v = c\sigma_u + d\sigma_v, \end{cases} \quad a, b, c, d \in \mathbb{R}. \tag{2.12} \]

The matrix of the Weingarten map in the basis $\{\sigma_u, \sigma_v\}$ is then $W = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Applying the first fundamental form now yields the following equations:
\[ \begin{align*} I(-N_u, \sigma_u) &= -N_u \cdot \sigma_u = a(\sigma_u \cdot \sigma_u) + b(\sigma_u \cdot \sigma_v) \\ I(-N_v, \sigma_u) &= -N_v \cdot \sigma_u = c(\sigma_u \cdot \sigma_u) + d(\sigma_u \cdot \sigma_v) \\ I(-N_u, \sigma_v) &= -N_u \cdot \sigma_v = a(\sigma_u \cdot \sigma_v) + b(\sigma_v \cdot \sigma_v) \\ I(-N_v, \sigma_v) &= -N_v \cdot \sigma_v = c(\sigma_u \cdot \sigma_v) + d(\sigma_v \cdot \sigma_v) \end{align*} \]

Which, written in matrix form with the use of the coefficients of the first- and second fundamental forms, is equivalent to
\[ \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \]

Matrix multiplication of $F_I^{-1}$ from the left gives us the expression in (2.11). 

Now, since $W$ is self-adjoint and assuming that $S$ is a smooth oriented surface, $\sigma$ can be chosen in such a way that there exists a basis $\{t_1, t_2\}$ of the tangent plane such that $t_1$ and $t_2$ are orthogonal eigenvectors of $W$, see A.0.5.

Proposition 2.4.5. For each point $p$ of a surface $S$ there exists a basis $\{t_1, t_2\}$ of $T_pS$ such that $t_1$ and $t_2$ are the eigenvectors of the matrix of the Weingarten map, i.e.
\[ W = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad \kappa_1, \kappa_2 \in \mathbb{R}. \tag{2.13} \]

The eigenvectors $t_1$ and $t_2$ of the Weingarten map are called the principal vectors and the corresponding eigenvalues $\kappa_1, \kappa_2$ are called the principal curvatures of the surface $S$.

Let $\gamma$ be unit-speed curve through a point $p$ on $S$. Then $\dot{\gamma}$ is a tangent vector to $S$ at $p$. Thus $\gamma, N$ and $N \times \gamma$ are mutually perpendicular unit vectors. We can deduce the following relationship between the curvature of curves on a surface and the second fundamental form.
Definition 2.4.6 (Normal- and geodesic curvature). Let \( \gamma \) be a unit-speed curve through a point \( p \) on a surface \( S \) and let \( N \) be the standard unit normal at that point. Then \( \ddot{\gamma} = \kappa_n N + \kappa_g (N \times \dot{\gamma}) \), where \( \kappa_n \) is the normal curvature and \( \kappa_g \) is the geodesic curvature of \( \gamma \). That is,

\[
\begin{align*}
\kappa_n &= \dot{\gamma} \cdot N \\
\kappa_g &= \dot{\gamma} \cdot (N \times \dot{\gamma}) \\
\kappa &= \kappa_n^2 + \kappa_g^2,
\end{align*}
\]

where \( \kappa = \|\ddot{\gamma}\| \) is the curvature of \( \gamma \) and \( \kappa_n = \text{II}(\dot{\gamma}, \dot{\gamma}) \). The principal curvatures \( \kappa_1 \) and \( \kappa_2 \) are the maximum and minimum values of the normal curvatures of curves through \( p \).

Definition 2.4.7 (Mean- and Gaussian curvature). We define the mean- and Gaussian curvature of a surface as

\[
H = \frac{1}{2} \text{trace}(\mathcal{W}) \quad \text{and} \quad K = \text{det}(\mathcal{W}),
\]

respectively.

The mean curvature is a measure of the curvature of a surface at a certain point. By combining [2.4.5] and [2.4.7] the mean- and Gaussian curvature can be expressed in terms of the principal curvatures as

\[
H = \frac{1}{2}(\kappa_1 + \kappa_2) \quad \text{and} \quad K = \kappa_1 \kappa_2.
\]

This together with [2.4.4] implies that the mean and Gaussian curvature can be expressed in terms of the coefficients of the first and second fundamental forms as

\[
H = \frac{LG - 2MF + NE}{2(EG - F^2)} \quad \text{and} \quad K = \frac{LN - M^2}{EG - F^2}.
\]

Example 2.4.8 (Monge patch). Let \( p \) be a point on a surface \( S \). Then there is a patch \( V \subseteq S \) containing \( p \) such that \( \sigma : U \to V \) is given by \( \sigma(u, v) = (u, v, f(u, v)) \), where \( f : U \to \mathbb{R} \) is a differentiable function. We calculate the
first and second fundamental forms of this surface patch, called a Monge patch.

\[ \sigma_u = (1, 0, f_u), \quad \sigma_v = (0, 1, f_v) \]

\[ E = \|\sigma_u\|^2 = 1 + f_u^2, \quad G = \|\sigma_v\|^2 = 1 + f_v^2, \quad F = \sigma_u \cdot \sigma_v = f_u f_v \]

\[ \therefore F_I = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix} \]

\[ \sigma_{uu} = (0, 0, f_{uu}), \quad \sigma_{vv} = (0, 0, f_{vv}), \quad \sigma_{uv} = (0, 0, f_{uv}) \]

\[ \sigma_u \times \sigma_v = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} (-f_u, -f_v, 1) \]

\[ N = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} (-f_u, -f_v, 1) \]

\[ L = N \cdot \sigma_{uu} = \frac{f_u}{\sqrt{1 + f_u^2 + f_v^2}}, \quad N = N \cdot \sigma_{vv} = \frac{f_v}{\sqrt{1 + f_u^2 + f_v^2}}, \]

\[ M = N \cdot \sigma_{uv} = \frac{f_u}{\sqrt{1 + f_u^2 + f_v^2}} \]

\[ \therefore F_{II} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} f_u & f_v \\ f_u & f_v \end{pmatrix} \]

We also make some preparatory work by calculating the mean and Gaussian curvature introduced in proposition [2.4.7]

\[ H = \frac{f_{uu}(1 + f_v^2) - 2f_{uv}(f_u f_v) + f_{vv}(1 + f_u^2)}{2(1 + f_u^2 + f_v^2)^{3/2}} \quad (2.18) \]

\[ K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^{3/2}}. \quad (2.19) \]

In fact, by the inverse function theorem stated in [A.0.3], every surface can locally be seen as a Monge patch. This turns out to be quite useful. If we, for a moment, regard \( U \) in example [2.4.8] as being a thin elastic plate, then \( V \) is the shape \( U \) takes as the deformation \( f \) is being applied to it. This deformation strains the plate and generates stresses in the material. The energy required to produce this deformation is called the strain energy. The coordinate system of \( \mathbb{R}^3 \) can be chosen in such a way that \( p = \sigma(0, 0) = (0, 0, f(0, 0)) \) is the origin and \( f(0, 0) = f_u(0, 0) = f_v(0, 0) = 0 \). According to the theory of elasticity, the strain energy is then proportional to the expression \( f_{uu}(0, 0)^2 + 2f_{uv}(0, 0)^2 + f_{vv}(0, 0)^2 \).
2.4. The Second Fundamental Form

We relate this expression to the first and second fundamental form by

$$
\kappa_1^2 + \kappa_2^2 = (\kappa_1 + \kappa_2)^2 - 2\kappa_1\kappa_2 = 4H^2 - 2K
$$

$$
= \frac{(f_{uu}(1 + f_u^2) - 2f_{uv}(f_u f_v) + f_{vv}(1 + f_v^2))^2}{(1 + f_u^2 + f_v^2)^3} - 2 \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^{3/2}},
$$

which at the point \(p\) is

$$
\kappa_1^2 + \kappa_2^2(0, 0) = (f_{uu}(0, 0) + f_{vv}(0, 0))^2 - 2(f_{uu}(0, 0)f_{vv}(0, 0) - f_{uv}(0, 0)^2)
$$

$$
= f_{uu}(0, 0)^2 + 2f_{uv}(0, 0)^2 + f_{vv}(0, 0)^2.
$$

Since the shape of a surface is the same regardless of choice of parametrization, the principal curvatures at each point of the surface is independent of the choice of parametrization. This means that the strain energy of a surface can be calculated for any given parametrization.

**Proposition 2.4.9** (Strain Energy of a Surface). Let \(V\) be a patch of a surface \(S\) and let \(\sigma : U \rightarrow V\) be the parametrization of \(V\). The strain energy \(W\) of \(V\) is then given by

$$
W = \int\int_U (\kappa_1^2 + \kappa_2^2) \sqrt{EG - F^2} \, dudv, \tag{2.20}
$$

where \(\kappa_1, \kappa_2\) are the principal curvatures and \(E, G\) and \(F\) are the coefficients of the first fundamental form.

We remind the reader of proposition 2.3.8. In 2.4.9 above, \((\kappa_1^2 + \kappa_2^2) \sqrt{EG - F^2}\) is the strain energy of the area element \(dA = \sqrt{EG - F^2} \, dudv\).

**Remark 2.4.10.** The theory of elasticity is not within the scope of this thesis but a brief account can be found in [Bonneau].
Chapter 3

Bézier Surfaces

We now focus on a specific kind of surfaces known as Bézier surfaces. These surfaces have certain interesting properties which makes them easy to manipulate according to one’s preferences. The convex hull property contains the surface within a set of points in $\mathbb{R}^3$. The recursive property substantially reduces the amount of calculations required to find the first and second fundamental forms which, in turn, makes calculating the strain energy a rather straightforward procedure. In this chapter we enumerate and study these properties. In section 4.2 the reader finds a brief introduction to additional properties which are not studied in full detail in this thesis, but are important nonetheless. At the end of this chapter the reader finds an example which covers the calculations required to find the strain energy at a point of a Bézier surface. This example serves as a complement to the results in section 4.1.

3.1 Bézier Curves

As already mentioned in the introduction, a Bézier curve is defined by its associated control polygon and the Bernstein basis polynomials. This means that the components of a Bézier curve of degree $n$ are linear combinations of the Bernstein basis polynomials of degree $\leq n$.

**Definition 3.1.1** (Bernstein basis polynomials). Let $n \in \mathbb{N}$. The polynomial

$$B_i^n(t) = \begin{cases} \binom{n}{i} t^i (1-t)^{n-i} & \text{if } i = 0, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

is called the $i^{th}$ Bernstein basis polynomial of degree $n$. Together the Bernstein
basis polynomials \( \{B^n_i\} \) form a basis for the vector space \( \mathbb{P}_n \) of polynomials of degree \( \leq n \).

**Proposition 3.1.2** (Properties of Bernstein basis polynomials). The Bernstein basis polynomials has the following properties:

\[
\sum_{i=0}^{n} B^n_i(t) = 1, \quad t \in [0,1] \tag{3.1}
\]

\[
B^n_i(t) \geq 0, \quad t \in [0,1] \tag{3.2}
\]

\[
B^n_{n-i}(t) = B^n_i(1-t) \tag{3.3}
\]

\[
B^n_i(t) = (1-t)B^n_{i-1}(t) + tB^n_{i+1}(t) \tag{3.4}
\]

**Example 3.1.3** (Bernstein polynomials as a basis). The Bernstein polynomials of degree 0,1,2 and 3 are given below.

\[
B^0_0(t) = 1 \\
B^1_0(t) = 1 - t, \quad B^1_1(t) = t \\
B^2_0(t) = (1-t)^2, \quad B^2_1(t) = 2t(1-t), \quad B^2_2(t) = t^2 \\
B^3_0(t) = (1-t)^3, \quad B^3_1(t) = 3t(1-t)^2, \quad B^3_2(t) = 3t^2(1-t), \quad B^3_3(t) = t^3
\]

As a consequence of (3.4), the derivative of a Bernstein polynomial of degree \( n \) is recursively defined by Bernstein polynomials of degree \( n-1 \).

**Proposition 3.1.4** (Derivatives of Bernstein polynomials). The first derivative of a Bernstein polynomial of degree \( n \) is recursively defined as

\[
(B^n_i)'(t) = n(B^n_{i-1}(t) - B^n_i(t))
\]

We will illustrate proposition 3.1.4 with an example.

**Example 3.1.5** (Derivatives of cubic Bernstein basis polynomials). The first derivatives of the cubic Bernstein basis polynomials are

\[
(B^3_0)'(t) = -3(1-t)^2 = -3B^2_0(t) \tag{3.5}
\]

\[
(B^3_1)'(t) = 3(1-t)^2 - 6t(1-t) = 3(B^2_0(t) - B^2_1(t)) \tag{3.6}
\]

\[
(B^3_2)'(t) = 6t(1-t) - 3t^2 = 3(B^2_1(t) - B^2_2(t)) \tag{3.7}
\]

\[
(B^3_3)'(t) = 3t^2 = 3B^2_2(t). \tag{3.8}
\]
Note that we in equations 3.5 and 3.8 used the fact that $B_{-1}^2(t) = 0$ and $B_3^2(t) = 0$ as per definition 3.1.1.

**Definition 3.1.6 (Bézier curve).** Let $n \in \mathbb{N}$ and let $b_0, \ldots, b_n$ be $n+1$ points in $\mathbb{R}^3$. The parametric curve $B$ defined by

$$B : [0, 1] \rightarrow \mathbb{R}^3$$

$$t \mapsto B(t) = \sum_{i=0}^{n} b_i B_i^n(t)$$

is called the *Bézier curve of degree* $n$, with *control points* $b_0, \ldots, b_n$. The polygon with vertices $b_0, \ldots, b_n$ is called the *control polygon* of $B$ and $B_i^n(t)$ is the $i^{th}$ Bernstein basis polynomial of degree $n$.

**Example 3.1.7 (Cubic Bézier curve).** Let $b_0, b_1, b_2, b_3$ be points in $\mathbb{R}^2$. The parametrized curve $B$ given by

$$B(t) = \sum_{i=0}^{3} b_i B_i^3(t), \quad t \in [0, 1],$$

is then the cubic Bézier curve with control points $b_0, b_1, b_2$ and $b_3$. If we, for example, let $b_0 = (-1, 0)$, $b_1 = (-\frac{1}{2}, \frac{1}{2})$, $b_2 = (\frac{1}{2}, -\frac{1}{2})$ and $b_3 = (1, 0)$, then

$$B(t) = \sum_{i=0}^{3} b_i B_i^3(t)$$

$$= (-1, 0)(1-t)^3 + 3(-\frac{1}{2}, \frac{1}{2})(1-t)^2t + 3(\frac{1}{2}, -\frac{1}{2})(1-t)t^2 + (1, 0)t^3$$

$$= (-t^3 + \frac{3}{2}t^2 + \frac{3}{2}t - 1 , 3t^3 - \frac{9}{2}t^2 + \frac{3}{2}t).$$

Note that $B(0) = (-1, 0) = b_0$ and $B(1) = (1, 0) = b_3$.

The following algorithm is yet another consequence of the recursive property of Bernstein polynomials.

**Proposition 3.1.8 (The de Casteljau Algorithm).** Let $B(t)$, $t \in [0, 1]$, be a Bézier curve of degree $n$ with control points $b_0, \ldots, b_n$. If we define

$$\begin{align*}
\left\{ 
\begin{array}{ll}
\hat{b}_i^0(t) &= b_i, & i = 0, \ldots, n \\
\hat{b}_i^r(t) &= (1-t)\hat{b}_i^{r-1}(t) + t\hat{b}_{i+1}^{r-1}(t), & 1 \leq r \leq n, & i = 0, \ldots, n-r
\end{array}
\right.
\end{align*}$$

then $B(t) = \hat{b}_0^n(t)$. 

Example 3.1.9 (The de Casteljau algorithm). Let $B(t), t \in [0, 1]$, be a Bézier curve of degree 2 with control points $(b_0, b_1, b_2)$.

\[
B(t) = \sum_{i=0}^{2} b_i B_i^2(t) = (1-t)^2 b_0 + 2t(1-t)b_1 + t^2 b_2.
\]

Following the de Casteljau algorithm we get the same result:

\[
b_i^0 = b_i, \quad i = 0, 1, 2
\]

\[
b_0^2(t) = (1-t)b_0^1 + tb_1^1 = (1-t)((1-t)b_0 + tb_1) + t((1-t)b_1 + tb_2) = (1-t)^2 b_0 + 2b_1 t(1-t) + b_2 t^2 = B(t)
\]

The figure to the left illustrates the de Casteljau algorithm when evaluating the point $B(0.5) = b_0^2(0.5)$. The figure to the right shows the quadric Bézier curve $B(t), t \in [0, 1]$.

Proposition 3.1.10 (Properties of Bézier curves). Let $B(t)$ be a Bézier curve with control points $b_0, \ldots, b_n$ and let $t \in [0, 1]$.

- (Convex hull property) $B(t)$ is contained within the convex hull of its control polygon, i.e.

\[
B(t) \in CH\{b_0, \ldots, b_n\}, \quad (3.9)
\]

see \[A.0.6\]
3.1. Bézier Curves

• (Invariance under affine transformation) If \( T \) is an affine transformation, then

\[
T\left(\sum_{i=0}^{n} b_i B_i^n(t)\right) = \sum_{i=0}^{n} T(b_i) B_i^n(t). \tag{3.10}
\]

• (Endpoint interpolation property) The endpoints \( \mathbf{B}(0) \) and \( \mathbf{B}(1) \) of \( \mathbf{B}(t) \) coincide with \( b_0 \) and \( b_n \), respectively. That is,

\[
\mathbf{B}(0) = b_0 \quad \text{and} \quad \mathbf{B}(1) = b_n \tag{3.11}
\]

• (Endpoint tangent property) The tangent of \( \mathbf{B}(t) \) in its endpoints is given by

\[
\dot{\mathbf{B}}(0) = n(b_1 - b_0) \quad \text{and} \quad \dot{\mathbf{B}}(1) = n(b_n - b_{n-1}) \tag{3.12}
\]

Remark 3.1.11. When calculating the strain energy of a few different Bézier surfaces in chapter 4, we make use of property (3.10) in order to manipulate the shape of the surfaces.

Example 3.1.12 (Derivatives of a Cubic Bézier Curve). If \( \mathbf{B} \) is a cubic Bézier curve then the first- and second derivative of \( \mathbf{B} \) are given by:

\[
\dot{\mathbf{B}}(t) = \sum_{i=0}^{3} b_i \dot{B}_i^3(t) \\
= \sum_{i=0}^{3} 3b_i (B_{i-1}^2(t) - B_i^2(t)) \\
= -3b_0 B_0^2(t) + 3b_1 (B_0^2(t) - B_1^2(t)) + 3b_2 (B_1^2(t) - B_2^2(t)) + 3b_3 B_2^2(t) \\
= 3(b_1 - b_0) B_0^2(t) + 3(b_2 - b_1) B_1^2(t) + 3(b_3 - b_2) B_2^2(t) \\
\ddot{\mathbf{B}}(t) = 6(b_0 - 2b_1 + b_2) B_0^1(t) + 6(b_1 - 2b_2 + b_3) B_1^1(t).
\]

Now, if we let \( b_{0}^{(1)} = 3(b_1 - b_0) \), \( b_{1}^{(1)} = 3(b_2 - b_1) \), \( b_{2}^{(1)} = 3(b_3 - b_2) \) and \( b_{0}^{(2)} = 2(b_{1}^{(1)} - b_{0}^{(1)}) \), \( b_{1}^{(2)} = 2(b_{2}^{(1)} - b_{1}^{(1)}) \), it follows that

\[
\dot{\mathbf{B}}(t) = \sum_{i=0}^{2} b_i^{(1)} B_i^2(t) \quad \text{and} \quad \ddot{\mathbf{B}}(t) = \sum_{i=0}^{1} b_i^{(2)} B_i^1(t).
\]

The results in example 3.1.12 suggests the following proposition.
**Proposition 3.1.13** (Derivatives of Bézier curves). Let \( B(t) \) be a Bézier curve of degree \( n \) with control points \( b_0, \ldots, b_n \). The \( r \)-th derivative of \( B(t) \) is given by
\[
B^{(r)}(t) = \sum_{i=0}^{n-r} b_i^{(r)} B_{n-r}^{i}(t),
\]
where \( b_i^{(r)} = \frac{n(n-1) \cdots (n-r-1)}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} b_{i+j}. \)

This means that the derivatives of a Bézier curve are recursively defined by its control points by the use of the de Castiljau algorithm. In other words, for any given control polygon the appropriate Bézier curve and its derivatives are already given.

### 3.2 Bézier Surfaces

A Bézier surface of degree \((m, n)\) can be seen as a "product" of two Bézier curves of degree \( m \) and \( n \), respectively, and is defined by its control polygon consisting of points in \( \mathbb{R}^3 \).

**Definition 3.2.1** (Bézier surface). Let \((b_{i,j}), i = 0, \ldots, m, j = 0, \ldots, n\), be \((m+1)(n+1)\) points in \( \mathbb{R}^3 \) and let \( s, t \in [0,1] \). The parametric surface \( X \) defined by
\[
(s, t) \mapsto X(s, t) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{i,j} B_i^m(s) B_j^n(t)
\]
is a Bézier surface of degree \((m, n)\), with control points \((b_{i,j})\). \( B_i^m(s) \) and \( B_j^n(t) \) are the \( i \)-th and \( j \)-th Bernstein basis polynomials of degree \( m \) and \( n \), respectively.

**Remark 3.2.2.** Strictly speaking, a Bézier surface is a tensor product of two Bézier curves. This notion is, however, not vital for the material in this thesis and will therefore be left out.

Some of the interesting properties of Bézier curves mentioned in proposition 3.1.10 such as the convex hull property, is inherited by the Bézier surfaces. This means that a point of a Bézier surface can be evaluated using the de Castiljau algorithm, as the following example illustrates.

**Example 3.2.3** (The de Castiljau algorithm for Bézier surface). Let \( m = 1 \), \( n = 2 \) and let \((b_{i,j}), i = 0, 1, j = 0, 1, 2\), be the 6 control points of the Bézier surface \( X(s, t) \) of degree \((1, 2)\), \( s, t \in [0,1] \). The figure below illustrates how the de Castiljau algorithm can be used to evaluate the points of \( X \); here at \( s = t = 0.5 \).
Proposition 3.2.4 (Derivative of a Bézier surface). Let $X(s,t)$, with $s,t \in [0,1]$, be a Bézier surface of degree $(m,n)$ with control points $b_{i,j}$. The partial derivative of order $\alpha$ and $\beta$ with respect to $s$ and $t$, respectively, is given by

$$X_{\alpha\beta}(s,t) = \sum_{i=0}^{m-\alpha} \sum_{j=0}^{n-\beta} b_{i,j}^{(\alpha,\beta)} B_i^{m-\alpha}(s) B_j^{n-\beta}(t),$$

(3.14)

where

$$b_{i,j}^{(\alpha,\beta)} = \frac{n!}{(n-\alpha)! (m-\beta)!} \sum_{k=0}^{\alpha} \sum_{l=0}^{\beta} (-1)^{k+l} \binom{\alpha}{k} \binom{\beta}{l} b_{i+k,j+l+\alpha-\beta}.$$ 

(3.15)

Using definition 2.3.2 and 2.4.1 regarding the first and second fundamental forms, as well as proposition 3.2.4 stated above, we can formulate the fundamental forms of a Bézier surface.

Proposition 3.2.5 (Fundamental forms of a Bézier surface). The fundamental forms of a Bézier surface can be written in matrix form as

$$F_I = \begin{pmatrix} X_{10} \cdot X_{10} & X_{10} \cdot X_{01} \\ X_{10} \cdot X_{01} & X_{01} \cdot X_{01} \end{pmatrix} \quad \text{and} \quad F_{II} = \begin{pmatrix} X_{20} \cdot N & X_{11} \cdot N \\ X_{11} \cdot N & X_{02} \cdot N \end{pmatrix},$$

respectively, where $N = \frac{X_{10} \times X_{01}}{||X_{10} \times X_{01}||}$ is the standard unit normal.

We end this chapter with an extensive example where the strain energy at a point of a Bézier surface is calculated. The plot of the surface, along with the associated strain energy surface, can be seen in chapter 4.
Example 3.2.6 (Strain energy of a Bézier surface). Let \( X(s,t) \), \( s,t \in [0,1] \), be a Bézier surface of degree \((3,3)\) such that \( X : [0,1] \times [0,1] \rightarrow \mathbb{R}^3 \). Let \((b_{i,j})\), where \( i = 0, \ldots, 3 \) and \( j = 0, \ldots, 3 \), be the 16 control points of \( X \) listed in \( CP \).

\[
CP = \begin{pmatrix}
 b_{0,0} & b_{0,1} & \cdots & b_{1,0} & b_{1,1} & \cdots & b_{3,0} & b_{3,1} & \cdots & b_{3,3} \\
 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0.8 & 0.8 & 0.8 & 0.8 & 1 & 1 & 1 & 1 \\
 0 & 0.2 & 0.8 & 1 & 0 & 0.2 & 0.8 & 1 & 0 & 0.2 & 0.8 & 1 & 0.2 & 0.2 & 0 \\
 0 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0 & 0.2 & 0 & 0
\end{pmatrix}
\]

We calculate the strain energy in the point \( X(0.1, 0.1) \).

\[
X(s,t) = \sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,j} B_i^3(s) B_j^3(t) = b_{0,0} B_0^3(s) B_0^3(t) + b_{0,1} B_0^3(s) B_1^3(t) + \ldots \\
\ldots + b_{1,0} B_1^3(s) B_0^3(t) + b_{1,1} B_1^3(s) B_1^3(t) + \ldots \\
\ldots + b_{3,0} B_3^3(s) B_0^3(t) + b_{3,1} B_3^3(s) B_1^3(t) + \cdots + b_{3,3} B_3^3(s) B_3^3(t).
\]

\[
X(0.1, 0.1) = \cdots = \begin{pmatrix}
 0.0712 \\
 0.0712 \\
 0.0934 
\end{pmatrix}
\]

\[
X_{10}(s,t) = \sum_{i=0}^{2} \sum_{j=0}^{3} b_{i-1,j}^{(1,0)} B_i^2(s) B_j^3(t) = \sum_{i=0}^{2} \sum_{j=0}^{3} 3(b_{i+1,j} - b_{i,j}) B_i^2(s) B_j^3(t)
\]

\[
X_{01}(s,t) = \sum_{i=0}^{3} \sum_{j=0}^{2} b_{i,j-1}^{(0,1)} B_i^3(s) B_j^2(t) = \sum_{i=0}^{3} \sum_{j=0}^{2} 3(b_{i,j+1} - b_{i,j}) B_i^3(s) B_j^2(t)
\]

\[
X_{10}(0.1, 0.1) = \cdots = \begin{pmatrix}
 0.8160 \\
 0 \\
 0.3504 
\end{pmatrix}
\]

\[
X_{01}(0.1, 0.1) = \cdots = \begin{pmatrix}
 0 \\
 0.8160 \\
 0.3504 
\end{pmatrix}
\]

\[
N(0.1, 0.1) = \frac{X_{10}(0.1, 0.1) \times X_{01}(0.1, 0.1)}{\|X_{10}(0.1, 0.1) \times X_{01}(0.1, 0.1)\|} = \begin{pmatrix}
 -0.3670 \\
 -0.3670 \\
 0.8547 
\end{pmatrix}
\]
3.2. Bézier Surfaces

\[ \mathbf{X}_{20}(s, t) = \sum_{i=0}^{1} \sum_{j=0}^{3} \mathbf{b}_{i,j}^{(2,0)} B_i^1(s) B_j^3(t) = \sum_{i=0}^{1} \sum_{j=0}^{3} 6(b_{i+2,j} - 2b_{i+1,j} + b_{i,j}) B_i^1(s) B_j^3(t) \]

\[ \mathbf{X}_{02}(s, t) = \sum_{i=0}^{3} \sum_{j=0}^{1} \mathbf{b}_{i,j}^{(0,2)} B_i^3(s) B_j^1(t) = \sum_{i=0}^{3} \sum_{j=0}^{1} 6(b_{i,j+2} - 2b_{i,j+1} + b_{i,j}) B_i^3(s) B_j^1(t) \]

\[ \mathbf{X}_{11}(s, t) = \sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{b}_{i,j}^{(1,1)} B_i^2(s) B_j^2(t) \]

\[ = \sum_{i=0}^{2} \sum_{j=0}^{2} 9(b_{i+1,j+1} - b_{i+1,j} - b_{i,j+1} + b_{i,j}) B_i^2(s) B_j^2(t) \]

\[ \mathbf{X}_{20}(0.1, 0.1) = \cdots = \begin{pmatrix} 1.9200 \\ 0 \\ -0.8760 \end{pmatrix} \]

\[ \mathbf{X}_{02}(0.1, 0.1) = \cdots = \begin{pmatrix} 0 \\ 1.9200 \\ -0.8760 \end{pmatrix} \]

\[ \mathbf{X}_{11}(0.1, 0.1) = \cdots = \begin{pmatrix} 0 \\ 0 \\ -1.1520 \end{pmatrix} \]

\[ \mathcal{F}_I(0.1, 0.1) = \begin{pmatrix} \mathbf{X}_{10}(0.1, 0.1) \cdot \mathbf{X}_{10}(0.1, 0.1) & \mathbf{X}_{10}(0.1, 0.1) \cdot \mathbf{X}_{01}(0.1, 0.1) \\ \mathbf{X}_{10}(0.1, 0.1) \cdot \mathbf{X}_{01}(0.1, 0.1) & \mathbf{X}_{01}(0.1, 0.1) \cdot \mathbf{X}_{01}(0.1, 0.1) \end{pmatrix} \]

\[ = \begin{pmatrix} 0.7886 & 0.1228 \\ 0.1228 & 0.7886 \end{pmatrix} \]

\[ \mathcal{F}_{II}(0.1, 0.1) = \begin{pmatrix} \mathbf{X}_{20}(0.1, 0.1) \cdot \mathbf{N}(0.1, 0.1) & \mathbf{X}_{11}(0.1, 0.1) \cdot \mathbf{N}(0.1, 0.1) \\ \mathbf{X}_{11}(0.1, 0.1) \cdot \mathbf{N}(0.1, 0.1) & \mathbf{X}_{02}(0.1, 0.1) \cdot \mathbf{N}(0.1, 0.1) \end{pmatrix} \]

\[ = \begin{pmatrix} -1.4535 & -0.9847 \\ -0.9847 & -1.4535 \end{pmatrix} \]

\[ H = \frac{(\mathbf{X}_{20} \cdot \mathbf{N})(\mathbf{X}_{01} \cdot \mathbf{X}_{01}) - 2(\mathbf{X}_{11} \cdot \mathbf{N})(\mathbf{X}_{10} \cdot \mathbf{X}_{01}) + (\mathbf{X}_{02} \cdot \mathbf{N})(\mathbf{X}_{10} \cdot \mathbf{X}_{10})}{\det(\mathcal{F}_I)} \]

\[ K = \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)} \]

\[ W = (4H^2 - 2K)\sqrt{\det(\mathcal{F}_I)} \]
\[ W(0.1, 0.1) = \cdots = 5.9603 \]

A plot of this Bézier surface can be found in figure 4.1. In table 4.1 the strain energy in three different points on the surface can be found. In this case \( a = 0.2 \) and according to the table, the strain energy in the point \( X(0.1, 0.1) \) is 0.2980. Note that there is a scaling factor 1/20 applied to the values in the table, so in this case \( \frac{1}{20} W(0.1, 0.1) \approx 0.2980 \).
Chapter 4

Results and Future Work

One rather important aspect to consider when constructing for example an airplane, is the strain energy in the material. If the strain energy is too great in some part of the airplane body it might break, causing a disaster. It is therefore desirable to be able to calculate the strain energy. In this chapter we do this for four Bézier surfaces of different shapes to illustrate how the shape contributes to the overall durability of, for example, a segment of an airplane body.

4.1 Strain Energy of a Bézier Surface

We analyze the strain energy of Bézier surfaces of degree \((3, 3)\) with control points \(b_{i,j}(a), 0 < a < 0.5\), represented in \(CP(a)\) below.

\[
CP(a) = \begin{pmatrix}
    b_{0,0}(a) & b_{0,1}(a) & \cdots & b_{3,0}(a) & b_{3,1}(a) & \cdots & b_{3,3}(a) \\
    0 & 0 & 0 & a & a & a & 1-a & 1-a & 1-a & 1-a & 1 & 1 & 1 & 1 \\
    0 & a & 1-a & 1 & 0 & a & 1-a & 1 & 0 & a & 1-a & 1 & 0 & a & 1-a & 1 \\
    0 & a & 0 & a & a & a & a & a & a & a & 0 & a & a & 0 \\
\end{pmatrix}
\]

The parameter \(a\) determines the shape of the Bézier surface. If \(X : [0, 1] \times [0, 1] \to \mathbb{R}^3\) is the Bézier surface, the strain energy \(W(s,t)\) is plotted as the surface given by \(X(s,t) + \frac{1}{w} W(s,t)N(s,t)\). However, the fact that the standard unit normal of the surface is undefined in the points at the very edges, i.e. for \(s = 0, 1\) and \(t = 0, 1\), makes plotting the strain energy there impossible. Hence, the strain energy surface is redefined as \(X + \frac{1}{w} WN : (0, 1) \times (0, 1) \to \mathbb{R}^3\).
Figure 4.1: Strain energy surface of Bézier surface with parameter values $a = 0.1$ and $w = 200$

Figure 4.2: Strain energy surface of Bézier surface with parameter values $a = 0.2$ and $w = 20$
4.1. Strain Energy of a Bézier Surface

Figure 4.3: Strain energy surface of Bézier surface with parameter values $a = 0.3$ and $w = 20$

Figure 4.4: Strain energy surface of Bézier surface with parameter values $a = 0.4$ and $w = 20$
Figures 4.1-4.4 illustrate the resulting Bézier surfaces, along with the appropriate strain energy surface, for \( a = 0.1, a = 0.2, a = 0.3 \) and \( a = 0.4 \), respectively. Since \( W > 0 \), this means that the strain energy surface will, with respect to the tangent plane, be situated above the Bézier surface and if \( W = 0 \) for some point the two surfaces are tangent at this point. The factor \( \frac{1}{w} \) is applied to scale the strain energy and is chosen in such a way as to make the illustrations easier to interpret. The coloring of the strain energy surfaces range from purple (low strain energy) to orange (high strain energy) and the color of the Bézier surface is uniformly purple. Note that, for the sake of comparability, the value of \( w \) is 10 times larger in figure 4.1 in order to compensate for the relatively high strain energy in the corners in this case compared to the other three.

For each one of the four cases, the strain energy is measured in the same set of three different points; one close to the edge in a corner, one in the middle, close to the edge, and one in the center of the surface. The results are listed in table 4.1. Note that \( w = 20 \) for all the listed values.

<table>
<thead>
<tr>
<th>( X(s,t) )</th>
<th>( a = 0.1 )</th>
<th>( a = 0.2 )</th>
<th>( a = 0.3 )</th>
<th>( a = 0.4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X(0.1, 0.1) )</td>
<td>0.3362</td>
<td>0.2980</td>
<td>0.1844</td>
<td>0.1528</td>
</tr>
<tr>
<td>( X(0.5, 0.5) )</td>
<td>0.0012</td>
<td>0.0062</td>
<td>0.0184</td>
<td>0.0444</td>
</tr>
<tr>
<td>( X(0.9, 0.5) )</td>
<td>0.0451</td>
<td>0.0534</td>
<td>0.0861</td>
<td>0.2292</td>
</tr>
</tbody>
</table>

As the value of \( a \) increases from 0.1 to 0.4 the curvature in the four corners is decreased while the curvature along the sides is increased. The increase in strain energy at the point \( X(0.1, 0.1) \) for the increasing values of \( a \) illustrates this phenomenon. The fact that the strain energy in \( X(0.9, 0.5) \), for increasing values of \( a \), is increased, further illustrate this.

For small values of \( a \) the center of the surface is almost flat, as the strain energy at \( X(0.5, 0.5) \) indicates. As the value of \( a \) increases, however, the overall shape of the surface becomes more curved and as a result the strain energy at the center of the surface is increased.

### 4.2 Future work

In the examples in section 4.1 we changed the shape of the surfaces by changing the associated control polygon. There is, however, a more practical way of doing this, namely by using a rational Bézier surface. A rational Bézier surface
is similar to a non-rational one with the exception that each point comes with a corresponding weight, giving it the following parametrization:

\[
X(s, t) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} \omega_{i,j} b_{i,j} B_i^m(s) B_j^n(t)}{\sum_{i=0}^{m} \sum_{j=0}^{n} \omega_{i,j} B_i^m(s) B_j^n(t)}
\]

Changing the values of the weights \( \omega_{i,j} \) alters the shape of the surface while the control polygon remains unchanged. Increasing the value of one of the weights would result in the surface being "dragged" towards the corresponding control point. This way we can find the optimal values of the weights such that, for example, the strain energy of the surface is minimized.

While it is theoretically possible to model a Bézier surface into any desired shape, most of the time it requires a Bézier surface of a high degree, which isn't desireable from a computational complexity point of view. However, by the use of the end point tangent property stated in (3.12), it is possible to use a multitude of lower degree Bézier surfaces that together make up the whole of the desired surface. With each Bézier surface being of low degree, the computational complexity decreases dramatically. This, along with the recursive property, is what makes Bézier surfaces the highly versatile and powerful tool that it is.
Bibliography


Appendix A

Fundamental Concepts

Here follows some definitions of mathematical concepts used in this thesis.

Definition A.0.1 (Smooth map). A map \( f : \mathbb{R}^m \to \mathbb{R}^n \) is smooth if all of its components have continuous partial derivatives of all orders.

Definition A.0.2 (Homeomorphism). A map \( \Phi : U \to \tilde{U} \) is a homeomorphism if \( \Phi \) and \( \Phi^{-1} \) are continuous and \( \Phi \) is bijective. The sets \( U \) and \( \tilde{U} \) are then said to be homeomorphic.

Theorem A.0.3 (Inverse function theorem). Let \( f : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be a differential mapping and suppose that the differential \( df_p : \mathbb{R}^n \to \mathbb{R}^n \) at a point \( p \in U \) is an isomorphism. Then there exists a neighborhood \( V \) of \( p \) in \( U \) and a neighborhood \( W \) of \( f(p) \) in \( \mathbb{R}^n \) such that \( f : V \to W \) has a differential inverse \( f^{-1} : W \to V \).

Definition A.0.4 (Bilinear form). Let \( \mathbb{V} \) be a vector space of finite dimension \( n \) over \( \mathbb{R} \). A map

\[
\begin{align*}
\mathbb{V} \times \mathbb{V} & \to \mathbb{R} \\
(u, w) & \mapsto \langle u, w \rangle
\end{align*}
\]

is called a bilinear form if, \( \forall \lambda_1, \lambda_2 \in \mathbb{R}, \) and \( u_1, u_2, w \in \mathbb{V}, \) we have

\[
\begin{align*}
\langle \lambda_1 u_1 + \lambda_2 u_2, w \rangle &= \lambda_1 \langle u_1, w \rangle + \lambda_2 \langle u_2, w \rangle \\
\langle w, \lambda_1 u_1 + \lambda_2 u_2 \rangle &= \lambda_1 \langle w, u_1 \rangle + \lambda_2 \langle w, u_2 \rangle
\end{align*}
\]

Thus, \( \langle u, w \rangle \) is a linear function of \( u \) for each fixed \( w \), and a linear function of \( w \) for each fixed \( u \). If \( \{v_1, \ldots, v_n\} \) is a basis of \( \mathbb{V} \), any bilinear form \( \langle , \rangle \) on \( \mathbb{V} \) is
determined by the \( n \times n \) matrix whose \((i, j)\)-entry is \( \langle v_i, v_j \rangle \) for \( i, j = 1, \ldots, n \). For, if

\[
\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i, \quad \mathbf{w} = \sum_{j=1}^{n} \mu_j \mathbf{v}_j
\]

are any two vectors in \( \mathbb{V} \), then

\[
\langle \mathbf{u}, \mathbf{w} \rangle = \sum_{i,j=1}^{n} \lambda_i \mu_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle.
\]

A bilinear form \( \langle \mathbf{,} \rangle \) is called \textit{symmetric} if

\[
\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle \quad \forall \mathbf{u}, \mathbf{w} \in \mathbb{V}.
\]

Equivalently, the matrix of \( \langle \mathbf{,} \rangle \) with respect to any basis of \( \mathbb{V} \) is symmetric.

\textbf{Theorem A.0.5} (Self-adjoint map). Let \( \mathbf{M} : \mathbb{V} \to \mathbb{V} \) be a self-adjoint linear map. Then, \( \mathbb{V} \) has a basis \( \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) consisting of eigenvectors of \( \mathbf{M} \). Moreover, if \( \mathbf{u}_i \) and \( \mathbf{u}_j \) are eigenvectors corresponding to distinct eigenvalues, then \( \mathbf{u}_i \) and \( \mathbf{u}_j \) are orthogonal.

\textbf{Definition A.0.6} (Convex Hull). Let \( \mathbf{X} = \{\mathbf{x}_0, \ldots, \mathbf{x}_n\} \) be a finite set of points. The \textit{convex hull} of \( \mathbf{X} \) is defined to be

\[
CH\{\mathbf{X}\} = \left\{ \alpha_0 \mathbf{x}_0 + \cdots + \alpha_n \mathbf{x}_n \left| \sum_{i=0}^{n} \alpha_i = 1, \alpha_i \geq 0 \right. \right\}.
\]
Appendix B

Matlab

Code 1: Binomial Coefficients

function f = Binomial(n,i)

f = factorial(n)/((factorial(n-i))*factorial(i));
end

Code 2: Bernstein Polynomial

function g = Bernstein_Polynom(n,i,t)

g = Binomial(n,i)*((1-t)^(n-i))*t^i;
end

Code 3: Control Polygon

function cp=CP_matrix(a)

p00=[0; 0; 0]; %fixed
p01=[0; a; a];
p02=[0; (1-a); a];
p03=[0; 1; 0]; %fixed

p10=[a; 0; a];
p11=[a; a; a];
p12=[a; (1-a); a];
\[ \begin{align*}
\mathbf{p}_{13} &= [a; 1; a]; \\
\mathbf{p}_{20} &= [(1-a); 0; a]; \\
\mathbf{p}_{21} &= [(1-a); a; a]; \\
\mathbf{p}_{22} &= [(1-a); (1-a); a]; \\
\mathbf{p}_{23} &= [(1-a); 1; a]; \\
\mathbf{p}_{30} &= [1; 0; 0]; \text{ fixed} \\
\mathbf{p}_{31} &= [1; a; a]; \\
\mathbf{p}_{32} &= [1; (1-a); a]; \\
\mathbf{p}_{33} &= [1; 1; 0]; \text{ fixed}
\end{align*} \]

\[
\mathbf{c p} = \\
\begin{bmatrix}
\mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\
\mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\
\mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\
\mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33}
\end{bmatrix};
\]

end

Code 4: A function used by the Bézier Coordinates function.

function \text{cpx}=\text{CP}\_coord(r,m,n,\text{cp},i,j,\text{dx,dy})

cpx=0;

for k=0:dx
    for l=0:dy
        cpx = cpx \\
            + (factorial(m)/factorial(m-dx)) \\
            * (factorial(n)/factorial(n-dy)) \\
            *((-1)^{(k+l)}) \\
            * Binomial(dx,k) \\
            * Binomial(dy,l) \\
            * \text{cp}(r,((i+dx-k)\ast(n+1)+(j+dy-1)+1));
    end
end

end

Code 5: Bézier Coordinates
function xr=Bezier_Coordinates(r,m,n,cp,s,t,dx,dy)

xr=0;

for i = 0:(m-dx)
    for j = 0:(n-dy)
        xr = xr
            + CP_coord(r,m,n,cp,i,j,dx,dy)
                * Bernstein_Polynom((m-dx),i,s)
                * Bernstein_Polynom((n-dy),j,t);
    end
end

end

Code 6: First Fundamental Form

function f=I_Form(m,n,cp,s,t)

f=[];

Xu=[
    Bezier_Coordinates(1,m,n,cp,s,t,1,0)
    Bezier_Coordinates(2,m,n,cp,s,t,1,0)
    Bezier_Coordinates(3,m,n,cp,s,t,1,0)
    ];

Xv=[
    Bezier_Coordinates(1,m,n,cp,s,t,0,1)
    Bezier_Coordinates(2,m,n,cp,s,t,0,1)
    Bezier_Coordinates(3,m,n,cp,s,t,0,1)
    ];

f=[
    sum(Xu.*Xu) sum(Xu.*Xv);
    sum(Xu.*Xv) sum(Xv.*Xv)
    ];

end

Code 7: Standard Unit Normal

function un=Standard_Unit_Normal(m,n,cp,s,t)
un=[];
N=[];

Na=[
    Bezier_Coordinates(1,m,n,cp,s,t,1,0)
    Bezier_Coordinates(2,m,n,cp,s,t,1,0)
    Bezier_Coordinates(3,m,n,cp,s,t,1,0)
];

Nb=[
    Bezier_Coordinates(1,m,n,cp,s,t,0,1)
    Bezier_Coordinates(2,m,n,cp,s,t,0,1)
    Bezier_Coordinates(3,m,n,cp,s,t,0,1)
];

N=cross(Na,Nb);
un=(1/sqrt(sum(N.*N))).*N;

end

Code 8: Second Fundamental Form

function ff=II_Form(m,n,cp,s,t)

ff=[];

Xuu=[
    Bezier_Coordinates(1,m,n,cp,s,t,2,0)
    Bezier_Coordinates(2,m,n,cp,s,t,2,0)
    Bezier_Coordinates(3,m,n,cp,s,t,2,0)
];

Xvv=[
    Bezier_Coordinates(1,m,n,cp,s,t,0,2)
    Bezier_Coordinates(2,m,n,cp,s,t,0,2)
    Bezier_Coordinates(3,m,n,cp,s,t,0,2)
];

Xuv=[
    Bezier_Coordinates(1,m,n,cp,s,t,1,1)
    Bezier_Coordinates(2,m,n,cp,s,t,1,1)
Bezier_Coordinates(3,m,n,cp,s,t,1,1)
]

ff=[
    sum(Xuu.*Standard_Unit_Normal(m,n,cp,s,t))
    sum(Xuv.*Standard_Unit_Normal(m,n,cp,s,t))
    sum(Xuv.*Standard_Unit_Normal(m,n,cp,s,t))
    sum(Xvv.*Standard_Unit_Normal(m,n,cp,s,t))
]
end

Code 9: Strain Energy

function E=Strain_Energy(m,n,cp,s,t,w)

FI=I_Form(m,n,cp,s,t);
FII=II_Form(m,n,cp,s,t);

detFI=(FI(1,1)*FI(2,2))-(FI(1,2)*FI(1,2));
detFII=(FII(1,1)*FII(2,2))-(FII(1,2)*FII(1,2));

q=(FI(1,1)*FII(2,2))-(2*FI(1,2)*FII(1,2))+(FI(2,2)*FII(1,1));
E=(1/w)*(((q/detFI)^2)-(2*(detFII/detFI)))*sqrt(detFI);
end

Code 10: A function used by the Surface Coloring function.

function scx=s_c_x(m,n,cp,G,w,a)

scx=[];
lx=(1/G);
ly=(1/G);

for i=1:(ly-1)
    scx=[scx Strain_Energy(m,n,cp,G*i,G*a,w)];
end

end

Code 11: Surface Coloring – generates a matrix consisting of strain energy values which are plotted as the strain energy surface in the Bézier Surface function.
function sc=Surface_Colouring(m,n,cp,G,w)

sc=[];
lx=(1/G);
ly=(1/G);

for j=1:(ly-1)
    sc=[sc; s_c_x(m,n,cp,G,w,j)];
end

end

Code 12: A function used by the z-coordinate function.

function zx=zx_coord(m,n,cp,G,a,e,w,lx)

zx=[];

if e==1
    for k=1:(lx-1)
        zx=[
            zx
            (Bezier_Coordinates(3,m,n,cp,G*k,G*a,0,0)
            + sum(
                [0 0 Strain_Energy(m,n,cp,G*k,G*a,w)]
                .* (Standard_Unit_Normal(m,n,cp,G*k,G*a))
            )
            ];
    end
else
    for i=0:lx
        zx=[zx Bezier_Coordinates(3,m,n,cp,G*i,G*a,0,0)];
    end
end

end

Codes 13, 14 and 15 are functions which generate the x, y and z-coordinates of the strain energy surface.

Code 13: x-coord
function x=x_coord(m,n,cp,G,e,w)

x=[];

if e==1
    for k=G:G:1-G
        x=[
            x
            (Bezier_Coordinates(1,m,n,cp,k,0,0,0)
            + sum(
                [Strain_Energy(m,n,cp,k,0,w) 0 0]
                .* (Standard_Unit_Normal(m,n,cp,k,0))
            ))
        ];
    end
else
    for j=0:G:1
        x=[x (Bezier_Coordinates(1,m,n,cp,j,0,0,0))];
    end
end

end

Code 14: y-coord

function y=y_coord(m,n,cp,G,e,w)

y=[];

if e==1
    for k=G:G:1-G
        y=[
            y
            (Bezier_Coordinates(2,m,n,cp,0,k,0,0)
            + sum(
                [0 Strain_Energy(m,n,cp,0,k,w) 0]
                .* (Standard_Unit_Normal(m,n,cp,0,k))
            ))
        ];
    end
end
Code 15: z-coord

function z=z_coord(m,n,cp,G,e,w)

z=[ ];

lx=(1/G);
ly=(1/G);

if e==1
    for j=1:(ly-1)
        z=[z; zx_coord(m,n,cp,G,j,e,w,lx)];
    end
else
    for j=0:ly
        z=[z; zx_coord(m,n,cp,G,j,e,w,lx)];
    end
end

Code 16: Bézier Surface and Strain Energy Surface

function S = Bezier_Surface(m,n,a,G,e,w)

%Parameter a: Determines the shape of the Bézier surface.
% 0<a<1
%Parameter e:
% *If e=0, plot only Bézier surface.
% *If e=1, plot Bézier surface and the associated strain energy surface.
%Parameter w: Weight associated with the strain energy.
% w>1

cp=CP_matrix(a);
C=zeros(((1/G)+1),((1/G)+1));
Sx=x_coord(m,n,cp,G,0,w);
Sy=y_coord(m,n,cp,G,0,w);
Sz=z_coord(m,n,cp,G,0,w);

if e==1
    SEx=x_coord(m,n,cp,G,e,w);
    SEy=y_coord(m,n,cp,G,e,w);
    SEz=z_coord(m,n,cp,G,e,w);
    SC=Surface_Colouring(m,n,cp,G,w)
    S=surf(Sx,Sy,Sz,C); hold on
    SE=surf(SEx,SEy,SEz,SC)

    print -deps graph.eps

elseif e==0
    S=surf(Sx,Sy,Sz,C);

    print -deps graph.eps
else
    disp 'Error: e must be 0 or 1'
end
end
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