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Ghostpeakons

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Abstract

In this thesis we study *peakons* (peaked solitons), a class of solutions which occur in certain wave equations, such as the Camassa–Holm shallow water equation and its mathematical relatives, the Degasperis–Procesi, Novikov and Geng– Xue equations. These four non-linear partial differential equations are all integrable systems in the sense of having Lax pairs, infinitely many conservation laws, and multipeakon solutions given by explicitly known formulas.

In the first paper, we develop a method which uses so-called *ghostpeakons* (peakons with amplitude zero) to find explicit formulas for the characteristic curves associated with the multipeakon solutions of the Camassa–Holm, Degasperis–Procesi and Novikov equations.

In the second paper, we use the ghostpeakon method to derive explicit formulas for arbitrary multipeakon solutions of the two-component Geng–Xue equation. The general case involves many inequivalent peakons configurations, depending on the order in which the peakons occur in the two components of the solution, and previously the solution was known only in the so-called interlacing case where the peakons lie alternatingly in one component and in the other. To obtain the solution formulas for an arbitrary configuration, we introduce auxiliary peakons to make the configuration interlacing. By taking suitable limits, we then drive the amplitudes of the auxiliary peakons to zero, leaving the solution formulas for the remaining ordinary peakons.

Sammanfattning på svenska

I denna avhandling studeras *peakoner* (från engelskans *peakons*, en förkortning för *peaked solitons*, dvs. spetsiga solitoner). Detta är en klass av lösningar som förekommer i vissa vågekvationer, t.ex. Camassa–Holm-ekvationen från teorin för vågor i grunt vatten, samt de matematiskt närbesläktade Degasperis–Procesi-, Novikov- och Geng–Xue-ekvationerna. Samtliga dessa fyra ickelinjära partiella differentialekvationer är integrabla system i den meningen att de har Laxpar, oändligt många konserveringslagar, samt multipeakonlösningar som ges av explicita formler.

I den första artikeln utvecklar vi en metod som använder s.k. *spökpeakoner* (peakoner med amplituden noll) för att finna explicita formler för de karaktäristiska kurvorna som hör till multipeakonlösningarna till Camassa–Holm-, Degasperis–Procesi- och Novikov-ekvationerna.

I den andra artikeln använder vi spökpeakonmetoden för att härleda explicita formler för godtyckliga multipeakonlösningar till Geng–Xue-ekvationen, som är en tvåkomponentsekvation. Det allmänna fallet innefattar många icke-ekvivalenta peakonkonfigurationer, beroende på i vilken ordning peakonerna kommer i de två komponenterna i lösningen, och tidigare var lösningen känd enbart i det s.k. sammanflätade fallet där peakonerna ligger växelvis i den ena och den andra komponenten. För att erhålla lösningsformlerna för en godtycklig konfiguration inför vi hjälppeakoner som gör konfigurationen sammanflätad. Genom lämpliga gränsövergångar tvingar vi sedan amplituden för dessa hjälppeakoner att gå mot noll, och kvar blir då lösningsformlerna för de resterande vanliga peakonerna.

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First, I would like to thank my supervisor, Hans Lundmark, for the patient guidance, encouragement and advice he has provided throughout my PhD study and for keeping his doors open whenever I needed. I have been extremely lucky to have a supervisor like him.

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Budor Shuaib
Linköping, 2018

List of papers

This thesis is based on the following papers, which from now on will be referred to by their roman numbers:

I. H. Lundmark and B. Shuaib. *Ghostpeakons and characteristic curves for the Camassa–Holm, Degasperis–Procesi and Novikov equations.*

II. B. Shuaib and H. Lundmark. *Non-interlacing peakon solutions of the Geng–Xue equation.*

Both papers were joint research projects conducted in close collaboration.

Paper I has undergone several revisions over a period of many years, with both authors actively discussing the formulation of the paper. We are especially grateful to Krzysztof Marciniak, who has given very helpful suggestions.

As the first author of Paper II, I have been responsible for formulating the results, writing the first drafts, and making all of the graphics. The final outcome is the result of many discussions between both authors.

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Paper I

Ghostpeakons and characteristic curves for the Camassa–Holm, Degasperis–Procesi and Novikov equations

Hans Lundmark and Budor Shuaib

Paper II

Non-interlacing peakon solutions of the Geng–Xue equation

Budor Shuaib and Hans Lundmark

1 Introduction

We present two papers in this thesis. The first paper studies so-called ghostpeakons and characteristic curves for three nonlinear partial differential equations, namely the Camassa–Holm, Degasperis–Procesi and Novikov equations. In the second paper we present the arbitrary multipeakon solution of the Geng–Xue equation. Both papers are coauthored with Hans Lundmark.

This study is part of the subject of *integrable systems*, and we begin with a brief introduction to this topic. Then we present the equations of our interest, which all admit a special kind of solutions called *peakons*. We recall the method of characteristics, using the example of Burgers equation, and at the end, an overview of the papers will be given.

2 Integrable partial differential equations

In the beginning of the history of ordinary differential equations, the aim was to find the explicit solution by integrating the equation. At the end of the 19th century, Henri Poincaré realized when studying the three-body problem that this is in general an impossible task, and it is only very special *integrable* systems that can be exactly solved (in principle). Non-integrable systems have to be studied qualitatively by other methods.

It is not easy to give a precise definition of integrability which covers all imaginable situations, but for example in classical mechanics, a Hamiltonian system with n degrees of freedom is called integrable (or sometimes *completely integrable*) if it has n functionally independent constants of motion which commute with respect to the Poisson bracket.

When it comes to partial differential equations (PDEs), it is usually even more difficult to find exact solutions. It is a surprising discovery from the 1960s that some nonlinear PDEs can be solved (more or less) exactly, and that they can be viewed as infinite-dimensional integrable systems, for example in the sense of having infinitely many conserved quantities which generate commuting flows. The most famous of these integrable PDEs is the Korteweg–de Vries (KdV) equation which comes from water wave theory and has a special kind of wave solutions known as *soliton* solutions.

The history of the soliton phenomenon starts with the amazing observation in 1834 by John Scott Russell [19]. Here is his original description:

> *I was observing the motion of a boat which was rapidly drawn along a narrow canal by a pair of horses, when the boat suddenly stopped – not so the mass of water in the canal which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation,*

then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the canal apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the canal. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

The water wave theory of the time wasn't able to explain this, but an explanation was proposed after investigations by Boussinesq and Rayleigh in 1870 and Korteweg and de Vries in 1895, in the form of the KdV equation

$$u_t + 6\,u\,u_x + u_{xxx} = 0, \tag{2.1}$$

which is a nonlinear PDE of third order. The unknown function $u(x,t)$ describes the elevation of the wave at position x and time t, with the subscripts denoting partial derivatives. Korteweg and de Vries found the traveling wave solution

$$u(x,t) = \frac{c/2}{\cosh^2(x - ct)} = \frac{2c}{\left(e^{ct-x} + e^{x-ct}\right)^2}, \tag{2.2}$$

which was interpreted as Scott Russell's solitary wave. Note that the velocity c is proportional to the amplitude of the wave, which is a nonlinear phenomenon.

Later, in 1965, Zabusky and Kruskal [22] observed in numerical simulations of the Korteweg–de Vries equation that several such waves with different amplitudes can coexist,and when a faster wave catches up with a slower wave, they are not destroyed, but reemerge after the interaction. They coined the name *soliton* for this type of stable nonlinear wave.

Not long afterwards, Gardner, Greene, Kruskal, and Miura developed a method of solution for the KdV equation which has become known as the inverse scattering transform (IST), see for example the book by Ablowitz and Clarkson [1]. In particular, this made it possible to find explicit formulas for the multisoliton solutions. An example of a two-soliton solution is

$$u(x,t) = 18\,\frac{4e^{4x-10t} + e^{5x-17t} + 18e^{3x-9t} + 9e^{x-t} + 36e^{2x-8t}}{(9 + 9e^{x-t} + 9e^{2x-8t} + e^{3x-9t})^2}. \tag{2.3}$$

Clearly this is not simply a superposition of two waves of the form (2.2); it is only as $t \to \pm\infty$ that two separate waves emerge. This was the beginning of the modern theory of integrable systems, and the KdV equation has been much studied since then.

Some other well-known examples of integrable nonlinear PDEs are the Harry Dym equation

$$u_t = u^3\, u_{xxx} \tag{2.4}$$

together with the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{2.5}$$

and the nonlinear Schrödinger (NLS) equation

$$i\, u_t + u_{xx} + \kappa\, u\, |u|^2 = 0. \tag{2.6}$$

All of these equations have the mathematical attributes that are typically associated with complete integrability, such as Lax pairs, an infinite number of conservation laws, Hamiltonian structures, and multisoliton solutions which can be explicitly computed using inverse spectral techniques similar to the inverse scattering transform [1].

3 Peakons

In this thesis we are interested in integrable systems similar to the KdV equation (2.1) but with solitons which are peak-shaped rather than smooth. Such peaked solitons are usually called *peakons*, and the first and most famous equation with peakon solutions is the Camassa–Holm (CH) equation

$$m_t + m_x u + 2\, m\, u_x = 0, \qquad m = u - u_{xx}, \tag{3.1}$$

which was derived as a model for shallow water waves in 1993 [4, 5]. We also consider some other PDEs which are mathematically related to the Camassa–Holm equation, namely the Degasperis–Procesi (DP) equation discovered in 1999 in a search for integrable systems similar to the CH equation [8, 7],

$$m_t + m_x u + 3\, m\, u_x = 0, \qquad m = u - u_{xx}, \tag{3.2}$$

and the Novikov equation, found in a similar way in 2008 [12, 18],

$$m_t + u(m_x u + 3 m u_x) = 0, \qquad m = u - u_{xx}. \tag{3.3}$$

The CH and DP equations have peaked travelling wave solutions (**peakons**) of the simple form

$$u(x,t) = c\, e^{-|x-ct|}, \tag{3.4}$$

where the velocity c is equal to the amplitude. Note that the absolute value term $|x - ct|$ is not differentiable at the point $x = ct$, so these solutions must be interpreted in a weak sense. The amplitude c may be negative; in this case one talks about **antipeakons**, which move to the left. The Novikov equation is similar, except that the velocity is the square of the amplitude,

$$u(x,t) = \pm c\, e^{-|x-c^2 t|}, \tag{3.5}$$

so that the wave travels to the right regardless of the sign of c.

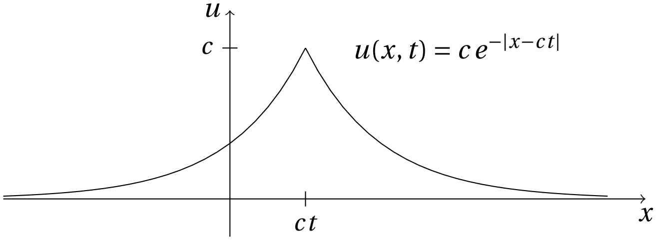

Figure 1. A single-peakon solution of the CH and DP equations – a peak-shaped travelling wave.

The multisoliton (in this case **multipeakon**) solutions of these equations have a remarkably simple form, namely a superposition of several peakons, each with its own time-dependent amplitude m_i and position x_i:

$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}. \tag{3.6}$$

See Figure 2 for an example with $N = 5$. This is not the usual superposition of solutions familiar from linear wave equations, but a more complicated kind of superposition, since in order for a function $u(x,t)$ of this form to be a weak solution of the CH equation, the positions $x_k(t)$ and amplitudes $m_k(t)$ have to satisfy the following system of $2N$ nonlinear coupled ODEs:

$$\begin{aligned} \dot{x}_k &= \sum_{i=1}^{N} m_i e^{-|x_k-x_i|}, \\ \dot{m}_k &= \sum_{i=1}^{N} m_k m_i \operatorname{sgn}(x_k-x_i)\, e^{-|x_k-x_i|}. \end{aligned} \tag{3.7}$$

Here sgn denotes the signum function, with the convention $\operatorname{sgn}(0) = 0$, and it is assumed that all x_k are distinct. In fact, we will always number the peakons in increasing order,

$$x_1(t) < x_2(t) < \cdots < x_N(t). \tag{3.8}$$

This constraint is preserved by the ODEs (3.7), at least locally in time simply because of continuity, and actually globally in time if all amplitudes $m_k(t)$ have the same sign [3, 15].

The multipeakon solutions of the DP and Novikov equations are governed by similar ODEs; see Section 2 in Paper I. In all three cases, the peakon ODEs have been solved by using inverse spectral methods [2, 3, 14, 15, 11]. Lundmark [13] showed that the DP equation also admits *shockpeakon* solutions with jump discontinuities, which form when a peakon collides with an antipeakon.

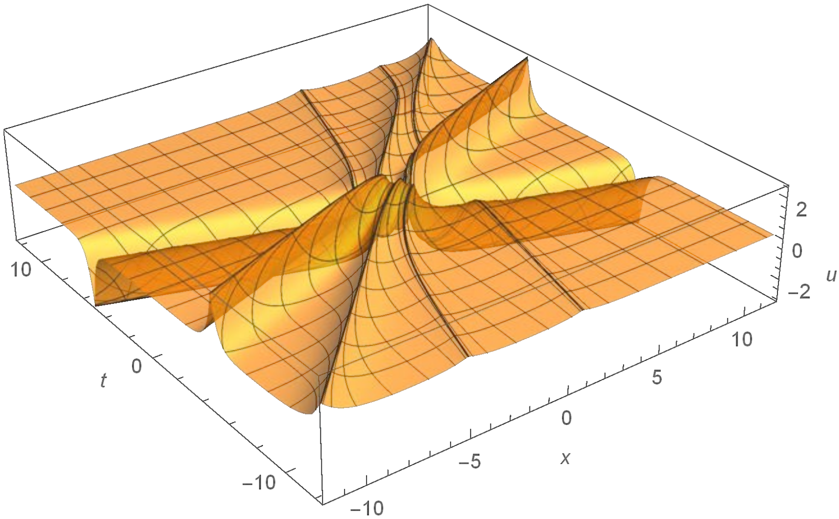

Figure 2. Plot of a multipeakon solution $u(x,t) = \sum_{i=1}^{5} m_i(t)\, e^{-|x-x_i(t)|}$ of the Camassa–Holm equation, with four peakons and one antipeakon.

In recent years many other integrable nonlinear partial differential equations with multipeakon solutions have been found. In Paper II we study the Geng–Xue equation introduced in 2009 [10],

$$\begin{aligned} m_t + (m_x u + 3 m u_x) v &= 0, \\ n_t + (n_x v + 3 n v_x) u &= 0, \\ m = u - u_{xx}, \qquad n &= v - v_{xx}. \end{aligned} \tag{3.9}$$

which is a generalization of the Novikov equation, since if we let $u = v$, both equations in (3.9) reduce to (3.3).

Other examples include the modified Camassa–Holm equation

$$m_t + \left((u^2 - u_x^2) m\right)_x = 0, \quad m = u - u_{xx} \tag{3.10}$$

which has attracted much attention very recently (see [6] and references therein), and the following two-component system with cubic nonlinearity and linear dispersion, found in 2015 by Xia and Qiao [21],

$$
\begin{aligned}
m_t &= b\,u_x + \frac{1}{2}\,[m(uv - u_x\,v_x)]_x - \frac{1}{2}m(uv_x - u_x v),\\
n_t &= b\,v_x + \frac{1}{2}\,[n(uv - u_x\,v_x)]_x - \frac{1}{2}n(uv_x - u_x v),\\
m &= u - u_{xx}, \qquad n = v - v_{xx},
\end{aligned}
\tag{3.11}
$$

where b is an arbitrary real constant.

4 Characteristic curves

The method of characteristics is a technique for solving certain types of PDEs by determining the so-called characteristic curves. The PDE is replaced by a family of ODEs, which can be solved along the characteristics curves, and subsequently the solutions of these ODEs can be related to the solution of the original PDE.

A typical textbook example is the *inviscid Burgers equation*, a simple model of nonlinear wave motion, where $u(x,t)$ is the height of the wave:

$$
\begin{aligned}
u_t + u\,u_x &= 0, \quad t > 0,\\
u(x,0) &= u_0(x), \quad x \in \mathbf{R}.
\end{aligned}
\tag{4.1}
$$

The characteristic curves associated to a solution $u(x,t)$ of (4.1) are defined by

$$
\begin{aligned}
\dot{\xi}(t) &= u(\xi(t),t), \quad t > 0,\\
\xi(0) &= \xi_0.
\end{aligned}
\tag{4.2}
$$

If u is a solution of (4.1) and $\xi(t)$ satisfies (4.2), then $u(\xi(t),t)$ satisfies

$$
\frac{d}{dt}u(\xi(t),t) = u_x(\xi(t),t)\frac{d\xi}{dt} + u_t(\xi(t),t) = u(\xi(t),t)\,u_x(\xi(t),t) + u_t(\xi(t),t) = 0,
\tag{4.3}
$$

which means that $u(x,t)$ is constant along the characteristic curve $x = \xi(t)$.

Now using the initial condition $u(x,0) = u_0(x)$ we can find the characteristic curve starting at the point $(x,t) = (\xi_0, 0)$, since

$$
\dot{\xi}(t) = u(\xi(t),t) = u(\xi(0),0) = u_0(\xi_0),
\tag{4.4}
$$

so that the characteristic curve is a straight line,

$$
x = \xi(t) = \xi_0 + u_0(\xi_0)\,t, \quad t > 0.
\tag{4.5}
$$

As long as the characteristic curves do not cross, this relation determines ξ_0 as a function of (x,t), and the solution of (4.1) takes the form

$$
u(x,t) = u_0(\xi_0(x,t)).
\tag{4.6}
$$

At points where characteristic curves with different values of u cross, a *shock* (jump discontinuity) will form, and one must consider weak solutions. Whether or not this happens depends on the initial data $u_0(x)$. See Whitham [20], or Chapter 3 of Evans [9].

Characteristic curves are also an important theoretical tool for hyperbolic PDEs, for example in the study of breakdown of regularity of solutions (formation of singularities). Concerning the Camassa–Holm equation, see the review in Section 1.5 in Paper I.

5 Overview of Paper I

In Paper I we use so-called *ghostpeakons* to derive explicit formulas for the characteristic curves for the CH, DP and Novikov multipeakon solutions. We will illustrate the idea for the Camassa–Holm equation by studying some examples here. For general results, including formulas for the DP and Novikov equations, see Paper I.

By inverting the operator $1-\partial_x^2$, the CH equation (3.1) can be written as

$$u_t + u\,u_x + \frac{1}{2}\,e^{-|x|} * \left(u^2 + \frac{1}{2}\,u_{xx}\right)_x = 0, \tag{5.1}$$

which is like the Burgers equation (4.1) with an additional nonlocal term (the star denotes convolution with respect to x). As for the Burgers equation, one defines the characteristic curves $x = \xi(t)$ associated to a solution $u(x,t)$ of the Camassa–Holm equation as the solutions of the ODE

$$\dot{\xi}(t) = u\,(\xi(t), t)\,. \tag{5.2}$$

Comparison to the peakon ODEs (3.7) shows that the peakons in a multipeakon solution follow characteristic curves. Moreover, and this is our main idea, it turns out that the *other* characteristic curves for a multipeakon solution can be thought of as trajectories of peakons with amplitude zero, which is what we call ghostpeakons. Before turning to the examples, let us first give some necessary background material about Camassa–Holm peakons.

Explicit formulas for Camassa–Holm N-peakon solutions when $N = 1$ and $N = 2$ were computed by direct integration of the ODEs (3.7) by Camassa and Holm in their original paper [4]. Beals, Sattinger and Szmigielski [2, 3] derived the solution for arbitrary N using the inverse spectral method (IST). They first define the forward spectral map, which is a change of variables which takes the $2N$ physical variables, namely the peakons' positions and amplitudes

$$x_1 < x_2 < \cdots < x_N, \quad m_1, m_2, \ldots, m_N \in \mathbf{R} \setminus \{0\}, \tag{5.3}$$

to a set of $2N$ so-called spectral variables

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N, \quad b_1, b_2, \ldots, b_N \in \mathbf{R}^+. \tag{5.4}$$

The variables λ_k are defined as the eigenvalues of a certain symmetric $N \times N$ matrix involving the positions and amplitudes as parameters, and they can be shown to be nonzero and distinct, while the variables b_k are related to the eigenvectors in a way which we will not go into here (they are the residues in the partial fraction expansion of the so-called Weyl function). Somewhat miraculously, this change of variables transforms the nonlinear coupled CH peakon ODEs into the linear decoupled system

$$\dot{\lambda}_k = 0, \qquad \dot{b}_k = b_k / \lambda_k,$$

so that the eigenvalues λ_k are constant, while the residues have the simple time-dependence

$$b_k = b_k(t) = b_k(0)\, e^{t/\lambda_k}. \tag{5.5}$$

Then the inverse spectral map, which expresses the physical variables in terms of the spectral variables, gives explicit formulas for the N-peakon solution of the Camassa–Holm equation, namely

$$x_{N+1-k}(t) = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2}, \qquad m_{N+1-k}(t) = \frac{\Delta_k^0\, \Delta_{k-1}^2}{\Delta_k^1\, \Delta_{k-1}^1}, \qquad k = 1, \ldots, N, \tag{5.6}$$

where the expressions Δ_k^a are certain functions of the spectral variables (see examples below, and Paper I for the general definitions).

When talking about N-peakon solutions $u(x,t) = \sum_{i=1}^N m_i(t)\, e^{-|x - x_i(t)|}$, it is understood that we really have N peakons, i.e., that all the amplitudes $m_1, \ldots, m_N$ are nonzero. Thus, knowing the general N-peakon solution of the PDE is the same thing as knowing the general solution of the Camassa–Holm N-peakon ODEs (3.7) in the case where all $m_k \neq 0$, and this is what the formulas (5.6) give us.

Example 5.1. The Camassa–Holm two-peakon ODEs are

$$\begin{aligned} \dot{x}_1 &= m_1 + m_2\, E_{12}, \\ \dot{x}_2 &= m_1\, E_{12} + m_2, \\ \dot{m}_1 &= -m_1 m_2\, E_{12}, \\ \dot{m}_2 &= m_2 m_1\, E_{12}, \end{aligned} \tag{5.7}$$

where $E_{12} = e^{|x_1 - x_2|} = e^{x_1 - x_2}$ (we assume $x_1 < x_2$, as explained earlier). The general solution of this system with *nonzero* amplitudes m_1 and m_2 is

$$\begin{aligned} x_1(t) &= \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}, \qquad & m_1(t) &= \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)}, \\ x_2(t) &= \ln(b_1 + b_2), & m_2(t) &= \frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2}, \end{aligned} \tag{5.8}$$

with the time dependence $b_k(t) = b_k(0)\, e^{\frac{t}{\lambda_k}}$. The constants λ_1, λ_2, $b_1(0)$ and $b_2(0)$ are uniquely determined by the initial conditions $x_1(0)$, $x_2(0)$, $m_1(0)$ and $m_2(0)$. By substituting this into the peakon ansatz (3.6)

$$u(x,t) = m_1(t)\, e^{-|x - x_1(t)|} + m_2(t)\, e^{-|x - x_2(t)|}, \tag{5.9}$$

we get the two-peakon solution of the CH equation.

In the CH peakon ODEs (3.7) it is of course possible to have some $m_k(0) = 0$, but then we will have $m_k(t) = 0$ for all t, since m_k is a factor in the right-hand sido of the equation for m_k:

$$\frac{dm_k}{dt} = m_k \sum_{i=1}^{N} m_i \operatorname{sgn}(x_k - x_i)\, e^{-|x_k - x_i|}. \tag{5.10}$$

With m_k identically zero, the system reduces to the $(N-1)$-peakon ODEs, whose solution we know (if all the other amplitudes are nonzero), plus a remaining nontrivial ODE for the position $x_k(t)$. This is the equation for a ghostpeakon at site number k, and if we can solve it, we obtain an explicit formula for a characteristic curve between peakons number $k-1$ and $k+1$. However, the inverse spectral method is not able to directly handle ghostpeakons, since amplitudes which are zero have no influence on the spectrum. In other words, the solution formulas (5.6) *only* cover the case when all amplitudes are nonzero. It also seems difficult to solve the ODE for the ghostpeakon's position by direct integration. Instead, we will find the solution via a limiting procedure.

Let us illustrate this in the case $N = 3$. One can find the general results in Paper I in this thesis.

The Camassa–Holm three-peakon ODEs (3.7), if $m_3 = 0$, are given by

$$\begin{aligned} \dot{x}_1 &= m_1 + m_2 E_{12} + 0 E_{13}, \\ \dot{x}_2 &= m_1 E_{12} + m_2 + 0 E_{23}, \\ \dot{x}_3 &= m_1 E_{13} + m_2 E_{23} + 0, \\ \dot{m}_1 &= m_1 (-m_2 E_{12} - 0 E_{13}), \\ \dot{m}_2 &= m_2 (m_1 E_{12} - 0 E_{23}), \\ 0 &= 0(m_1 E_{13} + m_2 E_{23}), \end{aligned} \tag{5.11}$$

where we use the abbreviation

$$E_{ij} = e^{|x_i - x_j|} = e^{x_i - x_j}, \tag{5.12}$$

for $i < j$ (assuming $x_1 < x_2 < x_3$). Here the equations for x_1, x_2, m_1 and m_2 are the two-peakon ODEs, for which we know the solution; see Example 5.1. But there is also an equation for $x_3(t)$ which remains to be solved.

If we instead let $m_2 = 0$ in the case $N = 3$, then the ODEs (3.7) become

$$\begin{aligned}
\dot{x}_1 &= m_1 + 0\,E_{12} + m_3\,E_{13}, \\
\dot{x}_2 &= m_1\,E_{12} + 0 + m_3\,E_{23}, \\
\dot{x}_3 &= m_1\,E_{13} + 0\,E_{23} + m_3, \\
\dot{m}_1 &= m_1\,(-0\,E_{12} - m_3\,E_{13}), \\
0 &= 0\,(m_1\,E_{12} - 0\,E_{23}), \\
\dot{m}_3 &= m_3(m_1\,E_{13} + 0\,E_{23}).
\end{aligned} \tag{5.13}$$

Up to relabeling, these are again the two-peakon ODEs, together with an additional equation for $x_2(t)$. Note that the equations for $x_3(t)$ in (5.11) and $x_2(t)$ in (5.13) are not equivalent; as it turns out, the former is fairly easy to solve directly since the variables x_3 and t can be separated, while the latter is much harder.

Let us first state our results for these cases, and afterwards we will show how they were derived. For the case $m_3(0) = 0$, the solution of the CH 3-peakon ODEs (5.11) is

$$\begin{aligned}
x_1(t) &= \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}, \\
x_2(t) &= \ln(b_1 + b_2), \\
x_3(t) &= \ln(b_1 + b_2 + \theta), \\
m_1(t) &= \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)}, \\
m_2(t) &= \frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2} \\
m_3(t) &= 0,
\end{aligned} \tag{5.14}$$

where λ_k and $b_k(t)$ are as before, while the equation for the ghostpeakon $x_3(t)$ contains an additional time-independent positive parameter θ which is in one-to-one correspondence with the value $x_3(0)$. By varying $0 < \theta < \infty$, we obtain the family of characteristic curves $\xi(t) = x_3(t)$ covering the region to the right of x_2 in the two-peakon solution. See Figure 3.

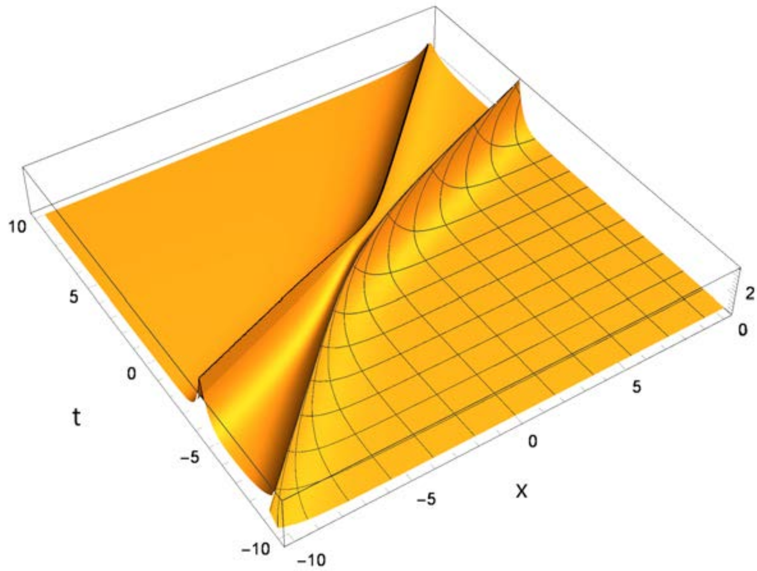

Figure 3. Plot of a pure two-peakon solution $u(x,t) = \sum_{i=1}^{2} m_i(t)\, e^{-|x-x_i(t)|}$ of the Camassa–Holm equation, with the family of characteristics covering the region to the right of x_2.

For the case $m_2(0) = 0$, the solution of the ODEs (5.13) is

$$\begin{aligned}
x_1(t) &= \ln \frac{(\lambda_1-\lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2},\\
x_2(t) &= \ln \frac{(\lambda_1-\lambda_2)^2\, b_1\, b_2 + \theta\,(b_1+b_2)}{(\lambda_1^2\, b_1 + \lambda_2^2\, b_2) + \theta},\\
x_3(t) &= \ln(b_1+b_2)\\
m_1(t) &= \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1\lambda_2(\lambda_1 b_1 + \lambda_2 b_2)},\\
m_2(t) &= 0\\
m_3(t) &= \frac{b_1+b_2}{\lambda_1\, b_1 + \lambda_2\, b_2}.
\end{aligned} \tag{5.15}$$

If we rename $x_2(t)$ to $\xi(t)$, and then (x_3, m_3) to (x_2, m_2), we get the ordinary two-peakon solution with the family of characteristic curves $\xi(t)$ covering the region between the peakons as θ varies in the range $0 < \theta < \infty$. See Figure 4.

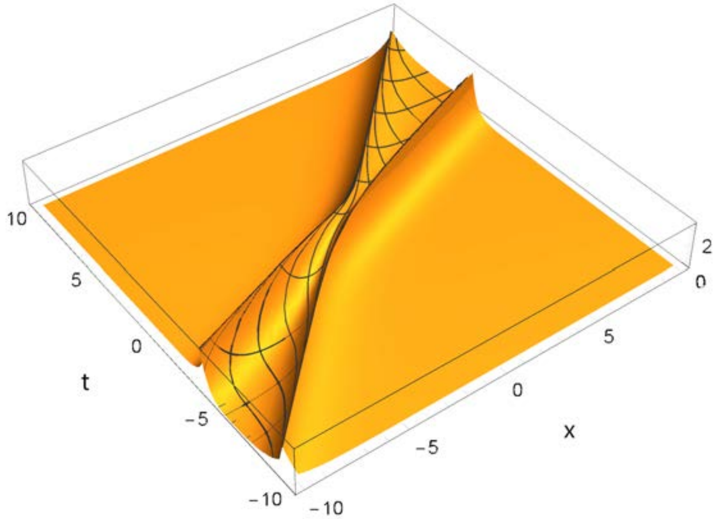

Figure 4. Plot of a pure two-peakon solution $u(x,t) = \sum_{i=1}^{2} m_i(t)\, e^{-|x-x_i(t)|}$ of the Camassa–Holm equation, with the family of characteristics covering the region between x_1 and x_2.

We will now sketch how to derive these results. The idea is to find the solution formula for the ghostpeakon from the known (non-ghost) formulas through some limiting procedure. We reparametrize the spectral data in the N-peakon solution formulas, such that we keep all the parameters $\{\lambda_k, b_k(0)\}_{k=1}^{N-1}$ and replace the parameters λ_N an $b_N(0)$ by two new parameters $\varepsilon \neq 0$ and $\theta > 0$. These new parameters are defined in such a way that we get the formula for the ghostpeakon when we substitute in the N-peakon solution formulas and let $\varepsilon \to 0$.

The formulas for the 3-peakon (non-ghost) solutions, which are parametrized by the constants λ_1, λ_2, λ_3 and $b_1(0)$, $b_2(0)$, $b_3(0)$, are

$$\begin{aligned}
x_1(t) &= \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\, b_1\, b_2\, b_3}{\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2 b_j\, b_k} \\
x_2(t) &= \ln \frac{\sum_{j<k}(\lambda_j-\lambda_k)^2 b_j\, b_k}{\lambda_1^2\, b_1+\lambda_2^2\, b_2+\lambda_3^2\, b_3} \\
x_3(t) &= \ln(b_1+b_2+b_3) \\
m_1(t) &= \frac{\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2 b_j\, b_k}{\lambda_1\,\lambda_2\,\lambda_3\sum_{j<k}\lambda_j\,\lambda_k(\lambda_j-\lambda_k)^2 b_j\, b_k} \\
m_2(t) &= \frac{(\lambda_1^2\, b_1+\lambda_2^2\, b_2+\lambda_3^2\, b_3)\sum_{j<k}(\lambda_j-\lambda_k)^2 b_j\, b_k}{(\lambda_1\, b_1+\lambda_2\, b_2+\lambda_3\, b_3)\sum_{j<k}\lambda_j\,\lambda_k(\lambda_j-\lambda_k)^2 b_j\, b_k} \\
m_3(t) &= \frac{b_1+b_2+b_3}{\lambda_1\, b_1+\lambda_2\, b_2+\lambda_3\, b_3},
\end{aligned} \tag{5.16}$$

where $b_k(t)=b_k(0)\,e^{t/\lambda_k}$ as usual. Substitute

$$\lambda_3=\varepsilon^{-1}, \quad b_3(0)=\varepsilon^2\theta, \tag{5.17}$$

so that $b_3(t)=b_3(0)e^{\frac{t}{\lambda_3}}=\varepsilon^2\theta\, e^{\varepsilon t}$. This means that we parametrize the same solution space with the constants λ_1, λ_2, $b_1(0)$, $b_2(0)$, θ, ε instead, and when we let $\varepsilon\to 0$ we obtain the formulas (5.15), as indicated below, where "$\cdots$" stands for expressions which are a little too large to write out here:

$$\begin{aligned}
x_1(t) &= \ln(\cdots) \to \quad \ln \frac{(\lambda_1-\lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1+\lambda_2^2 b_2}, \\
x_2(t) &= \ln \frac{(\lambda_1-\lambda_2)^2 b_1\, b_2+(\lambda_1-\varepsilon^{-1})^2 b_1\,\varepsilon^2\theta e^{\varepsilon t}+(\lambda_1-\varepsilon^{-1})^2 b_2\,\varepsilon^2\theta e^{\varepsilon t}}{\lambda_1^2\, b_1+\lambda_2^2\, b_2+(\varepsilon^{-1})^2\,\varepsilon^2\,\theta e^{\varepsilon t}} \\
&\to \ln \frac{(\lambda_1-\lambda_2)^2 b_1\, b_2+\theta(b_1+b_2)}{\lambda_1^2\, b_1+\lambda_2^2\, b_2+\theta}, \\
x_3(t) &= \ln(b_1+b_2+\varepsilon^2\theta e^{\varepsilon t}) \quad \to \quad \ln(b_1+b_2), \\
m_1(t) &= (\cdots) \quad \to \quad (\cdots), \\
m_2(t) &= (\cdots)\varepsilon \quad \to \quad 0, \\
m_3(t) &= \frac{b_1+b_2+\varepsilon^2\theta e^{\varepsilon t}}{\lambda_1\, b_1+\lambda_2\, b_2+\varepsilon^{-1}\varepsilon^2\theta e^{\varepsilon t}} \quad \to \quad \frac{b_1+b_2}{\lambda_1\, b_1+\lambda_2\, b_2}, \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{5.18}$$

This particular reparametrization was chosen to make m_2 reduce to zero, so that we obtain the ghostpeakon solution formulas with $m_2=0$. To get the ghostpeakon solution (5.14) with $m_3 = 0$, do the same but take $b_3(0) = \theta$ instead of $b_3(0) = \varepsilon^2 \theta$. For the ghostpeakon solution with $m_1 = 0$, take $b_3(0) = \varepsilon^4 \theta$.

The general ghostpeakon formulas for the Camassa–Holm equation are given in Paper I, Theorem 3.4 and Corollary 3.5. The corresponding results for the Degasperis–Procesi equation and Novikov equations are in Theorem 4.1, Corollary 4.2 and Theorem 5.1, Corollary 5.2, respectively.

6 Overview of Paper II

In Paper II, we deal with the Geng–Xue equation [10]

$$\begin{aligned} m_t + (m_x u + 3 m u_x) v &= 0, \\ n_t + (n_x v + 3 n v_x) u &= 0, \\ m = u - u_{xx}, \qquad n &= v - v_{xx}. \end{aligned} \tag{6.1}$$

In this system, peakons in u are not allowed to occupy the same sites as peakons in v. The *interlacing* peakon configuration (with first a peakon in u, then a peakon in v, then a peakon in u, and so on) has been solved by Lundmark and Szmigielski [16, 17], by the inverse spectral transform method.

Our contribution is to find the solution for and *arbitrary* peakon configuration in the Geng–Xue equation using ideas that originated during the study of ghostpeakons in Paper I. Since we know the interlacing peakon solutions, to find the peakon solution for any noninterlacing case, we introduce auxiliary peakons to make the configuration interlacing, and then make their amplitudes tend to zero by reparametrizing the spectral data in the interlacing solution formulas. In the limit, when the auxiliary peakons vanish, we obtain the solution formulas for the desired configuration.

Here we will illustrate the idea of Paper II by examples, since the general theorems are rather technical.

Example 6.1. The $2+2$ interlacing peakon solution of the GX equation, with $x_1(t) < y_1(t) < x_2(t) < y_2(t)$ for all t, and all amplitudes $m_k(t)$ and $n_k(t)$ positive for all t, has the form

$$\begin{aligned} u(x,t) &= m_1(t)\, e^{-|x - x_1(t)|} + m_2(t)\, e^{-|x - x_2(t)|}, \\ v(x,t) &= n_1(t)\, e^{-|x - y_1(t)|} + n_2(t)\, e^{-|x - y_2(t)|}, \end{aligned} \tag{6.2}$$

where the positions and amplitudes depend on time as described by the explicit formulas found by Lundmark and Szmigielski [16]. In terms of the notation from

Paper II (Definitions 2.3 and 2.4), these formulas are

$$\begin{aligned} \frac{1}{2}e^{2x_1} &= \frac{J_{21}^{00}}{J_{10}^{11} + C\,J_{11}^{10}}, & \frac{1}{2}e^{2y_1} &= \frac{J_{21}^{00}}{J_{10}^{11}}, \\ \frac{1}{2}e^{2x_2} &= J_{11}^{00}, & \frac{1}{2}e^{2y_2} &= J_{11}^{00} + D\,J_{10}^{00} \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} 2m_1 \exp(-x_1) &= \frac{\mu_1}{\lambda_1 \lambda_2}\left(\frac{J_{10}^{11}}{J_{11}^{10}} + C\right), & 2n_1 \exp(-y_1) &= \frac{J_{10}^{11}\,J_{11}^{10}}{J_{10}^{01}\,J_{21}^{01}}, \\ 2m_2 \exp(-x_2) &= \frac{J_{10}^{01}}{J_{11}^{10}}, & 2n_2 \exp(-y_2) &= \frac{1}{J_{10}^{00}}, \end{aligned} \tag{6.4}$$

where the expressions J_{ij}^{rs} depend on the constant parameters λ_1, λ_2, μ_1 and on the time-dependent quantities $a_1(t) = a_1(0)\,e^{t/\lambda_1}$, $a_2(t) = a_2(0)\,e^{t/\lambda_2}$, $b_1(t) = b_1(0)\,e^{t/\mu_1}$. The formulas also contain two additional constant parameters C and D. See Figure 5 for an illustration of how the peakons move for a certain choice of the parameter values.

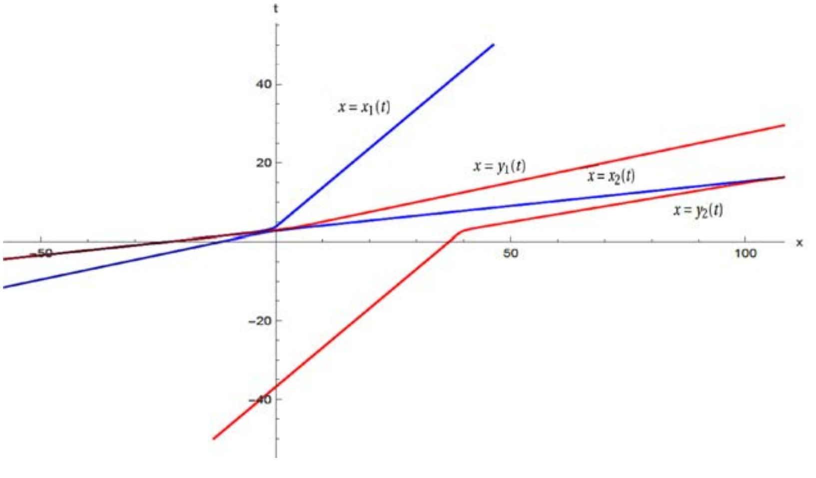

Figure 5. Plot in the (x, t) plane of the positions of a $2+2$-interlacing peakon solution $x = x_1(t)$ and $x = x_2(t)$ (blue curves from the left to the right), and $x = y_1(t)$ and $x = y_2(t)$ (red curves from the left to the right), given by the formulas in Example 6.1.

Remark 6.2. For comparison to the non-interlacing case, let us briefly explain how to find the spectral data λ_k, a_k, μ_1, b_1, C and D from the physical variables x_1, x_2, m_1, m_2 and y_1, y_2, n_1, n_2 according to the inverse spectral method used by Lundmark and Szmigielski [16]. We refer to their paper for details. By considering one of the two Lax pairs for the Geng–Xue equation, they were led to define polynomials

$$\begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} = T(y_2, n_2, \lambda)\, S(x_2, m_2, \lambda)\, T(y_1, n_1, \lambda)\, S(x_1, m_1, \lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.5}$$

where

$$\begin{aligned} S(x, m, \lambda) &= \begin{pmatrix} 1 & 0 & 0 \\ me^x & 1 & \lambda m e^{-x} \\ 0 & 0 & 1 \end{pmatrix}, \\ T(x, m, \lambda) &= \begin{pmatrix} 1 & -2\lambda m e^{-x} & 0 \\ 0 & 1 & 0 \\ 0 & 2me^x & 1 \end{pmatrix}. \end{aligned} \tag{6.6}$$

Explicitly, this gives

$$\begin{aligned} A(\lambda) = 1 - 2\lambda \left(m_1 n_1 e^{x_1 - y_1} + m_1 n_2 e^{x_1 - y_2} + m_2 n_2 e^{x_2 - y_2} \right) \\ + (2\lambda)^2 \left(m_1 n_1 e^{x_1 - y_1} (1 - e^{2(y_1 - x_2)}) m_2 n_2 e^{x_2 - y_2} \right) \end{aligned} \tag{6.7}$$

and

$$B(\lambda) = \left(m_1 e^{x_1} + m_2 e^{x_2} \right) - 2\lambda m_1 n_1 e^{x_1 - y_1} \left(1 - e^{2(y_1 - x_2)} \right) m_2 e^{x_2}. \tag{6.8}$$

It turns out that $A(\lambda)$ is time-independent (its coefficients are constants of motion), while $B(\lambda)$ depends on time in a known way. The eigenvalues $0 < \lambda_1 < \lambda_2$ are then defined as the zeros of $A(\lambda)$ (which implies that they are constant in time), while a_1 and a_2 are defined as the residues in the partial fraction decomposition of the so-called Weyl function

$$-\frac{B(\lambda)}{A(\lambda)} = \frac{a_1}{\lambda - \lambda_1} + \frac{a_2}{\lambda - \lambda_2}, \tag{6.9}$$

and from the time-dependence of $B(\lambda)$ one can show that $\dot{a}_k = a_k / \lambda_k$.

The other Lax pair produces polynomials

$$\begin{pmatrix} \tilde{A}(\lambda) \\ \tilde{B}(\lambda) \\ \tilde{C}(\lambda) \end{pmatrix} = S(y_2, n_2, \lambda)\, T(x_2, m_2, \lambda)\, S(y_1, n_1, \lambda)\, T(x_1, m_1, \lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.10}$$

i.e.,

$$\begin{aligned} \tilde{A}(\lambda) &= 1 - 2\lambda n_1 m_2 e^{y_1 - x_2}, \\ \tilde{B}(\lambda) &= \left(n_1 e^{y_1} + n_2 e^{y_2} \right) - 2\lambda m_2 n_1 e^{y_1 - x_2} \left(1 - e^{2(x_2 - y_2)} \right) n_2 e^{y_2}, \end{aligned} \tag{6.11}$$

and the eigenvalue μ_1 is defined as the zero of time-independent polynomial $\tilde{A}(\lambda)$, while b_1 and D are given by

$$-\frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda)} = -D + \frac{b_1}{\lambda - \mu_1}, \tag{6.12}$$

from which one can show that $\dot{b}_1 = b_1/\mu_1$ and that

$$D = \lim_{\lambda\to\infty} \frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda)} = n_2 e^{y_2}(1 - e^{2(x_2-y_2)}) \tag{6.13}$$

is constant. (This constant of motion D was denoted by b_∞ in the papers by Lundmark and Szmigielski.)

The remaining constant parameter C is given by

$$C = 2b^*_\infty \frac{\lambda_1\lambda_2}{\mu_1}, \tag{6.14}$$

where the constant of motion

$$b^*_\infty = m_1\, e^{-x_1}\left(1 - e^{2(x_1-y_1)}\right) \tag{6.15}$$

comes from the so-called adjoint spectral problem (or from the symmetry of the problem).

Example 6.3. Suppose next that we seek the solution formulas for a non-interlacing peakon configuration

$$\begin{aligned} u(x,t) &= m_1(t)\, e^{-|x-x_1(t)|} + m_2(t)\, e^{-|x-x_2(t)|}, \\ v(x,t) &= n_{1,1}(t)\, e^{-|x-y_{1,1}(t)|} + n_{1,2}(t)\, e^{-|x-y_{1,2}(t)|} \\ &\quad + n_{1,3}(t)\, e^{-|x-y_{1,3}(t)|} + n_2(t)\, e^{-|x-y_2(t)|}, \end{aligned} \tag{6.16}$$

where

$$x_1 < y_{1,1} < y_{1,2} < y_{1,3} < x_2 < y_2. \tag{6.17}$$

We can schematically represent this configuration with a diagram

●○○○●○

where the black dots represent the X-peakons in $u(x,t)$ and the white dots are the Y-peakons in $v(x,t)$. We assume that all of the amplitudes are positive.

In order to find solutions for configuration (6.17), we consider an interlacing configuration with two auxiliary X-peakons

$$\begin{array}{c} \quad\ \downarrow\ \ \downarrow \\ ●○●○●○●○, \end{array}$$

and write down the (already known) solution formulas for this case. Then we kill these two extra peakons, one at a time, by making substitutions and takings limits in a way similar to what we did for the CH equation in the overview of Paper I above. In the end, we are left with the solution formulas for the desired configuration, which turn out to be

$$\begin{aligned}
\frac{1}{2}e^{2x_1} &= \frac{\mathrm{J}_{21}^{00}}{\mathrm{J}_{10}^{11}+C\,\mathrm{J}_{11}^{10}},\\
\frac{1}{2}e^{2y_{1,1}} &= \frac{\tau_1\,\mathrm{J}_{21}^{00}}{\mathrm{J}_{11}^{11}+\tau_1\,\mathrm{J}_{10}^{11}},\\
\frac{1}{2}e^{2y_{1,2}} &= \frac{(\tau_1+\tau_2)\,\mathrm{J}_{21}^{00}+\sigma_1\tau_2\,\mathrm{J}_{11}^{00}}{\mathrm{J}_{11}^{11}+(\tau_1+\tau_2)\,\mathrm{J}_{10}^{11}+\sigma_1\tau_2},\\
\frac{1}{2}e^{2y_{1,3}} &= \frac{\mathrm{J}_{21}^{00}+\sigma_2\,\mathrm{J}_{11}^{00}}{\mathrm{J}_{10}^{11}+\sigma_2},\\
\frac{1}{2}e^{2x_2} &= \mathrm{J}_{11}^{00},\\
\frac{1}{2}e^{2y_2} &= \mathrm{J}_{11}^{00}+D\,\mathrm{J}_{10}^{00},
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
2m_1\exp(-x_1) &= \frac{\mu_1}{\lambda_1\lambda_2}\left(\frac{\mathrm{J}_{10}^{11}}{\mathrm{J}_{11}^{10}}+C\right),\\
2n_{1,1}\exp(-y_{1,1}) &= \frac{\sigma_1\,\mathrm{J}_{11}^{10}\left(\mathrm{J}_{11}^{11}+\tau_1\,\mathrm{J}_{10}^{11}\right)}{\left(\mathrm{J}_{21}^{01}+\sigma_1\left(\mathrm{J}_{11}^{01}+\tau_1\,\mathrm{J}_{10}^{01}\right)\right)\mathrm{J}_{21}^{01}},\\
2n_{1,2}\exp(-y_{1,2}) &= \frac{(\sigma_2-\sigma_1)\,\mathrm{J}_{11}^{10}\left(\mathrm{J}_{11}^{11}+(\tau_1+\tau_2)\,\mathrm{J}_{10}^{11}+\sigma_1\tau_2\right)}{\left(\mathrm{J}_{21}^{01}+\sigma_1\left(\mathrm{J}_{11}^{01}+\tau_1\,\mathrm{J}_{10}^{01}\right)\right)\left(\mathrm{J}_{21}^{01}+\sigma_1\,\mathrm{J}_{11}^{01}+(\sigma_2(\tau_1+\tau_2)-\sigma_1\tau_2)\,\mathrm{J}_{10}^{01}\right)},\\
2n_{1,3}\exp(-y_{1,3}) &= \frac{\mathrm{J}_{11}^{10}\left(\sigma_2+\mathrm{J}_{10}^{11}\right)}{\mathrm{J}_{10}^{01}\left(\mathrm{J}_{21}^{01}+\sigma_1\,\mathrm{J}_{11}^{01}+(\sigma_2(\tau_1+\tau_2)-\sigma_1\tau_2)\,\mathrm{J}_{10}^{01}\right)},\\
2m_2\exp(-x_2) &= \frac{\mathrm{J}_{10}^{01}}{\mathrm{J}_{11}^{10}},\\
2n_2\exp(-y_2) &= \frac{1}{\mathrm{J}_{10}^{00}}.
\end{aligned} \tag{6.19}$$

Note that the formulas for x_1, x_2, y_2, m_1, m_2 and n_2 are identical to the corresponding formulas in Example 6.1; it is only the formulas for the group of consecutive Y-peakons that are different. In Remark 6.4 below, we have tried to briefly indicate the reason for this. Also note that the parameters $0 < \tau_1$, $0 < \tau_2$ and $0 < \sigma_1 < \sigma_2$ only appear in the formulas for $y_{1,i}$ and $n_{1,i}$, so they are "internal parameters" associated with that Y-group. See Figure 6 for an illustration.

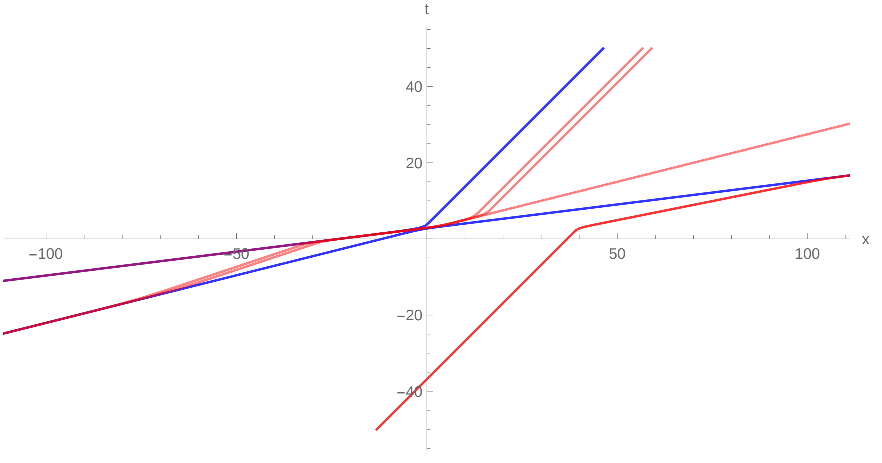

Figure 6. Plot of the positions of a non-interlacing peakon solutions x_1, x_2, (blue curves from the left to the right), and $y_{1,1}$, $y_{1,2}$, $y_{1,3}$ and y_2 (red curves from the left to the right), given by the formulas in Example 6.3.

Remark 6.4. Suppose we want to solve the non-interlacing peakon ODEs using the inverse spectral method, and try the same thing as we did in Remark 6.2. If we multiply the matrices for the adjacent Y-peakons,

$$T(y_{1,3}, n_{1,3}, \lambda)\, T(y_{1,2}, n_{1,2}, \lambda)\, T(y_{1,1}, n_{1,1}, \lambda), \tag{6.20}$$

we get

$$\begin{pmatrix} 1 & -2\lambda n_{1,1} e^{-y_{1,1}} - 2\lambda n_{1,2} e^{-y_{1,2}} - 2\lambda n_{1,3} e^{-y_{1,3}} & 0 \\ 0 & 1 & 0 \\ 0 & 2n_{1,1} e^{y_{1,1}} + 2n_{1,2} e^{y_{1,2}} + 2n_{1,3} e^{y_{1,3}} & 1 \end{pmatrix}, \tag{6.21}$$

which happens to be of the form $T(\tilde{y}_1, \tilde{n}_1, \lambda)$, where

$$\tilde{n}_1\, e^{\tilde{y}_1} = \sum_{i=1}^{3} n_{1,i} e^{y_{1,i}}, \qquad \tilde{n}_1\, e^{-\tilde{y}_1} = \sum_{i=1}^{3} n_{1,i} e^{-y_{1,i}}. \tag{6.22}$$

This means that

$$\begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} = T(y_2, n_2, \lambda)\, S(x_2, m_2, \lambda)\, T(\tilde{y}_1, \tilde{n}_1, \lambda)\, S(x_1, m_1, \lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.23}$$

and we get the time-invariant polynomial

$$
\begin{aligned}
A(\lambda) &= 1 - 2\lambda \left(m_1\, \tilde{n}_1\, e^{x_1 - \tilde{y}_1} + m_1\, n_2 e^{x_1 - y_2} + m_2\, n_2\, e^{x_2 - y_2} \right) \\
&\quad + (2\lambda)^2 \left(m_1\, \tilde{n}_1\, e^{x_1 - \tilde{y}_1} (1 - e^{2(\tilde{y}_1 - x_2)})\, m_2\, n_2\, e^{x_2 - y_2} \right),
\end{aligned}
\tag{6.24}
$$

and an auxiliary polynomial

$$
B(\lambda) = \left(m_1\, e^{x_1} + m_2\, e^{x_2} \right) - 2\lambda m_1\, \tilde{n}_1\, e^{x_1 - \tilde{y}_1} \left(1 - e^{2(\tilde{y}_1 - x_2)} \right) m_2 e^{x_2}. \tag{6.25}
$$

From the other Lax pair we get, since also the matrices S satisfy

$$
S(y_{1,3}, n_{1,3}, \lambda)\, S(y_{1,2}, n_{1,2}, \lambda)\, S(y_{1,1}, n_{1,1}, \lambda) = S(\tilde{y}_1, \tilde{n}_1, \lambda), \tag{6.26}
$$

that

$$
\begin{pmatrix} \tilde{A}(\lambda) \\ \tilde{B}(\lambda) \\ \tilde{C}(\lambda) \end{pmatrix} = S(y_2, n_2, \lambda)\, T(x_2, m_2, \lambda)\, S(\tilde{y}_1, \tilde{n}_1, \lambda)\, T(x_1, m_1, \lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.27}
$$

which gives

$$
\begin{aligned}
\tilde{A}(\lambda) &= 1 - 2\lambda\, \tilde{n}_1\, m_2\, e^{\tilde{y}_1 - x_2}, \\
\tilde{B}(\lambda) &= \left(\tilde{n}_1\, e^{\tilde{y}_1} + n_2\, e^{y_2} \right) - 2\lambda m_2\, \tilde{n}_1\, e^{\tilde{y}_1 - x_2} \left(1 - e^{2(x_2 - y_2)} \right) n_2\, e^{y_2}.
\end{aligned}
\tag{6.28}
$$

As before, the eigenvalues λ_1 and λ_2 are the zeros of $A(\lambda)$, and μ_1 is the zero of $\tilde{A}(\lambda)$. The residues a_1 and a_2 are given by

$$
-\frac{B(\lambda)}{A(\lambda)} = \frac{a_1}{\lambda - \lambda_1} + \frac{a_2}{\lambda - \lambda_2}, \tag{6.29}
$$

while b_1 and D are defined by

$$
-\frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda)} = -D + \frac{b_1}{\lambda - \mu_1}. \tag{6.30}
$$

And finally there is $C = 2b_\infty^* \lambda_1 \lambda_2 / \mu_2$ where the constant of motion

$$
b_\infty^* = m_1\, e^{-x_1} \left(1 - e^{2(x_1 - \tilde{y}_1)} \right) \tag{6.31}
$$

comes from the adjoint spectral problem.

These formulas are exactly as for the $2+2$ interlacing case in Remark 6.2, except that y_1 and n_1 are replaced with the quantities $\tilde{y}_1$ and $\tilde{n}_1$ from (6.22), which can be interpreted as the *effective position* and the *effective mass* of the group of three consecutive Y-peakons. Therefore, the $2+2$ interlacing solution formulas will give us the solution for $x_1(t)$, $\tilde{y}_1(t)$, $x_2(t)$, $y_2(t)$ and $m_1(t)$, $\tilde{n}_1(t)$, $m_2(t)$, $n_2(t)$.

This explains the phenomenon mentioned above, which was perhaps surprising at first, that the solution formulas for *singletons* ("groups" with just one single peakon, such as x_1, x_2 and y_2 above) are exactly the same as the solution formulas for the interlacing case. However, from this inverse spectral approach we do not obtain the formulas for the *individual* peakons in the Y-group, i.e., $y_{1,1}(t)$, $y_{1,2}(t)$, $y_{1,3}(t)$ and $n_{1,1}(t)$, $n_{1,2}(t)$, $n_{1,3}(t)$. Our ghostpeakon-inspired method overcomes this limitation, and we get the solution formulas for all peakons. Any choice of the internal parameters τ_i and σ_i in the solution formulas for $y_{1,i}$ and $n_{1,i}$ will give the same effective position $\tilde{y}_1$ and effective mass $\tilde{n}_1$, so these parameters are additional constants of motion which determine how the peakons are distributed within the group.

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