ON AN EXTREMAL PROPERTY OF JORDAN ALGEBRAS OF CLIFFORD TYPE

VLADIMIR G. TKACHEV

Abstract. If V is a finite-dimensional unital commutative (maybe nonassociative) algebra carrying an associative positive definite bilinear form $\langle,\rangle$ then there exist a nonzero idempotent $c \neq e$ (e being the algebra unit) of the shortest possible length $|c|^2 := \langle c, c\rangle$. In particular, there always holds $2|c|^2 \le |e|^2$. We prove that the equality for some idempotent $c \in V$ holds exactly when V is a Jordan algebra of Clifford type.

1. Introduction

Throughout this paper, V denotes a finite-dimensional commutative nonassociative algebra over $\mathbb{R}$ carrying an associating nonsingular bilinear form:

$$\langle xy, z\rangle = \langle x, yz\rangle, \quad \forall x, y, z \in V, \tag{1}$$

where $(x, y) \to xy = yx$ is the algebra multiplication. The nonsingularity means that the radical $\{x \in V : \langle x, y\rangle = 0, \forall y \in V\}$ is trivial. Following M. Bordemann [1], an algebra satisfying (1) is called *metrised*. We shall always assume that V is a *Euclidean* metrised algebra, i.e. the associative bilinear form $\langle\cdot, \cdot\rangle$ is positive definite. The classical example is formal real (Euclidean) Jordan algebras with the invariant trace form [3], [10]. Another important example is the Griess algebra $\mathfrak{G}$ appearing in connection with the Monster sporadic simple group [15] or, in general, many axial algebras [4], [17], [8]. More precisely, it was established in [7] that any primitive axial algebra of Jordan type admits a symmetric associating form, maybe with nontrivial radical. A more recent example of a metrised algebra is the class of nonassociative algebras decoding the geometric structure of cubic minimal cones and cubic polynomial solutions to certain elliptic PDEs [14, Chapter 6], [13], [20], [21]. Some related questions as well as geometry of idempotents are also discussed in [18], [5], [6], [2].

It is known [21], [11] (see also Proposition 2.1 below) that if V is a Euclidean metrised algebra then the set of nonzero idempotents $\mathrm{Idm}(V)$ is nonempty and there exists an idempotent $c \neq 0$ such that

$$|c|^2 \le |c'|^2, \qquad \forall c' \in \mathrm{Idm}(V), \tag{2}$$

where $|x|^2 = \langle x, x\rangle$. Such an idempotent is called *extremal*, denoted by $c \in \mathrm{Idm}_1(V)$. In other words, $\mathrm{Idm}_1(V)$ denotes the subset of shortest nonzero idempotents in V.

If additionally V is a *unital* algebra with the unit e then the conjugation $c \to \bar{c} := e - c$ is a natural involution on $\mathrm{Idm}(V)$, indeed,

$$\bar{c}^2 = e - 2ec + c = e - c = \bar{c}, \tag{3}$$

and similarly $c\bar{c} = 0$. Therefore, using (1) one obtains

$$\langle c, \bar{c}\rangle = \langle c^2, \bar{c}\rangle = \langle c, c\bar{c}\rangle = 0, \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 17A99, 17C27.

Key words and phrases. Commutative nonassociative algebras; Associative bilinear form; Idempotents; Jordan algebras; Jordan algebras of Clifford type; Axial algebras .

thus, c and $\bar{c}$ are orthogonal. This together with (2) immediately yields

$$|e|^2 = |c|^2 + |\bar{c}|^2 \ge 2|c|^2, \qquad \forall c \in \mathrm{Idm}_1(V). \tag{5}$$

Definition 1.1. A unital Euclidean metrised algebra such that the equality in (5) attains for some $c \in \mathrm{Idm}_1(V)$ is said to be *minimal*.

It is the main purpose of the present paper to completely characterize the class of minimal algebras. Let us recall some standard definitions, see [9, p. 13-14], [3], [12]. Let U be a vector space over a field k of characteristic $\ne 2$, $f(u, v)$ be a nonsingular symmetric bilinear form from U to k. Then $U_f := k \oplus U$ together with the multiplication law

$$(a \oplus u) \bullet (b \oplus v) = (ab + f(u, v)) \oplus (av + bu) \tag{6}$$

is a Jordan algebra, called the Jordan algebra of bilinear form f and denoted by U_f (also known as Jordan algebra of Clifford type or a spin-factor [12]). Then $\epsilon = 1 \oplus 0$ is the unit in U_f and any element $z = a \oplus u \in U_f$ satisfies the quadratic relation

$$z^2 - z \operatorname{tr} z + d(z)\epsilon = 0, \tag{7}$$

where $\operatorname{tr} z = 2a$ is the generic trace and $d(z) = a^2 - f(u, u)$ is the generic norm of z. It is well-known that the trace form $t(z, w) := \operatorname{tr}(z \bullet w)$ is associative in the sense of (1). In particular, U_f is a metrised algebra with respect to the trace form.

Our main result states that any minimal algebra is a Jordan algebra of Clifford type. More precisely, we shall prove

Theorem 1.2. *If V is a minimal algebra then*

$$V = \mathrm{span}(\mathrm{Idm}_1(V)). \tag{8}$$

Furthermore, V is isomorphic to the Jordan algebra $(e^{\perp})_f$ of the symmetric bilinear form

$$f(x, y) = \frac{1}{|e|^2}\langle x, y\rangle,$$

where $e^{\perp} = \{x \in V : \langle x, e\rangle = 0\}$.

The paper is organized as follows. In Section 2 we give a short overview of metrised algebras and discuss variational properties of extremal idempotents in more details. The spectral inequality (12) in Proposition 2.1 and Proposition 2.8 are key ingredients in the classification of minimal algebras. The proof of Theorem 1.2 is given in Section 3.

2. Extremal idempotents

2.1. **Euclidean metrised algebras.** Let V be a finite dimensional algebra over $\mathbb{R}$ with multiplication denoted by juxtaposition $(x, y) \to xy \in V$. A symmetric $\mathbb{R}$-bilinear form $\langle x, y\rangle : V \times V \to \mathbb{R}$ is called nonsingular if $\langle x, y\rangle = 0$ for all $y \in V$ implies $x = 0$. In what follows, we use the standard squared norm notation

$$|x|^2 = \langle x, x\rangle.$$

The bilinear form $\langle\cdot, \cdot\rangle$ is called associative [19], [16] if (1) holds. An algebra carrying a non-singular symmetric bilinear form is called metrised [1].

Recall that an endomorphism $A : V \to V$ is called self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$. Given two arbitrary self-adjoint endomorphisms A, B of V, we use the standard notation $A \le B$ to denote that $\langle Ax, x\rangle \le \langle Bx, x\rangle$ holds for any $x \in V$. For any nonzero $z \in V$,

$$(z \otimes z)(x) = z\langle z, x\rangle : V \to \mathrm{Span}(z)$$

is a self-adjoint endomorphism.

An important corollary of (1) is that in a metrised algebra V, the operator of left (=right) multiplication $L_x : y \to xy$ is self-adjoint:

$$\langle L_x y,\, z\rangle = \langle y,\, L_x z\rangle, \qquad \forall x, y, z \in V. \tag{9}$$

Since $L_c c = c$, $\mathrm{Span}(c)$ is an invariant subspace of L_c. Then (9) implies that the orthogonal complement $c^\perp = \{x \in V : \langle x,\, c\rangle = 0\}$ is an invariant subspace of L_c too.

Let V be a Euclidean metrised algebra. We denote by E_0 the set of solutions of the variational problem

$$f(x) = \langle x,\, x^2\rangle \to \max \quad \text{subject to a constraint} \quad \langle x,\, x\rangle = 1. \tag{10}$$

Since $\langle\cdot,\,\cdot\rangle$ is positive definite, the unit sphere $S = \{x \in V : \langle x,\, x\rangle = 1\}$ is compact in the standard Euclidean norm topology on V induced by $|x|^2$, hence E_0 is a nonempty set. We recall the following result, see [21], [22].

Proposition 2.1. *Let V be a nonzero Euclidean metrised algebra. If $z \in E_0$ then $\langle z^2,\, z\rangle > 0$ and $c = z/\langle z^2,\, z\rangle \in \mathrm{Idm}(V)$. Furthermore, L_c satisfies the extremal property*

$$L_c \le \frac{1}{|c|^2} c \otimes c + \frac{1}{2}. \tag{11}$$

In particular,

$$L_c \le \frac{1}{2} \text{ on } c^\perp := \{x \in V : \langle x,\, c\rangle = 0\}, \tag{12}$$

hence the eigenvalue 1 of L_c is simple.

Proof. Using the associating property (1), it is easy to see that for

$$f(x + yt) = \langle x,\, x^2\rangle + 3t\langle y,\, x^2\rangle + 3t^2\langle x,\, y^2\rangle + t^3\langle y,\, y^2\rangle, \qquad x, y \in V,\ t \in \mathbb{R}. \tag{13}$$

If $f \equiv 0$ on V, by (13) we have $\langle y,\, x^2\rangle = 0$ for all $x, y \in V$, therefore $x^2 = 0$ for any $x \in V$. The latter implies by the commutativity of V that $xy = \frac{1}{2}((x+y)^2 - x^2 - y^2) = 0$ for all $x, y \in V$, thus $VV = 0$, a contradiction.

Therefore, $f \not\equiv 0$. In particular, the maximum in (10) is strongly positive. Let z with $|z| = 1$ be an arbitrary point where the maximal value of $f(z) = \langle z,\, z^2\rangle > 0$ is attained. It follows from the homogeneity of (10) that

$$\langle x,\, x^2\rangle \le \langle z,\, z^2\rangle |x|^3, \quad \forall x \in V, \tag{14}$$

with the equality at z. Let $x = z + yt$ with $\langle z,\, y\rangle = 0$ and $|y| = 1$. Since

$$|x|^3 = |z + yt|^3 = (1 + t^2)^{3/2},$$

(13) and (14) yield for any $t \in \mathbb{R}$ that

$$\langle z,\, z^2\rangle + 3t\langle y,\, z^2\rangle + 3t^2\langle z,\, y^2\rangle + t^3\langle y,\, y^2\rangle \le \langle z,\, z^2\rangle(1 + t^2)^{3/2}.$$

Hence, for any positive t

$$3\langle y,\, z^2\rangle + 3t\langle z,\, y^2\rangle + t^2\langle y,\, y^2\rangle \le \langle z,\, z^2\rangle g(t), \tag{15}$$

where

$$g(t) = \frac{1}{t}((1 + t^2)^{3/2} - 1).$$

Since $\lim_{t\to 0+} g(0) = 0$, we have $\langle y,\, z^2\rangle \le 0$. The same inequality is obviously true for $-y$, therefore $\langle y,\, z^2\rangle = 0$. By the homogeneity, the latter equality holds for all y orthogonal to z. Thus, $z^2 = \alpha z$ for some $\alpha \in \mathbb{R}$. We have $f(z) = \langle z^2,\, z\rangle = \alpha|z|^2 = \alpha$, hence by our choice of z we have $\alpha > 0$. It follows that $z/\alpha = z/\langle z,\, z^2\rangle$ is a nonzero idempotent of V. This proves the first claim of the proposition.

Next, since $\langle y, z^2\rangle = 0$, (15) amounts to

$$3\langle z, y^2\rangle + t\langle y, y^2\rangle \le \langle z, z^2\rangle \frac{g(t)}{t}, \quad \forall t > 0.$$

Passing to the limit as $t \to 0+$ yields

$$3\langle z, y^2\rangle \le \langle z, z^2\rangle \lim_{t\to 0+} \frac{g(t)}{t} = g'(0)\langle z, z^2\rangle = \frac{3}{2}\langle z, z^2\rangle,$$

therefore

$$\langle c, y^2\rangle \le \frac{1}{2}$$

where $c = z/\langle z, z^2\rangle$ is an idempotent. Since the latter inequality holds for any y such that $\langle y, z\rangle = 0$ and $\langle y, y\rangle = 1$, we obtain

$$\langle y, L_c y\rangle = \langle y, cy\rangle = \langle c, y^2\rangle \le \frac{1}{2} = \frac{1}{2}\langle y, y\rangle. \tag{16}$$

By the homogeneity, (16) holds also for all $y \in V$ and $y \in c^\perp$. Since $c^\perp$ is an invariant subspace of L_c, (16) implies (12). Finally, (11) follows from (12) by using the invariant subspace decomposition

$$V = \mathrm{Span}(c) \oplus c^\perp \tag{17}$$

and the fact that L_c acts as the identity on $\mathrm{Span}(c)$. □

We shall also need the following generalization.

Corollary 2.2. *Let V be as in Proposition 2.1 and let $V = U \oplus W$ be an orthogonal decomposition, where U, W are nontrivial subspaces of V. If $UU \ne 0$ (i.e. U is not zero subalgebra of V) then there exists $0 \ne u \in U$ such that*

$$u^2 = \lambda u + w, \quad \text{where} \quad \lambda \in \mathbb{R}, \ w \in W.$$

Proof. If $\dim U = 1$ then the claim is trivial, therefore assume that $\dim U \ge 2$. We consider the variational problem of maximizing of the cubic form $f(x) := \langle x^2, x\rangle$ under two constraints: $|x|^2 = 1$ and $x \in U$. If U is a nonzero subalgebra of V then $f(x) \not\equiv 0$ on U, hence arguing as in Proposition 2.1 we conclude that the (positive) maximum attains at some $u \in U$, $|u| = 1$. Therefore we similarly obtain that $\langle u^2, y\rangle = 0$ for any $y \in U$ and $\langle u, y\rangle = 0$, thus

$$u^2 \in (U \cap u^\perp)^\perp = \mathrm{Span}(u) \oplus W,$$

and the desired conclusion follows. □

2.2. Variational properties.

Definition 2.3. An idempotent $c \in V$ is called *extremal*, or $c \in \mathrm{Idm}_1(V)$, if the function $\langle x^2, x\rangle$ attains its global maximum value at c for all $x \in V$ satisfying $|x| = |c|$.

Proposition 2.1 ensures that $\mathrm{Idm}_1(V)$ is nonempty for any Euclidean metrised algebra. We shall see that elements of $\mathrm{Idm}_1(V)$ have distinguished spectral properties. It follows from the above definition and for homogeneity reasons that if c is an extremal idempotent then

$$\langle x, x^2\rangle \le \frac{1}{|c|}|x|^3, \qquad \forall x \in V, \tag{18}$$

with equality for $x = c$. By the definition, all extremal idempotent have the same length. In particular, the extremal idempotents are the shortest among nonzero idempotents, because it follows from (18) that

$$|c_1| \le |c|, \qquad \forall c_1 \in \mathrm{Idm}_1(V), \ \forall c \in \mathrm{Idm}(V). \tag{19}$$

Remark 2.4. It is well-known, see for example [3], that if J is an Euclidean Jordan algebra equipped with the trace form $\langle x, y\rangle = \operatorname{tr}(xy)$ then for any idempotent $c \in \operatorname{Idm}(J)$: $|c|^2 = n \in \mathbb{Z}^+$, i.e. the squared length takes only positive integer values, and an idempotent $c \in J$ is primitive if and only it has minimal possible squared length. Furthermore, if J is a spin factor (a Jordan algebra associated with symmetric bilinear form) then all (nonzero) idempotents have the same square length $|c|^2 = 1$.

Taking into account the previous remark, it is convenient to scale the inner product such that all extremal idempotents have the unit length. To this end, note that if $\langle\cdot, \cdot\rangle$ is an associative positive definite bilinear form on V then so also is $k\langle\cdot, \cdot\rangle$ for any $k > 0$. We have the following definition.

Definition 2.5. Let V be a Euclidean metrised algebra with an associative inner product $\langle\cdot, \cdot\rangle$. The inner product is said to be *normalized* if

$$\langle x, x^2\rangle \le |x|^3 \tag{20}$$

holds for all $x \in V$ and the equality holds for some $x \neq 0$. By abuse of terminology, we call V normalized if its inner product is so.

In other words, the inner product is normalized if and only if any extremal idempotent has length $|c| = 1$. Therefore, in a normalized algebra

$$\operatorname{Idm}_1(V) = \{c \in V : c^2 = c \quad \text{and} \quad |c|^2 = 1\}.$$

As a corollary of (19),

$$|c| \ge 1, \quad \forall c \in \operatorname{Idm}(V). \tag{21}$$

If the inner product normalized, Proposition 2.1 can be reformulated as follows.

Proposition 2.6. *The equality in (20) is obtained for $x \neq 0$ if and only if $x/|x| \in \operatorname{Idm}_1(V)$. If $c \in \operatorname{Idm}_1(V)$ then $L_c \le \frac{1}{2}$ on $c^\perp$.*

2.3. **Unital algebras.** Recall that an algebra is called unital if there exists $e \in V$ such that $ex = xe = x$ for all x. If a unit exists then it is necessarily unique.

Proposition 2.7. *If V is a unital Euclidean metrised algebra and $\dim V \ge 2$ then there exist at least two different idempotents in V distinct from the unit.*

Proof. First note that a unital algebra is obviously nonzero, because $ee = e \neq 0$, where $e \in V$ is the algebra unit. Therefore by Proposition 2.1 there exists a (nonzero) extremal idempotent $c \in V$. By the extremal property, one also has $L_c \le \frac{1}{2}$ on $c^\perp$. Note that the subspace $c^\perp$ is nontrivial because by the assumption $\dim c^\perp = \dim V - 1 \ge 1$. The algebra unit e is an idempotent and $e \neq c$ because $L_e \equiv 1$ on V. By (3), $\bar{c} = e - c$ is also an idempotent and $c \neq e - c$ (because otherwise $e = 2c$, hence $e = e^2 = 4c = 2e$, implying a contradiction). This proves the proposition. □

In the rest of this section we assume that V is a unital algebra and $\dim V \ge 2$. Recall that an idempotent distinct from 0 and the unit e is called *nontrivial*.

Given an idempotent $c \in \operatorname{Idm}(V)$, let $\bar{c} = e - c$ denote its conjugate. Then

$$c\bar{c} = c(e - c) = c - c = 0,$$

and it follows from $c\bar{c} = 0$ and c and $\bar{c}$ are orthogonal, see (4).

By the above, $\operatorname{Idm}_1(V) \neq \emptyset$. If $c \in \operatorname{Idm}_1(V)$ then the orthogonal complement $c^\perp$ is nontrivial, hence $c \neq e$ (since $L_e = 1$ on the whole V). We have by the orthogonality and (21)

$$|e|^2 = |c|^2 + |\bar{c}|^2 \ge 2. \tag{22}$$

In the next section we classify all algebras where the equality in (22) is obtained.

2.4. **Minimal algebras.** For convenience reasons, we shall assume in the rest of the paper that V is a normalized algebra. Then rephrasing the Definition 1.1 yields that a unital normalized algebra is *minimal* if and only if

$$|e|^2 = 2. \tag{23}$$

This also implies that the unit e is not an extremal idempotent. The next proposition shows that it is the only (distinct from zero) non-extremal idempotent.

Proposition 2.8. *Let V be a minimal algebra. Then the set of nontrivial idempotents of V is exactly the set of extremal idemptoents. In other words,*

$$\mathrm{Idm}(V) = \mathrm{Idm}_1(V) \cup \{e\}.$$

Furthermore, if $c \in \mathrm{Idm}_1(V)$ then $c^\perp \cap \bar{c}^\perp$ is an invariant subspace of L_c and

$$L_c = \frac{1}{2} \quad \text{on } c^\perp \cap \bar{c}^\perp. \tag{24}$$

Proof. Let $c \in \mathrm{Idm}(V)$ be a nontrivial idempotent, i.e. $0 \ne c \ne e$. Then

$$2 = |e|^2 = |\bar{c}|^2 + |c|^2 \ge 2,$$

therefore $|c|^2 = 1$, which implies the first claim. Next, if $c \in \mathrm{Idm}_1(V)$ then by (22) $|\bar{c}|^2 = 1$, hence $\bar{c} \in \mathrm{Idm}_1(V)$ too. By (11),

$$L_c \le \frac{1}{2} + c \otimes c, \quad L_{\bar{c}} \le \frac{1}{2} + \bar{c} \otimes \bar{c}.$$

Since $L_{\bar{c}} = L_e - L_c = 1 - L_c$, we have

$$\frac{1}{2} - \bar{c} \otimes \bar{c} \le L_c \le \frac{1}{2} + c \otimes c. \tag{25}$$

Since $c\bar{c} = 0$, both $\mathrm{Span}(c, \bar{c})$ and its orthogonal complement $c^\perp \cap \bar{c}^\perp$ are invariant subspaces of L_c. We have $\bar{c} \otimes \bar{c} = c \otimes c = 0$ on $c^\perp \cap \bar{c}^\perp$, hence (25) implies (24). □

The identity (24) shows that the multiplication by an extremal idempotent is essentially $\frac{1}{2}$. This implies that the multiplication structure on a minimal algebra is quite special. More precisely we have

Proposition 2.9. *If V is a normalized minimal algebra then for any $c_1, c_2 \in \mathrm{Idm}_1(V)$ there holds*

$$2c_1c_2 = c_1 + c_2 - \langle \bar{c}_1, c_2 \rangle e. \tag{26}$$

Proof. First note that for any idempotent $c \in \mathrm{Idm}_1(V)$ there holds

$$\langle e, c \rangle = \langle e, cc \rangle = \langle ec, c \rangle = |c|^2 = 1, \tag{27}$$

hence if $c_1, c_2 \in \mathrm{Idm}_1(V)$ then

$$\langle c_1, \bar{c}_2 \rangle = \langle c_1, e - c_2 \rangle = \langle c_1, e \rangle - \langle c_1, c_2 \rangle = 1 - \langle c_1, c_2 \rangle. \tag{28}$$

Next note that (26) trivially holds if $c_2 = c_1$ or $c_2 = \bar{c}_1$, therefore we assume that c_2 is distinct from c_1 and $\bar{c}_1$. Let us decompose c_2 as

$$c_2 = \langle c_1, c_2 \rangle c_1 + \langle \bar{c}_1, c_2 \rangle \bar{c}_1 + z, \quad z \in c_1^\perp \cap \bar{c}_1^\perp.$$

By (24), $c_1 z = \frac{1}{2} z$, hence we obtain by virtue of (28)

$$\begin{aligned} c_1c_2 &= \langle c_1, c_2 \rangle c_1 + \tfrac{1}{2} z = \langle c_1, c_2 \rangle c_1 + \tfrac{1}{2}(c_2 - \langle c_1, c_2 \rangle c_1 - \langle \bar{c}_1, c_2 \rangle \bar{c}_1) \\ &= \tfrac{1}{2} c_2 + \tfrac{1}{2} \langle c_1, c_2 \rangle c_1 - \tfrac{1}{2} \langle \bar{c}_1, c_2 \rangle (e - c_1) = \tfrac{1}{2} c_2 + \tfrac{1}{2} c_1 - \tfrac{1}{2} \langle \bar{c}_1, c_2 \rangle e, \end{aligned}$$

as desired. □

Corollary 2.10. *If $S \subset \mathrm{Idm}_1(V)$ is such that $e \in \mathrm{span}(S)$ then $\mathrm{span}(S)$ is a subalgebra of V.*

Proof. Follows readily from (26). □

3. The proof of the main results

First we establishes (8) in Theorem 1.2.

Proposition 3.1. *If V is a minimal algebra then* $V = \operatorname{span}(\operatorname{Idm}_1(V))$.

Proof. Without loss of generality we may assume that V is a normalized minimal algebra. Recall that $\operatorname{Idm}_1(V)$ is nonempty by Proposition 2.1. Define

$$W := \operatorname{span}(\operatorname{Idm}_1(V)).$$

Then W is a subalgebra by Corollary 2.10. Since $c + \bar{c} = e$ for any $c \in \operatorname{Idm}_1(V)$, we have $e \in W$. Assume by contradiction that $W \neq V$, hence $V = W \oplus W^\perp$ with $W^\perp \neq 0$. Since $WW \subset W$, the associativity of the inner product implies that $WW^\perp \subset W^\perp$. Given an arbitrary $z \in W^\perp$, let us consider the orthogonal decomposition

$$z^2 = x + y, \quad x \in W, y \in W^\perp. \tag{29}$$

Let $c \in \operatorname{Idm}_1(V)$ be chosen arbitrarily. By Proposition 2.8,

$$L_c = \tfrac{1}{2} \quad \text{on} \quad W^\perp, \tag{30}$$

hence

$$\langle z^2, c\rangle = \langle z, zc\rangle = \tfrac{1}{2}|z|^2,$$

therefore it follows from (29) that $\langle x, c\rangle = \frac{1}{2}|z|^2$. Let $x_0 = x - \frac{1}{2}|z|^2 e$. Then $x_0 \in W$ and using (27) we obtain

$$\langle x_0, c\rangle = \langle x, c\rangle - \tfrac{1}{2}|z|^2\langle e, c\rangle = \tfrac{1}{2}|z|^2 - \tfrac{1}{2}|z|^2 = 0,$$

which by virtue of the arbitrariness of $c \in \operatorname{Idm}(V)$ and the definition of W implies that $x_0 \in W^\perp$. Therefore $x_0 \in W^\perp \cap W$, i.e. $x_0 = 0$. This proves that for any $z \in W^\perp$

$$z^2 = \tfrac{1}{2}|z|^2 e + y, \quad y \in W^\perp. \tag{31}$$

In particular, $z^2 \neq 0$ if $z \neq 0$, hence $W^\perp W^\perp \neq 0$. This implies by Corollary 2.2 that there exists a nonzero vector $z \in W^\perp$ such that

$$z^2 = \tfrac{1}{2}|z|^2 e + \lambda z, \quad \lambda \in \mathbb{R}. \tag{32}$$

With this z in hand, we claim that the idempotent equation $p^2 = p$ for $p = ae + bz$ with a, b being some real numbers has a solution distinct from e. Note that we may assume that $b \neq 0$, because otherwise $p = e$. Then expanding $p^2 = p$ by virtue of (32) yields the system

$$\tfrac{1}{2}b^2|z|^2 + a^2 = a \quad \text{and} \quad 2ab + \lambda b^2 = b.$$

From the second equation we have $\lambda b = 1 - 2a$, therefore

$$\tfrac{1}{2}b^2|z|^2 = a - a^2 = \tfrac{1}{4} - (a - \tfrac{1}{2})^2 = \tfrac{1}{4} - \tfrac{1}{4}\lambda^2 b^2.$$

This yields $b^2 = 1/(2|z|^2 + \lambda^2)$. Since the latter equation is solvable, this proves that there exists an idempotent $p = ae + bz \in \operatorname{Idm}(V)$ with $b \neq 0$, i.e. $p \neq e$. Therefore by Proposition 2.8, $p \in \operatorname{Idm}_1(V) \subset W$, which obviously contradicts to the assumption that $0 \neq z \in W^\perp$. This proves that assumption $W^\perp \neq 0$ is wrong, thus $W = V$. The proposition follows. □

Now we are ready to finish the proof of Theorem 1.2.

Proposition 3.2. *If V is a minimal algebra then V is a Jordan algebra of the symmetric bilinear form*

$$f(x, y) = \frac{1}{|e|^2}\langle x, y\rangle.$$

Proof. First let V be a normalized minimal algebra. Note that (8) readily implies that there exists a basis in V consisting of e and idempotents with the unit norm. Let $\{e, c_1, \dots, c_n\}$ be such a basis, where $n+1 = \dim V \geq 2$. It is easy to see that if c_i is in the basis then $\bar{c}_i = e - c_i$ is not. This implies that the new set $\{e, e_1, \dots, e_n\}$ with $e_i = c_i - \bar{c}_i = 2c_i - e$, $1 \leq i \leq n$, is also an basis of V.

Let $x \in V$ and let

$$x = a_0 e + \sum_{i=1}^{n} a_i e_i \tag{33}$$

be the corresponding basis decomposition. By (27)

$$\langle e,\, e_i\rangle = \langle e,\, c - \bar{c}\rangle = 1 - 1 = 0$$

for all $i \geq 1$, hence

$$a_0 = \frac{\langle x,\, e\rangle}{\langle e,\, e\rangle} = \frac{1}{2}\langle x,\, e\rangle.$$

Note also that for all $1 \leq i, j \leq n$, we have by virtue of (26)

$$e_i e_j = 4c_i c_j - 2c_i - 2c_j + e = (1 - 2\langle \bar{c}_i,\, c_j\rangle)e. \tag{34}$$

Rewrite (33) as $x = \frac{1}{2}\langle x,\, e\rangle e + y$, where $y = \sum_{i=1}^{n} a_i e_i$. Since y is orthogonal to e we find that $|x|^2 = \frac{1}{2}\langle x,\, e\rangle^2 + |y|^2$. Also from (34) it follows that $y^2 = \lambda e$ for some λ, hence

$$y^2 = \frac{\langle y^2,\, e\rangle}{\langle e,\, e\rangle} e = \frac{|y|^2}{2} e.$$

By polarizing the latter identity we obtain

$$uv = \frac{\langle u,\, v\rangle}{2} e, \qquad \forall u, v \in e^{\perp}. \tag{35}$$

It remains to establish an isomorphism between V and the Jordan algebra $(e^{\perp})_f$ with $f(x, y) = \frac{1}{2}\langle x,\, y\rangle$ (recall that in a normalized minimal algebra $|e|^2 = 2$). To this end, note that $(e^{\perp})_f = \mathbb{R} \oplus e^{\perp}$ with the product (6) given in the present notation by

$$z \bullet w = (a \oplus u) \bullet (b \oplus v) = (ab + \tfrac{1}{2}\langle u,\, v\rangle) \oplus (av + bu),$$

where $z = a \oplus u$ and $w = b \oplus v$ with $u, v \in e^{\perp}$ and $a, b \in \mathbb{R}$. Let us define a vector space isomorphism

$$\phi(z) = ae + u : (e^{\perp})_f \to V.$$

Then for we have by (35)

$$\begin{aligned} \phi(z \bullet w) &= (ab + \tfrac{1}{2}\langle u,\, v\rangle)e + (av + bu) \\ &= abe + uv + av + bu \\ &= (ae + u)(be + v) \\ &= \phi(z)\phi(w), \end{aligned}$$

hence ϕ is an algebra isomorphism.

Finally, if V is an arbitrary unital minimal algebra with the inner product $\langle \cdot,\, \cdot\rangle$, let us define the new inner product by

$$\langle x,\, y\rangle_1 = \frac{2\langle x,\, y\rangle}{\langle e,\, e\rangle},$$

so that $\langle e,\, e\rangle_1 = 2$. Then $(V, \langle \cdot,\, \cdot\rangle_1)$ is a normalized minimal algebra. Thus $V \cong (e^{\perp})_f$, where $f(x, y) = \frac{1}{2}\langle x,\, y\rangle_1 = \frac{\langle x,\, y\rangle}{\langle e,\, e\rangle}$, as desired. The proposition is proved completely. $\square$

Acknowledgements

I thank the anonymous referee for many valuable suggestions and an important hint which helped me give a shorter proof of Proposition 3.2.

References

[1] M. Bordemann. Nondegenerate invariant bilinear forms on nonassociative algebras. *Acta Math. Univ. Comenian.*, 66(2):151–201, 1997.
[2] A. Castillo-Ramirez. Associative subalgebras of low-dimensional Majorana algebras. *J. Algebra*, 421:119–135, 2015.
[3] J. Faraut and A. Korányi. *Analysis on symmetric cones*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
[4] J. I. Hall, F. Rehren, and S. Shpectorov. Primitive axial algebras of Jordan type. *J. Algebra*, 437:79–115, 2015.
[5] J. I. Hall, F. Rehren, and S. Shpectorov. Universal axial algebras and a theorem of Sakuma. *J. Algebra*, 421:394–424, 2015.
[6] J.I. Hall, Y. Segev, and S. Shpectorov. Miyamoto involutions in axial algebras of Jordan type half. *Israel J. Math.*, 223(1):261–308, 2018.
[7] J.I. Hall, Y. Segev, and S. Shpectorov. On primitive axial algebras of Jordan type. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 13, 2018. to appear.
[8] A. A. Ivanov. Majorana representation of the Monster group. In *Finite simple groups: thirty years of the atlas and beyond*, volume 694 of *Contemp. Math.*, pages 11–17. Amer. Math. Soc., Providence, RI, 2017.
[9] N. Jacobson. *Structure and representations of Jordan algebras*. American Mathematical Society Colloquium Publications, Vol. XXXIX. American Mathematical Society, Providence, R.I., 1968.
[10] M. Koecher. *The Minnesota notes on Jordan algebras and their applications*, volume 1710 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999. Edited, annotated and with a preface by Aloys Krieg and Sebastian Walcher.
[11] Ya. Krasnov and V.G. Tkachev. Variety of idempotents in nonassociative algebras. In *Modern Trends in Hypercomplex Analysis*, Trends Math. Birkhäuser/Springer, Cham, 2018, *to appear*
[12] K. McCrimmon. *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, 2004.
[13] N. Nadirashvili, V.G. Tkachev, and S. Vlăduţ. A non-classical solution to a Hessian equation from Cartan isoparametric cubic. *Adv. Math.*, 231(3-4):1589–1597, 2012.
[14] N. Nadirashvili, V.G. Tkachev, and S. Vlăduţ. *Nonlinear elliptic equations and nonassociative algebras*, volume 200 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.
[15] S. Norton. The Monster algebra: some new formulae. In *Moonshine, the Monster, and related topics (South Hadley, MA, 1994)*, volume 193 of *Contemp. Math.*, pages 297–306. Amer. Math. Soc., Providence, RI, 1996.
[16] S. Okubo and J.M. Osborn. Algebras with nondegenerate associative symmetric bilinear forms permitting composition. *Comm. Algebra*, 9(12):1233–1261, 1981.
[17] F. Rehren. Generalised dihedral subalgebras from the Monster. *Trans. Amer. Math. Soc.*, 369(10):6953–6986, 2017.
[18] L. Rowen and Y. Segev. Associative and Jordan algebras generated by two idempotents. *Algebr. Represent. Theory*, 20(6):1495–1504, 2017.
[19] R.D. Schafer. *An introduction to nonassociative algebras*. Pure and Applied Mathematics, Vol. 22. Academic Press, New York, 1966.
[20] V.G. Tkachev. A Jordan algebra approach to the cubic eiconal equation. *J. of Algebra*, 419:34–51, 2014.
[21] V.G. Tkachev. On the non-vanishing property for real analytic solutions of the p-Laplace equation. *Proc. Amer. Math. Soc.*, 144(6):2375–2382, 2016.
[22] V.G. Tkachev. A correction of the decomposability result in a paper by Meyer-Neutsch. *J. Algebra*, 504:432–439, 2018.

Department of Mathematics, Linköping University, Linköping, 58183, Sweden
E-mail address: `vladimir.tkatjev@liu.se`