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Practical Inlet Boundary Conditions for Internal Flow Calculations

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Abstract

To impose boundary conditions, data at the boundaries must be known, and consequently measurements of the imposed quantities must be available. In this paper, we consider the two most commonly used inflow boundary conditions with available data for internal flow calculations: the specification of the total temperature and total pressure. We use the energy method to prove that the specification of the total temperature and the total pressure together with the tangential velocity at an inflow boundary lead to well-posedness for the linearized compressible Euler equations. Next, these equations are discretized in space using high-order finite-difference operators on summation-by-parts form, and the boundary conditions are weakly imposed. The resulting numerical scheme is proven to be stable and the implementation of the corresponding nonlinear scheme is verified with the method of manufactured solutions. We also derive the spectrum for the continuous and discrete problems and show how to predict the convergence rate to steady state. Finally, nonlinear steady-state computations are performed, and they confirm the predicted convergence rates.

Keywords: Internal flow, inlet boundary conditions, steady state, Euler equations, well-posedness, eigenmode analysis

1. Introduction

Internal flow calculations are widely used in gas turbine modeling and construction of turbocharged engines [1, 2]. The influence of boundary conditions on these calculations is significant and several studies have been pub-

lished on how the boundary conditions affect these simulations [3, 4, 5]. To impose boundary conditions, data at the boundaries must be known, and consequently measurements of the imposed quantities must be available. In this paper, we consider the two most commonly used inflow boundary conditions with available data for internal flow calculations [1, 2, 3, 4, 5, 6]: the specification of the total temperature and total pressure (later referred to as the total variables). The two-dimensional Euler equations govern the flow, and they require three boundary conditions at a subsonic inflow boundary. We investigate if a third boundary condition can be added on to yield a well-posed initial-boundary value problem (IBVP) [7].

Solvers that rapidly converge to steady state are important in many applications [8, 9]. Different numerical techniques such as local time-stepping and multigrid are often used to enhance the convergence rate. A requirement for these techniques to work is that the spectrum of the IBVP resides in the appropriate half of the complex plane. This requirement is also necessary for well-posedness of the problem [10]. We use the Laplace transform method and compute the spectrum of the IBVP, which determines the convergence rate to steady state [11, 12]. See [10, 13] for more information about well-posedness, the energy method and the Laplace transform method.

Once well-posedness is investigated, and the continuous spectrum is derived, we proceed to the discrete approximation. High-order finite-difference operators on summation-by-parts (SBP) form are used for the discretization in space. The boundary conditions are imposed weakly using the Simultaneous-Approximation-Terms (SAT) technique and the resulting linear SBP-SAT scheme is proven to be stable, using the discrete energy method [14]. Finally, we compute the discrete spectrum of the linearized IBVP and show how to predict the convergence rate to steady state for the nonlinear scheme.

The rest of the paper proceeds as follows: The governing equations and the total variables are introduced in Section 2. In Section 3, sufficient conditions for well-posedness for a general boundary operator are derived using the energy method. Later in the same section, the specific boundary conditions mentioned above are investigated. The continuous spectrum is derived in Section 4. The provably stable discrete scheme is derived in Section 5 together with the discrete spectrum. Nonlinear computations are presented in Section 6, where the implementation is verified with the method of manufactured solutions. Additionally, results from steady-state computations are compared with the predicted convergence rates from linear theory. Finally conclusions are drawn in Section 7.

Figure 1: Illustration of the domain. The north and south boundaries are solid walls and the east (Γ_e) and west (Γ_w) are artificial.

2. Problem description and the governing equations

Consider the Euler equations on a rectangular domain as exemplified in Figure 1. The north and south boundaries of the domain are solid walls, i.e. the normal velocity is zero. The west and east boundaries are artificial, where the investigated boundary conditions are imposed. The flow upstream and downstream of the computational domain is assumed constant and denoted by the subscript ∞ . We assume that the changes in the flow are small and gradual, which makes the flow isentropic [15] and amenable for linearization.

In a non-dimensional form, the governing nonlinear Euler equations are

$$\vec{u}_t + F(\vec{u})_x + G(\vec{u})_y = 0, \quad (1)$$

where $\vec{u} = (\rho, \rho u, \rho v, e)^\top$ and the fluxes are

$$F(\vec{u}) = (\rho u, \rho u^2 + p, \rho uv, (e+p)u)^\top \quad \text{and} \quad G(\vec{u}) = (\rho v, \rho uv, \rho v^2 + p, (e+p)v)^\top.$$

The quantities ρ, u, v, p are the density, the velocities in the x, y -direction, the pressure and $e = p/(\gamma - 1) + \rho(u^2 + v^2)/2$ the total energy, respectively. We have used the subscripts t, x, y to denote the partial derivatives with respect to time and space. Furthermore, the perfect gas law takes the form $M_\infty^2 \gamma p = \rho T$, where T is the temperature and γ is the heat capacity ratio. We have introduced the free-stream Mach number $M_\infty = U_\infty/c_\infty$, where U_∞ and c_∞ are the free-stream speed and speed of sound, respectively. The local speed of sound is $c^2 = T/M_\infty^2$. The first two boundary conditions specified at the inflow (assuming isentropic flow) are

$$\begin{aligned} P_0 &= p \left[1 + (u^2 + v^2)(\gamma - 1)M_\infty^2/(2T) \right]^{\gamma/(\gamma-1)} = g_1, \\ T_0 &= T + (u^2 + v^2)(\gamma - 1)M_\infty^2/2 = g_2, \end{aligned} \quad (2)$$

which are the total pressure and temperature, respectively.

Remark 1. Preceding the upcoming analysis we stress that boundary conditions of the general form $Lu = g$ must have *i)* an appropriate boundary operator (L) for boundedness *and ii)* available data (g) for accuracy. Without these two requirements, meaningful calculations are not possible.

3. The continuous linearized problem

The energy method require symmetric matrices [16], which can be obtained from the linearized frozen-coefficient representation of (1). We make the change of variables $(\rho, \rho u, \rho v, e)^\top \rightarrow (\rho, u, v, p)^\top$ [17] and linearize around a constant free-stream flow [13]. Both the north and the south boundaries are solid walls (see Figure 1), and therefore $\bar{v} = 0$. The constant state (indicated by an overbar) we consider is

$$\bar{\rho} = \bar{u} = \bar{T} = 1, \quad \bar{v} = 0, \quad \bar{c} = 1/M_\infty. \quad (3)$$

The constant state (3) is used in all upcoming computations unless otherwise explicitly specified. Further, for dry air (the typical working fluid for gas turbines) $\gamma = 1.4$, which leaves M_∞ as the only parameter. We postulate that the direction of the main flow is in the positive x -direction and that the speed is subsonic, $0 < \bar{u} < \bar{c}$, since $M_\infty < 1$.

By using the symmetrizers from [18], the linear symmetric system becomes

$$W_t + AW_x + BW_y = 0, \quad (4)$$

where $W = [\rho\bar{c}/(\sqrt{\gamma}\bar{\rho}), u, v, T/(\bar{c}M_\infty^2\sqrt{\gamma(\gamma-1)})]^\top$. The quantities ρ, u, v, T are small perturbations from the constant state and the symmetrized matrices are

$$A = \begin{pmatrix} \bar{u} & \bar{c}/\sqrt{\gamma} & 0 & 0 \\ \bar{c}/\sqrt{\gamma} & \bar{u} & 0 & \bar{c}\sqrt{(\gamma-1)/\gamma} \\ 0 & 0 & \bar{u} & 0 \\ 0 & \bar{c}\sqrt{(\gamma-1)/\gamma} & 0 & \bar{u} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \bar{c}/\sqrt{\gamma} & 0 \\ 0 & 0 & 0 & 0 \\ \bar{c}/\sqrt{\gamma} & 0 & 0 & \bar{c}\sqrt{(\gamma-1)/\gamma} \\ 0 & 0 & \bar{c}\sqrt{(\gamma-1)/\gamma} & 0 \end{pmatrix}.$$

In the upcoming analysis, all boundaries except the inflow boundary Γ_w with its outward-pointing unit normal $n_w = (-1, 0)$ will be disregarded and are assumed to be treated in a correct manner.

3.1. The energy method and well-posedness

The IBVP that we will analyze is (4) augmented with initial and boundary conditions

$$\begin{aligned} W_t + AW_x + BW_y &= 0, & (x, y) \in \Omega, & \quad t \geq 0, \\ LW &= g, & (x, y) \in \partial\Omega, & \quad t \geq 0, \\ W &= f, & (x, y) \in \Omega, & \quad t = 0. \end{aligned} \quad (5)$$

In (5), $\Omega = [0, 1]^2$, $\partial\Omega$ denotes the boundary to Ω , L is the boundary operator and f and g are smooth functions that are considered to be known. We introduce inner products and the corresponding norm

$$\langle U, V \rangle_\Omega = \iint_\Omega U^\top V d\Omega, \quad \langle U, V \rangle_\Gamma = \int_\Gamma U^\top V dS, \quad \|U\|^2 = \langle U, U \rangle_\Omega,$$

where $U, V \in \mathcal{L}^2(\Omega)$ and $\Gamma \subset \partial\Omega$.

Definition 1. *The IBVP (5) with $g = 0$ is well posed if a unique solution exists satisfying*

$$\|W(\cdot, t)\|^2 \leq K_c e^{\alpha_c t} \|f(\cdot)\|^2,$$

where K_c and α_c are constants.

Applying the energy method to (5) results in

$$\|W\|_t^2 - \langle W, AW \rangle_{\Gamma_w} = 0, \quad (6)$$

where all boundary terms except the west one's have been neglected. Since A is symmetric, it can be decomposed as $A = X\Lambda X^\top$, where

$$X = \begin{pmatrix} 1/\sqrt{2\gamma} & 1/\sqrt{2\gamma} & -\sqrt{(\gamma-1)/\gamma} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{(\gamma-1)/(2\gamma)} & \sqrt{(\gamma-1)/(2\gamma)} & 1/\sqrt{\gamma} & 0 \end{pmatrix}, \quad (7)$$

$$\Lambda = \text{diag}(\bar{u} + \bar{c}, \bar{u} - \bar{c}, \bar{u}, \bar{u}).$$

The corresponding characteristic variables become $q = (q_1, q_2, q_3, q_4)^\top = X^\top W$. Let $\Lambda^+ = \text{diag}(\bar{u} + \bar{c}, \bar{u}, \bar{u})$ and $\Lambda^- = \bar{u} - \bar{c}$ denote the positive and negative eigenvalues respectively, then (6) takes the form

$$\|W\|_t^2 - \left\langle \begin{pmatrix} q^+ \\ q^- \end{pmatrix}, \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix} \begin{pmatrix} q^+ \\ q^- \end{pmatrix} \right\rangle_{\Gamma_w} = 0,$$

where $q^+ = (q_1, q_3, q_4)^\top$ and $q^- = q_2$.

Following [7], the boundary conditions are written in a general form where the ingoing characteristics are specified in terms of the outgoing ones and data as

$$q^+ - Rq^- = g, \quad (8)$$

where $R \in \mathbb{R}^{3 \times 1}$. Imposing the homogeneous ($g = 0$) form of (8) yields $\|W\|_t^2 - \langle q^-, (\Lambda^- + R^\top \Lambda^+ R) q^- \rangle_{\Gamma_w} = 0$. Integrating in time, we find that

$$\|W(t)\|^2 - \int_0^t \langle q^-, (\Lambda^- + R^\top \Lambda^+ R) q^- \rangle_{\Gamma_w} d\tau = \|f\|^2. \quad (9)$$

A suitable energy estimate is hence obtained if $\Lambda^- + R^\top \Lambda^+ R$ is non-positive. The energy estimate leads directly to uniqueness and existence is given by the fact that we use a minimal number of boundary conditions. The minimal number of boundary conditions is equal to the number of nonzero entries in Λ^+ . We have proven

Proposition 1. *The continuous problem (5) with homogeneous boundary conditions on the form (8) is well posed if*

$$\Lambda^- + R^\top \Lambda^+ R \leq 0. \quad (10)$$

Remark 2. The result and proof of Proposition 1 are given in [7] and included here for clarification and completeness. In [7], a strong estimate for the non-homogeneous form of (8) is also given.

3.2. The boundary operator

We linearize the nonlinear boundary conditions (2) to obtain

$$\begin{aligned} \rho \bar{u}^2 / 2 + u \bar{\rho} \bar{u} + p [1 - \bar{u}^2 / (2 \bar{c}^2)] &= g_1, \\ -\rho \gamma \bar{p} / \bar{\rho}^2 + u \bar{u} (\gamma - 1) + p \gamma / \bar{\rho} &= g_2. \end{aligned} \quad (11)$$

The third boundary condition (we need three since Λ^+ has three nonzero entries for subsonic flow) that we consider is given by

$$\sigma_1 \bar{c}^2 \rho + \sigma_2 \bar{\rho} \bar{c} u + \sigma_3 \bar{\rho} \bar{c} v + \sigma_4 p = g_3, \quad (12)$$

which is a linear combination of the state variables with dimensionless weights $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Our ambition is to choose the weights $\sigma_i, i = 1, \dots, 4$ such that an energy estimate is obtained. We combine (11) and (12) to obtain the boundary operator in (5) as

$$\underbrace{\begin{pmatrix} \frac{\bar{\rho}[2\bar{c}^2+(\gamma-1)\bar{u}^2]}{2\bar{c}\sqrt{\gamma}} & \bar{\rho}\bar{u} & 0 & \bar{\rho}\bar{c}\left(1-\frac{\bar{u}^2}{(2\bar{c}^2)}\right)\sqrt{\frac{(\gamma-1)}{\gamma}} \\ 0 & (\gamma-1)\bar{u} & 0 & \bar{c}\sqrt{(\gamma-1)\gamma} \\ \frac{\bar{\rho}\bar{c}(\gamma\sigma_1+\sigma_4)}{\sqrt{\gamma}} & \sigma_2\bar{\rho}\bar{c} & \sigma_3\bar{\rho}\bar{c} & \bar{\rho}\bar{c}\sigma_4\sqrt{\frac{(\gamma-1)}{\gamma}} \end{pmatrix}}_L W = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

Provided that $\sigma_3 \neq 0$, the boundary matrix in (8) can be obtained by the transformations

$$LW = g \iff (L_X^+)^{-1}(L_X^+q^+ + L_X^-q^-) = (L_X^+)^{-1}g \iff q^+ - Rq^- = (L_X^+)^{-1}g.$$

The matrices L_X^+ and L_X^- are the columns in LX that correspond to q^+ and q^- respectively and

$$R = -(L_X^+)^{-1}L_X^- = \begin{pmatrix} (\bar{u} - \bar{c})/(\bar{u} + \bar{c}) \\ 0 \\ \sqrt{2}[\bar{c}\sigma_2 - \bar{u}(\sigma_1 + \sigma_4)]/(\sigma_3(\bar{u} + \bar{c})) \end{pmatrix}. \quad (13)$$

Condition (10) now becomes

$$\Lambda^- + R^\top \Lambda^+ R = f(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \leq 0, \quad (14)$$

where

$$f = \frac{2[(\sigma_1 + \sigma_4)^2\bar{u}^3 + \sigma_2^2\bar{u}\bar{c}^2 + \sigma_3^2\bar{u}(\bar{u}^2 - \bar{c}^2) - 2\sigma_2(\sigma_1 + \sigma_4)\bar{c}\bar{u}^2]}{[\sigma_3(\bar{u} + \bar{c})]^2}.$$

We aim for weights $\sigma_i, i = 1, \dots, 4$ such that the inequality in (14) holds. Let $(\nabla f)_i = \partial f / \partial \sigma_i, i = 1 \dots 4$ denote the gradient of f . Stationary points to f are given by

$$\nabla f = \frac{4}{\sigma_3^2(\bar{u} + \bar{c})^2} \begin{pmatrix} (\sigma_1 + \sigma_4)\bar{u}^3 - \sigma_2\bar{c}\bar{u}^2 \\ \sigma_2\bar{u}\bar{c}^2 - (\sigma_1 + \sigma_4)\bar{c}\bar{u}^2 \\ \frac{[2\sigma_2(\sigma_1 + \sigma_4)\bar{c}\bar{u}^2 - (\sigma_1 + \sigma_4)^2\bar{u}^3 - \sigma_2^2\bar{u}\bar{c}^2]}{(\sigma_1 + \sigma_4)\bar{u}^3 - \sigma_2\bar{c}\bar{u}^2} \end{pmatrix} = 0, \quad (15)$$

which yields $\sigma_2 = (\sigma_1 + \sigma_4)(\bar{u}/\bar{c})$. To classify the stationary points, we examine the difference $\delta f = f[\sigma_1 + i, (\sigma_1 + \sigma_4)\bar{u}/\bar{c} + j, \sigma_3 + k, \sigma_4 + l] - f[\sigma_1, (\sigma_1 + \sigma_4)\bar{u}/\bar{c}, \sigma_3, \sigma_4]$. If δf is non-negative for all i, j, k, l sufficiently small, then the stationary points are all local minima. We get

$$\delta f = 2\bar{u}[\bar{c}j + (i + l)\bar{u}]^2 / [(\sigma_3 + k)(\bar{u} + \bar{c})]^2 \geq 0,$$

which means that all stationary points are minima. One convenient choice is therefore $\sigma_1 = \sigma_4 = \sigma_2 = 0$ and $\sigma_3 = 1/(\bar{\rho}\bar{c})$, which turns (12) into

$$v = 0. \tag{16}$$

Hence, the final set of inflow conditions that we will consider is:

$$\begin{aligned} \rho\bar{u}^2/2 + u\bar{\rho}\bar{u} + p[1 - \bar{u}^2/(2\bar{c}^2)] &= g_1, \\ -\rho\gamma\bar{p}/\bar{\rho}^2 + u\bar{u}(\gamma - 1) + p\gamma/\bar{\rho} &= g_2, \\ v &= 0. \end{aligned} \tag{17}$$

With (17), we find $R = [(\bar{u} - \bar{c})/(\bar{u} + \bar{c}), 0, 0]^\top$ and condition (14) becomes

$$\Lambda^- + R^\top \Lambda^+ R = 2\bar{u}(\bar{u} - \bar{c})/(\bar{u} + \bar{c}) < 0.$$

Thus, the three inflow conditions (17) forms a well-posed boundary formulation.

Remark 3. Note that (17) corresponds to a set of inflow conditions for the nonlinear problem for which the boundary data are physically measurable and hence possible to use in practice.

4. The spectrum

A solution to a hyperbolic problem may vanish at the boundaries but remain non-zero in the interior of the domain. Hence, the energy method does not provide sufficient information about the convergence rate to steady state. A necessary condition is that the solution vanishes as $t \rightarrow \infty$ for any initial data f when $g = 0$ in (5). We use the Laplace transform method [10] in order to investigate if that is the case.

Consider the linear system (5) with homogeneous inflow and outflow conditions:

$$\begin{aligned} W_t + AW_x + BW_y &= 0, & (x, y) \in \Omega, & \quad t \geq 0, \\ L_0 W &= 0, & (x, y) \in \Gamma_w, & \quad t \geq 0, \\ L_1 W &= 0, & (x, y) \in \Gamma_e, & \quad t \geq 0, \\ W &= f, & (x, y) \in \Omega, & \quad t = 0, \end{aligned} \tag{18}$$

posed on the domain $\Omega = [0, 1] \times [-\pi, \pi]$. The inflow and outflow boundaries are denoted $\Gamma_w = \{(x, y) \in \Omega : x = 0\}$ and $\Gamma_e = \{(x, y) \in \Omega : x = 1\}$, respectively. The inflow conditions (17) are imposed by L_0 , while L_1 imposes the static pressure at the outflow boundary (a well-posed subsonic outflow condition [19]).

We consider the solution to be 2π -periodic in the y -direction, such that a Fourier expansion can be used, and apply the Laplace transform in time:

$$W(x, y, t) = \sum_{\omega=-\infty}^{\infty} \tilde{W}(x, \omega, t) e^{i\omega y}, \quad \hat{W}(x, \omega, s) = \int_0^{\infty} e^{-st} \tilde{W}(x, \omega, t) dt. \tag{19}$$

In (19), $s = \eta + i\xi$ and $\eta, \xi \in \mathbb{R}$. All integrals exist since only smooth and bounded solutions are considered. Applying (19) to (18) yields for every ω

$$\begin{aligned} (sI + i\omega B)\hat{W} + A\hat{W}_x &= \tilde{f}(x, \omega), & x \in [0, 1], \\ L_0 \hat{W} &= 0, & x = 0, \\ L_1 \hat{W} &= 0, & x = 1, \end{aligned} \tag{20}$$

where $\tilde{f}(x, \omega)$ are the Fourier coefficients of f and I is the identity matrix. The solution $\hat{W} = \hat{W}_h + \hat{W}_p$ to (20) consists of a homogeneous and a particular solution. The particular solution depends on the initial data and can be considered known.

By making the ansatz $\hat{W}_h = \psi e^{\kappa x}$, where $\psi \in \mathbb{C}^{4 \times 1}$ and $\kappa \in \mathbb{C}$, we find that

$$(sI + i\omega B + \kappa A)\psi = 0 \tag{21}$$

holds. For non-trivial solutions,

$$\det(sI + i\omega B + \kappa A) = (s + \kappa \bar{u})^2 [(s + \kappa \bar{u}^2)^2 + \bar{c}^2 (\omega^2 - \kappa^2)] = 0 \tag{22}$$

must be satisfied, where $\det(\cdot)$ denotes the determinant. The roots to (22) with respect to κ are

$$\kappa_{1,2} = (\bar{u}s \pm \bar{c}\alpha) / (\bar{c}^2 - \bar{u}^2), \quad \kappa_3 = \kappa_4 = -s/\bar{u},$$

where $\alpha = \sqrt{s^2 + \omega^2(\bar{c}^2 - \bar{u}^2)}$. The two distinct roots κ_1 and κ_2 inserted in (21) yield

$$\begin{aligned}\psi_1 &= \left(1, \kappa_1 \beta_+, i\omega(\bar{c}^2 - \bar{u}^2)\beta_+, \sqrt{\gamma - 1}\right)^\top, \\ \psi_2 &= \left(1, \kappa_2 \beta_-, i\omega(\bar{c}^2 - \bar{u}^2)\beta_-, \sqrt{\gamma - 1}\right)^\top,\end{aligned}$$

where $\beta_\pm = -\sqrt{\gamma}[\bar{c}s \pm \bar{u}\alpha]/[\bar{c}^2 s^2 + \bar{u}^2 \alpha^2 \pm 2s\bar{c}\alpha]$. For the repeated root $\kappa_3 = \kappa_4 = -s/\bar{u}$, we reformulate the ansatz to $\hat{W}_h = (\psi + x\phi)e^{\kappa_3 x}$, where $\phi \in \mathbb{C}^{4 \times 1}$, and get the system of linearly independent equations

$$(sI + i\omega B + \kappa_3 A)\phi = 0, \quad (23a)$$

$$(sI + i\omega B + \kappa_3 A)\psi + A\phi = 0. \quad (23b)$$

We first solve (23a) to obtain ϕ , which is inserted into (23b). The solution to the whole system (23) becomes $\phi = (0, 0, 0, 0)^\top$ and $\psi = \sigma_3 \psi_3 + \sigma_4 \psi_4$, where $\psi_3 = (-\sqrt{\gamma - 1}, 0, 0, 1)^\top$ and $\psi_4 = (0, i\omega \bar{u}, s, 0)^\top$. Finally, the homogeneous solution to (20) is $\hat{W}_h = \Psi K(x) \vec{\sigma}$, where $\Psi = [\psi_1, \psi_2, \psi_3, \psi_4]$, $K(x) = \text{diag}(e^{\kappa_1}, e^{\kappa_2}, e^{\kappa_3}, e^{\kappa_3})$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)^\top$ contain the solution vectors, the exponential functions and the parameters that will be determined from the boundary conditions, respectively.

Inserting $\hat{W}$ into the boundary conditions yields for every ω and M_∞

$$E(s) \vec{\sigma} = \begin{pmatrix} L_0 \Psi K(0) \\ L_1 \Psi K(1) \end{pmatrix} \vec{\sigma} = H. \quad (24)$$

In (24), $E(s) \in \mathbb{C}^{4 \times 4}$ (three inflow conditions and one outflow condition) and H depends on the particular solution. The spectrum to the problem is given by $s^* = \{s \in \mathbb{C} : \det(E(s)) = 0\}$. We have a well-posed problem if all s^* are located in the left half of the complex plane [10]. Let $\eta_C^* = \max \Re(s^*)$ denote the real part of the eigenvalues closest to the imaginary axis. If $\Re(s) > \eta_C^*$, we can solve for $\vec{\sigma}$ and formally transform back to the time domain by taking the inverse Laplace transform:

$$\tilde{W}(x, \omega, t) = e^{\eta_C^* t} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \hat{W}(x, \omega, \eta_C^* + i\xi) e^{i\xi t} d\xi.$$

The solution will converge to steady state independent of the initial condition if $\eta_C^* < 0$.

The singularities to (24) are found with the Secant method. Initial values on a grid are used such that all roots within a certain domain around the

Figure 2: The continuous spectrum for $M_\infty = 0.5$ and different ω .

w	$M_\infty = 0.1$	$M_\infty = 0.5$	$M_\infty = 0.9$
0	-0.9933	-0.8240	-0.3108
1	-0.9009	-0.7206	-0.1558
5	-0.2784	-0.1737	-0.0134
10	-0.0881	-0.0518	-0.0031

Table 1: The largest real part of the spectrum (η_C^*) for different M_∞ and ω .

origin are found. Spectra for $M_\infty = 0.5$ and different ω are illustrated in Figure 2. Similar results are obtained for $M_\infty = 0.1$ and $M_\infty = 0.9$. All spectra resides in the left half of the complex plane. As $|\omega|$ increases, the spectrum moves closer to the imaginary axis, but stays in the left part of the complex plane, a behavior also observed in [20]. Moreover, the spectrum also moves closer to the imaginary axis as M_∞ increases, but stays in the left half of the complex plane for $M_\infty < 1$. Table 1 presents the η_C^* extracted from the spectra. As can be seen from the table, all cases converge to steady state.

5. The semi-discrete scheme

Let the domain $\Omega = [0, 1]^2$ be discretized with $N + 1$ and $M + 1$ grid points; $x_i = i/N$, $i = 0, \dots, N$ and $y_j = j/M$, $j = 0, \dots, M$. Further, let the solution field be $\mathbf{W}^{ijk}(t) \approx W(x_i, y_j, k, t)$, where i, j correspond to the grid point and k to the corresponding dependent variable.

The SBP approximation of the spatial derivatives are given by

$$\mathbf{D}_x \mathbf{W} = (P_x^{-1} Q_x \otimes I_y \otimes I_4) \mathbf{W} \approx \vec{W}_x, \quad \mathbf{D}_y \mathbf{W} = (I_x \otimes P_y^{-1} Q_y \otimes I_4) \mathbf{W} \approx \vec{W}_y,$$

where $\otimes$ denotes the Kronecker product and $\vec{W}_{x,y}$ contain the partial derivatives evaluated on the grid. The matrices $P_{x,y}$ are diagonal and positive definite, such that $\mathbf{P} = P_x \otimes P_y \otimes I_4$ forms a quadrature rule that defines the norm $\|\mathbf{W}\|_{\mathbf{P}}^2 = \mathbf{W}^\top \mathbf{P} \mathbf{W} \approx \iint_{\Omega} W^\top W d\Omega$. Furthermore, the matrices $Q_{x,y}$ satisfy the SBP-property

$$Q_x + Q_x^\top = E_N - E_0, \quad Q_y + Q_y^\top = E_M - E_0,$$

where $E_0 = \text{diag}(1, 0, 0, \dots, 0)$ and $E_{N,M} = \text{diag}(0, 0, 0, \dots, 1)$ are matrices of appropriate sizes.

We will also need the matrices

$$\mathbf{A} = I_x \otimes I_y \otimes A, \quad \mathbf{B} = I_x \otimes I_y \otimes B, \quad \mathbf{X} = I_x \otimes I_y \otimes X, \quad \mathbf{\Lambda} = I_x \otimes I_y \otimes \Lambda,$$

where $\mathbf{A} = \mathbf{X} \mathbf{\Lambda} \mathbf{X}^\top$. Let $\mathbf{\Lambda}^\pm = (\mathbf{\Lambda} \pm |\mathbf{\Lambda}|)/2$ and $\mathbf{X}^\pm = I_x \otimes I_y \otimes X^\pm$, where

$$X^+ = \begin{pmatrix} 1/\sqrt{2\gamma} & 0 & -\sqrt{(\gamma-1)/\gamma} & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{(\gamma-1)/(2\gamma)} & 0 & 1/\sqrt{\gamma} & 0 \end{pmatrix},$$

$$X^- = \begin{pmatrix} 0 & 1/(\sqrt{2\gamma}) & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{(\gamma-1)/(2\gamma)} & 0 & 0 \end{pmatrix}.$$

The characteristic variables in semi-discrete form are $\mathbf{q} = \mathbf{X}^\top \mathbf{W}$ and $\mathbf{q}^\pm = (\mathbf{X}^\pm)^\top \mathbf{W}$.

Finally, the semi-discrete formulation of (5) becomes

$$\begin{aligned} \mathbf{W}_t + (\mathbf{D}_x \mathbf{A} + \mathbf{D}_y \mathbf{B}) \mathbf{W} &= \mathbf{P}^{-1} \mathbf{P}_y^0 \Sigma(\mathbf{q}^+ - \mathbf{R} \mathbf{q}^- - \bar{\mathbf{g}}) + \mathbf{Pen}, & t \geq 0, \\ \mathbf{W} &= \mathbf{f} & t = 0. \end{aligned} \quad (25)$$

All penalty terms except the west one's are included in the term $\mathbf{Pen}$ and will not be discussed further. We have introduced $\mathbf{P}_y^0 = E_0 \otimes P_y \otimes I_4$ and $\mathbf{f}$, which is the quadrature formula over the west boundary and the initial data

evaluated on the grid, respectively. The inflow conditions (17) are imposed weakly on the form (8) and $\mathbf{R} = I_x \otimes I_y \otimes R$, where R is a matrix corresponding to the continuous boundary conditions. Furthermore, Σ is a penalty matrix that will be determined for stability. We make the following definition

Definition 2. *The semi-discrete scheme (25) with $\mathbf{g} = \mathbf{0}$ is stable if*

$$\|\mathbf{W}(t)\|_{\mathbf{P}}^2 \leq K_d e^{\alpha_d t} \|\mathbf{f}\|_{\mathbf{P}}^2,$$

where K_d, α_d are constants.

Following the procedure outlined in [7], we choose $\Sigma = -\mathbf{X}^+ \Lambda^+$ and apply the discrete energy method to (25) with $\mathbf{g} = \mathbf{0}$ to obtain the estimate

$$\begin{aligned} \|\mathbf{W}(t)\|_{\mathbf{P}}^2 - \int_0^t (\mathbf{q}^-)^\top \mathbf{P}_y^0 (\Lambda^- + R^\top \Lambda^+ R) \mathbf{q}^- d\tau = \\ \|\mathbf{f}\|_{\mathbf{P}}^2 - \int_0^t (\mathbf{q}^+ - \mathbf{R} \mathbf{q}^-)^\top \mathbf{P}_y^0 \Lambda^+ (\mathbf{q}^+ - \mathbf{R} \mathbf{q}^-) d\tau. \end{aligned} \quad (26)$$

Since Proposition 1 holds, $(\Lambda^- + R^\top \Lambda^+ R) \leq 0$, and we have proved

Proposition 2. *The scheme (25) with $\Sigma = -\mathbf{X}^+ \Lambda^+$ is stable.*

The estimate in (26) mimics the continuous estimate (9) and the last term in (26) adds dissipation to the scheme.

The provably stable linear scheme (25) must be related to the nonlinear scheme. Let the solution field $\vec{\mathbf{u}}(t)$ hold the conservative variables, ordered in the same way as $\mathbf{W}(t)$ in (25). Further, let $\mathbf{M}$ be the transformation matrix between the conservative and symmetric variables such that $\vec{\mathbf{u}} = \mathbf{M} \mathbf{W}$ and $\mathbf{W} = \mathbf{M}^{-1} \vec{\mathbf{u}}$. Multiplying (25) by $\mathbf{M}$ yields

$$\begin{aligned} \vec{\mathbf{u}}_t + \mathbf{D}_x \vec{\mathbf{F}} + \mathbf{D}_y \vec{\mathbf{G}} &= \mathbf{P}^{-1} \mathbf{P}_y^0 \bar{\Sigma} (\mathbf{L}_x^+)^{-1} (\mathbf{L} \vec{\mathbf{u}} - \mathbf{g}) + \mathbf{M} \mathbf{Pen} & t \geq 0, \\ \vec{\mathbf{u}} &= \mathbf{M} \mathbf{f} & t = 0, \end{aligned} \quad (27)$$

where $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are the flux vectors. The penalty matrix is $\bar{\Sigma} = \mathbf{M} \Sigma$, where $\Sigma = -\mathbf{X}^+ \Lambda^+$ as in (25). More details are found in the Appendix.

5.1. The discrete spectrum

Let $\Omega' = [0, 1]$ be discretized with $N+1$ grid points; $x_i = i/N, i = 0, \dots, N$ and consider the discrete scheme

$$\begin{aligned} \mathbf{W}_t + \mathbf{M}\mathbf{W} &= \mathbf{0}, & t \geq 0, \\ \mathbf{W} &= \mathbf{f}, & t = 0, \end{aligned} \quad (28)$$

where $\mathbf{M} = D_x \otimes A + i\omega(I_x \otimes B) - P_x^{-1}E_0 \otimes X^+\Lambda^+H_0 + P_x^{-1}E_N \otimes X^-\Lambda^-H_1$ and where D_x is an SBP operator [14]. Further, H_0 imposes (17) at the inflow boundary and H_1 the pressure at the outflow boundary. By applying the Laplace transform in time to (28), we obtain

$$(s(I_x \otimes I_4) + \mathbf{M})\hat{\mathbf{W}} = \mathbf{f}, \quad (29)$$

which is the discrete formulation of (20). As in the continuous case, the solution $\hat{\mathbf{W}} = \hat{\mathbf{W}}_h + \hat{\mathbf{W}}_p$ consists of a homogeneous and particular solution, where the particular solution depends on the initial data and can be considered known. Only the homogeneous part is of interest and (29) turns into the eigenvalue problem $(s(I_x \otimes I_4) + \mathbf{M})\hat{\mathbf{W}} = 0$. The discrete spectrum is defined as $\{s \in \mathbb{C} : \det(s(I_x \otimes I_4) + \mathbf{M}) = 0\}$ and it is obtained by computing the eigenvalues to $\mathbf{M}$.

Figure 3-6 illustrates the discrete spectra for different number of grid points for the 4th order SBP approximation [14] when $M_\infty = 0.5$ and ω is varied. The continuous spectra is included and, as can be seen in the figures, the discrete spectra converge to its continuous analogue as the number of grid points is increased. All spectra resides in the left half of the complex plane. Similar results are obtained for $M_\infty = 0.1$ and $M_\infty = 0.9$. In the continuous case, the singularity with the largest real part (η_C^*) determines the decay rate. For the discrete scheme, it is similarly the eigenvalue to $\mathbf{M}$ with largest real part (denoted η_D^*) that determines the decay rate. From Figure 3-6, we can see that η_C^* is well approximated by η_D^* for all cases.

6. Nonlinear Computations

The linear equations (5) and the corresponding discrete approximation (25) serve as analytical tools, while it is the nonlinear equations (1) that must be used to obtain realistic results. A nonlinear compressible Euler solver based on the SBP-SAT technique is used for the following computations on the domain $\Omega = [0, 1]^2$. The north and south boundaries are (as in Figure

Figure 3: Comparison between the continuous and the discrete spectrum for $M_\infty = 0.5$ and $\omega = 0$.

Figure 4: Comparison between the continuous and the discrete spectrum for $M_\infty = 0.5$ and $\omega = 1$.

Figure 5: Comparison between the continuous and the discrete spectrum for $M_\infty = 0.5$ and $\omega = 5$.

Figure 6: Comparison between the continuous and the discrete spectrum for $M_\infty = 0.5$ and $\omega = 10$.

1) solid walls [21] and the pressure (also typically a measured quantity in experiments) is imposed at the outflow [19]. We use the nonlinear inflow conditions (2) together with (16) at the west boundary.

6.1. The Order of Accuracy

For verification, the method of manufactured solutions is used for the SBP-operators SBP21, SBP42, SBP63 and SBP84 with expected global convergence rate 2, 3, 4 and 5 respectively [14]. The errors from the time-integration are small enough to be neglected and the simulations are terminated at $t = 5$.

A manufactured solution that satisfy the solid wall boundary conditions is

$$\begin{aligned}\rho &= 1 + 0.05[\sin(\pi(x - t)) + \sin(\pi(y - t))], \\ u &= 1 + 0.05[\sin(\pi(x - t)) + \sin(\pi(y - t))], \\ v &= 0.05 \sin(2\pi y) \sin(\pi x), \\ e &= 3 + 0.05[\sin(\pi(x - t)) + \sin(\pi(y - t))].\end{aligned}$$

The error is measured separately for $\rho, \rho u, \rho v$ and e in the L_2 -norm by $\|\mathbf{err}\|^2 = \mathbf{err}^\top P \mathbf{err}$, where $\mathbf{err}$ is the pointwise error and P is the quadrature corresponding to the SBP-operator. Furthermore, the convergence rate is computed as $\log(\|\mathbf{err}\|_i / \|\mathbf{err}\|_{i-1}) / \log(h_i / h_{i-1})$, where h is the grid spacing and i refers to a specific number of grid points that are used in the computations. The error and rate of convergence for $\rho, \rho u, \rho v$ and e are presented in Table 2, 3, 4 and 5 respectively, and agrees well with theory.

Operator	SBP21		SBP42		SBP63		SBP84	
N	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate
21	2.73e-04	-	8.46e-05	-	6.74e-05	-	2.86e-05	-
41	6.99e-05	1.97	1.06e-05	3.00	2.93e-06	4.53	1.04e-06	4.79
81	1.80e-05	1.96	1.31e-06	3.02	2.12e-07	3.79	3.03e-08	5.10
161	4.65e-06	1.95	1.63e-07	3.01	1.44e-08	3.88	9.39e-10	5.01

Table 2: Error and convergence rate for ρ .

6.2. The convergence rate to steady state

As previously mentioned, for a small initial disturbance, the solution should decay with the rate η_D^* , which in turn should converge to η_C^* in Table

Operator	SBP21		SBP42		SBP63		SBP84	
N	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate
21	3.16e-04	-	9.65e-05	-	6.66e-05	-	3.99e-05	-
41	8.26e-05	1.94	1.18e-05	3.03	4.03e-06	4.05	1.32e-06	4.92
81	2.21e-05	1.90	1.44e-06	3.03	2.83e-07	3.83	3.58e-08	5.21
161	5.80e-06	1.93	1.78e-07	3.02	1.89e-08	3.90	1.04e-09	5.10

Table 3: Error and convergence rate for ρu .

Operator	SBP21		SBP42		SBP63		SBP84	
N	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate
21	1.06e-04	-	8.20e-05	-	6.63e-05	-	1.75e-05	-
41	2.69e-05	1.98	1.15e-05	2.84	3.37e-06	4.30	9.34e-07	4.23
81	6.75e-06	1.99	1.42e-06	3.01	1.72e-07	4.30	2.93e-08	4.99
161	1.69e-06	2.00	1.78e-07	3.00	9.13e-09	4.23	1.00e-09	4.87

Table 4: Error and convergence rate for ρv .

Operator	SBP21		SBP42		SBP63		SBP84	
N	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate	$\ \mathbf{err}\ $	rate
21	9.76e-04	-	2.91e-04	-	2.75e-04	-	8.14e-05	-
41	2.46e-04	1.99	3.80e-05	2.94	1.10e-05	4.64	3.32e-06	4.61
81	6.17e-05	1.99	4.76e-06	3.00	6.11e-07	4.17	1.05e-07	4.98
161	1.56e-05	1.99	5.94e-07	3.00	3.94e-08	3.95	3.44e-09	4.93

Table 5: Error and convergence rate for e .

1. The domain in Section 4, i.e. $\Omega = [0, 1] \times [-\pi, \pi]$, is considered for the computations in this section and periodic boundary conditions are imposed at $y = \pm\pi$. As initial state, $T = u = 1$, $v = 0$, $c = 1/M_\infty$ and $\rho = 1 + 0.1e^{-(x-0.5)^2/0.02} \cos(\omega y)$ and the parameters M_∞ and ω are varied. We measure the deviation of ρ from its steady-state value $\rho = 1$ in the L_2 -norm. Figure 7-9 presents the convergence rates for different M_∞ and ω . The SBP63 operator is used on a grid containing 80×80 grid points for all computations. The computed convergence rates agree well with the results from Section 4 and 5.

Figure 7: Comparison between the computed and theoretical convergence rates to steady state for $M_\infty = 0.1$ and different ω . The dashed lines are η_C^* taken from Table 1.

Figure 8: Comparison between the computed and theoretical convergence rates to steady state for $M_\infty = 0.5$ and different ω . The dashed lines are η_C^* taken from Table 1.

7. Conclusions

We have studied a set of practical inlet boundary conditions for internal flow calculations. Well-posedness for the imposition of the total temperature and total pressure together with the tangential velocity at the inflow boundary for subsonic flow was proved. The corresponding semi-discrete scheme was also proved to be stable. Furthermore, these inflow conditions were implemented in a nonlinear compressible Euler solver and the implementation

Figure 9: Comparison between the computed and theoretical convergence rates to steady state for $M_\infty = 0.9$ and different ω . The dashed lines are η_C^* taken from Table 1.

was verified by the method of manufactured solutions. The continuous and discrete spectra were computed and the convergence rate to steady state for the inflow conditions estimated. The obtained convergence rates from the nonlinear computations agreed well with the result predicted by the linear theory.

Appendix

The relation between the linearized symmetric scheme and the fully non-linear scheme is discussed below.

The continuous case

Let T be the transformation between the conservative variables, $\vec{u} = (\rho, \rho u, \rho v, e)$, and primitive variables, $\vec{v} = (\rho, u, v, p)^\top$, such that $\vec{u} = T\vec{v}$ and $\vec{v} = T^{-1}\vec{u}$. The Euler equations are $\vec{u}_t + A_u \vec{u}_x + B_u \vec{u}_y = 0$. Multiplying by T^{-1} from the left yields $\vec{v}_t + A_v \vec{v}_x + B_v \vec{v}_y = 0$, where $A_v = M^{-1}A_u M$ and $B_v = M^{-1}B_u M$. The symmetric system (4) is obtained by multiplying with the symmetrizer S^{-1} . The transformation between the conservative and symmetric variables becomes $\vec{u} = MW$ and $W = M^{-1}\vec{u}$, where

$$M = TS \quad \text{and} \quad M^{-1} = S^{-1}T^{-1}. \quad (30)$$

Let X be the matrix defined in (7) and let $q = X^\top W$. We obtain (8) by the relations

$$\begin{aligned} (L_X^+)^{-1}LW = (L_X^+)^{-1}g &\iff (L_X^+)^{-1}(L_X^+q^+ + L_X^-q^-) = (L_X^+)^{-1}g \\ &\iff q^+ - Rq^- = (L_X^+)^{-1}g, \end{aligned} \quad (31)$$

where $L_X = LX$ and $L_X^\pm$ are the columns in L_X that correspond to $q^\pm$ respectively. Further, the boundary matrix is $R = -(L_X^+)^{-1}L_X^-$.

The semi-discrete case

The transformations between the conservative variables $\vec{\mathbf{u}}$ and the symmetric variables $\mathbf{W}$ are $\vec{\mathbf{u}} = \mathbf{M}\mathbf{W}$ and $\mathbf{W} = \mathbf{M}^{-1}\vec{\mathbf{u}}$. The transformation matrices are $\mathbf{M} = I_x \otimes I_y \otimes M$ and $\mathbf{M}^{-1} = I_x \otimes I_y \otimes M^{-1}$, where M and M^{-1} are defined in (30). Multiplying (25) by $\mathbf{M}$ yields

$$\vec{\mathbf{u}}_t + (\mathbf{D}_x\mathbf{A}_u + \mathbf{D}_y\mathbf{B}_u)\vec{\mathbf{u}} = \mathbf{M}\mathbf{P}^{-1}\mathbf{P}_y^0\Sigma(\mathbf{q}^+ - \mathbf{R}\mathbf{q}^- - \bar{\mathbf{g}}) + \mathbf{M}\mathbf{P}\mathbf{e}\mathbf{n}, \quad (32)$$

where $\mathbf{A}_u = \mathbf{M}\mathbf{A}\mathbf{M}^{-1}$ and $\mathbf{B}_u = \mathbf{M}\mathbf{B}\mathbf{M}^{-1}$. Since $\mathbf{M}$ and $\mathbf{P}^{-1}\mathbf{P}_y^0$ commute, it follows from (31) and (32) that

$$\vec{\mathbf{u}}_t + (\mathbf{D}_x\mathbf{A}_u + \mathbf{D}_y\mathbf{B}_u)\vec{\mathbf{u}} = \mathbf{P}^{-1}\mathbf{P}_y^0\bar{\Sigma}(\mathbf{L}_X^+)^{-1}(\bar{\mathbf{L}}\vec{\mathbf{u}} - \mathbf{g}) + \mathbf{M}\mathbf{P}\mathbf{e}\mathbf{n}, \quad (33)$$

where $\bar{\Sigma} = \mathbf{M}\Sigma$. The boundary operator is $\bar{\mathbf{L}} = \mathbf{L}\mathbf{M}^{-1}$, where $\mathbf{L} = I_x \otimes I_y \otimes L$ is the boundary operator to the symmetric system. Lastly, we have the relation $\bar{\mathbf{g}} = (\mathbf{L}_X^+)^{-1}\mathbf{g}$.

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