Applicability of solutions for periodic intralaminar crack distributions to non-uniformly damaged laminates

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Abstract

Stiffness reduction simulation in laminates with intralaminar cracks is usually performed assuming that cracks are equidistant and crack density is the only parameter needed. However, the crack distribution in the damaged layer is very non-uniform, especially in the initial stage of multiple cracking. In this article, the earlier developed model for general symmetric laminates is generalized to account for non-uniform crack distribution. This model, in which the normalized average crack-opening and crack-sliding displacements are the main characteristics of the crack, is used to calculate the axial modulus of cross-ply laminates with cracks in internal and surface layers. In parametric analysis, the crack-opening displacement and crack-sliding displacement are calculated using finite element method, considering the smallest versus the average crack spacing ratio as non-uniformity parameter. It is shown that assuming uniform distribution, we obtain lower bond to elastic modulus. A 'double-periodic' approach presented to calculate the crack-opening displacement of a crack in a non-uniform case as the average of two solutions for periodic crack systems is very accurate for cracks in internal layers of cross-ply laminates, whereas for high crack density in surface layers, it underestimates the modulus reduction.

Keywords

Laminates, transverse cracking, crack-opening displacement, elastic properties, damage mechanics

Introduction

Intralaminar cracking in laminates is the most typical mode of damage in laminates. Initiation, evolution and effect of these cracks on laminate stiffness has been discussed in many papers, see for example review papers.^{1,2} Intralaminar cracks (called also matrix cracks, transverse cracks, inclined cracks, etc.) are orthogonal to the laminate midplane, they run parallel to fibres in the layer, usually cover the whole thickness and width of the layer in the specimen.

In the presence of cracks, the average stress in the damaged layer is lower than that in the same layer of the undamaged laminate. The average stress between two cracks depends on the distance between them (normalized spacing). Usually, the extent of cracking (number of cracks and distance between them) is characterized in an average sense by average crack spacing and crack density (cracks/mm). Most of the existing stiffness models, for example, Lundmark and Varna,³ Hashin,⁴ Talreja⁵ and Kashtalyan and Soutis⁶ use this

term. It is convenient to use and is expected to give sufficient accuracy.

However, the crack distribution in the layer may be highly non-uniform as schematically shown in Figure 1. This is more pronounced in the beginning of the cracking process when the average crack density is relatively low. At high crack density close to saturation, the cracks are more equidistant. The reason is the random distribution of transverse failure properties along the transverse direction of the layer. At low crack density, the stress distribution between two existing cracks has a large plateau region and any position there is a site of possible failure. At high crack density, there is a distinct maximum in the stress distribution

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Figure 1. Non-uniform distribution of M cracks in damaged layer shown in its local coordinate system.

and a new crack most likely will be created in the middle between existing cracks.

The discussion in this study is focused on the possible inaccuracy introduced in laminate stiffness prediction using assumption of uniform spacing between cracks in a layer. Numerical results presented here are for two cases: (a) when the system of cracks is 'non-interactive' in average (low crack density), but some cracks are close to each other and interact and (b) the crack density is high and cracks interact 'in average'.

There are only a few studies where the effect of non-uniformity is addressed, see for example McCartney and Schoeppner,⁷ Silberschmidt⁸ and Vinogradov and Hashin.⁹ In McCartney and Schoeppner,⁷ hypothesis was introduced that for a non-uniformly cracked laminate, the deformation field in the 'element' between two neighbouring ply cracks separated by a distance l_k is identical to that in a uniformly cracked laminate where the crack spacing is l_k . Then, for example, the axial strain of the whole representative volume element (RVE) can be calculated by the 'rule of mixtures' of average strains of 'elements' leading to simple expressions for RVE axial modulus. The high accuracy of this approach was demonstrated in McCartney and Schoeppner⁷ comparing results with another high-accuracy, semi-analytical methodology applied to the RVE. This assumption is reexamined in this article analysing crack-opening displacements (CODs) of both crack faces and showing that the average stress in the 'element' on one side, where the distance to the next crack is smaller, is overestimated by this assumption, whereas on the other side it is underestimated. In the study of Silberschmidt⁸ and Vinogradov and Hashin,⁹ the non-uniform damage evolution is analysed in a probabilistic way, not discussing the effect of non-uniform distribution on stiffness.

The reduced average transverse stress and in-plane shear stress in the damaged layer are responsible for laminate stiffness changes. The average stress change between two cracks is proportional to the COD and crack-sliding displacement (CSD) normalized with far-field stress.^{10,11} The far-field stress at the given load is calculated using classical laminate theory

(CLT). Therefore, the damaged laminate stiffness can also be expressed in terms of density of cracks and two parameters: average COD and CSD as done in the GLOB-LOC model.^{3,12} These two rather robust parameters depend on the normalized distance to neighbouring cracks. Therefore, for non-uniform crack distribution, they are different for each individual crack. The values of COD and CSD in the commonly assumed uniform crack distribution case correspond to average spacing between cracks and are different than the calculated average over CODs and CSDs of all individual cracks.

In this article, we first generalize the previously developed expressions for stiffness reduction in symmetric laminates (GLOB-LOC model³) for non-uniform spacing case. Then, parametric analysis of the effect of geometrical non-uniformity in terms of COD and the laminate axial modulus will be performed for particular cases of $[90m/0n]_s$ and $[0n/90m]_s$ cross-ply laminates with cracks in 90-layers. Cases when sliding displacement CSD affects the stiffness are included in the stiffness expressions in 'Stiffness model' section, but they are not numerically analysed in this article. Extreme layer thickness ratios and different material anisotropy levels comparing carbon fibre (CF) and glass fibre (GF) composites will be discussed. To simplify stiffness calculations for an arbitrary non-uniform distribution, routine allowing determination of CODs for any crack as a sum of solutions for two periodic systems of cracks will be formulated and its accuracy discussed (one solution is for periodic system with spacing as on the '+' side of the crack and another one for a periodic system with spacing as on the '-' side of the crack, see Figure 1).

Material model of damaged symmetric laminates with intralaminar cracks

Distances between cracks

We consider RVE of a layer with M cracks as shown in Figure 1. The RVE length is L , the

average distance between cracks (spacing) is l_{av} , the crack density is ρ

$$L = \sum_{m=0}^{M-1} l_m \quad l_{av} = \frac{L}{M} \quad \rho = \frac{1}{l_{av}} \quad (1)$$

Stress state between two cracks in a layer, see Figure 1 where the cracked layer is shown in its local coordinates, and also the opening and sliding displacements of crack faces depend on the normalized distance between cracks. Normalization is with respect to the layer thickness t

$$l_{mn} = \frac{l_m}{t} \quad m = 0, 1, \dots, M-1, \quad l_{avn} = \frac{l_{av}}{t} \quad (2)$$

Index k , used in following sections to identify k th layer in the laminate, is omitted here for simplicity. Crack with index m has two neighbours located at different distances l_{m-1} and l_m from this crack. Using notation u_{2an}, u_{1an} for the average normalized COD and CSD defined by equations (25) to (27), we can write for the m th crack

$$u_{2an}^m = u_{2an}^m(l_{(m-1)n}, l_{mn}), \quad u_{1an}^m = u_{1an}^m(l_{(m-1)n}, l_{mn}) \quad (3)$$

If $l_m > l_{m-1}$, the displacements on the ‘-’ side will be larger than on the ‘+’ side.

If the part of the layer shown in Figure 1 is smaller than the RVE, the methodology of this article can still be applied, but the unknown displacements of the outmost to the left ($m=1$) and the outmost to the right ($m=M$) positioned cracks are affecting the calculated homogenized stiffness. The uncertainty is because COD and CSD of these two cracks depend on the distance to the next cracks, not shown in Figure 1, or, in other words, on boundary conditions. The uncertainty is avoided if the shown distribution is considered as ‘repeating super-element’ with M cracks in it. In this case, symmetry conditions can be applied on $x=0$ and $x=L$. To model this periodic structure, we have to assume $l_0 = l_M$.

Stiffness model

The upper part of symmetric N -layer laminate with intralaminar cracks is shown in Figure 2. The k th layer of the laminate has thickness t_k , fibre orientation angle with respect to the global x -axis θ_k and stiffness matrix $[Q]$ in the material symmetry axes, calculated from elastic constants $E_1, E_2, G_{12}, \nu_{12}$. The total thickness of the laminate, $h = \sum_{k=1}^N t_k$. The crack density in a layer ρ_k is calculated using equation (1) where the average distance between cracks l_{av}^k is measured

Figure 2. Representative volume element of the damaged laminate with intralaminar cracks in layers. RVE: representative volume element.

transverse to the fibre direction in the k th layer. Dimensionless crack density ρ_{kn} in the layer is introduced as

$$\rho_{kn} = t_k \rho_k \quad (4)$$

It is assumed that in the damaged state, the laminate is still symmetric; in other words, the crack density in corresponding symmetrically placed layers is the same.

The stiffness matrixes of the damaged, $[Q]^{LAM}$, and the undamaged laminates, $[Q]_0^{LAM}$, are defined by the stress-strain relationships

$$\begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \sigma_{xy0} \end{Bmatrix}^{LAM} = [Q]^{LAM} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{LAM}, \quad \begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \sigma_{xy0} \end{Bmatrix}^{LAM} = [Q]_0^{LAM} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}^{LAM} \quad (5)$$

The compliance matrix of the undamaged laminate is $[S]_0^{LAM} = ([Q]_0^{LAM})^{-1}$.

The expressions below for thermo-elastic constants of the damaged laminate with non-uniform crack distribution are derived in Appendixes 1 and 2

$$[Q]^{LAM} = \left([I] + \sum_{k=1}^N \rho_{kn} \frac{t_k}{h} [K]_k [S]_0^{LAM} \right)^{-1} [Q]_0^{LAM} \quad (6)$$

$$[S]^{LAM} = [S]_0^{LAM} \left([I] + \sum_{k=1}^N \rho_{kn} \frac{t_k}{h} [K]_k [S]_0^{LAM} \right) \quad (7)$$

$$\begin{aligned} \{\alpha\}^{LAM} &= \left([I] + \sum_{k=1}^N \frac{t_k}{h} \rho_{kn} [S]_0^{LAM} [K]_k \right) \{\alpha\}_0^{LAM} \\ &\quad - \sum_{k=1}^N \frac{t_k}{h} \rho_{kn} [S]_0^{LAM} [K]_k \{\bar{\alpha}\}_k \end{aligned} \quad (8)$$

Figure 3. ‘Super-element’ models for crack-opening displacement studies with non-uniformly cracked 90-layers: (a) cracks in inside layer and (b) cracks in surface layer.
COD: crack-opening displacement.

In equations (6), (7), and (8), the matrix function $[K]_k$ for a layer with index k is defined as

$$[K]_k = \frac{1}{E_2} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k \quad (9)$$

The involved matrices $[T]_k$ and $[\bar{Q}]_k$ are defined according to CLT, upper index T denotes transposed matrix and bar over stiffness matrix indicates that it is written in global coordinates. The influence of cracks in k th layer is represented by matrix $[U]_k$.

$$[U]_k = 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & u_{2an}^k & 0 \\ 0 & 0 & \frac{E_2}{G_{12}} u_{1an}^k \end{bmatrix} \quad (10)$$

Elements of this matrix u_{2an}, u_{1an} are defined in Appendix 2. They are calculated, see equation (30), using normalized and averaged crack face opening (COD) and sliding displacements (CSD) of all cracks as affected by varying spacing between them

$$\begin{aligned} u_{1an} &= \frac{1}{M} \sum_{m=1}^M u_{1an}^m(l_{(m-1)n}, l_{mn}), \\ u_{2an} &= \frac{1}{M} \sum_{m=1}^M u_{2an}^m(l_{(m-1)n}, l_{mn}) \end{aligned} \quad (11)$$

Index for layer k is omitted in equation (11). Certainly, since u_{2an}, u_{1an} in k th layer depends also on neighbouring layer properties, they are different in different layers. The methodology used in appendices is exactly the same as in Lundmark and Varna.³ The main difference is in Appendix 2 where the two crack faces of any crack may have different displacements due to non-uniformity. Appendix 1, which in a compact form, contains the same information as given in Lundmark and Varna³ is included to ensure consistency of explanation.

Elastic modulus of balanced laminates with cracks in 90-layer

In case of balanced laminates with damage in 90-layers only, expressions for $[K]_k$ and $[S]^{LAM}$ have been obtained by calculating the matrix products in equations (6) to (9) analytically. For example, the obtained expression for laminate normalized axial modulus is

$$\frac{E_x}{E_x^0} = \frac{1}{1 + 2\rho_{90n} \frac{l_{90}}{h} u_{2an}^{90} c}, \quad c = \frac{E_2}{E_x^0} \left(\frac{1 - \nu_{12} \nu_{xy}^0}{1 - \nu_{12} \nu_{21}} \right)^2 \quad (12)$$

Index 90 is used to indicate 90-layer. The quantities with lower index x, y are laminate constants, quantities with additional upper index 0 are undamaged laminate constants and equation (11) has to be used to calculate u_{2an}^{90} . In the case of uniform crack distribution, all CODs in equation (11) are equal and equation (12) is just a different form of equation (31) in Lundmark and Varna,³ leading to numerically identical results. In the following parametric analysis, we consider COD-related properties only and validation is based on axial modulus. Therefore, shear modulus expression of the damaged laminate which is related to CSD only is not presented here.

Results and discussion

Formulation of calculation examples

The effect of the non-uniform crack distribution on u_{2an}^{90} was analysed using finite element method (FEM) for damaged $[0_n/90_8]_S$ and $[90_8/0_n]_S$ laminates ($n = 1, 8$) at fixed dimensionless crack density ρ_{90n} , see Figure 3 where the repeating ‘super-element’ with two cracks is shown. Parameter K is introduced as the ratio

$$K = \frac{l_0}{l_{av}} \quad (13)$$

Table 1. Material properties used in simulations

Material	E_1 (GPa)	E_2 (GPa)	ν_{12}	ν_{23}	G_{12} (GPa)	G_{23} (GPa)
GF/EP	45	15	0.3	0.4	5	5.36
CF/EP	150	10	0.3	0.4	5	3.57

GF: glass fibre; EP: epoxy; and CF: carbon fibre.

to characterize the non-uniformity of the spatial distribution. This parameter has value 1 for uniform crack distribution. The average crack density, l_{av} , was kept constant (two cracks over fixed distance $L = l_0 + l_1$ give $l_{av} = \frac{l_0 + l_1}{2}$) with average normalized spacing calculated according to equation (1) $l_{avn} = \frac{1}{2}(l_{0n} + l_{1n})$.

To cover large variation in elastic constants both CF/epoxy (EP) and GF/EP composites with constants given in Table 1 were analysed. All results are presented in terms of dimensionless crack spacing and crack density. Results depend on layer thickness ratio, not on absolute value of ply thickness.

For laminates with cracks in surface layers, staggered crack system, where the crack in the bottom layer is located in the middle between cracks in the top layer, could be analysed instead of symmetric damage state shown in Figure 3. This case analysed in Nairn and Hu¹³ is relevant when the failure analysis is deterministic and the small variation in stress state points on the locus of the next failure (always exactly in the middle between the two existing cracks). However, the strength (or the fracture toughness) is not one single value, but follows certain statistical distribution. The variation of failure properties along the 90-layer transverse direction is often much larger than the stress perturbation in the bottom layer of the laminate due to crack in the top layer. Therefore, the assumption of staggered cracks is as far from reality as the assumed symmetry of the damage with respect to the midplane used in this article. Starting with symmetric damage state in the stiffness analysis, we are trying to create a simple reference case. The interaction effects between systems of cracks in different layers have to be separately analysed. Otherwise, the number of parameters changing is too large to draw conclusions.

In all FE calculations, the commercial code ABAQUS was used. In order to model the left half of the 'super-element' (Figure 3), a 3D model was created. Three-dimensional (3D) continuum elements (C3D8) 8-node linear brick were used in order to mesh volumes. The same fine mesh with total number of elements 86,400 was used in each FE model. The (x, z) plane consisted of 21,600 elements, with refined mesh near the crack surfaces. In the ply with cracks, the number of elements in the thickness direction was 120. The number of elements in y-direction was four which, as

described below, is more than sufficient for the used edge conditions. The problem was solved by applying to the right boundary $x = 0$ of the model a given constant displacement in x-direction corresponding to 1% average strain and keeping at the left boundary $u_x = 0$. The top surface was free of tractions. On the front edge ($y = 0$) and the edge $y = w$, coupling conditions were applied for normal displacements ($u_y = \text{unknown constant}$). In this way, edge effects are avoided and the solution does not depend on y-coordinate. It corresponds to solution for an infinite structure in the width direction. Obviously, these conditions lead to generalized plane strain case and corresponding finite elements could be used obtaining the same results. The displacements in direction x for the nodes at the crack surface were used to calculate the average value of the crack face displacement COD.

All plies are considered to be transversally isotropic with $E_2 = E_3$, $G_{12} = G_{13}$ and $\nu_{12} = \nu_{13}$.

In calculations, two values of the average normalized spacing were used: (a) $l_{avn} = 10$ corresponding to $\rho_{90n} = 0.1$ where the interaction between uniformly spaced cracks would be negligible and (b) $l_{avn} = 2$ corresponding to interactive crack region with crack density $\rho_{90n} = 0.5$.

Studying the effect of non-uniform distribution, the normalized spacing l_{0n} , see Figure 3, was used as a parameter which was lower or equal to the average spacing. In case (a) $l_{0n} \in [0.5; 10]$ and in case (b) $l_{0n} \in [0.5; 2]$. It is worth to remind here that at very high crack density (in the so-called crack saturation region), the normalized average spacing may be close to 1 (the distance between cracks is equal to the crack size (layer thickness)). Straight intralaminar cracks are almost never observed closer to each other than half of the cracked layer thickness.

COD parametric analysis at low crack density

In this section, results for average normalized spacing $l_{avn} = 10$ ($\rho_{90n} = 0.1$) are presented.

Internal cracks. For internal cracks, the profiles of normalized crack face displacements ($u_{2n}^-(z_n)$, $u_{2n}^+(z_n)$) defined in Appendix 2) along the thickness coordinate $z_n = z \frac{2}{t_{90}} + 1$ are shown in Figures 4 and 5. The '+' face of the crack has smaller displacements than the '-' face, and the difference is larger when the l_{0n} is smaller than 1 (the neighbouring crack to the left is very close). The neighbour to the '-' face is at larger distance than the average spacing and therefore the displacement profile is almost unaffected. For the same geometry, the CODs in CF composites are always significantly smaller.

Figure 4. Crack-opening displacement profiles of cracks in $[0/90_8]_s$ laminate with normalized crack density $\rho_{90n} = 0.1$. COD: crack-opening displacement.

Figure 5. Crack-opening displacement profiles of cracks in $[0/90]_s$ laminate with normalized crack density $\rho_{90n} = 0.1$. COD: crack-opening displacement.

Figure 6. Effect of non-uniform spacing on crack-opening displacement of internal cracks in cross-ply laminates with normalized crack density $\rho_{90n} = 0.1$. COD: crack-opening displacement.

The displacements of both crack faces are significantly smaller when the relative thickness of the neighbouring layer is higher, Figure 5. This effect is more pronounced for GF composite where the 0-layer versus 90-layer modulus ratio is not very large.

Using crack face displacements, the average normalized CODs u_{2an}^{90} are calculated by numerical integration using the expressions in Appendix 2. The obtained dependence on the non-uniformity parameter K is shown in Figure 6. The average normalized COD is larger if the spacing is uniform. However, the effect is negligible for $K > 0.2$ ($l_0 > 2t_{90}$).

Surface cracks. For surface cracks in $[90_8/0]_s$ and $[90/0]_s$ laminates, the profiles of normalized crack face displacements, $u_{2n}^-(z_n)$, $u_{2n}^+(z_n)$ along the thickness coordinate $z_n = \frac{z-t_0/2}{t_{90}}$ (t_0 is 0-layer thickness) are shown in Figures 7 and 8. The trends are the same as for internal cracks, but the face displacements are larger, and the shape of the profile, especially on the non-interactive side, is different, not becoming vertical at outer surface, $z_n = 1$. Since the outer surface is not a symmetry surface, this result was expected.

The average normalized CODs, u_{2an}^{90} , are calculated as described in ‘Internal cracks’ section. The obtained

Figure 7. Crack-opening displacement profiles of surface cracks in $[90_8/0]_s$ laminate with normalized crack density $\rho_{90n} = 0.1$. COD: crack-opening displacement.

Figure 8. Crack-opening displacement profiles of surface cracks in $[90/0]_s$ laminate with normalized crack density $\rho_{90n} = 0.1$. COD: crack-opening displacement.

dependence on the non-uniformity parameter K is shown in Figure 9. The average normalized COD is smaller if the spacing is non-uniform. According to equation (12), it will lead to smaller axial modulus reduction. For $[90_8/0]_s$ laminate, this effect becomes negligible for $K > 0.3$ ($l_0 > 3t_{90}$), whereas for $[90/0]_s$ laminate, the transition values are slightly larger ($K > 0.5$, $l_0 > 5t_{90}$).

Approximate COD determination from periodic solutions

The average normalized COD u_{2an} of a crack in a layer with non-uniform crack distribution can be found considering separately the average normalized COD of the ‘-’ face of the crack and ‘+’ of the crack in Figure 1

$$u_{2an} = \frac{1}{2}(u_{2an}^+ + u_{2an}^-) \quad (14)$$

In this section, the following hypothesis will be analysed

The COD of “-” face depends on the distance to the closest neighbouring crack on the right only and can be calculated considering the region between these two cracks as a periodic element. The COD of the “+”

face is obtained in a similar manner, considering the region on the left as periodic element

This ‘double-periodic’ approach with considering two periodic solutions states that

$$u_{2an} \approx u_{2an}^p, u_{2an}^p = \frac{1}{2}(u_{2an}^{p-} + u_{2an}^{p+}) \quad (15)$$

The two values, u_{2an}^{p+} and u_{2an}^{p-} , are solutions of the two periodic models.

This hypothesis is equivalent to saying that in Figure 3 symmetry conditions on the plane $x = \pm l_1/2$ can be applied. This would mean that even in the deformed state the line $x = \pm l_1/2$ in the 0-layer remains straight. Unfortunately, there is no symmetry in Figure 3 and this line will be deformed. The accuracy of the symmetry condition used in the ‘double-periodic’ approach can be estimated only numerically. Hypothesis that the deviation can be neglected was used by Joffe et al.¹¹ in calculating the work to close the crack for fracture mechanics based damage growth analysis.

If the ‘double-periodic’ approach is accurate enough, the u_{2an} for any crack location with respect to other cracks could be calculated from a master curve for uniform crack distribution. This curve, which is the

Figure 9. Effect of non-uniform spacing on crack-opening displacement of surface cracks in cross-ply laminates with normalized crack density $\rho_{90n} = 0.1$.

COD: crack-opening displacement.

Figure 10. Calculated crack-opening displacement profiles: (a) of internal cracks in $[0/90]_s$ laminate (b) of surface cracks in $[90/0]_s$ laminate with normalized crack density $\rho_{90n} = 0.1$.

COD: crack-opening displacement.

expression of u_{2an} as a function of crack spacing in layer with uniformly distributed cracks, would be used twice to read the u_{2an}^+ and u_{2an}^- values of the left and the right face of the cracks.

In order to check the accuracy and validity of the ‘double-periodic’ assumption, the u_{2an} for each value of non-uniformity parameter was calculated in two different ways: (a) directly applying FEM to the non-uniform geometry and (b) applying FEM two times and using equation (15), first, for periodic distribution with spacing as on the left from the crack and, second, for periodic distribution with spacing as on the right of the crack.

From Figure 10, where displacement profiles according to (a) and (b) are presented, we conclude that the trends in the double-periodic approach are described correctly, but the values of face displacements are not

accurate. On the left face, where the interaction is strongest, the u_{2n}^+ is too small; but, on the right face, where the next crack is further away, u_{2n}^- is too large. It seems that this result questions the validity of used hypothesis at low crack density.

However, for stiffness predictions, the average of the COD of both faces, u_{2an}^p , given by equation (15) is requested and not the value for each face separately. The values of u_{2an} and u_{2an}^p can be compared using results presented in Tables 2 and 3 for all lay-ups, materials and non-uniformity parameter values. A very good agreement between values exists for all cases which validates the use of the ‘double-periodic’ hypothesis.

The validity of enforcing symmetry in positions like $x = \pm l_1/2$ in Figure 3 is the basic assumption also in McCartney and Schoepner.⁷ In this article, we have

Table 2. Average normalized COD of internal cracks from FEM and from ‘double-periodic’ approach at normalized crack density $\rho_{90n} = 0.1$

K	[0/90 ₈] _s				[0/90] _s			
	GF/EP		CF/EP		GF/EP		CF/EP	
	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p
0.50	1.1027	1.1030	0.6927	0.6927	0.6941	0.6943	0.5721	0.5722
0.30	1.1027	1.1030	0.6928	0.6931	0.6928	0.6935	0.5712	0.5716
0.20	1.1039	1.1045	0.6905	0.6907	0.6771	0.6786	0.5600	0.5607
0.10	1.0510	1.0537	0.6189	0.6230	0.5860	0.5910	0.4857	0.4894
0.05	0.8588	0.8811	0.4927	0.5091	0.4841	0.4890	0.3977	0.4035

COD: crack-opening displacement; FEM: finite element method; GF: glass fibre; EP: epoxy; and CF: carbon fibre.

Table 3. Average normalized COD of surface cracks from FEM and from ‘double-periodic’ approach at normalized crack density $\rho_{90n} = 0.1$

K	[90 ₈ /0] _s				[90/0] _s			
	GF/EP		CF/EP		GF/EP		CF/EP	
	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p	u_{2an}	u_{2an}^p
0.50	2.0314	2.0378	1.3912	1.3930	1.5649	1.5661	1.3327	1.3318
0.30	2.0164	2.0169	1.3357	1.3364	1.4432	1.4471	1.2252	1.2256
0.20	1.9062	1.9221	1.2230	1.2268	1.2856	1.2958	1.0910	1.0964
0.10	1.6367	1.6421	1.0146	1.0184	1.0650	1.0798	0.9035	0.9141
0.05	1.3707	1.3915	0.8675	0.8744	0.9368	0.9477	0.7949	0.8040

COD: crack-opening displacement; FEM: finite element method; GF: glass fibre; EP: epoxy; and CF: carbon fibre.

shown, see Figure 10, that it can lead to noticeable inaccuracy in the displacement of each crack face. However, the average COD of the crack presented in Tables 2 and 3 is very accurate. Since the COD values of the ‘double-periodic’ approach coincide with exact values when the spacing is uniform, the accuracy increases with higher values of the non-uniformity parameter. To understand the real boundary conditions at $x = \pm l_1/2$, the intralaminar shear stress, σ_{xz} , distribution in this cross-section along z -coordinate was calculated using FEM for different values of non-uniformity parameter K . For example, for GF/EP [0/90]_s laminate with $\rho_{90n} = 0.1$ and 1% applied strain, the σ_{xz} in 0-layer at a distance from the crack tip equal to 10% of the 0-layer thickness changes from 0 to 3.8 to 33 MPa when the non-uniformity parameter changes from 1 to 0.5 to 0.05 showing that the boundary conditions are noticeably violated only for high non-uniformity (low value of parameter K). The effect is stronger for cracks in surface layers.

Rephrasing the above in terms of average stresses and strains between two cracks calculated using similar assumption in McCartney and Schoeppner,⁷ the accuracy in each element is reduced at high non-uniformity,

but the average values and the total values calculated in this way have, as shown in McCartney and Schoeppner,⁷ very high accuracy.

Elastic modulus prediction and validation with FEM

The effect of the non-uniform crack distribution on axial modulus of cross-ply laminates is shown in Figure 11 for GF/EP laminates and in Figure 12 for CF/EP laminates. All results are for the same normalized crack density $\rho_{90n} = 0.1$. The normalized axial modulus of the laminate is calculated in three different ways as follows.

- Calculating the average applied stress using FEM and using definition of E_x .
- Applying equation (12) and using for u_{2an}^{90} values of u_{2an} obtained from FEM and presented in Tables 2 and 3.
- Applying (12) and using for u_{2an}^{90} values of u_{2an}^p obtained from ‘double-periodic’ approach presented in Tables 2 and 3.

Figure 11. Effect of non-uniform crack distribution on axial modulus of GF/EP cross-ply laminates with normalized crack density $\rho_{90n} = 0.1$.

Figure 12. Effect of non-uniform crack distribution on axial modulus of CF/EP cross-ply laminates with normalized crack density $\rho_{90n} = 0.1$.

The elastic modulus of the RVE with two non-equidistant cracks directly calculated from force (FEM) and the applied average strain (notation FEM in Figure 11) is equal to that calculated using equation (12) with u_{2an} input from the same FEM solution. Since equation (12) is an exact analytical expression, this result was expected and some numerical discrepancy is possible only if the elastic modulus of the RVE is calculated with a different mesh than the u_{2an} . Since these two

calculations always lead to coinciding results, only one of them (called FEM) is shown in the following figures.

On the other side, FEM values practically coincide with the ones where the 'double-periodic' approach is used, proving the accuracy and potential of this approach for simulation of systems with multiple non-uniformly spaced cracks.

For the used crack density and all investigated materials and lay-ups, the axial modulus reduction is

Table 4. Axial modulus of cross-ply laminates with non-uniform and uniform ($K = 1$) crack distributions

Material	Lay-up	$\rho_{90n} = 0.1$			$\rho_{90n} = 0.5$		
		$K = 0.05$		Ratio	$K = 0.25$		Ratio
		$l_{0n} = 0.5$	$K = 1 \quad l_{0n} = 10$		$l_{0n} = 0.5$	$K = 1 \quad l_{0n} = 2$	
GF/EP	$[0/90_8]_s$	0.8894	0.8624	1.0312	0.6165	0.5556	1.1098
GF/EP	$[90_8/0]_s$	0.8345	0.7725	1.0804	0.5016	0.4337	1.1565
GF/EP	$[0/90]_s$	0.9768	0.9673	1.0098	0.8950	0.8619	1.0384
GF/EP	$[90/0]_s$	0.9566	0.9285	1.0303	0.8305	0.8057	1.0308
CF/EP	$[0/90_8]_s$	0.9671	0.9541	1.0136	0.8537	0.8070	1.0579
CF/EP	$[90_8/0]_s$	0.9428	0.9116	1.0342	0.7756	0.7321	1.0594
CF/EP	$[0/90]_s$	0.9950	0.9928	1.0022	0.9759	0.9672	1.0090
CF/EP	$[90/0]_s$	0.9902	0.9835	1.0068	0.9580	0.9510	1.0074

GF: glass fibre; EP: epoxy; and CF: carbon fibre.

Figure 13. Effect of non-uniform crack distribution on axial modulus of GF/EP cross-ply laminates with normalized crack density $\rho_{90n} = 0.5$.

the highest if cracks have uniform distribution. For the reasons described in ‘Introduction’ section, the experimental crack distribution at crack density $\rho_{90n} = 0.1$ is expected to be rather non-uniform and the axial modulus is higher than predicted by models based on periodic crack distribution. The normalized axial modulus value at the highest considered non-uniformity ($K=0.05$) and at uniform distribution ($K=1$) are given in Table 4.

The axial modulus of laminates with relatively thick damaged layers is more sensitive to non-uniform crack distribution: the highest value is 1.077 for GF/EP composite with lay-up $[90_8/0]_s$.

The non-uniform distribution of internal cracks does not affect laminate modulus if the non-uniformity

parameter $K > 0.2$. For laminates with cracks in surface layers, the corresponding value is between 0.3 and 0.5. Note that K values given here are the same as the values when the non-uniformity stops to affect the average normalized COD.

Similar calculations, as described above, were performed for higher crack density $\rho_{90n} = 0.5$. Results are presented in Table 4 and in Figures 13 and 14. Due to limitations for minimum possible spacing, the range of the considered non-uniformity in calculations is narrower. Nevertheless, the effect of non-uniform distribution is even larger than at low crack density. In contrast to low crack density case, there is no plateau region in Figures 13 and 14. The ‘double-periodic’ approach at high crack density is still highly accurate

Figure 14. Effect of non-uniform crack distribution on axial modulus of CF/EP cross-ply laminates with normalized crack density $\rho_{90n} = 0.5$.

Table 5. Thermo-elastic properties of damaged cross-ply laminates according to McCartney and Schoeppner⁷ and this article with normalized crack density $\rho_{90n} = 1.016$

Method	FEM	LRAM ⁷	Double-periodic	Error (%)	Analytical ⁷	Error ⁷ (%)
E_x (GPa)	50.9606	50.9602	50.9224	-0.075	50.9111	-0.096
ν_{xy}	0.01746	0.01748	0.01736	-0.620	0.01733	-0.872
$\alpha_x (10^{-6} \cdot 1/^\circ\text{C})$	1.09622	1.10161	1.08213	-1.285	1.07685	-2.248

FEM: Finite element method and LRAM: left rectangular approximation method.

for internal cracks. For surface cracks with high non-uniformity, this approach underestimates the modulus reduction.

From ‘Stiffness model’ section, it follows that all elastic constants of the cross-ply laminate except the shear modulus can be analysed using the calculated u_{2an} . As shown in McCartney and Schoeppner,⁷ the reduction of all properties in damaged laminate is linked and the same accuracy and trends as demonstrated for axial modulus apply for other constants. The shear modulus in our formulation depends on sliding displacements which have to be studied separately.

Calculations were also performed for CF/EP [0/90]₂s laminate with non-uniformly distributed cracks in the 90-layer analysed by McCartney and Schoeppner.⁷ The used unidirectional composite properties are

$$\begin{aligned}
 E_1 &= 144.78 \text{ GPa}, E_2 = 9.58 \text{ GPa}, G_{12} = 4.785 \text{ GPa}, \\
 G_{23} &= 3.090 \text{ GPa}, \nu_{12} = 0.31 \\
 \alpha_1 &= -0.72 \cdot 10^{-6} / ^\circ\text{C}, \alpha_2 = 27.0 \cdot 10^{-6} / ^\circ\text{C}, t_0 = 0.127 \text{ mm}
 \end{aligned}$$

One of the numerical examples in McCartney and Schoeppner⁷ is for RVE between two cracks of length $2L = 1 \text{ mm}$ (notation as in McCartney and Schoeppner⁷). The third crack is embedded in between these two and the distance to the closest crack $2L_1 = 0.1 \text{ mm}$. Using our terminology, the average spacing in the RVE is 0.5 mm and the non-uniformity parameter $K = 0.2$. Elastic modulus of the RVE in McCartney and Schoeppner⁷ was calculated using left rectangular approximation method (LRAM) and subdividing each layer in seven to eight sub-layers. In the analytical model which is based on similar assumptions as our ‘double-periodic’ model, each layer was subdivided into five to eight layers of equal thickness. Their results are compared with our results (FEM for the RVE and the ‘double-periodic’ model) in Table 5. Our FEM results and LRAM results for E_x of the RVE are almost coinciding (our mesh had 120 elements in each layer in the thickness direction). In addition, also the ν_{xy} and the axial thermal expansion coefficient were calculated and for the latter the difference is slightly larger. Analytical expressions for these constants of

the damaged cross-ply laminate one can find in Loukil et al.¹⁴

The results of the analytical method⁷ and the ‘double-periodic’ approach are very similar. In all cases, they give slightly lower values than the values obtained by FEM or LRAM. Since our values are slightly higher, the accuracy is slightly better than for the analytical model.⁷ We believe that the accuracy of results in McCartney and Schoeppner⁷ could be improved by sub-division of layers in more sub-layers. Another observation from McCartney and Schoeppner⁷ is that even in the case when the third crack is exactly in the middle, between the two cracks, the results of LRAM and the analytical model⁷ slightly differ. It can be explained only by different sub-divisions in sub-layers in both methods which has affected the accuracy.

Conclusions

Earlier developed model for elastic properties of damaged symmetric laminates was generalized for the case when the intralaminar crack distribution is non-uniform, and due to interactions, each crack may have different opening (COD) and sliding (CSD) displacements. These displacements and the number of cracks per unit length in the layer are governing the laminate properties reduction. The obtained analytical expressions for elastic constants are exact.

This model was applied to cross-ply laminates with cracks in 90-layers located in the middle or on the surface. The dependence of the damaged cross-ply laminate axial modulus (it depends on COD only) on the non-uniformity parameter in a repeating element containing two cracks was numerically analysed. The non-uniformity parameter is defined as the ratio of the smallest and the average spacing between cracks. COD values needed as an input in the model were calculated using FEM in generalized plane strain formulation and stiffness calculations were performed for GF/EP as well as CF/EP laminates with low and also high crack densities.

The trend is the same for all crack densities and lay-ups: assuming uniform crack distribution, the damaged laminate modulus is underestimated.

An approximate ‘double-periodic’ approach was proposed stating that the COD of a crack with different distances to the closest neighbours can be calculated as the average of two solutions for equidistant cracks. It was numerically shown for cross-ply laminates that in internal layers very accurate COD values for cracks with non-uniform spacing and elastic modulus values can be obtained using this approach. For cracks in surface layers, this approach is accurate only for low crack densities. The applicability of the ‘double-periodic’

approach to sliding displacement of non-uniformly distributed cracks has not been investigated.

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Conflict of interest

None declared.

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Appendix 1

Homogenized stiffness of damaged laminate in global coordinates

Using divergence theorem, it is easy to show that for stress states that satisfy equilibrium equations, the average stress applied to external boundary is equal to volume averaged stress.¹⁵ This statement is correct under assumption that stresses at internal boundaries (cracks) are zero. For laminated composites with applied average stress $\{\sigma_0\}^{LAM}$, this equality can be written as

$$\{\sigma_0\}^{LAM} = \{\bar{\sigma}\}^a = \sum_{k=1}^N \{\bar{\sigma}\}_k^a \frac{t_k}{h} \quad (16)$$

In equation (16), the volume average is calculated expressing the integral over the laminate volume as a sum of volume integrals over N layers. Upper index a is used to indicate volume averages. Using Hook's law and averaging it over a layer, we have for averages the same form as for arbitrary point

$$\begin{Bmatrix} \sigma_x^a \\ \sigma_y^a \\ \sigma_{xy}^a \end{Bmatrix}_k = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x^a \\ \varepsilon_y^a \\ \gamma_{xy}^a \end{Bmatrix}_k \quad (17)$$

Substituting equation (17) in equation (16) and using the relationship between volume averaged strain in a layer and the displacements applied to external and internal boundaries^{15,16}

$$\begin{Bmatrix} \varepsilon_x^a \\ \varepsilon_y^a \\ \gamma_{xy}^a \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{LAM} + \begin{Bmatrix} \beta_x \\ \beta_y \\ 2\beta_{xy} \end{Bmatrix}_k \quad (18)$$

we obtain

$$\begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \sigma_{xy0} \end{Bmatrix}^{LAM} = [Q]_0^{LAM} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{LAM} + \sum_{k=1}^N [\bar{Q}]_k \frac{t_k}{h} \begin{Bmatrix} \beta_x \\ \beta_y \\ 2\beta_{xy} \end{Bmatrix}_k \quad (19)$$

In equation (19), $\{\bar{\beta}\}$ is the Vakulenko–Kachanov¹⁶ tensor written in Voigt notation. In Cartesian coordinates

$$\beta_{ij} = \frac{1}{V} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS \quad (20)$$

Integration in equation (20) involves the total crack surface S_C in the layer; u_i are the displacements of the points on the crack surface, n_i the outer normal to the crack surface and V the volume of the layer. Obviously, equation (20) represents the effect on stiffness of the crack face displacements (opening and sliding). Since β_{ij} and strain are tensors for both of them, we have the same transformation expressions between local and global coordinates

$$\{\bar{\beta}\}_k = [T]_k^T \{\beta\}_k \quad (21)$$

Expression for $\{\beta\}_k$ in local coordinates is given by equation (31) in Appendix 2.

The laminate theory stress $\{\sigma_0\}_k$ in the k th layer in local coordinates can be expressed through the applied laminate stress as follows

$$\begin{aligned} \{\sigma_0\}_k &= [T]_k \begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \sigma_{xy0} \end{Bmatrix}_k = [T]_k [\bar{Q}]_k \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}^{LAM} \\ &= [T]_k [\bar{Q}]_k [S]_0^{LAM} \begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \sigma_{xy0} \end{Bmatrix} \end{aligned} \quad (22)$$

Substituting equation (31) with equation (22) in equation (21) and further in equation (19), we obtain, after arranging the result in the form of equation (5), the form of stiffness matrix of the damaged laminate given by equation (6).

Appendix 2

Incorporation of COD and CSD in Vakulenko–Kachanov tensor in local coordinates

We consider a RVE of a layer with M cracks. Schematic picture of a non-uniform crack distribution with varying spacing between cracks, l_m , $m=0,1,2,\dots,M$ is shown in Figure 1. Index denoting k th layer is omitted to simplify explanation. The cracked layer is considered in its local coordinates with indexes 1, 2 and 3 corresponding to longitudinal, transverse and thickness directions. For transverse cracks, the coordinates of the normal vector to the two faces of crack surface are

$$n_1 = n_3 = 0 \quad n_2 = \pm 1 \quad (23)$$

If the crack density is high, the stress perturbation zones of individual cracks overlap and the crack face displacements depend on the distance between cracks. Using the definition (20) for β_{ij} , we see that the matrix contains only two non-zero elements: β_{12} and β_{22}

$$\begin{aligned}\beta_{22} &= \frac{1}{Lt} \sum_{m=1}^M \int_{-t/2}^{+t/2} [u_2^{m+}(z) - u_2^{m-}(z)] dz \\ \beta_{12} &= \frac{1}{Lt} \sum_{m=1}^M \int_{-t/2}^{+t/2} \frac{1}{2} [u_1^{m+}(z) - u_1^{m-}(z)] dz\end{aligned}\quad (24)$$

In equation (24), t is the cracked layer thickness, $u_1^m(z)$ and $u_2^m(z)$ are the sliding and opening displacements of the m th crack and symbols '+' or '-' denote the particular crack face according to Figure 1.

As in previous papers^{3,12}, for uniform crack distribution, we also introduce here normalized opening and sliding displacements of crack faces (σ_{20} and σ_{120} are CLT stresses)

$$u_{1n} = u_1 \frac{G_{12}}{t\sigma_{120}} \quad u_{2n} = u_2 \frac{E_2}{t\sigma_{20}} \quad (25)$$

Introducing average values of displacements of each crack surface over ply thickness

$$\begin{aligned}u_{1a}^{m+} &= -\frac{1}{t} \int_{-t/2}^{+t/2} u_1^{m+}(z) dz & u_{1a}^{m-} &= \frac{1}{t} \int_{-t/2}^{+t/2} u_1^{m-}(z) dz \\ u_{2a}^{m+} &= -\frac{1}{t} \int_{-t/2}^{+t/2} u_2^{m+}(z) dz & u_{2a}^{m-} &= \frac{1}{t} \int_{-t/2}^{+t/2} u_2^{m-}(z) dz\end{aligned}\quad (26)$$

The average value of average displacements on both surfaces is

$$u_{1a}^m = \frac{1}{2} (u_{1a}^{m+} + u_{1a}^{m-}) \quad u_{2a}^m = \frac{1}{2} (u_{2a}^{m+} + u_{2a}^{m-}) \quad (27)$$

Using equations (26) and (27), the expressions for β_{12} and β_{22} are

$$\begin{aligned}\beta_{12} &= -\frac{1}{L} \sum_{m=1}^M u_{1a}^m(l_{(m-1)n}, l_{mn}) \\ \beta_{22} &= -\frac{2}{L} \sum_{m=1}^M u_{2a}^m(l_{(m-1)n}, l_{mn})\end{aligned}\quad (28)$$

We indicate here that the displacements will be mostly affected by normalized distances to the two closest neighbouring cracks. These expressions can be rewritten in terms of average crack density and average (over all cracks) displacements

$$\beta_{12} = -\rho u_{1a} \quad \beta_{22} = -2\rho u_{2a} \quad (29)$$

$$\begin{aligned}u_{1a} &= \frac{1}{M} \sum_{m=1}^M u_{1a}^m(l_{(m-1)n}, l_{mn}) \\ u_{2a} &= \frac{1}{M} \sum_{m=1}^M u_{2a}^m(l_{(m-1)n}, l_{mn})\end{aligned}\quad (30)$$

Normalization (25) can be applied also to u_{1a}^m and u_{2a}^m using notation u_{1an}^m and u_{2an}^m for the result. Expressions for β_{ij} in equation (29) in the result of normalization are slightly modified. It is easy to check that they can be presented in the following matrix form

$$\begin{aligned}\{\beta\} &= \begin{Bmatrix} 0 \\ \beta_{22} \\ 2\beta_{12} \end{Bmatrix} = -\frac{\rho_n}{E_2} [U] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} \\ [U] &= 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & u_{2an} & 0 \\ 0 & 0 & \frac{E_2}{G_{12}} u_{1an} \end{bmatrix}\end{aligned}\quad (31)$$

In equation (31), ρ_n is the normalized crack density in the layer defined by equation (4).