Linköping Studies in Science and Technology.
Licenciate Thesis No. 1813

# Kähler–Poisson Algebras

**Ahmed Al-Shujary**

LINKÖPING UNIVERSITY

Department of Mathematics
Linköping, 2018

Linköping Studies in Science and Technology. Licenciate Thesis No. 1813

**Kähler–Poisson Algebras**



*Department of Mathematics*
*Linköping University*
*SE-581 83 Linöping, Sweden*

Email: ahmed.al-shujary@liu.se

ISSN 0280-7971
ISBN 978-91-7685-245-3
Printed by LiU-Tryck, Linköping, Sweden, 2018

# Abstract

The focus of this thesis is to introduce the concept of Kähler-Poisson algebras as analogues of algebras of smooth functions on Kähler manifolds. We first give here a review of the geometry of Kähler manifolds and Lie-Rinehart algebras. After that we give the definition and basic properties of Kähler-Poisson algebras. It is then shown that the Kähler type condition has consequences that allow for an identification of geometric objects in the algebra which share several properties with their classical counterparts. Furthermore, we introduce a concept of morphism between Kähler-Poisson algebras and show its consequences. Detailed examples are provided in order to illustrate the novel concepts.

# Acknowledgments

First of all, I would like to thank my supervisor Joakim Arnlind. Thank you for many interesting discussions, thoughtful comments, and your patience and support. I also want to thank my co-supervisor Milagros Izquierdo for ideas, discussions and for useful comments in the write up of this thesis. The department of Mathematics at Linköping University is a very nice place to work at and therefore I would like to thank all of my colleagues, the administration and all the PhD students for making a pleasant working environment. I would like to thank the Ministry of Higher Education and Scientific Research in Iraq for the financial support. Finally, I am grateful to my family for their support. Thank you my dear father for your love, courage and prayers. My late mother I will never forget you.Thank you my wife and my children Fatima, Narjis and Adam for your encouragement.

# Contents

# Notation

The following lists the symbols that are mostly used in this thesis.

| | | |
|---|---|---|
| $\mathcal{A}$ | Algebra | 7 |
| $\mathbb{K}$ | Field (either $\mathbb{R}$ or $\mathbb{C}$) | 7 |
| $\mathfrak{g}$ | Module of (inner derivations) | 7 |
| $\Sigma$ | Manifold | 8 |
| $g$ | Metric | 8 |
| $\mathrm{End}_{\mathbb{K}}(M)$ | Module of endomorphism of the module $M$ | 9 |
| $\{.,.\}$ | Poisson bracket | 11 |
| $\mathfrak{g}^*$ | Dual module of $\mathfrak{g}$ | 14 |
| $\frac{\partial p}{x^i}$,$\partial_{x^i} P$ | Formal derivative of the polynomial $p$ with respect to the variable $x^i$ | 24 |

# 1 – Introduction

## Background

In algebraic formulations of geometry, focus is usually shifted from a set of points to an algebra of functions. Geometric properties can in many cases be translated into algebraic properties of this algebra; an idea which has been taken far in algebraic geometry, but also in other contexts. However, metric aspects of geometry have not been investigated to the same extent. Motivated by the results in [1, 2], where it is shown that one may reformulate the Riemannian geometry of an embedded Kähler manifold $\Sigma$ entirely in terms of the Poisson structure on the algebra of smooth functions of $\Sigma$, we set out to find structures that resemble algebras of smooth functions on Kähler manifolds.

In this thesis, it is shown that any Poisson algebra, fulfilling an "almost Kähler condition", enjoys many properties similar to those of the algebra of smooth functions on an almost Kähler manifold, opening up for a more metric treatment of Poisson algebras. Such algebras will be called Kähler-Poisson algebras, and we show that one may associate a Kähler-Poisson algebra to every algebra in a large class of Poisson algebras. In particular, we prove the existence of a unique Levi-Civita connection on the module generated by the inner derivations, and show that the curvature operator has all the classical symmetries.

Note that the methods of algebraic geometry may be readily extended to Poisson algebras (see e.g. [4]); however, this will not be directly relevant to us as we shall start by focusing on metric aspects. Furthermore, the starting point of our approach is quite similar to that of [8] (although metric aspects were not considered there). More precisely, we will use the language of Lie-Rinehart algebras, which we will extend to a metric setting.

As shown in [1], the Riemannian geometry of embedded almost Kähler manifolds can be reformulated entirely in terms of the Poisson algbera of smooth functions. In particular, one obtains Poisson algebraic expressions for geometric quantities by using the algbera generated by the embedding coordinates. A trivial, but for our purposes crucial, observation is that on a Kähler manifold $(\Sigma, g)$ isometrically embedded in the Riemannian manifold $(M, \bar{g})$, the compatibility between the metric and the symplectic form (and, hence, the Poisson structure) implies that

$$\sum_{i,j,k,l=1}^{m} \{f_1, x^i\}\bar{g}_{ij}\{x^j, x^k\}\bar{g}_{kl}\{x^l, f_2\} = -\{f_1, f_2\}$$

for all $f_1, f_2 \in C^\infty(\Sigma)$, where $x^1, \ldots, x^m$ is a set of smooth functions providing an isometric embedding of $\Sigma$ into $M$. Note that this situation is generic in the sense that any Riemannian manifold can be isometrically embedded in Euclidean space [14]. Surprisingly, it turns out that if one imposes the above relation in an abstract Poisson algebra (i.e. not necessarily the function algebra of a manifold), many classical results in Riemannian geometry may be worked out in a purely algebraic setting.

# Preliminaries

The concept of Kähler manifolds has been widely studied. A Kähler manifold is a manifold with three mutually compatible structures, a complex structure, a Riemannian structure, and a symplectic structure. The concept was first studied by Jan Arnoldus Schouten and David van Dantzig in 1930, and then introduced by Erich Kähler in 1933 [12]. The terminology has been fixed by André Weil.

Let $V$ be a vector space over a ring $R$. A complex structure on $V$ is an endomorphism $J : V \to V$ such that $J^2 = -1$. Such a structure turns $V$ into a complex vector space by defining multiplication with $i$ by $iv := Jv$. Vice versa, multiplication by $i$ in a complex vector space provides a complex structure on the underlying real vector space. Let $M$ be a smooth manifold of real dimension $2m$. We say that a smooth atlas $A$ of $M$ is holomorphic if for any two coordinate charts $z : U \to U' \subset \mathbb{C}^m$ and $\omega : V \to V' \subset \mathbb{C}^m$ in $A$, the coordinate transition map $z \circ \omega^{-1}$ is holomorphic. We say that $M$ is a complex manifold of complex dimension $m$ if $M$ comes equipped with a holomorphic atlas. Any coordinate chart of the corresponding complex structure will be called a holomorphic coordinate chart of $M$. A Riemann surface or complex curve is a complex manifold of complex dimension 1. For example, $\mathbb{C}^n$ is a complex manifold. Let $M$ be a complex manifold with corresponding complex structure $J$. We say that a Riemannian metric $g = \langle ., . \rangle$ is compatible with $J$ if

$$\langle JX, JY \rangle = \langle X, Y \rangle \qquad (1.0.1)$$

for all vector fields $X, Y$ on $M$. A complex manifold together with a compatible Riemannian metric is called a Hermitian manifold. See, e.g. [19] for more details on complex manifolds.

Let $M$ be Hermitian manifold with complex structure $J$ and compatible Riemannian metric $g = \langle ., . \rangle$ as in (1.0.1). The alternating 2-form $\omega(X, Y) := g(JX, Y)$ is called the associated Kähler form. We can retrieve $g$ from $\omega$ by $g(X, Y) = \omega(X, JY)$. Also we note that $\omega$ is an anti-symmetric form. We say that $g$ is a Kähler metric and that $M$ (together with $g$) is a Kähler manifold if  is closed [12].

Any submanifold of a Kähler manifold is a Kähler manifold. In particular, all non-singular projective complex algebraic varieties are Kähler manifolds, and, moreover, their Kähler metric is induced by the Fubini-Study metric on the complex projective space. A symplectic form on a manifold $M$ is a closed 2-form on $M$ which is nondegenerate at every point of $M$. A symplectic manifold is a pair $(M, \omega)$ where $M$ is a manifold and $\omega$ is a symplectic form on $M$. Kähler manifolds and their properties have been extensively studied, see e.g. [5],[13] and [18].

We introduce a few results from [1], in order to motivate and understand the introduction of a Kähler type condition for Poisson algebras, and explains how the theory of Lie-Rinehart algebras can be extended to include metric aspects. We define Kähler-Poisson algebras and investigate their basic properties as well as showing that one may associate a Kähler-Poisson algebra to an arbitrary Poisson algebra in a large class of algebras. We derive a compact formula for the Levi-Civita connection as well as introducing curvature. We present a number of examples together with detailed computations.

# Outline

- In Chapter 2 we review the concept of Lie-Rinehart algebras.
- In Chapter 3 we introduce the main definition of our work which is called Kähler-Poisson algebras with basic properties and construction of Kähler-Poisson algebras. Also we will define Levi-Civita connection with curvature and give examples.
- In Chapter 4 we define morphisms of Kähler-Poisson algebras with examples and future research.

# 2 – Lie-Rinehart Algebras

In this chapter we introduce the concept of a metric Lie-Rinehart algebra and recall a few results on the Levi-Civita connection. We assume that a (commutative) algebra $\mathcal{A}$ is given (corresponding to the algebra of functions on a Kähler manifold), together with an $\mathcal{A}$-module $\mathfrak{g}$ (corresponding to the module of vector fields on a Kähler manifold) which is also a Lie algebra and acts on $\mathcal{A}$ as derivations. For more details and proofs, we refer to [3, 8].

A Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g})$ consists of a commutative algebra $\mathcal{A}$ and a Lie algebra $\mathfrak{g}$ with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold.

Let $R$ be a commutative ring. Recall that a *Lie algebra* $(\mathfrak{g}, [.,.])$ over $R$ consists of an $R$-module $\mathfrak{g}$ and a pairing $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a Lie bracket, satisfying the relations of antisymmetry and Jacobi identity. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ , a morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras over $R$ is a morphisms of $R$-modules which is compatible with the Lie brackets. Let $\mathcal{A}$ be an algebra over $R$.
Recall that a *derivation* of $\mathcal{A}$ (over $R$) is a morphism $\delta : \mathcal{A} \to \mathcal{A}$ of algebras so that $\delta(ab) = (\delta(a))b + a\delta(b)$. It is well known that the module $\mathrm{Der}(\mathcal{A})$ of derivations of $\mathcal{A}$, with bracket given by $[\alpha, \beta](a) = \alpha(\beta(a)) - \beta(\alpha(a))$, where $\alpha, \beta \in \mathrm{Der}\,\mathcal{A}$, $a \in \mathcal{A}$, is again a Lie algebra. If $\mathfrak{g}$ is a Lie algebra over $R$, an action of $\mathfrak{g}$ on $\mathcal{A}$ is a morphism $\omega : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$. Given two algebras $\mathcal{A}$ and $\mathcal{A}'$, over $R$, and two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over $R$, where $\mathfrak{g}$ and $\mathfrak{g}'$ act on $\mathcal{A}$ and $\mathcal{A}'$, respectively, a morphism of actions $(\phi, \psi) : (\mathcal{A}, \mathfrak{g}) \to (\mathcal{A}', \mathfrak{g}')$ consists of a morphism $\phi : \mathcal{A} \to \mathcal{A}'$ and a morphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras over $R$ so that, for every $a \in \mathcal{A}$, $\alpha \in \mathfrak{g}$, $\phi(\alpha(a)) = (\psi(\alpha))(\phi(a))$. More details can be found in [7, 16, 17].

We start by introducing the concept of Lie-Rinehart algebras with results which will applied to "Kähler-Poisson algebras" in Chapter 3.

## 2.1 Lie-Rinehart algebras

**Definition 2.1.1. (Lie-Rinehart algebra)**. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}$ be a commutative $\mathbb{K}$-algebra and let $(\mathfrak{g}, [.,.])$ be an $\mathcal{A}$-module which is also a Lie algebra over $\mathbb{K}$. Given a map $\omega : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$, The pair $(\mathcal{A}, \mathfrak{g})$ is called a Lie-Rinehart algebra if

$$\omega(a\alpha)(b) = a(\omega(\alpha)(b)) \tag{2.1.1}$$
$$[\alpha, a\beta] = a[\alpha, \beta] + (\omega(\alpha)(a))\beta, \tag{2.1.2}$$

for $\alpha, \beta \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. (In most cases, we will leave out $\omega$ and write $\alpha(a)$ instead of $\omega(\alpha)(a)$.)

Let us point out some immediate examples of Lie-Rinehart algebras.

**Example 2.1.2.** *Let $\mathcal{A}$ be a commutative algebra and let $\mathfrak{g} = \mathrm{Der}(\mathcal{A})$ be the $\mathcal{A}$-module of derivations of $\mathcal{A}$. As we mentioned before $\mathrm{Der}(\mathcal{A})$ is a Lie algebra with Lie-bracket, defined as*

$$[\alpha, \beta](a) = \alpha(\beta(a)) - \beta(\alpha(a)).$$

*The pair $(\mathcal{A}, \mathrm{Der}(\mathcal{A}))$ is a Lie-Rinehart algebra where $\mathrm{Der}(\mathcal{A})$ acts as derivations.*

**Example 2.1.3.** *Let $\mathcal{A} = C^\infty(\Sigma)$ be the algebra (over $\mathbb{R}$) of smooth functions on a manifold $\Sigma$, and let $\mathfrak{g} = \chi(\mathcal{A})$ be the $\mathcal{A}$-module of vector fields on $\Sigma$. With respect to the standard action of a vector field as a derivation of $C^\infty(\Sigma)$, the pair $(C^\infty(\Sigma), \chi(\mathcal{A}))$ is a Lie-Rinehart algebra.*

In differential geometry, consider a smooth manifold $\Sigma$ of dimension $n$; for instance a surface. At each point $p \in M$ there is a vector space $T_p\Sigma$, called the tangent space, consisting of all tangent vectors to the manifold at the point $p$. A metric at $p$ is a function $g_p(X_p, Y_p)$ which takes as inputs a pair of tangent vectors $X_p$ and $Y_p$ at $p$, and produces as output a real number (scalar). Globally, a metric maps two vector fields to a function.

## 2.2 Metric Lie-Rinehart algebras

In this section we introduce metrics on Lie-Rinehart algebras, as well as the corresponding Levi-Civita connection.

**Definition 2.2.1.** Let $(\mathcal{A}, \mathfrak{g})$ be a Lie-Rinehart algebra and let $M$ be an $\mathcal{A}$-module. An $\mathcal{A}$-bilinear form $g : M \times M \to \mathcal{A}$ is called a metric on $M$ if it holds that

1. $g(m_1, m_2) = g(m_2, m_1)$ for all $m_1, m_2 \in M$,
2. the map $\hat{g} : M \to M^*$, given by $(\hat{g}(m_1))(m_2) = g(m_1, m_2)$, is an $\mathcal{A}$-module isomorphism, where $M^*$ denotes the dual of $M$.

We shall often refer to property (2) as the metric being non-degenerate.

**Example 2.2.2.** *In the definition above let $M = \mathcal{A}$ and let $g(a, b) = ab$. consider $\hat{g} : \mathcal{A} \to \mathcal{A}^*$ then*

$$\hat{g}(a)(b) = g(a, b) = ab$$

*If $\mathcal{A}$ is unital, choose $b = 1$ then $ab = 0$ $\forall b \in \mathcal{A}(= M)$ implies that $a = 0$. Hence $g$ is non-degenerate.*

**Definition 2.2.3.** A *metric* Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g}, g)$ is a Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g})$ together with a metric $g : \mathfrak{g} \times \mathfrak{g} \to \mathcal{A}$.

In Example 2.1.3, if we choose $M$ to be the sections of a vector bundle over the manifold then we obtain a metric on the vector bundle, i.e. on the set of smooth sections.

We proceed by introducing connections on modules over Lie-Rinehart algebras.

**Definition 2.2.4.** Let $(\mathcal{A}, \mathfrak{g})$ be a Lie-Rinehart algebra over $\mathbb{K}$ and let $M$ be an $\mathcal{A}$-module. A **connection** $\nabla$ on $M$ is a map $\nabla : \mathfrak{g} \to End_{\mathbb{K}}(M)$ (i.e. $\mathbb{K}$-linear endomorphisms), written as $\alpha \to \nabla_\alpha$, such that

1. $\nabla_{a\alpha+\beta} = a\nabla_\alpha + \nabla_\beta$
2. $\nabla_\alpha(am) = a\nabla_\alpha m + \alpha(a)m$, for all $a \in \mathcal{A}, \alpha, \beta \in \mathfrak{g}$ and $m \in M$.

**Example 2.2.5** (Continuation of 2.1.3)**.** *In Example 2.1.3, if we choose $M = \mathfrak{g}$ to be the sections of a vector bundle over the manifold $\Sigma$, then any connection on the vector bundle is a connection in the sense of Definition 2.2.4.*

**Definition 2.2.6.** Let $(\mathcal{A}, \mathfrak{g})$ be a Lie-Rinehart algebra and let $M$ be an $\mathcal{A}$-module with connection $\nabla$ and metric $g$. The connection is called metric if

$$\alpha(g(m_1, m_2)) = g(\nabla_\alpha m_1, m_2) + g(m_1, \nabla_\alpha m_2) \tag{2.2.1}$$

for all $\alpha \in \mathfrak{g}$ and $m_1, m_2 \in M$.

**Definition 2.2.7.** Let $(\mathcal{A}, \mathfrak{g})$ be a Lie-Rinehart algebra and let $\nabla$ be a connection on $\mathfrak{g}$. The connection is called torsion-free if

$$\nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta] = 0 \tag{2.2.2}$$

for all $\alpha, \beta \in \mathfrak{g}$.

On any manifold there are infinitely many affine connections. If the manifold is further endowed with a Riemannian metric $g$ then there is a natural choice of affine connection, called the Levi-Civita connection.

As in differential geometry, one can show that there exists a unique torsion-free and metric connection associated to the Riemannian metric. The first step involves Koszul's formula, which in the setting of Lie-Rinehart algebras is as follows:

**Proposition 2.2.8.** *Let $(\mathcal{A}, \mathfrak{g}, g)$ be a metric Lie-Rinehart algebra. If $\nabla$ is a metric and torsion-free connection on $\mathfrak{g}$ then it holds that*

$$\begin{aligned} 2g(\nabla_\alpha \beta, \gamma) = \alpha(g(\beta, \gamma)) + \beta(g(\gamma, \alpha)) - \gamma(g(\alpha, \beta)) \\ + g(\beta, [\gamma, \alpha]) + g(\gamma, [\alpha, \beta]) - g(\alpha, [\beta, \gamma]) \end{aligned}$$

*for all $\alpha, \beta, \gamma \in \mathfrak{g}$.*

This is Koszul's formula for a Lie-Rinehart algebra. In general for a Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g}, g)$, with $g$ metric on $\mathfrak{g}$, one can also show the following.

**Proposition 2.2.9.** *Let $(\mathcal{A}, \mathfrak{g}, g)$ be a metric Lie-Rinehart algebra. Then there exists a unique metric and torsion-free connection on $\mathfrak{g}$.*

The unique connection in will be referred to as the Levi-Civita connection of a metric Lie-Rinehart algebra.

We shall recall some of the properties satisfied by a metric and torsion-free connection. The differential geometric proofs goes through with only a change in notation needed. We refer to [11, 15] for a nice overview of differential geometric constructions in modules over general commutative algebras.

We define *curvature* as

$$R(\alpha,\beta)\gamma = \nabla_\alpha \nabla_\beta \gamma - \nabla_\beta \nabla_\alpha \gamma - \nabla_{[\alpha,\beta]}\gamma.$$

and with (abuse of notation) we define

$$\begin{aligned} R(\alpha,\beta,\gamma) &= R(\alpha,\beta)\gamma \\ R(\alpha,\beta,\gamma,\delta) &= g(\alpha, R(\gamma,\delta)\beta). \end{aligned}$$

**Example 2.2.10** (Continuation of 2.1.2)**.** *If $g$ is a metric on* $\mathrm{Der}(\mathcal{A})$ *in Example 2.1.2, then there exists a unique Levi-Civita connection on* $\mathrm{Der}(\mathcal{A})$*.*

Let us also consider the canonical extention of $\nabla$ to multilinear maps $T : \mathfrak{g}^k \to \mathcal{A}$

$$(\nabla_\beta T)(\alpha_1, ..., \alpha_k) = \beta(T(\alpha_1, ..., \alpha_k)) - \sum_{i=1}^k T(\alpha_1, ..., \nabla_\beta \alpha_i, ..., \alpha_k),$$

as well as to $\mathfrak{g}$-valued multilinear maps $T : \mathfrak{g}^k \to \mathfrak{g}$

$$(\nabla_\beta T)(\alpha_1, ..., \alpha_k) = \nabla_\beta(T(\alpha_1, ..., \alpha_k)) - \sum_{i=1}^k T(\alpha_1, ..., \nabla_\beta \alpha_i, ..., \alpha_k).$$

As in classical geometry, one obtains a generalization of the Bianchi identities.

**Proposition 2.2.11.** *Let $\nabla$ be the Levi-Civita connection of a metric Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g}, g)$ and let $R$ denote corresponding curvature. Then it holds that*

$$R(\alpha,\beta,\gamma) + R(\gamma,\alpha,\beta) + R(\beta,\gamma,\alpha) = 0 \tag{2.2.3}$$

$$(\nabla_\alpha R)(\beta,\gamma,\delta) + (\nabla_\beta R)(\gamma,\alpha,\delta) + (\nabla_\gamma R)(\alpha,\beta,\delta) = 0 \tag{2.2.4}$$

*for all $\alpha, \beta, \gamma, \delta \in \mathfrak{g}$.*

One is also able to derive the classical symmetries of the curvature tensor.

**Proposition 2.2.12.** *Let $\nabla$ be the Levi-Civita connection of a metric Lie-Rinehart algebra $(\mathcal{A}, \mathfrak{g}, g)$ and let $R$ denote corresponding curvature. Then it holds that*

$$R(\alpha,\beta,\gamma,\delta) = -R(\beta,\alpha,\gamma,\delta) = -R(\alpha,\beta,\delta,\gamma). \tag{2.2.5}$$

$$R(\alpha,\beta,\gamma,\delta) = R(\delta,\gamma,\alpha,\beta) \tag{2.2.6}$$

*for all $\alpha, \beta, \gamma, \delta \in \mathfrak{g}$.*

We will see in Chapter 3 other examples of Lie-Rinehart algebras, these are called Kähler-Poisson algebras. We will show that Kähler-Poisson algebras are metric Lie-Rinehart algebras, which implies that the results of this Chapter can be applied; in particular, there exists a unique torsion-free metric connection on every Kähler-Poisson algebra.

# 3 – Kähler–Poisson Algebras

We shall introduce a type of Poisson algebras that resembles the smooth functions on an (isometrically) embedded almost Kähler manifold, in this way we can develope an analogue of Riemannian geometry for Poisson algebras. Namely, let us consider a unital Poisson algebra $(\mathcal{A}, \{.,.\})$, over a field $\mathbb{K}$ (which think of as either $\mathbb{R}$ or $\mathbb{C}$) and let $\{x^1, ..., x^m\}$ be a set of distinguished elements of $\mathcal{A}$. These elements play the role of functions providing an embedding into $\mathbb{R}^m$ in the geometrical case. We also will define the concept of "Kähler-Poisson algebras".

We will show that Kähler-Poisson algebras are, in a natural way, metric Lie-Rinehart algebras, which implies that the results of Chapter 2 can be applied; in particular, there exists a unique torsion-free metric connection on every Kähler-Poisson algebra. Note that Lie-Rinehart algebras related to Poisson algebras have been extensively studied by Huebschmann (see e.g. [8, 9]). At the end, detailed examples are provided in order to illustrate the novel concepts.

## 3.1 Introduction to Kähler-Poisson Algebras

A Poisson structure on a smooth manifold $\Sigma$ is a Lie bracket on the algebra of smooth functions, which is also a derivation. Poisson algebras appear naturally in Hamiltonian mechanics, and are also central in the study of quantum groups and deformation quantization, see e.g. [10]. Manifolds with a Poisson algebra structure are known as Poisson manifolds, of which the symplectic manifolds and the Poisson-Lie groups are a special case. The algebra is named in honour of Simon Denis Poisson (see e.g. [5, 18]).

In general, one defines the concept of Poisson algebras as follows:

**Definition 3.1.1.** A Poisson algebra $(\mathcal{A}, \{.,.\})$ is a pair consisting of an algebra $\mathcal{A}$ (over a field $\mathbb{K}$) together with a map $\{.,.\}$: $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that
(1) $\{a, b\} = -\{b, a\}$
(2) $\{\lambda a, \mu b\} = \lambda\mu\{a, b\}$
(3) $\{ab, c\} = a\{b, c\} + \{a, c\}b$
(4) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$, for all $a, b, c \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{K}$. The map $\{.,.\}$ is called the Poisson bracket.

In [2] it was shown that the geometry of embedded almost Kähler manifolds can be reformulated entirely in the Poisson algebra of smooth functions. We continue with the main definition of this thesis.

**Definition 3.1.2.** Let $(\mathcal{A}, \{\cdot,\cdot\})$ be a Poisson algebra over $\mathbb{K}$ and let $x^1, ..., x^m \in \mathcal{A}$. Given a symmetric $m \times m$ matrix $g = (g_{ij})$ with entrices $g_{ij} \in \mathcal{A}$, for $i, j = 1, ..., m$, we say that the triple $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ is a Kähler–Poisson algebra if there exists $\eta \in \mathcal{A}$ such that

$$\sum_{i,j,k,l}^{m} \eta\{a, x^i\} g_{ij} \{x^j, x^k\} g_{kl} \{x^l, b\} = -\{a, b\} \tag{3.1.1}$$

for all $a, b \in \mathcal{A}$. Moreover, we set $\mathcal{P}^{ij}{=}\{x^i, x^j\}$.

*Remark* 3.1.3. From now on, we shall use the differential geometric convention that repeated indices are summed over from 1 to $m$, and omit explicit summation symbols.

*Remark* 3.1.4. If $\mathcal{A}$ is generated by $x^1, ..., x^m$ (that is, every element of $\mathcal{A}$ is a polynomial in the variables $x^1, ..., x^m$) then the Poisson structure is completly determined by $\mathcal{P}^{ij}$.

Note that

$$\{a, b\} = \{x^i, x^j\} \tfrac{\partial a}{\partial x^i} \tfrac{\partial b}{\partial x^j} = \mathcal{P}^{ij} \tfrac{\partial a}{\partial x^i} \tfrac{\partial b}{\partial x^j}$$

due to the fact that the Poisson bracket is a derivation in both arguments see Definition 3.1.1(3). For example in dimensions 2, where the Poisson structure is determined by $\{x, y\}$,

$$\{x^2 + x, y\} = 2x\{x, y\} + \{x, y\}.$$

For an algebra generated by $x^1, ..., x^m$, condition (3.1.1) is satisfied if and only if

$$\eta \mathcal{P} g \mathcal{P} g \mathcal{P} = -\mathcal{P}, \tag{3.1.2}$$

where $\mathcal{P} = (\mathcal{P}^{ij})$ and $g = (g_{ij})$ (in Definition 3.1.2) are considered as $(m \times m)$-matrices and the product in the expression above is matrix multiplication.

Let us now consider a few examples of Kähler–Poisson algebras.

**Example 3.1.5.** *Let $\mathcal{A}$ be a Poisson algebra generated by two elements $x^1 = x$ and, $x^2 = y$. Let*

$$\mathcal{P} = (\mathcal{P}^{ij}) = \begin{pmatrix} 0 & \{x, y\} \\ -\{x, y\} & 0 \end{pmatrix}$$

*It is easy to check that for the matrix*

$$g = \begin{pmatrix} x & x + y \\ x + y & y \end{pmatrix}, \textit{ with } \det(g) = xy - (x + y)^2$$

*we obtain,*

$$\begin{aligned} \mathcal{P} g \mathcal{P} g &= \begin{pmatrix} (x + y)\{x, y\} & y\{x, y\} \\ -x\{x, y\} & -(x + y)\{x, y\} \end{pmatrix} \begin{pmatrix} (x + y)\{x, y\} & y\{x, y\} \\ -x\{x, y\} & -(x + y)\{x, y\} \end{pmatrix} \\ &= \begin{pmatrix} (x + y)^2\{x, y\}^2 - xy\{x, y\}^2 & y(x + y)\{x, y\}^2 - y(x + y)\{x, y\}^2 \\ x(x + y)\{x, y\}^2 - x(x + y)\{x, y\}^2 & (x + y)^2\{x, y\}^2 - xy\{x, y\}^2 \end{pmatrix} \end{aligned}$$

*Therefore,*

$$\begin{aligned}\mathcal{P}g\mathcal{P}g\mathcal{P} &= \begin{pmatrix} 0 & \{x,y\}((x+y)^2\{x,y\}^2 - xy\{x,y\}^2) \\ -\{x,y\}((x+y)^2\{x,y\}^2 - xy\{x,y\}^2) & 0 \end{pmatrix} \\ &= -\{x,y\}^2(xy-(x+y)^2)\mathcal{P} \\ &= -\{x,y\}^2 \det(g)\mathcal{P},\end{aligned}$$

*implying that* $\eta = (\{x,y\}^2 \det(g))^{-1} = (\{x,y\}^2(xy-(x+y)^2))^{-1}$.

*Thus, as long as* $\{x,y\}^2 \det(g)$ *is not a zero-divisor, one may localize* $\mathcal{A}$ *(i.e. formally adding the inverse of* $\{x,y\}^2 \det(g)$ *to* $\mathcal{A}$*) to obtain a Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[(\{x,y\}^2 \det(g))^{-1}], g, \{x,y\}).$$

**Example 3.1.6.** *Let* $\mathcal{A}$ *be a Poisson algebra generated by two elements* $x$ *and,* $y$*. Let* $x^1 = y$*,* $x^2 = y^2$ *and*

$$\mathcal{P} = \begin{pmatrix} 0 & \{y,y^2\} \\ -\{y,y^2\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

*and therefore,* $\mathcal{P}g\mathcal{P}g\mathcal{P} = 0$*. Then for arbitrary* $g$ *and* $\eta$ *it holds that* $\eta\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mathcal{P}$*. Hence,* $(\mathcal{A}, g, \{y,y^2\})$ *is a Kähler-Poisson algebra.*

**Example 3.1.7.** *Let* $\mathcal{A}$ *be a Poisson algebra generated by two elements* $x$ *and,* $y$*. Let* $x^1 = x+y$*,* $x^2 = x-y$ *and*

$$\mathcal{P} = \begin{pmatrix} 0 & \{x+y,x-y\} \\ -\{x+y,x-y\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2\{x,y\} \\ 2\{x,y\} & 0 \end{pmatrix}$$

*It is easy to check that for the matrix*

$$g = \begin{pmatrix} \{x,y\} & 1 \\ 1 & \{x,y\} \end{pmatrix}, \text{ where } \det(g) = \{x,y\}^2 - 1$$

*we obtain,*

$$\begin{aligned}\mathcal{P}g\mathcal{P}g &= \begin{pmatrix} \{x+y,x-y\} & \{x,y\}\{x+y,x-y\} \\ -\{x,y\}\{x+y,x-y\} & -\{x+y,x-y\} \end{pmatrix}\begin{pmatrix} \{x+y,x-y\} & \{x,y\}\{x+y,x-y\} \\ -\{x,y\}\{x+y,x-y\} & -\{x+y,x-y\} \end{pmatrix} \\ &= \begin{pmatrix} \{x+y,x-y\}^2 - \{x,y\}^2\{x+y,x-y\}^2 & 0 \\ 0 & \{x+y,x-y\}^2 - \{x,y\}^2\{x+y,x-y\}^2 \end{pmatrix}\end{aligned}$$

*Therefore,*

$$\begin{aligned}\mathcal{P}g\mathcal{P}g\mathcal{P} &= \begin{pmatrix} 0 & \{x+y,x-y\}(\{x+y,x-y\}^2 - \{x,y\}^2\{x+y,x-y\}^2) \\ -\{x+y,x-y\}(\{x+y,x-y\}^2 - \{x,y\}^2\{x+y,x-y\}^2) & 0 \end{pmatrix} \\ &= -\{x+y,x-y\}^2(\{x,y\}^2 - 1)\mathcal{P},\end{aligned}$$

*and*

$$\begin{aligned}\eta &= (\{x+y,x-y\}^2 \det(g))^{-1} \\ &= (\{x+y,x-y\}^2(\{x,y\}^2 - 1))^{-1} \\ &= ((-2\{x,y\})^2(\{x,y\}^2 - 1)^{-1}.\end{aligned}$$

*Thus, as long as* $\{x+y,x-y\}^2 \det(g)$ *is not a zero-divisor, one may localize to obtain a Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[(4\{x, y\}^2 \det(g))^{-1}], g, \{x, y\}).$$

Given a Kähler–Poisson algebra $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$, we let $\mathfrak{g}$ denote the $\mathcal{A}$-module generated by all *inner derivations* , i.e.

$$\mathfrak{g} = \{a_1\{c^1, .\} + ... + a_N\{c^N, .\} : a_i, c^i \in \mathcal{A} \text{ and } N \in \mathbb{N}\}.$$

It is standard fact that $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$ with respect to

$$[\alpha, \beta](a) = \alpha(\beta(a)) - \beta(\alpha(a)),$$

where $\alpha, \beta \in \mathfrak{g}$ and $a \in \mathcal{A}$.

We proved in [3] that $\mathfrak{g}$ is a projective module. Example 2.1.2 shows that $(\mathcal{A}, \mathfrak{g})$ is a Lie-Rinehart algebra. The matrix $g$ provides a metric on $\mathfrak{g}$ (in the sense of Definition 2.2.1), defined by

$$g(\alpha, \beta) = \alpha(x^i)g_{ij}\beta(x^i). \tag{3.1.3}$$

To the metric $g$ one may associate a map $\hat{g} : \mathfrak{g} \to \mathfrak{g}^*$ defined as

$$\hat{g}(\alpha)(\beta) = g(\alpha, \beta).$$

The following result shows that $g$ is a metric on $\mathfrak{g}$ in the sense of Definition 2.2.1.

**Proposition 3.1.8** ([3]). *If $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ is a Kähler–Poisson algebras then the metric $g$ is non-degenerate; i.e. the map $\hat{g} : \mathfrak{g} \to \mathfrak{g}^*$ is a module isomorphism.*

We have seen in example 2.1.2 that $(\mathcal{A}, \mathfrak{g})$ is indeed a Lie-Rinehart algebra and, furthermore, Proposition 3.1.8 implies that $(\mathcal{A}, \mathfrak{g}, g)$ is a metric Lie-Rinehart algebra.

Let us now introduce some notation for Kähler–Poisson algebras. We recall

$$\mathcal{P}^{ij} = \{x^i, x^j\}$$

and we set

$$\mathcal{P}^i(a) = \{x^i, a\}$$

for $a \in \mathcal{A}$, as well as

$$\mathcal{D}^{ij} = \eta\{x^i, x^l\}g_{lk}\{x^j, x^k\}$$
$$\mathcal{D}^i(a) = \eta\{x^k, a\}g_{kl}\{x^l, x^i\}.$$

Note that $\mathcal{D}^{ij} = \mathcal{D}^{ji}$. The metric $g$ will be used to lower indices in analogy with differential geometry. E.g.

$$\mathcal{P}^i{}_j = \mathcal{P}^{ik}g_{kj} \quad \mathcal{D}^i{}_j = \mathcal{D}^{ik}g_{kj} \quad \mathcal{D}_i = g_{ij}\mathcal{D}^j.$$

With respect to this notation, (3.1.1) can be stated as

$$\mathcal{D}_i(a)\mathcal{P}^i(b) = \{a, b\}. \tag{3.1.4}$$

Furthermore, one immediately derives the following identities

$$\mathcal{D}^{ij}\mathcal{P}_j(a) = \mathcal{P}^i(a),\ \mathcal{P}^{ij}\mathcal{D}_j(a) = \mathcal{P}^i(a) \text{ and } \mathcal{D}^i_j\mathcal{D}^{jk} = \mathcal{D}^{jk}. \tag{3.1.5}$$

There is a natural embedding $\iota : \mathfrak{g} \to \mathcal{A}^m$, given by

$$\iota(a_i\{b^i, .\}) = a_i\{b^i, x^k\}e_k,$$

where $\mathcal{A}^m$ is a free module and $\{e_k\}_{k=1}^m$ denotes the canonical basis of $\mathcal{A}^m$. Moreover, $g$ induces a bilinear form on $\mathcal{A}^m$ via

$$g(X, Y) = X^i g_{ij} Y^j$$

for $X = X^i e_i \in \mathcal{A}^m$ and $Y = Y^i e_i \in \mathcal{A}^m$. Finally, we introduce the map $\mathcal{D} : \mathcal{A}^m \to \mathcal{A}^m$ by setting

$$\mathcal{D}(X) = \mathcal{D}^i{}_j X^j e_i$$

for $X = X^i e_i \in \mathcal{A}^m$.

**Proposition 3.1.9** ([3]). *The map $\mathcal{D} : \mathcal{A}^m \to \mathcal{A}^m$ is an orthogonal projection, i.e.*

$$\mathcal{D}^2(X) = \mathcal{D}(X) \text{ and } g(\mathcal{D}(X), Y) = g(X, \mathcal{D}(Y))$$

*for all $X, Y \in \mathcal{A}^m$.*

From Proposition 3.1.9 we conclude that $T\mathcal{A} = im(\mathcal{D})$ is a finitely generated projective module. Moreover, $\mathfrak{g}$ is isomorphic to $T\mathcal{A} = im(\mathcal{D})$. In fact the map $\iota : \mathfrak{g} \to \mathcal{A}^m$ is an isomorphism from $\mathfrak{g}$ to $T\mathcal{A}$.

**Proposition 3.1.10.** *$\mathfrak{g}$ is a finitely generated projective module and the $\mathcal{A}$-module $\mathfrak{g}$ is generated by $\{\mathcal{D}^1, ..., \mathcal{D}^m\}$.*

Note that the above result is clearly not dependent on whether or not the underlying Poisson algebra has the structure of a Kähler-Poisson algebra, as the definition of $\mathfrak{g}$ involves only inner derivations. Hence, as soon as the Poisson algebra admits the structure of a Kähler-Poisson algebra, it follows that the module of inner derivations is projective.

## 3.2 Construction of Kähler–Poisson Algebras

Given a Poisson algebra $\{\mathcal{A}, \{., .\}\}$ one may ask if there exist $\{x^1, ..., x^m\}$ and $(g_{ij})$ such that $\{\mathcal{A}, \{x^1, ..., x^m\}, g\}$ is a Kähler-Poisson algebra? Let us consider the case when $\mathcal{A}$ is a finitely generated algebra, and let $\{x^1, ..., x^m\}$ be an arbitrary set of generators. If we denote by $\mathcal{P}$ the matrix with entries $\{x^i, x^j\}$ and by $g$ the symmetric matrix with entries $g_{ij}$, the Kähler-Poisson condition (3.1.1) may be written in matrix notation as

$$\eta\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mathcal{P}$$

In the following, we provide a rather general way to associate a localization $\mathcal{A}[\lambda^{-1}]$ and a metric $g$ to $\mathcal{A}$, such that $(\mathcal{A}[\lambda^{-1}], \{x^1, ..., x^m\}, g)$ is a Kähler-Poisson algebra.

Let $P$ be an arbitrary antisymmetric matrix with entries in a commutative ring $R$. We will start by writing $P$ as being similar to a block diagonal matrix. This is a well known result in linear algebra, in which case the eigenvalues appear in the diagonal blocks. For an antisymmetric matrix with entries in a commutative ring, a similar result holds.

**Proposition 3.2.1** ([3]). *Let $R$ be a commutative ring. Let $M_N(R)$ denote the set of $N \times N$ matrices with entries in $R$.*

1. *For $N \geq 2$, let $P \in M_N(R)$ be an antisymmetric matrix. Then there exist $V \in M_N(R)$, an antisymmetric $Q \in M_{N-2}(R)$ and $\lambda \in R$ such that*

$$V^T P V = \left(\begin{array}{cc|c} 0 & \lambda & \multirow{2}{*}{0} \\ -\lambda & 0 & \\ \hline \multicolumn{2}{c|}{0} & Q \end{array}\right).$$

2. *Let $P \in M_N(R)$ be an antisymmetric matrix, and let $\hat{N}$ denote the integer part of $N/2$. Then there exists $V \in M_N(R)$ and $\lambda_1, ..., \lambda_{\hat{N}} \in R$ such that*

$$V^T P V = diag(\Lambda_1, ..., \Lambda_{\hat{N}}) \text{ if } N \text{ is even}$$
$$V^T P V = diag(\Lambda_1, ..., \Lambda_{\hat{N}}, 0) \text{ if } N \text{ is odd,}$$

*where*

$$\Lambda_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}.$$

Returning to the case of a Poisson algebra generated by $x^1, ..., x^m$, assume for the moment that $m = 2N$ for a positive integer $N$. By Proposition (3.2.1), there exists a matrix $V$ such that

$$V^T \mathcal{P} V = \mathcal{P}_0$$

where $\mathcal{P}_0$ is a block diagonal matrix of the form

$$\mathcal{P}_0 = \mathrm{diag}(\Lambda_1, ..., \Lambda_N),$$

with

$$\Lambda_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}.$$

In the same way, defining $g_0 = \mathrm{diag}(g_1, ..., g_N)$ with

$$g_k = \frac{\lambda}{\lambda_k}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$\lambda = \lambda_1 \cdots \lambda_N,$$

for $k = 1, ..., N$ and we set $g = V g_0 V^T$. Noticing that

$$\mathcal{P}_0 g_0 \mathcal{P}_0 g_0 \mathcal{P}_0 = -\lambda^2 \mathcal{P}_0$$

one finds

$$0 = \mathcal{P}_0 g_0 \mathcal{P}_0 g_0 \mathcal{P}_0 + \lambda^2 \mathcal{P}_0 = V^T (\mathcal{P} g \mathcal{P} g \mathcal{P} + \lambda^2 \mathcal{P}) V.$$

It is a general fact that for an arbitrary matrix $V$ there exists a matrix $\tilde{V}$ such that $\tilde{V} V = V \tilde{V} = (\det V)\mathbb{1}$. Multiplying the above equation from the left by $\tilde{V}^T$ and from the right by $\tilde{V}$ yields

$$\det(V)^2 (\mathcal{P} g \mathcal{P} g \mathcal{P} + \lambda^2 \mathcal{P}) = 0.$$

As long as $\det(V)$ is not a zero divisor, this implies that

$$\mathcal{P} g \mathcal{P} g \mathcal{P} = -\lambda^2 \mathcal{P},$$

which implies that $(\mathcal{A}[\lambda^{-1}], g, \{x^1, ..., x^m\})$ is a Kähler-Poisson algebra. Note that the above argument, with only slight notation changes, also applies to the case when $m$ is odd, in which case an extra block of 0 will appear in $\mathcal{P}_0$.

# 3.3 Examples

In this section, we shall present more examples of an algebraic nature to further illustrate the fact that algebras of smooth functions are not the only examples of Kähler-Poisson algebras. As shown below all Poisson-algebras generated by two and three elements can be made into Kähler-Poisson algebras.

## 3.3.1 Poisson Algebras Generated by two elements

The following examples generalizes 3.1.5 and 3.1.7.

**Example 3.3.1.** *Let $\mathcal{A}$ be a Poisson algebra generated by two elements $x^1 = x \in \mathcal{A}$ and, $x^2 = y \in \mathcal{A}$. Let*

$$\mathcal{P} = \begin{pmatrix} 0 & \{x, y\} \\ -\{x, y\} & 0 \end{pmatrix}$$

*It is easy to check that for an arbitrary symmetric matrix (*$\det(g) \neq 0$*)*

$$g = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

*one obtains,*

$$\begin{aligned}\mathcal{P}g\mathcal{P}g &= \begin{pmatrix} c\{x,y\} & b\{x,y\} \\ -a\{x,y\} & -c\{x,y\}\end{pmatrix}\begin{pmatrix} c\{x,y\} & b\{x,y\} \\ -a\{x,y\} & -c\{x,y\}\end{pmatrix} \\ &= \begin{pmatrix} (c\{x,y\})^2 - ab\{x,y\}^2 & 0 \\ 0 & (c\{x,y\})^2 - ab\{x,y\}^2\end{pmatrix}\end{aligned}$$

*Therefore,*

$$\begin{aligned}\mathcal{P}g\mathcal{P}g\mathcal{P} &= \begin{pmatrix} 0 & \{x,y\}((c\{x,y\})^2 - ab\{x,y\}^2) \\ -\{x,y\}((c\{x,y\})^2 - ab\{x,y\}^2) & 0\end{pmatrix} \\ &= -\{x,y\}^2(ab - c^2)\mathcal{P} \\ &= -\{x,y\}^2 \det(g)\mathcal{P},\end{aligned}$$

*giving* $\eta = (\{x,y\}^2 \det(g))^{-1}$. *Thus, as long as* $\{x,y\}^2 \det(g)$ *is not a zero-divisor, one may localize to obtain a Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[(\{x,y\}^2 \det(g))^{-1}], \{x,y\}, g).$$

## 3.3.2 Poisson Algebras Generated by three elements.

**Example 3.3.2.** *Let* $\mathcal{A}$ *be a Poisson algebra generated by three elements* $x^1 = x, x^2 = y, x^3 = z \in \mathcal{A}$. *Writing* $\{x,y\} = a, \{y,z\} = b$ *and* $\{z,x\} = c$, *i.e.*

$$\mathcal{P} = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0\end{pmatrix}$$

*It is easy to check that for an arbitrary symmetric matrix* $g$

$$\mathcal{P}g\mathcal{P}g\mathcal{P} = -\tau\mathcal{P},$$

*where*

$$\tau = a^2|g|_{33} + b^2|g|_{11} + c^2|g|_{22} + 2ab|g|_{31} - 2ac|g|_{32} - 2bc|g|_{21},$$

*and* $|g|_{ij}$ *denotes the determinant of the matrix obtained from* $g$ *by deleting the* $i'$*th row and the* $j'$*th column. Thus one may construct the Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[\tau^{-1}], \{x,y,z\}, g).$$

*In particular, if* $g = \mathrm{diag}(\lambda,\lambda,\lambda)$, *then* $\tau = \lambda^2(a^2 + b^2 + c^2)$.

**Example 3.3.3.** *Let* $\mathcal{A}_C$ *be the Poisson algebra on three generators x,y,z constructed from a polynomial* $C = \frac{1}{n+1}(x^{n+1} + y^{n+1} + z^{n+1}) - \lambda xyz$, *where* $\lambda \in \mathbb{R}$ *and we define a Poisson structure on* $\mathcal{A}_C$ *via*

$\{x,y\} = z^n - \lambda xy = \partial_z C$, $\{y,z\} = x^n - \lambda yz = \partial_x C$ *and* $\{z,x\} = y^n - \lambda xz = \partial_y C$,

*i.e*

$$\mathcal{P} = \begin{pmatrix} 0 & \partial_z C & -\partial_y C \\ -\partial_z C & 0 & \partial_x C \\ \partial_y C & -\partial_x C & 0 \end{pmatrix}.$$

*Consider the Kähler-Poisson algebra obtained from $\mathcal{A}_C$ by choosing the metric $g = \operatorname{diag}(\mu, \mu, \mu)$. We obtain*

$$\begin{aligned}
\mathcal{P}g\mathcal{P}g &= \begin{pmatrix} 0 & \mu\partial_z C & -\mu\partial_y C \\ -\mu\partial_z C & 0 & \mu\partial_x C \\ \mu\partial_y C & -\mu\partial_x C & 0 \end{pmatrix} \begin{pmatrix} 0 & \mu\partial_z C & -\mu\partial_y C \\ -\mu\partial_z C & 0 & \mu\partial_x C \\ \mu\partial_y C & -\mu\partial_x C & 0 \end{pmatrix} \\
&= \begin{pmatrix} -(\mu\partial_z C)^2 - (\mu\partial_y C)^2 & \mu^2\partial_y C\partial_x C & \mu^2\partial_z C\partial_x C \\ \mu^2\partial_y C\partial_x C & -(\mu\partial_z C)^2 - (\mu\partial_x C)^2 & \mu^2\partial_z C\partial_y C \\ \mu^2\partial_x C\partial_z C & \mu^2\partial_y C\partial_z C & -(\mu\partial_y C)^2 - (\mu\partial_x C)^2 \end{pmatrix}
\end{aligned}$$

*Therefore,*

$$\begin{aligned}
\mathcal{P}g\mathcal{P}g\mathcal{P} &= \begin{pmatrix} 0 & -\partial_z C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) & -\partial_y C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) \\ \partial_z C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) & 0 & -\partial_x C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) \\ -\partial_y C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) & \partial_x C((\mu\partial_z C)^2+(\mu\partial_y C)^2+(\mu\partial_x C)^2) & 0 \end{pmatrix} \\
&= -((\mu\partial_z C)^2 + (\mu\partial_y C)^2 + (\mu\partial_x C)^2)\mathcal{P} \\
&= -\mu^2((\partial_z C)^2 + (\partial_y C)^2 + (\partial_x C)^2)\mathcal{P}.
\end{aligned}$$

*With $C$ given as above, one obtains*

$$\begin{aligned}
\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mu^2\Big(&(\partial_x(\frac{1}{n+1}(x^{n+1} + y^{n+1} + z^{n+1}) - \lambda xyz))^2 + \\
&(\partial_y(\frac{1}{n+1}(x^{n+1} + y^{n+1} + z^{n+1}) - \lambda xyz))^2 + \\
&(\partial_z(\frac{1}{n+1}(x^{n+1} + y^{n+1} + z^{n+1}) - \lambda xyz))^2\Big).
\end{aligned}$$

*We conclude that*

$$\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mu^2((x^n - \lambda yz)^2 + (y^n - \lambda xz)^2 + (z^n - \lambda xy)^2)\mathcal{P}.$$

*Therefore,*

$$\eta = (\mu^2((x^n - \lambda yz)^2 + (y^n - \lambda xz)^2 + (z^n - \lambda xy)^2))^{-1}.$$

*Thus one may construct the Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[(\mu^2((x^n - \lambda yz)^2 + (y^n - \lambda xz)^2 + (z^n - \lambda xy)^2))^{-1}], g, \{x, y, z\}).$$

## 3.4 Levi-Civita Connection and Curvature

In Riemannian geometry, the Levi-Civita connection is a specific connection on the tangent bundle of a manifold. More precisely, it is the torsion-free metric connection, i.e., the torsion-free connection on the tangent bundle preserving a given (pseudo-)Riemannian metric.

The fundamental theorem of Riemannian geometry states that there is a unique connection which satisfies these properties above (see e.g. [6]).

In the theory of Riemannian and pseudo-Riemannian manifolds the term covariant derivative is often used for the Levi-Civita connection. The components of this connection with respect to a system of local coordinates are called Christoffel symbols. In Chapter 2 we showed that we can generalize the Levi-Civita connection for Lie-Rinehart algebras and so for Kähler-Poisson algebras. In this section, we shall derive an explicit expression for the Levi-Civita connection of an arbitrary Kähler-Poisson algebra $\mathcal{K} = (\mathcal{A}, g, \{x^1, \cdots, x^m\})$. Recall that for a Kähler-Poisson algebra, the Levi-Civita connection is the unique torsion-free and metric connection on the module $\mathfrak{g}$. We define Levi-Civita (as Definition 2.2.4) $\nabla : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$.

In the following, it turns out to be convenient to reformulate the results in terms of the generators $\{\mathcal{D}^1, \cdots, \mathcal{D}^m\}$ of $\mathfrak{g}$. Applying Kozul's formula in Proposition 2.2.8 to $\nabla : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ gives the connection as

$$\begin{aligned} 2g(\nabla_{\mathcal{D}^i}\mathcal{D}^j, \mathcal{D}^k) = \mathcal{D}^i(g(\mathcal{D}^j, \mathcal{D}^k)) + \mathcal{D}^j(g(\mathcal{D}^k, \mathcal{D}^i)) - \mathcal{D}^k(g(\mathcal{D}^i, \mathcal{D}^j)) \\ -g([\mathcal{D}^j, \mathcal{D}^k], \mathcal{D}^i) + g([\mathcal{D}^k, \mathcal{D}^i], \mathcal{D}^j) + g([\mathcal{D}^i, \mathcal{D}^j], \mathcal{D}^k) \end{aligned} \tag{3.4.1}$$

and one notes that an element $\alpha = a\{b, .\}$ may be recovered from $g(\alpha, \mathcal{D}^i)$ as

$$g(\alpha, \mathcal{D}^i)\mathcal{D}_i(f) = a\{b, x^k\}\mathcal{D}^i_k\mathcal{D}_i(f) = a\{b, x^k\}\mathcal{D}_k(f) = a\{b, f\} = \alpha(f).$$

Thus, one immediately obtains $\nabla_{\mathcal{D}^i}\mathcal{D}^j = g(\nabla_{\mathcal{D}^i}\mathcal{D}^j, \mathcal{D}^k)\mathcal{D}_k$. However, it turns out that one can obtain a more compact formula for the connection. Let us start with the following result.

**Lemma 3.4.1** ([3]). $g([\mathcal{D}^i, \mathcal{D}^j], \mathcal{D}^k) = \mathcal{D}^i(\mathcal{D}^{jk}) - \mathcal{D}^j(\mathcal{D}^{ik})$.

The above result allows for a rather compact formulation of the Levi-Civita connection for a Kähler Poisson-algebra. Using Kozul's formula and Lemma 3.4.1 we obtain the following result.

**Proposition 3.4.2** ([3]). *If $\nabla$ denotes the Levi-Civita connection of a Kähler-Poisson algebra $\mathcal{K}$ then*

$$\nabla_{\mathcal{D}^i}\mathcal{D}^j = \frac{1}{2}\mathcal{D}^i(\mathcal{D}^{jk})\mathcal{D}_k - \frac{1}{2}\mathcal{D}^j(\mathcal{D}^{ik})\mathcal{D}_k + \frac{1}{2}\mathcal{D}^k(\mathcal{D}^{ij})\mathcal{D}_k \tag{3.4.2}$$

*or, equivalently, $\nabla_{\mathcal{D}^i}\mathcal{D}^j = \Gamma^{ij}{}_k\mathcal{D}^k$ where*

$$\Gamma^{ij}{}_k = \frac{1}{2}\mathcal{D}^i(\mathcal{D}^{jl})\mathcal{D}_{lk} - \frac{1}{2}\mathcal{D}^j(\mathcal{D}^{il})\mathcal{D}_{lk} + \frac{1}{2}\mathcal{D}_k(\mathcal{D}^{ij}). \tag{3.4.3}$$

For $\alpha = \alpha_i\mathcal{D}^i$ and $\beta = \beta_i\mathcal{D}^i$ arbitrary elements of $\mathfrak{g}$, one obtains

$$\nabla_\alpha\beta = \alpha(\beta_i)\mathcal{D}^i + \Gamma^{ij}{}_k\alpha_i\beta_j\mathcal{D}^k.$$

From Section 2.2 we recall *curvature*

$$R(\alpha,\beta)\gamma = \nabla_\alpha\nabla_\beta\gamma - \nabla_\beta\nabla_\alpha\gamma - \nabla_{[\alpha,\beta]}\gamma,$$

and with (abuse of notation) we define

$$R(\alpha,\beta,\gamma) = R(\alpha,\beta)\gamma$$
$$R(\alpha,\beta,\gamma,\delta) = g(\alpha, R(\gamma,\delta)\beta).$$

**Example 3.4.3.** *From Example 3.1.5 and 3.3.1 we get the Kähler-Poisson algebra*

$$\mathcal{K} = (\mathcal{A}[(\{x,y\}^2\det(g))^{-1}], g, \{x,y\}),$$

*where*

$$g = \tfrac{1}{\lambda}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

*with $\lambda$ being an invertible element of $\mathcal{A}$. From the above consideration, we know that $(\mathcal{A}[\{x,y\}^{-1}], g, \{x,y\})$ is a Kähler-Poisson algebra with $\eta = \frac{\lambda^2}{p^2}$ and $p = \{x,y\}$. We introduce $\gamma = \frac{p}{\lambda}$ such that $\eta = \frac{1}{\gamma^2}$. Let us start by computing the derivations $\mathcal{D}^x = \mathcal{D}^1$ and $\mathcal{D}^y = \mathcal{D}^2$, which generate the module $\mathfrak{g}$:*

$$\mathcal{D}^x = \eta\{x,x^i\}g_{ij}\{\cdot,y^i\} = \tfrac{\lambda}{p}\{\cdot,y\} = \tfrac{1}{\gamma}\{\cdot,y\}$$
$$\mathcal{D}^y = \eta\{y,x^i\}g_{ij}\{\cdot,x^i\} = \tfrac{\lambda}{p}\{\cdot,x\} = -\tfrac{1}{\gamma}\{\cdot,x\}$$

*as well as*

$$\mathcal{D}_x = g_{1k}\mathcal{D}^k = \tfrac{1}{\lambda}\mathcal{D}^x \text{ and } \mathcal{D}_y = g_{2k}\mathcal{D}^k = \tfrac{1}{\lambda}\mathcal{D}^y$$

*Moreover, they provide an orthogonal set of generators since*

$$\begin{aligned} g(\mathcal{D}^x,\mathcal{D}^x) &= \mathcal{D}^x(x^i)g_{ij}\mathcal{D}^x(x^j) = \frac{1}{\gamma}\{\cdot,y\}(x^i)g_{ij}\frac{1}{\gamma}\{\cdot,y\}(x^j)\\ &= \frac{1}{\gamma}\{x^i,y\}g_{ij}\frac{1}{\gamma}\{x^i,y\} = \frac{1}{\gamma}\{x,y\}g_{11}\frac{1}{\gamma}\{x,y\}\\ &= \frac{1}{\gamma^2}\frac{p^2}{\lambda} = \frac{\lambda^2}{p^2}\frac{p^2}{\lambda} = \lambda, \end{aligned}$$

*as well as $g(\mathcal{D}^y,\mathcal{D}^y) = \lambda$ and $g(\mathcal{D}^x,\mathcal{D}^y) = 0$. Let us define the derivation $\mathcal{D}^\lambda = \gamma^{-1}\{\lambda,\cdot\}$ and note that*

$$\mathcal{D}^\lambda = [\mathcal{D}^x,\mathcal{D}^y] = \tfrac{1}{\lambda}\mathcal{D}^x(\lambda)\mathcal{D}^y - \tfrac{1}{\lambda}\mathcal{D}^y(\lambda)\mathcal{D}^x.$$

*One can compute*

$$g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^x) = \tfrac{1}{2\gamma}\{\lambda,x\} \text{ and } g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^y) = \tfrac{1}{2\gamma}\{\lambda,y\}.$$

*Therefore,*

$$\begin{aligned} (\nabla_{\mathcal{D}^x}\mathcal{D}^y)(f) &= g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^i)\mathcal{D}_i(f)\\ &= g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^1)\mathcal{D}_1(f) + g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^2)\mathcal{D}_2(f)\\ &= g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^x)\mathcal{D}_x(f) + g(\nabla_{\mathcal{D}^x}\mathcal{D}^y,\mathcal{D}^y)\mathcal{D}_y(f)\\ &= \frac{1}{2\gamma}\{\lambda,x\}\frac{1}{\lambda}\mathcal{D}^x(f) + \frac{1}{2\gamma}\{\lambda,y\}\frac{1}{\lambda}\mathcal{D}^y(f)\\ &= \frac{1}{2}\mathcal{D}^x(\lambda)\mathcal{D}_y - \frac{1}{2}\mathcal{D}^y(\lambda)\mathcal{D}_x, \end{aligned}$$

*since* $\mathcal{D}^x(\lambda) = \frac{1}{\gamma}\{\lambda, y\}$, $\mathcal{D}^y(\lambda) = -\frac{1}{\gamma}\{\lambda, x\}$ *and* $\mathcal{D}_y = \frac{1}{\lambda}\mathcal{D}^y(\lambda)$. *Similarly,*

$$\begin{aligned}
\nabla_{\mathcal{D}^y}\mathcal{D}^y &= \frac{1}{2}\mathcal{D}^x(\lambda)\mathcal{D}_x + \frac{1}{2}\mathcal{D}^y(\lambda)\mathcal{D}_y \\
\nabla_{\mathcal{D}^x}\mathcal{D}^x &= \frac{1}{2}\mathcal{D}^x(\lambda)\mathcal{D}_x + \frac{1}{2}\mathcal{D}^y(\lambda)\mathcal{D}_y \\
\nabla_{\mathcal{D}^y}\mathcal{D}^x &= \frac{1}{2}\mathcal{D}^y(\lambda)\mathcal{D}_x - \frac{1}{2}\mathcal{D}^x(\lambda)\mathcal{D}_y
\end{aligned}$$

*Moreover, the curvature can be computed*

$$R(\mathcal{D}^x, \mathcal{D}^y)\mathcal{D}^x = \Big[\mathcal{D}_x(\lambda)^2 + \mathcal{D}_y(\lambda)^2 - \tfrac{1}{2}\mathcal{D}_x(\mathcal{D}^x(\lambda)) - \tfrac{1}{2}\mathcal{D}_y(\mathcal{D}^y(\lambda))\Big]\mathcal{D}^y,$$

*and*

$$R(\mathcal{D}^x, \mathcal{D}^y)\mathcal{D}^y = -\Big[\mathcal{D}_x(\lambda)^2 + \mathcal{D}_y(\lambda)^2 - \tfrac{1}{2}\mathcal{D}_x(\mathcal{D}^x(\lambda)) - \tfrac{1}{2}\mathcal{D}_y(\mathcal{D}^y(\lambda))\Big]\mathcal{D}^x.$$

# 4 – Morphisms of Kähler-Poisson Algebras

## 4.1 Homomorphisms of Kähler-Poisson Algebras

Let $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ and $\mathcal{K}' = (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$ be Kähler-Poisson algebras together with their corresponding modules of derivations $\mathfrak{g}$ and $\mathfrak{g}'$. In this chapter we would like to understand when $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic. As Kähler-Poisson algebras are also metric Lie-Rinehart algebras as we saw in Chapter 3, we shall require that a morphism of Kähler-Poisson algebras is also a morphism of metric Lie-Rinehart algebras. However, as the definition of a Kähler-Poisson algrbra also involves the choice of a set of distinguished elements, we will require a morphism to respect the subalgebra generated by these elements. To this end, we start by making the following definition.

**Definition 4.1.1.** Given a Kähler-Poisson algebra $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$, let $\mathcal{A}_{\text{fin}} \subseteq \mathcal{A}$ denote the subalgebra generated by $\{x^1, ..., x^m\}$.

Equipped with this definition, we introduce morphisms of Kähler-Poisson algebras in the following way.

**Definition 4.1.2.** Let $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ and $\mathcal{K}' = (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$ be Kähler-Poisson algebras together with their corresponding modules of derivations $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. A morphism of Kähler-Poisson algebras is a pair of maps $(\phi, \psi)$, with $\phi : \mathcal{A} \to \mathcal{A}'$ a Poisson algebra homomorphism and $\psi : \mathfrak{g} \to \mathfrak{g}'$ a Lie algebra homomorphism, such that

1. $\psi(a\alpha) = \phi(a)\psi(\alpha)$,
2. $\phi(\alpha(a)) = \psi(\alpha)(\phi(a))$,
3. $\phi(g(\alpha, \beta)) = g'(\psi(\alpha), \psi(\beta))$,
4. $\phi(\mathcal{A}_{\text{fin}}) \subseteq \mathcal{A}'_{\text{fin}}$,

for all $a \in \mathcal{A}$ and $\alpha, \beta \in \mathfrak{g}$.

Furthermore, an isomorphism of Kähler-Poisson algebras is a morphism $(\phi, \psi)$ of Kähler-Poisson algebras such that $\phi$ is a Poisson algebra isomorphism and $\phi(\mathcal{A}_{\text{fin}}) = \mathcal{A}'_{\text{fin}}$.

Let $(\mathcal{A}, \{., .\})$ be a Poisson algebra and let $x^i \in \mathcal{A}$ for $i = 1, ..., m$. If $p \in \mathcal{A}$ is a polynomial in $\{x^1, ..., x^m\}$ then, using Leibniz rule, one may compute

$$\{p, a\} = \frac{\partial p}{\partial x^i}\{x^i, a\} \tag{4.1.1}$$

where $\frac{\partial p}{\partial x^i}$ denotes the formal derivative of the polynomial $p$ with respect to the variable $x^i$. Note that, in general, $\frac{\partial p}{\partial x^i}$ is itself not well-defined in the algebra, since there might exist several different (but equivalent) representations of $p$ as a polynomial in $x^1, ..., x^m$, and the formal derivative then yields several, possibly non-equivalent, elements of the algebra. However, the combination in (4.1.1) is always well-defined, and gives the same result for all representations of $p$.

Given a matrix $M = (m_{ij})$ over $\mathcal{A}$, we set $\phi(M) = (\phi(m_{ij}))$. Given a morphism $(\phi, \psi) : (\mathcal{A}, g, \{x^1, ..., x^m\}) \to (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$, it will be convenient to introduce the notation

$$A^i{}_\alpha = \frac{\partial \phi(x^i)}{\partial y^\alpha}$$

(keeping in mind that this is not well-defined by itself); recall that if $(\phi, \psi)$ is a morphism of Kähler-Poisson algebras, then $\phi(\mathcal{A}_{\text{fin}}) \subseteq \mathcal{A}'_{\text{fin}}$, ensuring that $\phi(x^i)$ is indeed a polynomial in $y^1, ..., y^{m'}$. This notation allows us to write

$$\phi(\{x^i, x^j\}) = \{\phi(x^i), \phi(x^j)\}' = A^i{}_\alpha \{y^\alpha, y^\beta\}' A^j{}_\beta$$

in matrix notation as

$$\phi(\mathcal{P}) = A\mathcal{P}'A^T,$$

where $\mathcal{P} = (\{x^i, x^j\})$ and $\mathcal{P}' = (\{y^\alpha, y^\beta\}')$.

In the following, we shall consider a number of examples in order to explore when the Kähler-Poisson algebras are isomorphic. A standing assumption is that all the algebras have been properly localized to allow for the construction of a Kähler-Poisson algebra (cp. the examples in Section 3.1).

Now given two Kähler-Poisson algebras $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ and $\mathcal{K}' = (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$, we would like to understand when they are isomorphic. For instance, if $\mathcal{A} = \mathcal{A}'$ we consider different numbers of generators. Let us start with the following example.

**Example 4.1.3.** *Let $(\mathcal{A}, \{., .\})$ be a Poisson algebra generated by two elements x,y. From example 3.3.1 we know that $\mathcal{K} = (\mathcal{A}, g, \{x, y\})$ is a Kähler-Poisson algebra for arbitrary symmetric matrices*

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},$$

*with*

$$\eta = \left(\{x, y\}^2 \det(g)\right)^{-1}.$$

*Let $\mathcal{K}' = (\mathcal{A}, h, \{x, y, x\})$ with*

$$\mathcal{P}' = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix},$$

*where* $\lambda = \{x, y\}$*. It is easy to check that for the symmetric matrix* $h$

$$h = \begin{pmatrix} \frac{1}{4}g_{11} & \frac{1}{2}g_{12} & \frac{1}{4}g_{11} \\ \frac{1}{2}g_{12} & g_{22} & \frac{1}{2}g_{12} \\ \frac{1}{4}g_{11} & \frac{1}{2}g_{12} & \frac{1}{4}g_{11} \end{pmatrix},$$

*one obtains*

$$\mathcal{P}' h \mathcal{P}' h \mathcal{P}' = -\{x, y\}^2 \det(g) \mathcal{P}',$$

*giving* $\eta' = (\{x, y\}^2 \det(g))^{-1}$*. We conclude that* $\eta' = \eta$*.*

*Now let us check that* $\mathcal{K} \cong \mathcal{K}'$ *by using Definition 4.1.2. From*

$$\mathfrak{g} = \{a_1\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\}$$

*and*

$$\begin{aligned} \mathfrak{g}' &= \{a_1\{x, \cdot\} + a_2\{y, \cdot\} + a_3\{x, \cdot\} : a_1, a_2, a_3 \in \mathcal{A}\} \\ &= \{(a_1 + a_3)\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2, a_3 \in \mathcal{A}\}, \end{aligned}$$

*we conclude that* $\mathfrak{g} = \mathfrak{g}'$*. To prove that* $\mathcal{K} \cong \mathcal{K}'$*, we need to define the maps* $\phi : \mathcal{A} \to \mathcal{A}$ *and* $\psi : \mathfrak{g} \to \mathfrak{g}$*. By choosing* $\phi = id$ *and* $\psi = id$ *we will show that* $(\phi, \psi)$ *is an isomorphism of Kähler Poisson algebras.*

*Let us check properites (1.-4.) in Definition 4.1.2*

1. $\phi(a)\psi(\alpha) = a\alpha = \psi(a\alpha)$.

2. $\psi(\alpha)(\phi(a)) = \alpha(a) = \phi(\alpha(a))$,

3. *We need to show that* $\phi(g(\alpha, \beta)) = h(\psi(\alpha), \psi(\beta))$.

*Starting from the right hand side we get*

$$\begin{aligned} h(\psi(\alpha), \psi(\beta)) &= h(\alpha, \beta) \\ &= \alpha(x^i) h_{ij} \beta(x^j) \\ &= \alpha(x)h_{11}\beta(x) + \alpha(x)h_{12}\beta(y) + \alpha(y)h_{13}\beta(x) + \alpha(y)h_{21}\beta(x) + \alpha(y)h_{22}\beta(y) \\ &\quad + \alpha(y)h_{23}\beta(x) + \alpha(x)h_{31}\beta(x) + \alpha(x)h_{32}\beta(y) + \alpha(x)h_{33}\beta(x) \\ &= \frac{1}{4}\alpha(x)g_{11}\beta(x) + \frac{1}{2}\alpha(x)g_{12}\beta(y) + \frac{1}{4}\alpha(x)g_{11}\beta(x) + \frac{1}{2}\alpha(y)g_{12}\beta(x) + \alpha(y)g_{22}\beta(y) \\ &\quad + \frac{1}{2}\alpha(y)g_{12}\beta(x) + \frac{1}{4}\alpha(x)g_{11}\beta(x) + \frac{1}{2}\alpha(x)g_{12}\beta(y) + \frac{1}{4}\alpha(x)g_{11}\beta(x) \\ &= \alpha(x)g_{11}\beta(x) + \alpha(x)g_{12}\beta(y) + \alpha(y)g_{12}\beta(x) + \alpha(y)g_{22}\beta(y). \end{aligned}$$

*From the left hand side we get*

$$\begin{aligned} \phi(g(\alpha, \beta)) &= \alpha(x^i) g_{ij} \beta(x^j) \\ &= \alpha(x)g_{11}\beta(x) + \alpha(x)g_{12}\beta(y) + \alpha(y)g_{21}\beta(x) + \alpha(y)g_{22}\beta(y). \end{aligned}$$

*Therefore,* $\phi(g(\alpha, \beta)) = h(\psi(\alpha), \psi(\beta))$*, which shows that* $\mathcal{K} \cong \mathcal{K}'$*.*

Note that the above example extends to the case where we choose more (dependent) generators of the algebra, giving many possible presentations of the same Kähler-Poisson algebra. Next, let us explore the case when we choose another set of generators for the same algebra.

**Example 4.1.4.** *Let $(\mathcal{A}, \{.,.\})$ be a Poisson algebra generated by two elements x,y. From example 3.3.1 we know that $\mathcal{K} = (\mathcal{A}, g, \{x, y\})$ is a Kähler-Poisson algebra for arbitrary symmetric matrices*

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},$$

*with*

$$\eta = \big(\{x, y\}^2 \det(g)\big)^{-1}.$$

*Let $\mathcal{K}' = (\mathcal{A}, h, \{x, x + y\})$, we obtain*

$$\begin{aligned} \mathcal{P}' &= \begin{pmatrix} \{x, x\} & \{x, x + y\} \\ \{x + y, x\} & \{x + y, x + y\} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \{x, y\} \\ -\{x, y\} & 0 \end{pmatrix}. \end{aligned}$$

*It is easy to check that for the symmetric matrix h*

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix},$$

*one obtains*

$$\mathcal{P}' h \mathcal{P}' h \mathcal{P}' = -\{x, y\}^2 \det(h) \mathcal{P}', \tag{4.1.2}$$

*giving $\eta' = (\{x, y\}^2 \det(h))^{-1}$. Let us try to find h such that $\mathcal{K} \cong \mathcal{K}'$.*

*From*

$$\mathfrak{g} = \{a_1\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\}$$

*and*

$$\begin{aligned} \mathfrak{g}' &= \{a_1\{x, \cdot\} + a_2\{x + y, \cdot\} : a_1, a_2 \in \mathcal{A}\} \\ &= \{a_1\{x, \cdot\} + a_2\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\} \\ &= \{(a_1 + a_3)\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\}, \end{aligned}$$

*we conclude that $\mathfrak{g} = \mathfrak{g}'$. To prove that $\mathcal{K} \cong \mathcal{K}'$, we need to define the maps $\phi : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathfrak{g} \to \mathfrak{g}$. By choosing $\phi = id$ and $\psi = id$ we will show that $(\phi, \psi)$ is an isomorphism of Kähler Poisson algebras for a suitable choice of matrix h.*

*Let us check properites (1.-4.) in Definition 4.1.2*

1. *$\phi(a)\psi(\alpha) = a\alpha = \psi(a\alpha)$.*

2. $\psi(\alpha)(\phi(a)) = \alpha(a) = \phi(\alpha(a))$,

3. *We need to show that* $g(\alpha,\beta) = h(\alpha,\beta)$.

*Starting from the left hand side we get*

$$\begin{aligned}\phi(g(\alpha,\beta)) &= \alpha(x^i)g_{ij}\beta(x^j)\\ &= \alpha(x)g_{11}\beta(x) + \alpha(x)g_{12}\beta(y) + \alpha(y)g_{21}\beta(x) + \alpha(y)g_{22}\beta(y).\end{aligned}$$

*From the right hand side we get*

$$\begin{aligned}h(\alpha,\beta) &= \alpha(x^i)h_{ij}\beta(x^j)\\ &= \alpha(x)h_{11}\beta(x) + \alpha(x)h_{12}\beta(x+y) + \alpha(x+y)h_{21}\beta(x) + \alpha(x+y)h_{22}\beta(x+y)\\ &= \alpha(x)h_{11}\beta(x) + \alpha(x)h_{12}\beta(x) + \alpha(x)h_{12}\beta(y) + \alpha(x)h_{21}\beta(x) + \alpha(y)h_{21}\beta(x) +\\ &\quad (\alpha(x)+\alpha(y))h_{22}(\beta(x)+\beta(y))\\ &= \alpha(x)h_{11}\beta(x) + \alpha(x)h_{12}\beta(x) + \alpha(x)h_{12}\beta(y) + \alpha(x)h_{21}\beta(x) + \alpha(y)h_{21}\beta(x) +\\ &\quad + \alpha(y)h_{22}\beta(x) + \alpha(y)h_{22}\beta(x) + \alpha(x)h_{22}\beta(y) + \alpha(y)h_{22}\beta(y).\end{aligned}$$

*We conclude that*

$$\begin{gathered}\alpha(x)\beta(x) : h_{11} + h_{12} + h_{21} + h_{22} = g_{11}\\ \alpha(x)\beta(y) : h_{12} + h_{22} = g_{12}\\ \alpha(y)\beta(x) : h_{21} + h_{22} = g_{21}\\ \alpha(y)\beta(y) : h_{22} = g_{22},\end{gathered}$$

*from which we find* $h_{21} = g_{21} - g_{22}$, $h_{12} = g_{12} - g_{22}$, *and*

$$\begin{gathered}h_{11} + g_{12} - g_{22} + g_{21} - g_{22} + g_{22} = g_{11}\\ h_{11} + g_{12} - g_{22} + g_{21} = g_{11}\\ h_{11} = g_{11} - g_{12} - g_{21} + g_{22}\\ h_{11} = g_{11} - 2g_{12} + g_{22}\end{gathered}$$

*Now, the symmetric matrix* $h$ *becomes*

$$h = \begin{pmatrix} g_{11} - 2g_{12} + g_{22} & g_{12} - g_{22}\\ g_{21} - g_{22} & g_{22}\end{pmatrix}.$$

*We check that*

$$\begin{aligned}\det(h) &= g_{22}(g_{11} + g_{22} - 2g_{12}) - (g_{12} - g_{22})^2\\ &= g_{22}g_{11} + (g_{22})^2 - 2g_{12}g_{22} - ((g_{12})^2 + (g_{22})^2 - 2g_{12}g_{22})\\ &= (g_{11}g_{22} - (g_{12})^2)\\ &= \det(g).\end{aligned}$$

*Putting* $\det(h)$ *in equation 4.1.2 we get*

$$\mathcal{P}'h\mathcal{P}'h\mathcal{P}' == -\{x,y\}^2\det(g)\mathcal{P}'.$$

*giving* $\eta' = (\{x,y\}^2\det(g))^{-1}$. *Therefore,* $\eta' = \eta$. *We conclude that* $g(\alpha,\beta) = h(\alpha,\beta)$, *which shows that* $\mathcal{K} \cong \mathcal{K}'$.

**Example 4.1.5.** *Let $(\mathcal{A}, \{.,.\})$ be a Poisson algebra generated by two elements $x^1 = x$, $x^2 = y$. From example 3.3.1 we know that $\mathcal{K} = (\mathcal{A}, g, \{x, y\})$ is a Kähler-Poisson algebra for arbitrary symmetric matrices*

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},$$

*with*

$$\eta = \big(\{x, y\}^2 \det(g)\big)^{-1}.$$

*Let $\mathcal{K}' = (\mathcal{A}, h, \{x + y, x - y\})$, we obtain*

$$\begin{aligned} \mathcal{P}' &= \begin{pmatrix} \{x+y, x+y\} & \{x+y, x-y\} \\ \{x+y, x-y\} & \{x-y, x-y\} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2\{x, y\} \\ 2\{x, y\} & 0 \end{pmatrix}. \end{aligned}$$

*It is easy to check that for the symmetric matrix $h$*

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix},$$

*one obtains*

$$\mathcal{P}' h \mathcal{P}' h \mathcal{P}' = -4\{x, y\}^2 \det(h) \mathcal{P}', \tag{4.1.3}$$

*giving $\eta' = (4\{x, y\}^2 \det(h))^{-1}$. Let us try to find $h$ such that $\mathcal{K} \cong \mathcal{K}'$.*

*From*

$$\mathfrak{g} = \{a_1\{x, \cdot\} + a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\},$$

*and*

$$\begin{aligned} \mathfrak{g}' &= \{a_1\{x + y, \cdot\} + a_2\{x - y, \cdot\} : a_1, a_2 \in \mathcal{A}\} \\ &= \{a_1\{x, \cdot\} + a_1\{y, \cdot\} + a_2\{x, \cdot\} - a_2\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\} \\ &= \{(a_1 + a_2)\{x, \cdot\} + (a_1 - a_2)\{y, \cdot\} : a_1, a_2 \in \mathcal{A}\}, \end{aligned}$$

*we conclude that $\mathfrak{g} = \mathfrak{g}'$. To prove that $\mathcal{K} \cong \mathcal{K}'$, we need to define the maps $\phi : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathfrak{g} \to \mathfrak{g}$. By choosing $\phi = id$ and $\psi = id$ we will show that $(\phi, \psi)$ is an isomorphism of Kähler Poisson algebras for a suitable choice of matrix $h$.*

*Let us check properites (1.-4.) in Definition 4.1.2*

1. *$\phi(a)\psi(\alpha) = a\alpha = \psi(a\alpha)$.*

2. *$\psi(\alpha)(\phi(a)) = \alpha(a) = \phi(\alpha(a))$.*

3. *We need to show that $g(\alpha, \beta) = h(\alpha, \beta)$.*

*Starting from the left hand side we get*

$$\begin{aligned}\phi(g(\alpha,\beta)) &= \alpha(x^i)g_{ij}\beta(x^j)\\ &= \alpha(x)g_{11}\beta(x)+\alpha(x)g_{12}\beta(y)+\alpha(y)g_{21}\beta(x)+\alpha(y)g_{22}\beta(y).\end{aligned}$$

*From the right hand side we get*

$$\begin{aligned}h(\alpha,\beta) &= \alpha(x^i)h_{ij}\beta(x^j)\\ &= \alpha(x+y)h_{11}\beta(x+y)+\alpha(x+y)h_{12}\beta(x-y)+\alpha(x-y)h_{21}\beta(x+y)\\ &\quad+\alpha(x-y)h_{22}\beta(x-y)\\ &= \alpha(x)h_{11}\beta(x)+\alpha(x)h_{11}\beta(y)+\alpha(y)h_{11}\beta(x)+\alpha(y)h_{11}\beta(y)+\alpha(x)h_{12}\beta(x)+\\ &\quad-\alpha(x)h_{12}\beta(y)+\alpha(y)h_{12}\beta(x)-\alpha(y)h_{12}\beta(y)+\alpha(x)h_{21}\beta(x)+\alpha(x)h_{21}\beta(y)\\ &\quad-\alpha(y)h_{21}\beta(x)-\alpha(y)h_{21}\beta(y)+\alpha(x)h_{22}\beta(x)-\alpha(x)h_{22}\beta(y)-\alpha(y)h_{22}\beta(x)\\ &\quad+\alpha(y)h_{22}\beta(y)\\ &= \alpha(x)h_{11}\beta(x)+\alpha(x)h_{11}\beta(y)+\alpha(y)h_{11}\beta(x)+\alpha(y)h_{11}\beta(y)+2\alpha(x)h_{12}\beta(x)\\ &\quad-2\alpha(y)h_{12}\beta(y)+\alpha(x)h_{22}\beta(x)-\alpha(x)h_{22}\beta(y)-\alpha(y)h_{22}\beta(x)+\alpha(y)h_{22}\beta(y).\end{aligned}$$

*We conclude that*

$$\alpha(x)\beta(x): h_{11}+2h_{12}+h_{22}=g_{11} \tag{4.1.4}$$

$$\alpha(x)\beta(y): h_{11}-h_{22}=g_{12} \tag{4.1.5}$$

$$\alpha(y)\beta(x): h_{11}-h_{22}=g_{21} \tag{4.1.6}$$

$$\alpha(y)\beta(y): h_{11}-2h_{12}+h_{22}=g_{22} \tag{4.1.7}$$

*From* (4.1.7) *we obtain,* $h_{11}=2h_{12}-h_{22}+g_{22}$ *and putting this in* (4.1.4) *we get*

$$\begin{aligned}2h_{12}-h_{22}+g_{22}+2h_{12}+h_{22}&=g_{11}\\ 4h_{12}&=g_{11}-g_{22}\\ h_{12}&=\tfrac{g_{11}-g_{22}}{4}.\end{aligned}$$

*From* (4.1.5) *we get* $h_{22}=h_{11}-g_{12}$*, which in* (4.1.4) *gives*

$$\begin{aligned}h_{11}+2(\tfrac{g_{11}-g_{22}}{4})+h_{11}-g_{12}&=g_{11}\\ 2h_{11}+(\tfrac{g_{11}-g_{22}}{2})-g_{12}&=g_{11}\\ 4h_{11}+g_{11}-g_{22}&=2(g_{11}+g_{12})\\ 4h_{11}&=g_{11}+2g_{12}+g_{22}\\ h_{11}&=\tfrac{g_{11}+2g_{12}+g_{22}}{4},\end{aligned}$$

*which implies that*

$$\begin{aligned}h_{22}&=h_{11}-g_{12}\\ &=\frac{g_{11}+2g_{12}+g_{22}}{4}-g_{12}\\ &=\frac{g_{11}-2g_{12}+g_{22}}{4}.\end{aligned}$$

*Now, the symmetric matrix h becomes*

$$h = \begin{pmatrix} \frac{g_{11}+2g_{12}+g_{22}}{4} & \frac{g_{11}-g_{22}}{4} \\ \frac{g_{11}-g_{22}}{4} & \frac{g_{11}-2g_{12}+g_{22}}{4} \end{pmatrix},$$

*giving*

$$\begin{aligned}
\det(h) &= \frac{1}{16}\Big[(g_{11}+2g_{12}+g_{22})(g_{11}-2g_{12}+g_{22}) - (g_{11}-g_{22})^2\Big] \\
&= \frac{1}{16}\Big[(g_{11})^2 - 2g_{11}g_{12} + g_{11}g_{22} + 2g_{11}g_{12} - 4(g_{12})^2 + 2g_{12}g_{22} + g_{11}g_{22} \\
&\quad - 2g_{12}g_{22} + (g_{22})^2 - (g_{11})^2 - (g_{22})^2 + 2g_{11}g_{22}\Big] \\
&= \frac{1}{16}(4g_{11}g_{22} - 4(g_{12})^2) \\
&= \frac{1}{4}(g_{11}g_{22} - (g_{12})^2) \\
&= \frac{1}{4}\det(g).
\end{aligned}$$

*Putting* $\det(h)$ *in equation 4.1.3 we get*

$$\begin{aligned}
\mathcal{P}'h\mathcal{P}'h\mathcal{P}' &= -4\{x,y\}^2\frac{1}{4}\det(g)\mathcal{P}' \\
&= -\{x,y\}^2\det(g)\mathcal{P}'
\end{aligned}$$

*giving* $\eta' = (\{x,y\}^2\det(g))^{-1}$*. Therefore* $\eta' = \eta$*. We conclude that* $g(\alpha,\beta) = h(\alpha,\beta)$*, which shows that* $\mathcal{K} \cong \mathcal{K}'$*.*

## 4.2 Properties of isomorphisms

Let $\mathcal{K} = (\mathcal{A}, g, \{x^1,...,x^m\})$ and $\mathcal{K}' = (\mathcal{A}', g', \{y^1,...,y^{m'}\})$ be Kähler-Poisson algebras. Assume that there exists a Poisson-algebra isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$, when does there exist a map $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $(\phi,\psi)$ is an isomorphism of Kähler-Poisson algebras?. The following result provides an answer to this question.

**Proposition 4.2.1.** *Let* $\mathcal{K} = (\mathcal{A}, g, \{x^1,...,x^m\})$ *and* $\mathcal{K}' = (\mathcal{A}', g', \{y^1,...,y^{m'}\})$ *be Kähler-Poisson algebras.* $\mathcal{K}$ *and* $\mathcal{K}'$ *are isomorphic if and only if there exists a Poisson algebra isomorphism* $\phi : \mathcal{A} \to \mathcal{A}'$ *such that* $\phi(\mathcal{A}_{fin}) = \mathcal{A}'_{fin}$*, and*

$$\mathcal{P}'g'\mathcal{P}' = \mathcal{P}'A^T\phi(g)A\mathcal{P}',$$

*where* ${A^i}_\alpha = \frac{\partial\phi(x^i)}{\partial y^\alpha}$ *and* $(\mathcal{P}')^{\alpha\beta} = \{y^\alpha, y^\beta\}$*.*

*Proof.* Assume $\mathcal{K} \cong \mathcal{K}'$ then we need to show that

$$\mathcal{P}'g'\mathcal{P}' = \mathcal{P}'A^T\phi(g)A\mathcal{P}'.$$

Let us start by computing $\phi(\mathcal{P}^{ij})$

$$
\begin{aligned}
\phi(\mathcal{P}^{ij}) = \phi(\{x^i, x^j\}) = \{\phi(x^i), \phi(x^j)\}' = \frac{\partial \phi(x^i)}{\partial y^\alpha}\{y^\alpha, \phi(x^i)\} \\
= \frac{\partial \phi(x^i)}{\partial y^\alpha}\{y^\alpha, y^\beta\}\frac{\partial \phi(x^j)}{\partial y^\beta} = A^i{}_\alpha(\mathcal{P}')^{\alpha\beta}A^j{}_\beta.
\end{aligned}
$$

Now, let $\alpha = \{x^i, .\}$ and $\beta = \{x^j, .\}$, then

$$\phi(g(\alpha, \beta)) = \phi(\{x^i, x^k\}g_{kl}\{x^j, x^l\}) = \phi(\mathcal{P}^{ik}g_{kl}\mathcal{P}^{jl}).$$

From the second property of Definition 4.1.2 it follows that

$$\psi(\alpha)(b) = \phi(\{x^i, \phi^{-1}(b)\}) = \{\phi(x^i), b\}',$$

which implies that,

$$\psi(\alpha) = \{\phi(x^i), \cdot\}', \quad \psi(\beta) = \{\phi(x^j), \cdot\}'.$$

Furthermore,

$$\phi(\mathcal{P}^{ik}g_{kl}\mathcal{P}^{jl}) = \{\phi(x^i), y^\alpha\}'A^k{}_\alpha\phi(g_{kl})\{\phi(x^j), y^\beta\}'A^l{}_\beta,$$

and

$$g'(\psi(\alpha), \psi(\beta)) = \{\phi(x^i), y^\alpha\}'g'_{\alpha\beta}\{\phi(x^j), y^\beta\}'.$$

and from the third property of Definition 4.1.2 one obtains

$$\{\phi(x^i), y^\alpha\}'(A^k{}_\alpha\phi(g_{kl})A^l{}_\beta - g'_{\alpha\beta})\{\phi(x^j), y^\beta\}' = 0. \tag{4.2.1}$$

Note that if $\{\phi(x^i), y^\beta\}C_\beta = 0$, for $C_\beta \in \mathcal{A}'$, then

$$
\begin{aligned}
\{y^\alpha, y^\beta\}'C_\beta &= \phi(\{\phi^{-1}(y^\alpha), \phi^{-1}(y^\alpha)\})C_\beta \\
&= \phi\Big(\frac{\partial \phi^{-1}(y^\alpha)}{\partial x^i}\{x^i, \phi^{-1}(y^\beta)\}\Big)C_\beta \\
&= \phi\Big(\frac{\partial \phi^{-1}(y^\alpha)}{\partial x^i}\Big)\phi\Big(\{x^i, \phi^{-1}(y^\beta)\}\Big)C_\beta \\
&= \phi\Big(\frac{\partial \phi^{-1}(y^\alpha)}{\partial x^i}\Big)\{\phi(x^i), y^\beta\}'C_\beta \\
&= 0.
\end{aligned}
$$

Therefore, equation (4.2.1) yields

$$\{y^\gamma, y^\alpha\}'(A^k{}_\alpha\phi(g_{kl})A^l{}_\beta - g'_{\alpha\beta})\{\phi(x^j), y^\beta\}' = 0,$$

and furthermore

$$\{y^\gamma, y^\alpha\}'(A^k{}_\alpha\phi(g_{kl})A^l{}_\beta - g'_{\alpha\beta})\{y^\delta, y^\beta\}' = 0.$$

In matrix notation this becomes

$$\mathcal{P}' A\phi(g)A^T\mathcal{P}' = \mathcal{P}'g'\mathcal{P}'. \tag{4.2.2}$$

To prove the converse, we assume that $\phi$ is an isomorphism, $\phi(\mathcal{A}_{\text{fin}}) = \mathcal{A}'_{\text{fin}}$ and (4.2.2) holds. First, we need to define the map $\psi$. Since $\phi$ is an isomorphism we may define $\psi$ as:

$$\psi(\alpha)(a') := \phi(\alpha(\phi^{-1}(a'))),$$

which clearly fulfills:

$$\phi(\alpha(a)) = \psi(\alpha)(\phi(a)).$$

Next, let us show that $\psi(\alpha) \in \mathfrak{g}'$ if $\alpha \in \mathfrak{g}$. For $\alpha = a_i\{b^i, .\} \in \mathfrak{g}$ one obtains

$$\begin{aligned}
\psi(\alpha)(a') &= \phi(\alpha(\phi^{-1}(a'))) \\
&= \phi(a_i\{b^i, \phi^{-1}(a')\}) \\
&= \phi(a_i)\{\phi(b^i), a'\}',
\end{aligned}$$

which implies that

$$\psi(\alpha) = \phi(a_i)\{\phi(b^i), .\}' \in \mathfrak{g}'.$$

Secondly, we need to show that $\psi$ is a Lie algebra isomorphism. To show that $\psi$ is a homomorphism we need to show the following properties:

$$\begin{aligned}
[\psi(\alpha), \psi(\beta)](a') &= \psi(\alpha)\big(\psi(\beta)(a')\big) - \psi(\beta)\big(\psi(\alpha)(a')\big) \\
&= \psi(\alpha)\big(\phi(\beta(\phi^{-1}(a')))\big) - \psi(\beta)\big(\phi(\alpha(\phi^{-1}(a')))\big) \\
&= \phi\big(\alpha(\beta(\phi^{-1}(a')))\big) - \phi\big(\beta(\alpha(\phi^{-1}(a')))\big) \\
&= \phi\big(\alpha(\beta(\phi^{-1}(a'))) - \beta(\alpha(\phi^{-1}(a')))\big) \\
&= \phi\big([\alpha, \beta](\phi^{-1}(a')\big) \\
&= \psi([\alpha, \beta])(a').
\end{aligned}$$

Also

$$\begin{aligned}
\psi(\alpha)(a') + \psi(\beta)(a') &= \phi\big(\alpha(\phi^{-1}(a'))\big) + \phi\big(\beta(\phi^{-1}(a'))\big) \\
&= \phi\big(\alpha(\phi^{-1}(a') + \beta(\phi^{-1}(a'))\big) \\
&= \phi\big((\alpha + \beta)(\phi^{-1}(a'))\big) \\
&= \psi(\alpha + \beta)(a').
\end{aligned}$$

Therefore $\psi$ is a Lie algebra homomorphism. To show that $\psi$ is an isomorphism, we define the map $\psi^{-1}$ as

$$\psi^{-1}(\alpha')(a) := \phi^{-1}(\alpha'(\phi(a))),$$

for all $a \in \mathcal{A}$ and $\alpha' \in \mathfrak{g}'$ then one obtains

$$\begin{aligned}\psi^{-1}\big(\psi(\alpha)\big)(a) &= \phi^{-1}\Big(\psi(\alpha)\big(\phi(a)\big)\Big)\\ &= \phi^{-1}\Big(\phi\big(\alpha(\phi^{-1}\phi(a))\big)\Big)\\ &= \alpha(a)\end{aligned}$$

and

$$\begin{aligned}\psi\big(\psi^{-1}(\alpha')\big)(a') &= \phi\Big(\psi^{-1}(\alpha')\big(\phi^{-1}(a')\big)\Big)\\ &= \phi\Big(\phi^{-1}\big(\alpha'(\phi\phi^{-1}(a'))\big)\Big)\\ &= \alpha'(a').\end{aligned}$$

Finally, we show that $(\phi, \psi)$ is a morphism of Kähler-Poisson algebras.
1)

$$\begin{aligned}\phi(a)\psi(\alpha)(a') &= \phi(a)\phi(\alpha(\phi^{-1}(a')))\\ &= \phi(a\alpha(\phi^{-1}(a')))\\ &= \psi(a\alpha)(a').\end{aligned}$$

2)

$$\begin{aligned}\psi(\alpha)(\phi(a)) &= \phi(\alpha(\phi^{-1}(\phi(a))))\\ &= \phi(\alpha(a)).\end{aligned}$$

3) For $\alpha = \alpha_i\{x^i, .\}$ and $\beta = \beta_i\{x^i, .\}$ one gets

$$\begin{aligned}\phi(g(\alpha,\beta)) &= \phi(\alpha_i\{x^i, x^k\}g_{kl}\beta_j\{x^j, x^l\})\\ &= \phi(\alpha_i)\phi(\{x^i, x^k\})\phi(g_{kl})\phi(\beta_j)\phi(\{x^j, x^l\})\\ &= \phi(\alpha_i)\phi(\mathcal{P}^{ik})\phi(g_{kl})\phi(\beta_j)\phi(\mathcal{P}^{jl})\\ &= \phi(\alpha_i)(A\mathcal{P}'A^T)^{ik}\phi(g_{kl})\phi(\beta_j)(A\mathcal{P}'A^T)^{jl}\\ &= \phi(\alpha_i)\phi(\beta_j)\big(A\mathcal{P}'A^T\phi(g)A(-\mathcal{P}')A^T\big)^{ij}\\ &= -\phi(\alpha_i)\phi(\beta_j)\big(A\mathcal{P}'A^T\phi(g)A\mathcal{P}'A^T\big)^{ij}.\end{aligned}$$

By using

$$\mathcal{P}'g'\mathcal{P}' = \mathcal{P}'A^T\phi(g)A\mathcal{P}',$$

one obtains

$$\begin{aligned}\phi(g(\alpha,\beta)) &= -\phi(\alpha_i)\phi(\beta_j)(A\mathcal{P}'g'\mathcal{P}'A^T)^{ij}\\ &= -\phi(\alpha_i)\phi(\beta_j)A^i{}_\alpha\{y^\alpha, y^\beta\}g'_{\beta\gamma}\{y^\gamma, y^\delta\}A^j{}_\delta\\ &= -\phi(\alpha_i)\phi(\beta_j)\{\phi(x^i), y^\beta\}g'_{\beta\gamma}\{y^\gamma, \phi(x^j)\}\\ &= \psi(\alpha)(y^\beta)g'_{\beta\gamma}\psi(\beta)(y^\gamma)\\ &= g'(\psi(\alpha), \psi(\beta)).\end{aligned}$$

Note that 4. is true by assumption. □

As an illustration, let us apply this result to Example 4.1.5 to again find the metric $h$ such that $\mathcal{K} \cong \mathcal{K}'$.

**Example 4.2.2.** *(Continuation of 4.1.5)*
*From Proposition 4.2.1 it follows that if we set $h = A^T\phi(g)A$, then $\mathcal{P}'h\mathcal{P}' = \mathcal{P}'A^T\phi(g)A\mathcal{P}'$ implying that $\mathcal{K} \cong \mathcal{K}'$.*

*Let $y^1 = x + y$ and $y^2 = x - y$. Hence $x = \frac{1}{2}(y^1 + y^2)$ and $y = \frac{1}{2}(y^1 - y^2)$. We compute the matrix $A$ from*

$$A^i{}_\alpha = \frac{\partial\phi(x^i)}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha},$$

*recalling that $\phi = id$. Therefore,*

$$A^1{}_1 = \frac{\partial x^1}{\partial y^1} = \frac{\partial(\frac{1}{2}(y^1+y^2))}{\partial y^1} = \frac{1}{2}.$$

$$A^1{}_2 = \frac{\partial x^1}{\partial y^2} = \frac{\partial(\frac{1}{2}(y^1+y^2))}{\partial y^2} = \frac{1}{2}.$$

$$A^2{}_1 = \frac{\partial x^2}{\partial y^1} = \frac{\partial y}{\partial y^1} = \frac{\partial(\frac{1}{2}(y^1-y^2))}{\partial y^1} = \frac{1}{2}.$$

$$A^2{}_1 = \frac{\partial x^2}{\partial y^2} = \frac{\partial y}{\partial y^2} = \frac{\partial(\frac{1}{2}(y^1-y^2))}{\partial y^2} = -\frac{1}{2}.$$

*Therefore the matrix $A$ becomes*

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

*and*

$$\begin{aligned} h &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} g_{11} & g_{22} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(g_{11} + g_{21}) & \frac{1}{2}(g_{12} + g_{22}) \\ \frac{1}{2}(g_{11} - g_{21}) & \frac{1}{2}(g_{12} - g_{22}) \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}(g_{11} + g_{21}) + \frac{1}{4}(g_{12} + g_{22}) & \frac{1}{4}(g_{11} + g_{21}) - \frac{1}{4}(g_{12} + g_{22}) \\ \frac{1}{4}(g_{11} - g_{21}) + \frac{1}{4}(g_{12} - g_{22}) & \frac{1}{4}(g_{11} - g_{21}) - \frac{1}{4}(g_{12} - g_{22}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}(g_{11} + 2g_{12} + g_{22}) & \frac{1}{4}(g_{11} - g_{22}) \\ \frac{1}{4}(g_{11} - g_{22}) & \frac{1}{4}(g_{11} - 2g_{12} + g_{22}) \end{pmatrix}, \end{aligned}$$

*which agrees with the result in Example 4.1.5. Also $\phi(\mathcal{A}_{fin}) = \phi(\mathcal{A}'_{fin})$, since $\phi = id$. Therefore we can now use Proposition 4.2.1 to conclude that $\mathcal{K} \cong \mathcal{K}'$.*

In all the examples, we have seen that $\eta = \eta'$ if $\mathcal{K} \cong \mathcal{K}'$. This is a more general fact, which is stated in the following result.

**Proposition 4.2.3.** *Let $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ and $\mathcal{K}' = (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$ be Kähler-Poisson algebras together with their corresponding modules of derivations $\mathfrak{g}$ and $\mathfrak{g}'$. Let $(\phi, \psi) : \mathcal{K} \to \mathcal{K}'$ be an isomorphism of Kähler-Poisson algebras. If*

$$\eta\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mathcal{P} \text{ and } \eta'\mathcal{P}'g'\mathcal{P}'g'\mathcal{P}' = -\mathcal{P}'$$

*then* $(\phi(\eta)-\eta')\mathcal{P}'=0$.

*Proof.* By Proposition 4.2.1 one obtains

$$\mathcal{P}'g'\mathcal{P}'=\mathcal{P}'A^T\phi(g)A\mathcal{P}'.$$

Starting from $\eta\mathcal{P}g\mathcal{P}g\mathcal{P}=-\mathcal{P}$ and using that $\phi(\mathcal{P})=A\mathcal{P}'A^T$ and $\mathcal{P}'g'\mathcal{P}'=\mathcal{P}'A^T\phi(g)A\mathcal{P}'$, one obtains

$$-\phi(\mathcal{P})=\phi(\eta)\phi(\mathcal{P}g\mathcal{P}g\mathcal{P}).$$

By multiplying both sides by $\eta'$ one obtains

$$\begin{aligned}
-\eta'\phi(\mathcal{P})&=\phi(\eta)\phi(\mathcal{P}g\mathcal{P}g\mathcal{P})\eta'\\
&=\phi(\eta)A\mathcal{P}'A^T\phi(g)A\mathcal{P}'A^T\phi(g)A\mathcal{P}'A^T\eta'\\
&=\phi(\eta)A\mathcal{P}'g'\mathcal{P}'A^T\phi(g)A\mathcal{P}'A^T\eta'\\
&=\phi(\eta)A\eta'\mathcal{P}'g'\mathcal{P}'g'\mathcal{P}'A^T\\
&=-\phi(\eta)A\mathcal{P}'A^T,
\end{aligned}$$

by using that $\eta'\mathcal{P}'g'\mathcal{P}'g'\mathcal{P}'=-\mathcal{P}'$. Hence, one obtains

$$\eta'A\mathcal{P}'A^T=\phi(\eta)A\mathcal{P}'A^T,$$

which implies that

$$(\eta'-\phi(\eta))\mathcal{P}'=0,$$

by using the same argument as in the proof of Proposition 4.2.1 □

Note that if at least one of $\{y^\alpha,y^\beta\}$ is not a zero divisor, then Proposition 4.2.3 implies that $\phi(\eta)=\eta'$.

## 4.3 Outlook

In this thesis, Kähler-Poisson algebras have been introduced, along with a concept of homomorphism between such algebras. These algebras mimic algebras of smooth functions on Kähler manifolds, and contains metric information. Basic results and properties, as well as a manifold of examples, have been provided in order to better understand the novel concepts. There are many open questions that one would like to address in future work. Let us present a few of them here.

- We have seen that almost any (finitely generated) Poisson algebra may be localized in order to construct a Kähler-Poisson algebra. This procedure involves the choice of a metric, and it is natural to ask if it is possible to classify (up to isomorphism) all the possible metrics for a given Poisson algebra? Is it possible to understand the moduli space of such metrics? In general, the question is probably quite hard to answer, but a more tractable problem would be to solve this starting from Poisson algebras generated by two elements.

- In all the examples of isomorphism so far, we have chosen $\phi = id$ and $\psi = id$. What happens if we do not require this?
- The examples in this thesis have focused on the case of finitely generated algebras. For algebras that are not finitely generated, there are many inequivalent choices of generators to construct a Kähler-Poisson algebra. Can one find a way to classify (up to isomorphism) the possible choices of generators for the Kähler-Poisson algebra?
- From a more algebraic point of view, one would like to introduce direct sums and tensor products of Kähler-Poisson algebras. Can this be done in a straight-forward manner? What are the properties of these operations?
- For finitely generated algebras, one should compare the geometric approach for Kähler-Poisson algebras with the language of algebraic geometry. How does the Poisson structure interact with the standard definitions of algebraic geometry? Is it possible to introduce a Kähler-Poisson algebra structure on projective varieties?
- The symmetric algebra of any Lie algebra is a Poisson algebra which in many cases can be made into a Kähler-Poisson algebra. It would be interesting to compare the geometry of such algberas with the geometry of Lie groups. In what way does the Kähler-Poisson algebra reflect the geometry of the Lie group?

# References


[1] J. Arnlind and G. Huisken. Pseudo-Riemannian geometry in terms of multilinear brack- ets. Lett. Math. Phys., 104(12):15071521, 2014.

[2] J. Arnlind, J. Hoppe, and G. Huisken. Multi-linear formulation of differential geometry and matrix regularizations. J. Differential Geom., 91(1):139, 2012.

[3] J.Arnlind. A. Al-Shujary. Kähler-Poisson Algebras. arXiv:1712.08397.

[4] R. Berger. Géométrie algébrique de Poisson. C. R. Acad. Sci. Paris Sér. A-B, 289(11):A583A585, 1979.

[5] J.-L. Brylinski. A differential complex for Poisson manifolds. J. Differential Geom., 28(1):93114, 1988.

[6] S. Helgason. Differential geometry, Lie groups, and symmetric spaces, volume 34 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

[7] J.-C. Herz. Pseudo-algébres de Lie. I. C. R. Acad. Sci. Paris, 236:19351937, 1953.

[8] J. Huebschmann. Poisson cohomology and quantization. J. Reine Angew. Math., 408:57 113, 1990.

[9] J. Huebschmann. Extensions of Lie-Rinehart algebras and the Chern-Weil construction. In Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), volume 227 of Contemp. Math., pages 145176. Amer. Math. Soc., Provi- dence, RI, 1999.

[10] M. Kontsevich. Deformation quantization of Poisson manifolds. Lett. Math. Phys., 66(3):157216, 2003.

[11] J. L. Kozul. Lectures on fibre bundles and differential geometry. Tata Institute of Fun- damental Research, Bombay, 1960.

[12] E. Kähler, Uber eine bemerkenswerte Hermitesche Metrik, Abh. Math. Sem. Hamburg Univ. 9 (1933), 173186.

[13] A. Lichnerowicz. Les variétés de Poisson et leurs algébres de Lie associées. J. Differential Geometry, 12(2):253300, 1977.

[14] J. Nash. The imbedding problem for Riemannian manifolds. Ann. of Math. (2), 63:20 63, 1956.

[15] E. Nelson. Tensor Analysis. Princeton University Press, Princeton, New Jersey, 1967.

[16] R. S. Palais. The cohomology of Lie rings. In Proc. Sympos. Pure Math., Vol. III, pages 130137. American Mathematical Society, Providence, R.I., 1961.

[17] G. S. Rinehart. Differential forms on general commutative algebras. Trans. Amer. Math. Soc., 108:195222, 1963.

[18] A. Weinstein. The local structure of Poisson manifolds. J. Differential Geom., 18(3):523 557, 1983.

[19] R.O. Wells jr., Differential analysis on complex manifolds, Springer-Verlag, New York-Heidelberg-Berlin,1980.

# Paper I

The paper associated with this thesis have been removed for copyright reasons. For more details about these see:

http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-150620