Quantum Annealing

Continuous-variable quantum annealing with superconducting circuits

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Quantum annealing is expected to be a powerful generic algorithm for solving hard combinatorial optimization problems faster than classical computers. Finding the solution to a combinatorial optimization problem is equivalent to finding the ground state of an Ising Hamiltonian. In today's quantum annealers the spins of the Ising Hamiltonian are mapped to superconducting qubits. On the other hand, dissipation processes degrade the success probability of finding the solution. In this thesis we set out to explore a newly proposed architecture for a noise-resilient quantum annealer that instead maps the Ising spins to continuous variable quantum states of light encoded in the field quadratures of a two-photon pumped Kerr-nonlinear resonator based on the proposal by Puri et al. (2017). In this thesis we study the Wigner negativity for this newly proposed architecture and evaluate its performance based on the negativity of the Wigner function. We do this by determining an experimental value to when the presence of losses become too detrimental, such that the Wigner function of the quantum state during the evolution within the anneal becomes positive for all times.

Furthermore, we also demonstrate the capabilities of this continuous variable quantum annealer by simulating and finding the best solution of a small instance of the NP-complete subset sum problem and of the number partitioning problem.

Nyckelord
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Keywords: quantum annealing, adiabatic quantum computing, continuous variables, coherent states, Wigner function, superconducting circuits, Kerr-nonlinear resonator.
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Quantum Annealing (QA) is a quantum algorithm that is based on Adiabatic Quantum Computation (AQC), and aims at solving hard combinatorial optimization problems. A combinatorial optimization problem seeks to find the best answer to a given problem from a vast collection of configurations. A typical example of a combinatorial optimization problem is the travelling salesman problem, where a salesman seeks to find the shortest travel distance between different locations, such that all locations are visited once. The naive method to solve this problem would be to make a list of all the different routes between locations that the salesman could take and from that list find the best answer. This could work when the number of locations are few, but the naive method would fail if the number of locations grows large, since the number of possible configurations increases exponentially with the number of locations. Thus the naive method is not efficient in practice, and we should therefore develop algorithms.

A combinatorial optimization problem like the travelling salesman problem can be mapped onto finding the ground state of an Ising Hamiltonian, these problems are therefore also referred to as Ising problems. Finding the ground state of an Ising Hamiltonian can be computationally difficult \[1\]. The main idea of QA is to cleverly take advantage of adiabatic evolution to go from a simple non-degenerate ground state of an initial Hamiltonian to the ground state of an Ising Hamiltonian, that encodes the solution of the desired combinatorial optimization problem.

The uses of QA are ample and diverse, it can for example be used for finding the low-energy conformations of lattice protein, or for finding the optimal way of investing money for set assets, or even for the training of deep neural networks \[2\]–\[5\].

1.1 Adiabatic quantum computing & quantum annealing

In 2001 Farhi et al. \[6\] invented adiabatic quantum computation which is based on the adiabatic theorem\(^1\). The theorem states that a system that begins from a non-degenerate ground state of a time-dependent Hamiltonian \(\mathcal{H}(t)\) that is slowly changing from some initial form \(\mathcal{H}_0\) to some final form \(\mathcal{H}_1\), during time \(\tau\), will remain in its instantaneous ground state throughout the evolution, provided that the Hamiltonian varies sufficiently slow. Assuming a linear time

\(^1\)See appendix A.1 for proof of the adiabatic theorem.
dependence the AQC Hamiltonian is defined as

\[ \hat{H}(t) = \left( 1 - \frac{t}{\tau} \right) \hat{H}_0 + \frac{t}{\tau} \hat{H}_1, \quad (0 \leq t \leq \tau), \]  

(1.1)

where the coefficient in front of \( \hat{H}_0 \) changes from unity to zero, and the coefficient in front of \( \hat{H}_1 \) changes from zero to unity. Moreover it is crucial that \( \hat{H}_0 \) and \( \hat{H}_1 \) are two noncommuting Hamiltonians. In quantum annealing \( \hat{H}_0 \) is such that its ground state is easy to construct and \( \hat{H}_1 \) is such that its ground state encodes the solution of a combinatorial optimization problem.

We may logically ask: what is the minimum evolution time \( \tau \), such that the time-evolution in Eq. (1.1) is considered adiabatic? According to the adiabatic theorem it can be shown that for a non-degenerate ground state, the adiabatic evolution is assured if the evolution time \( \tau \) satisfies the following condition [7], [8]

\[ \frac{\max_{0 \leq s \leq 1} | \langle \psi_1(s) | \frac{d}{ds} \hat{H}(s) | \psi_0(s) \rangle |}{\min_{0 \leq s \leq 1} \Delta^2(s)} \geq \frac{t}{\tau}; \quad s \equiv \frac{t}{\tau} \]  

(1.2)

where \( | \psi_0(s) \rangle \) and \( | \psi_1(s) \rangle \) are the instantaneous ground and first excited eigenstate of the Hamiltonian in Eq. (1.1), and \( \Delta^2(s) = (E_1(s) - E_0(s))^2 \) is the instantaneous energy gap between the ground state and first excited state energies. If the criterion of Eq. (1.2) is fulfilled, then we can be certain that the system will evolve into the ground state of \( \hat{H}_1 \).

### 1.2 Quantum annealing over continuous variables

In a classical computer the fundamental unit of information is represented by a bit. The bit is based on the binary system and can be either 0 or 1. In contrast, a quantum computer uses quantum bits, or qubits for short. Qubits have the additional property to exist in a superposition of both \( |0\rangle \) and \( |1\rangle \) at the same time

\[ |\psi\rangle = a|0\rangle + b|1\rangle, \]  

(1.3)

where \( a \) and \( b \) are two complex numbers that satisfy the normalization condition \( |a|^2 + |b|^2 = 1 \). A qubit can be encoded onto any two-level quantum system, like the spin of an electron, the polarization of a photon or the ground and excited states of an atom. In the traditional implementations of quantum annealers, the Ising spins of the Ising Hamiltonian are mapped to qubits [7], [9]. However quantum computation and quantum annealing is not limited to a discrete variable setup, such as qubits. It is also possible to encode information into continuous variables, like position and momentum or the amplitude of the electromagnetic field [10]. The difference between a discrete and continuous variable quantum system is that the discrete variable quantum system is described by a finite Hilbert space, while continuous variable quantum system must be described by an infinite dimensional Hilbert space [11]. As an example, a spin-1/2 particle is described by a two dimensional Hilbert space, while a single-mode of the free electromagnetic field is modelled by a quantum harmonic oscillator, and associated to an infinite dimensional Hilbert space.

Physical realizations of qubit based quantum annealers have been demonstrated [12], [13]. The D-wave machine, which is the world’s first commercially available qubit based quantum annealer, has been shown to exhibit a constant speed up over classical algorithms for certain problems\(^3\) [14]. However, dissipation processes are unavoidable, and the presence of noise will ultimately lead to lower success probability of finding the optimal solution. So in order to

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\(^2\)An argument for this is given in appendix A.2.

\(^3\)By constant speed up we mean that the scaling of problem size versus time is the same for the D-wave machine and certain classical algorithms but that they are separated by a constant overhead pre factor.
1.3 Motivation & goal

In this thesis we set out to explore the continuous variable quantum annealing architecture proposed by Puri et al. (2017) [15], [16], by trying to understand the onset for efficient classical simulatability based on the Wigner function. The Wigner function is a phase-space quasi probability distribution, meaning that it can take on negative values. It has been shown that quantum circuits where the quantum state during the time evolution is described by a positive Wigner function, in combination with measurements that are described by positive Wigner function such as homodyne detection, can be classically efficiently simulated [17], [18]. The negativity of the Wigner function is therefore a requirement if a quantum annealer is to outperform its classical counterpart. In this thesis we aim at determining an experimental value to when the presence of losses become too detrimental in this architecture, such that the Wigner function of the quantum state during the anneal becomes positive for all times.

Furthermore, in order to demonstrate the capabilities of this architecture we will simulate the optimal solution of the NP-complete subset sum problem and the number partitioning problem.

1.4 Organization of the thesis

The organization of this thesis is as follows. Chapter 2 is devoted to theory and background. In this chapter we will begin by explaining how computational problems are classified based on their inherit “difficulty”. We will then introduce the Ising Hamiltonian, and show how the subset sum problem and the number partitioning problem can be mapped onto finding the ground state of an Ising Hamiltonian. We will then proceed with a review of the quantization of the electromagnetic field and from that present the very useful coherent states, which are states of the quantized electromagnetic field. After this we will introduce the Wigner function and discuss the meaning of non-classicality, and how it can be viewed as a resource in quantum computation. We will then introduce the Lindblad master equation that describes the time-evolution of an open quantum system in presence of dissipation processes.

In Chapter 3 we present the two-photon pumped Kerr-nonlinear resonator. This will be our main building block for constructing a continuous variable quantum annealer. We will also discuss how the continuous variable quantum annealer is physically realized in superconducting circuits.

In Chapter 4 we present and discuss the results, and finally we summarize and end with a conclusion in Chapter 5.
In this chapter we will begin by giving a description of computational complexity theory. We will then introduce the Ising Hamiltonian and show some examples of how combinatorial optimization problems can be mapped onto an Ising Hamiltonian. We will then provide a review of the quantization of the electromagnetic field and from that move on to present the coherent states, which will play an important role in this thesis. After that we will introduce the phase space representation that rests on the concept of the Wigner function. At the end of this chapter we will discuss the modelling of open quantum systems and lastly introduce the concept of fidelity.

2.1 Computational complexity theory

Not all combinatorial optimization problems are equally hard. Several problems in computer science can be considered as decision problems, i.e. problems with a “yes” or “no” answer. A decision problem for instance, would be to ask; “Is the greatest common divisor (gcd) between 30 and 8 equal to 2?” In this case the answer is “yes”.

Loosely speaking, in computational complexity theory the complexity class $P$ represents all decision problems that can be efficiently solved on a classical computer. By efficiently we mean it can be solved in a time that grows no faster than a polynomial in the size of the problem. Example of such a problem is finding the gcd between two integers.

$NP$ (which stands for Non deterministic Polynomial time) is the class of decision problems which solution when given a “yes” answer, can be efficiently verified on a classical computer. An example of this is prime factorization, which is to find two prime factors to an integer $n$. There exist today no known classical algorithm that can solve this problem in polynomial time, which therefore suggests that the problem is not in $P$. However once given a number $p$ one can quickly verify if $p$ is a divisor of $n$, by simply dividing $p$ into $n$.

A decision problem is said to be $NP$-complete if it lies in $NP$ and all other problems in $NP$ can be reduced to it in polynomial time. Thus the $NP$-complete problems are the only ones we need to study to understand the computational resources needed to solve all of the problems in $NP$ [19] (*one problem to rule them all!*). An example of a $NP$-complete problem is the subset sum problem which we will consider in section 2.3.

Lastly, $NP$-hard problems are the problems that are at least as hard as the $NP$-complete problems, but that are not necessarily contained in $NP$. In other words, a problem is $NP$-hard
2.2. Ising model & qubit based quantum annealing

The Ising model consists of $N$ Ising spins on a lattice that can take the values $s_i = +1$ or $s_i = -1$, which corresponds to the spin-$\uparrow$ and the spin-$\downarrow$ direction. Here $s_i$ denotes the Ising spin at site $i$ on the lattice. The Ising spins are coupled together through long-range magnetic interactions, which can be either ferromagnetic or antiferromagnetic, corresponding to the spins being encouraged to be aligned parallel or antiparallel respectively. Moreover, an external magnetic field can be applied at each individual spin site which will give a different energy to the spin-$\uparrow$ and spin-$\downarrow$ directions. The energy configuration of the Ising model with $N$ Ising spins is given by the Ising Hamiltonian

$$H(s_1, \ldots, s_N) = - \sum_{i,j=1}^{N} J_{ij} s_i s_j - \sum_{i=1}^{N} h_i s_i, \quad (s_i = \pm 1), \quad (2.1)$$

where $J_{ij}$ is the coupling strength between the $i$th spin and $j$th spin and $h_i$ is the magnetic field at the $i$th spin site. Note that for a given set of couplings, magnetic fields and spin-configurations, the Ising Hamiltonian will just “spit out” a number that represents the energy of that particular spin-configuration. For a given set of couplings and magnetic fields there always exists at least one spin-configuration that minimizes the energy of the Ising Hamiltonian.

To solve an Ising problem on a qubit based quantum annealer one defines a set of qubits $|q_1 q_2 \ldots q_N\rangle$ to store the answer to the problem and interpret $q_i = 0$ to mean $s_i = +1$ (spin-$\uparrow$) and $q_i = 1$ to mean $s_i = -1$ (spin-$\downarrow$) [19]. Then one chooses a pair of noncommuting Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_1$, such that the ground state of $\mathcal{H}_0$ is easy to prepare, and that the ground state of $\mathcal{H}_1$ encodes the solution to the Ising problem. To achieve this the initial Hamiltonian is

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1One of the Millennium Prize Problems is to prove that $P \neq NP$ [20].
2.3 Examples of combinatorial optimization problems

We now turn to a couple of specific examples of combinatorial optimization problems that can be solved on a quantum annealer, and how they can be mapped onto Ising Hamiltonians.

Subset sum problem

The subset sum problem is a famous combinatorial optimization problem that is known to be \textbf{NP}-complete. It is defined as the following: Given an integer \( m \) and a set of positive and negative integers \( n = \{n_1, n_2, \ldots, n_N\} \) of length \( N \), is there a subset of those integers that sums exactly to \( m \)?

\textbf{Example:} Consider the case when \( m = 7 \) and the set \( n = \{-5, -3, 1, 4, 9\} \). In this particular example answer is “\textit{yes}”, and the subset is \( \{-3, 1, 9\} \).

\textbf{Example:} Consider \( m = 13 \) and \( n = \{-3, 2, 8, 4, 20\} \). This time the answer is “\textit{no}”.

This problem can be framed as an energy minimization problem. The energy function for the subset sum problem can be formulated as [21]

\[
E'(\varepsilon_1, \ldots, \varepsilon_N) = \left( \sum_{i=1}^{N} n_i \varepsilon_i - m \right)^2, \quad \varepsilon_i \in \{0, 1\},
\]

where \( N \) corresponds to the size of the subset. Hence, if a configuration of \( \varepsilon_i \) exists such that \( E' = 0 \), then the answer is “\textit{yes}”. Likewise if \( E' > 0 \) for all possible configurations of \( \varepsilon_i \), then the answer is “\textit{no}”. To map this energy function onto an Ising Hamiltonian we introduce the Ising spins \( s_i = \pm 1 \) instead of \( \varepsilon_i \) as

\[
\varepsilon_i = \frac{1}{2}(1 + s_i),
\]
such that $s_i = +1$ (spin-$\uparrow$) corresponds to $\varepsilon_i = 1$, and $s_i = -1$ (spin-$\downarrow$) corresponds to $\varepsilon_i = 0$. Then the Hamiltonian is written as

$$H(s_1, \ldots, s_N) = \left( \sum_{i=1}^{N} n_i \frac{1}{2} (1 + s_i) - m \right)^2$$

$$= \left( \sum_{i=1}^{N} \frac{1}{2} n_i s_i + \frac{1}{2} \sum_{i=1}^{N} n_i - m \right)^2$$

$$= \frac{1}{4} \sum_{1 \leq i, j \leq N} n_i n_j s_i s_j + \sum_{i=1}^{N} h_i s_i + \left( \frac{1}{2} \sum_{j=1}^{N} n_j - m \right)^2.$$

We now introduce the coupling $J_{ij}$ and the magnetic field $h_i$ as

$$J_{ij} \equiv n_i n_j, \quad \text{and} \quad h_i = \left( \frac{1}{2} \sum_{j=1}^{N} n_j - m \right) n_i,$$

and finally obtain

$$H(s_1, \ldots, s_N) = \frac{1}{4} \sum_{1 \leq i, j \leq N} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i + \left( \frac{1}{2} \sum_{j=1}^{N} n_j - m \right)^2.$$

Notice that the last term is simply a constant. This Hamiltonian is an Ising Hamiltonian cf. Eq. (2.1). Furthermore, by observing that $J_{ij}$ is symmetric and that the sum of the diagonal elements $i = j$ is equal to the trace of $J_{ij}$, we get

$$H(s_1, \ldots, s_N) = \frac{1}{2} \sum_{1 \leq i < j \leq N} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i + \left( \frac{1}{2} \sum_{j=1}^{N} n_j - m \right)^2 + \frac{1}{4} \text{Tr}[J_{ij}]$$

$$= \sum_{1 \leq i < j \leq N} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i + \text{const},$$

where we have absorbed the $1/2$ into $J_{ij}$ in the second equality and made an implicit redefinition of the couplings $J_{ij} \equiv n_i n_j/2$.

Number partitioning problem

The number partitioning problem is another well-known combinatorial optimization problem that is also NP-complete [25, 26]. The number partitioning problem is defined as follows: Given a set $S$ of positive integers $\{n_1, n_2, \ldots, n_N \}$ of length $N$, can this set be partitioned into two sets $S_1$ and $S_2$ such that the sum of the sets are equal?

**Example:** Consider the set $\{1, 2, 3, 4, 6, 10\}$, can this set be partitioned into two sets, such that the sum of both sets are equal? The answer is “yes” and the partitions are $S_1 = \{1, 2, 4, 6\}$ and $S_2 = \{3, 10\}$.

**Example:** Consider the set $\{1, 2, 3, 4, 6, 7\}$, can this set be partitioned into two sets, such that the sum of both sets are equal? This time the answer is “no”, which you can try to convince yourself that it is.

The Ising Hamiltonian for the number partitioning problem can be straightforwardly written down as follows

$$H(s_1, \ldots, s_N) = \left( \sum_{i=1}^{N} n_i s_i \right)^2, \quad (s_i = \pm 1).$$

It is clear that the answer to the number partitioning problem is “yes” if $H = 0$, because then there exist a spin configuration where the sum of the $n_i$ for the $+1$ spins is the equal to the
2.4 Quantizing the electromagnetic field

In this section we will consider the quantization of a single-mode electromagnetic field following Ref. [27], [28]. To quantize the electromagnetic field we will begin by considering a closed cavity of volume $V$ with mirrors of perfect reflection located at $z = 0$ and $z = L$. We imagine that we have a monochromatic, single-mode electromagnetic field that is assumed to be polarized along the $x$-direction. Without any presence of sources and charges the Maxwell equations read

\begin{align}
\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\nabla \times B &= \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}, \\
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0.
\end{align}

Using Maxwell’s equations and the given boundary conditions, the electric field has the form

\[ E(z,t) = \left( \frac{2 \omega^2}{V \varepsilon_0} \right)^{1/2} q(t) \sin(kz) e_x, \]

where $V$ is the volume of the cavity, $q(t)$ is a function of time and $e_x$ is the unit-vector along the $x$-direction. From the boundary conditions the allowed frequencies are found to be

\[ \omega_n = \frac{c \pi n}{L}, \quad n = 1, 2, 3, \ldots \]

Figure 2.2: Cavity with two perfectly reflecting mirrors located at $z = 0$ and $z = L$. The electric field is assumed to be polarized along the $x$-direction.

sum of the $n_i$ for the $-1$ spins [2]. Likewise the answer is “no” if $(\mathcal{H}) > 0$ for all possible spin configurations. Expanding the square of Eq. (2.4) we get

\[ \mathcal{H}(s_1, \ldots, s_N) = \sum_{1 \leq i, j \leq N} J_{ij} s_i s_j = 2 \sum_{1 \leq i < j \leq N} J_{ij} s_i s_j + \text{Tr}[J_{ij}], \]

where we have introduced the couplings as

\[ J_{ij} = n_i n_j. \]

It should be noted that the ground state of Ising Hamiltonian for the number partitioning problem is always at least two-fold degenerate (the problem has at least two solutions). This has to do with changing $s_i$ to $-s_i$ does not change $\mathcal{H}$. What is also notable about the number partitioning problem compared to the subset sum problem is that the number partitioning problem does not require any additional magnetic fields on each spin site.

This concludes this section of examples.

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\[ E(z,t) = \left( \frac{2 \omega^2}{V \varepsilon_0} \right)^{1/2} q(t) \sin(kz) e_x, \]

where $V$ is the volume of the cavity, $q(t)$ is a function of time and $e_x$ is the unit-vector along the $x$-direction. From the boundary conditions the allowed frequencies are found to be

\[ \omega_n = \frac{c \pi n}{L}, \quad n = 1, 2, 3, \ldots \]

Figure 2.2: Cavity with two perfectly reflecting mirrors located at $z = 0$ and $z = L$. The electric field is assumed to be polarized along the $x$-direction.
with \( k_n = \omega_n / c \) as the corresponding wave number. In Eq. (2.10) we have assumed a specific frequency \( \omega \), i.e. a specific standing wave which is called a mode of the field. We find the corresponding magnetic field by substituting Eq. (2.10) into (2.7)

\[
B(z,t) = \frac{\mu_0 \varepsilon_0}{k} \left( \frac{2\omega^2}{V\varepsilon_0} \right)^{1/2} \dot{q}(t) \cos(kz) e_y.
\]

(2.11)

We now identify \( q(t) \) as a canonical coordinate, and \( p(t) \equiv \dot{q}(t) \) as the momenta canonically conjugate to \( q(t) \). The energy stored in the field of the single-mode is

\[
\mathcal{H} = \frac{1}{2} \int_V \left( \frac{1}{\varepsilon_0} |E|^2 + \mu_0 |B|^2 \right) dV = \frac{1}{2} (\dot{p}^2 + \omega^2 \dot{q}^2).
\]

We see that this is nothing but the energy of a harmonic oscillator with unit mass. To quantize the electromagnetic field we promote \( q \) and \( p \) to operators

\[
\hat{q} \quad \text{and} \quad \hat{p},
\]

and impose that they obey the canonical commutation relation

\[
[\hat{q}, \hat{p}] = i\hbar.
\]

Thus the electric and magnetic field are also promoted to operators

\[
\hat{E}_x(z,t) = \left( \frac{2\omega^2}{V\varepsilon_0} \right)^{1/2} \hat{q} \sin(kz),
\]

(2.12)

\[
\hat{B}_y(z,t) = \frac{\mu_0 \varepsilon_0}{k} \left( \frac{2\omega^2}{V\varepsilon_0} \right)^{1/2} \hat{p} \cos(kz).
\]

(2.13)

The subscript \( x \) and \( y \) denotes the constituent components of the fields. The Hamiltonian now reads

\[
\hat{\mathcal{H}} = \frac{1}{2} (\dot{\hat{p}}^2 + \omega^2 \dot{\hat{q}}^2).
\]

(2.14)

Next we define the very useful non-Hermitian ladder operators

\[
\hat{a} = (2\hbar \omega)^{-1/2} (\omega \hat{q} + i\hat{p}),
\]

(2.15)

\[
\hat{a}^\dagger = (2\hbar \omega)^{-1/2} (\omega \hat{q} - i\hat{p}),
\]

(2.16)

which obey the boson commutation relation

\[
[\hat{a}, \hat{a}^\dagger] = 1.
\]

Inverting Eq. (2.15) and (2.16) and substituting it into (2.14) we get a Hamiltonian entirely written in terms of the ladder operators

\[
\hat{\mathcal{H}} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right).
\]

This Hamiltonian has the following energy eigenvalues

\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots
\]

and the eigenstates of the Hamiltonian are known as Fock states, written in the Dirac notation as \( |n\rangle \), where \( n \) represents the number of quanta or photons in the single-mode field. The Fock states are eigenstates of the number operator \( \hat{n} = \hat{a}^\dagger \hat{a} \), satisfying

\[
\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle.
\]

The amplitude of the electric field in Eq. (2.10) was cleverly chosen to yield the energy of a harmonic oscillator of unit mass.
2.4. Quantizing the electromagnetic field

The vacuum state of the harmonic oscillator is defined by

\[ \hat{a} |0\rangle = 0. \]  

(2.17)

Acting with the creation and annihilation operators on the Fock state yields

\[ \hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \]

(2.18)

\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle. \]

(2.19)

Hence it is clear that the creation operator \( \hat{a}^\dagger \), creates a quanta of energy \( \hbar \omega \) and the annihilation operator \( \hat{a} \) destroys a quanta of energy \( \hbar \omega \) in the single-mode field. Any Fock state can be generated by acting on the vacuum state multiple times with the creation operator

\[ \frac{\hat{a}^n}{\sqrt{n!}} |0\rangle = |n\rangle. \]

Furthermore the number states are orthogonal

\[ \langle m | n \rangle = \delta_{mn} \]

and form a complete set

\[ \sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \]

So far we haven’t discussed the time-dependence of the operators. What we’ve done so far is assumed to hold at some time \( t \), for example \( t = 0 \). In the Schrödinger picture the states are time-dependent and the operators are time-independent. On the contrary, in the Heisenberg picture the operators are time-dependent and the states are time-independent. In the Heisenberg picture the time evolution of the annihilation operator is given by

\[ \frac{d\hat{a}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{a}] \]

\[ = \frac{i}{\hbar} [\hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}), \hat{a}] \]

\[ = i \omega (\hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger) \]

\[ = i \omega [\hat{a}, \hat{a}^\dagger] \hat{a} = -i \omega \hat{a}. \]  

(2.20)

Solving this equation yields

\[ \hat{a}(t) = \hat{a}_0 e^{-i \omega t}, \]

(2.21)

where \( \hat{a}(0) \equiv \hat{a} \). Taking the Hermitian adjoint of Eq. (2.21) we also find that

\[ \hat{a}^\dagger(t) = \hat{a}^\dagger_0 e^{i \omega t}. \]

After substituting Eq. (2.15) into (2.12) and Eq. (2.16) into (2.13), the electric and magnetic field with the inclusion of the time-dependence become, respectively

\[ \hat{E}_x(z, t) = \left( \frac{\hbar \omega}{V_{\infty}} \right)^{1/2} (\hat{a} e^{-i \omega t} + \hat{a}^\dagger e^{i \omega t}) \sin(kz), \]

(2.22)

\[ \hat{B}_y(z, t) = \frac{\mu_0 \varepsilon_0}{ik} \left( \frac{\hbar \omega^3}{V_{\infty}} \right)^{1/2} (\hat{a} e^{-i \omega t} - \hat{a}^\dagger e^{i \omega t}) \cos(kz). \]

(2.23)

**Quadrature operators**

It is convenient to introduce the two dimensionless quantities

\[ \hat{X} = \sqrt{\frac{\omega}{2\hbar}} \hat{q} = \frac{1}{2} (\hat{a} + \hat{a}^\dagger), \]

\[ \hat{X}_{\pi/2} = \frac{1}{\sqrt{\hbar \omega}} \hat{p} = \frac{1}{2i} (\hat{a} - \hat{a}^\dagger), \]

(2.24)
which satisfy the commutation relation

\[
[\hat{X}, \hat{X}_{x/2}] = \frac{i}{2}.
\]

From here it is easy to show that the electric field operator Eq. (2.22) can be written in terms of the dimensionless quantities \(\hat{X}\) and \(\hat{X}_{x/2}\) as

\[
\hat{E}_x(z, t) = 2 \left( \frac{\hbar \omega}{\varepsilon_0 V} \right)^{1/2} \left( \hat{X} \cos(\omega t) + \hat{X}_{x/2} \sin(\omega t) \right) \sin(kz).
\]

(2.25)

This expression shows that \(\hat{X}\) and \(\hat{X}_{x/2}\) are associated with the electric field amplitude, where the second term is offset by \(\pi/2\) compared to the \(\cos(\omega t)\) term. The operators \(\hat{X}\) and \(\hat{X}_{x/2}\) are therefore known as the quadrature operators.

The variance of an arbitrary operator is defined by

\[
\langle (\Delta \hat{A})^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2
\]

(2.26)

and can interpreted as the uncertainty of an observable. The expectation value of the quadratures

\[
\langle \hat{X} \rangle = \frac{1}{2} \langle n | (\hat{a} + \hat{a}^\dagger) | n \rangle = 0,
\]

\[
\langle \hat{X}_{x/2} \rangle = \frac{1}{2i} \langle n | (\hat{a} - \hat{a}^\dagger) | n \rangle = 0
\]

are evaluated to zero, which means that the expectation value of the electric field is also zero from Eq. (2.25). On the other hand the expectation value of the square is non-zero

\[
\langle \hat{X}^2 \rangle = \frac{1}{4} \langle n | (\hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^2) | n \rangle = \frac{1}{2} (1 + 2n),
\]

\[
\langle \hat{X}_{x/2}^2 \rangle = \frac{1}{4} \langle n | (\hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^2) | n \rangle = \frac{1}{2} (1 + 2n).
\]

Thus it follows from Eq. (2.26) that the uncertainty in both quadratures are equal, and when \(n = 0\) (corresponding to the vacuum state), the uncertainty is minimum

\[
\langle (\Delta \hat{X})^2 \rangle_{\text{vac}} = \frac{1}{4} = \langle (\Delta \hat{X}_{x/2})^2 \rangle_{\text{vac}}.
\]

(2.27)

This is known as the vacuum fluctuations. Moreover, in quantum optics the quadrature operators \(\hat{X}\) and \(\hat{X}_{x/2}\) are commonly labelled \(\hat{q}\) and \(\hat{p}\), since they are more used than the position and momentum operators.

### 2.5 Coherent states

In quantum optics, the coherent states are the states with most resemblance to classical states, in the sense that they give rise to expectation values that look like the classical electric field.

To construct a state with close resemblance to the classical electromagnetic field, one can observe that by replacing \(\hat{a}\) and \(\hat{a}^\dagger\) with a complex variable in Eq. (2.22) and (2.23) it would produce a “classical field”, i.e. a field that oscillates. In order to achieve this, one can define a coherent state to be an eigenstate to the annihilation operator

\[
\hat{a} | \alpha \rangle = \alpha | \alpha \rangle.
\]

(2.28)

\(^3\)Oscillating terms that are separated by \(90^\circ\) (\(\pi/2\)) are said to be in quadrature.

\(^4\)Roy J Glauber (1963) was the first person to define a coherent state in quantum optics.
Because $a$ is non-Hermitian $\alpha$ is usually complex. For the creation operator $a^\dagger$ we have for obvious reasons 

$$\langle \alpha | a^\dagger = \alpha^* \langle \alpha |.$$ 

Since the Fock states form a complete set, we’ll use them to express $\alpha$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$  \hspace{1cm} (2.29)

where $c_n = \langle n | \alpha \rangle$ denotes a complex number which is to be determined. Inserting Eq. (2.29) into (2.28) and using Eq. (2.17) and (2.18) we obtain

$$\sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} c_n \alpha |n\rangle.$$ 

Since the Fock-states form an orthogonal basis, we can multiply with an arbitrary state $\langle m |$ from left and use the orthogonality condition $\langle m | n \rangle = \delta_{mn}$ to obtain

$$c_{m+1} \sqrt{m+1} = \alpha c_m.$$ 

By the substitution $m \to n - 1$

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{n(1-n)}} c_{n-2} = \ldots = \frac{\alpha^n}{\sqrt{n!}} c_0,$$

we obtain a recursion formula. Hence Eq. (2.29) can be expressed as

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$ 

We use the normalization condition to determine the coefficient $|c_0|^2$,

$$\langle \alpha | \alpha \rangle = 1 = |c_0|^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^m \alpha^n}{m! n!} \langle m | n \rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} = |c_0|^2 e^{\alpha^* \alpha},$$

Therefore $|c_0| = e^{-|\alpha|^2/2}$ and our final expression for the coherent state $|\alpha\rangle$ expressed in terms of Fock states is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

which is a superposition of infinite many number of Fock states! Furthermore calculating the expectation value of the number operator $\hat{n}$

$$\langle \hat{n} \rangle = \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2,$$

we see that $|\alpha|^2$ is related to the mean number of photons in the field. Using this we can compute the probability of finding $n$ photons in the field

$$\langle |n\alpha|^2 \rangle = e^{-|\alpha|^2/2} \frac{|\alpha|^{2n}}{n!} = e^n \tilde{n}^n \frac{n!}{n!},$$

which we recognize as a Poisson distribution with a mean of $\tilde{n}$. This distribution arises when the probability that an event occurs is independent of earlier events.

Let us now consider the expectation value of the electric field given by Eq. (2.22)

$$\langle \hat{E}_x(z, t) \rangle_\alpha = \langle \alpha | \hat{E}_x(z, t) | \alpha \rangle = \left( \frac{\hbar \omega}{V_\varepsilon_0} \right)^{1/2} \alpha |(a e^{-i\omega t} + a^\dagger e^{i\omega t}) |\alpha\rangle \sin(kz)$$

$$= \left( \frac{\hbar \omega}{V_\varepsilon_0} \right)^{1/2} (\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}) \sin(kz).$$
2.5. Coherent states

The displacement from the origin is equal to $|\alpha|$ and the angle $\phi$ is the phase, measured from the $X$-axis.

Writing $\alpha$ in polar coordinates $\alpha = |\alpha|e^{i\phi}$ we get

$$\langle E_x(z,t) \rangle_\alpha = \left( \frac{\hbar \omega}{V_{z_0}} \right)^{1/2} 2|\alpha| \cos(\omega t - \phi) \sin(kz),$$

and we see that the field oscillates very much like the classical electric field. Likewise, for the quadrature operator Eq. (2.24), we have

$$\langle \hat{X} \rangle_\alpha = \frac{1}{2} \langle \alpha (\hat{a} + \hat{a}^\dagger) |\alpha\rangle = \frac{1}{2} (\alpha + \alpha^*) = \text{Re} \alpha = |\alpha| \cos(\phi)$$

and similarly $\langle \hat{X}_{x/2} \rangle_\alpha = \text{Im} \alpha = |\alpha| \sin(\phi)$, so the mean of the quadratures are related to the real and imaginary part of $\alpha$. It can be easily verified that the quantum uncertainty for both quadratures are

$$\langle (\Delta \hat{X})^2 \rangle_\alpha = \frac{1}{4} = \langle (\Delta \hat{X}_{x/2})^2 \rangle_\alpha,$$

which shows that they still exhibit the fluctuations of the vacuum, cf. Eq. (2.27).

A neat way to illustrate a coherent state is in phase space. More of it will be explained in section 2.6, but for now it is sufficient to think of phase space as a mathematical abstract space, where the state of a harmonic oscillator is represented in terms of its quadratures. Since position $\hat{X}$ and momentum $\hat{X}_{x/2}$ are non commutative operators in quantum physics, the state can’t be represented as a point in phase space as it would in classical physics, because both position and momentum aren’t allowed to have precise values. In Fig. 2.3 the phase space diagram of a coherent state $|\alpha\rangle$ is illustrated. The coherent state can be viewed as a displacement of the vacuum state with a distance $|\alpha|$ from the origin and an angle $\phi$ measured from the $X$-axis. The grey circle area represents the uncertainty of the coherent state and has a constant diameter of 1/2.

Lastly, in laboratories coherent states are produced by a laser that is operating far above threshold [30], where they form a useful basis for expanding the optical field in problems as laser physics and nonlinear optics.

Figure 2.3: Phase-space diagram of a coherent state $|\alpha\rangle$. The displacement from the origin is equal to $|\alpha|$ and the angle $\phi$ is the phase, measured from the $X$-axis. The quantum uncertainty is displayed as a grey circle with a diameter of 1/2.
Non-orthogonality

Coherent states are known to be quasi-orthogonal. For example, consider the scalar product $\langle \beta | \alpha \rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are different coherent states

$$
\langle \beta | \alpha \rangle = e^{-|\beta|^2/2} e^{-|\alpha|^2/2} \sum_n \frac{(\beta^* \alpha^n)}{\sqrt{n!n!}} (m|n)
$$

$$
= e^{-|(\beta^* \alpha)|^2/2} \sum_n \frac{\beta^* \alpha^n}{n!}
$$

$$
= e^{-|\beta|^2 + |\alpha|^2 - 2\beta^* \alpha}/2
$$

$$
= e^{-|\alpha - \beta|^2/2} e^{(\alpha \beta^* - \beta \alpha^*)}/2;
$$

taking the modulus square we get

$$
|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}.
$$

(2.30)

From Eq. (2.30) it is evident that two coherent states are non-orthogonal. Only when $|\alpha - \beta|^2$ is large, so that $|\langle \beta | \alpha \rangle|^2 \sim 0$, they become quasi-orthogonal.

#### 2.6 Phase space representation

In quantum mechanics a system can be fully described by its density operator $\hat{\rho}$, however the density operator can be a rather abstract object and it can be hard to read off its properties. Therefore we employ the phase space representation, that is based on the concept of the Wigner function, which provides useful means of visualizing a quantum state [31].

Because position and momenta are so closely related to the quadrature operators, the phase space representation is a useful tool for studying the behaviour of continuous variable systems. This can be done by changing from the representation of position and momentum to the representation of quadrature operators [32].

**Wigner function**

The Wigner function is defined by

$$
W(q,p) \equiv \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} \langle q + \frac{1}{2} x | \hat{\rho} | q - \frac{1}{2} x \rangle e^{ixp/\hbar} dx
$$

where $x$, $q$ and $p$ are now interpreted as quadratures, and $\hat{\rho}$ is the density operator for a quantum system. $W(q,p)$ is known as a quasi-probability distribution since it can take on negative values. Even though the Wigner function is not a “real” probability distribution it can still be associated to one, for example calculating the probability distribution (also referred to as the marginal distribution) over $p$

$$
Pr(q) = \int_{-\infty}^{\infty} W(q,p) dp = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \psi^*(q - \frac{1}{2} x) \psi(q + \frac{1}{2} x) e^{ixp/\hbar}
$$

$$
= \int_{-\infty}^{\infty} dx \psi^*(q - \frac{1}{2} x) \psi(q + \frac{1}{2} x) \delta(x)
$$

$$
= \psi^*(q) \psi(q) = |\psi(q)|^2
$$

returns the probability density of $q$. Similarly, one can show that

$$
Pr(p) = \int_{-\infty}^{\infty} W(q,p) dq = |\tilde{\psi}(p)|^2
$$

is the probability density of $p$, where $\tilde{\psi}(p)\,dq$ is the wave function in $p$-representation.

---

5It was Wigner (1932) [33] that introduced this function, in order to characterize a quantum state $|\psi\rangle$ in phase space.

6Note that the arguments $q$ and $p$ in the Wigner function are $c$-numbers and not operators.
Quantum tomography

It is possible to reconstruct the Wigner function from experimental data, a procedure known as *quantum tomography*. The probability distribution along an arbitrary quadrature-axis is,

$$ Pr(q, \theta) = \int_{-\infty}^{\infty} W(q \cos \theta - p \sin \theta, q \sin \theta + p \cos \theta) dp. $$

By measuring $P(q, \theta)$ for all possible phases $\theta$ the Wigner function and thus the complete information of the state $|\psi\rangle$ can be reconstructed [34], [35]. To experimentally measure the probability distribution $Pr(q, \theta)$ one can use *balanced homodyne detection*. A typical balanced homodyne detection setup is shown in Fig. 2.4. One arm is called the local oscillator (LO) that contains a strong field which is in a coherent state $|\beta\rangle = |\beta| e^{i\theta}$. The ingoing field $\hat{a}$ is mixed with the local oscillator $\beta$ in a 50:50 beam splitter. After the optical mixing of the signal with the local oscillator, each beam strikes a photodetector and the photocurrents are measured and electrically processed. The photocurrents are then subtracted from each other $I_{12} \equiv I_2 - I_1$ to yield a quantity that contains the interference term of the LO with the signal. We assume for simplicity that the currents are proportional to the photon numbers of the beams striking each detector. The photon numbers for each detector is given by

$$ \hat{n}_1 = \hat{a}_1^\dagger \hat{a}_1 \quad \text{and} \quad \hat{n}_2 = \hat{a}_2^\dagger \hat{a}_2, $$

which in terms of the mode operators are

$$ \hat{a}_1 = \frac{1}{\sqrt{2}} (\hat{a} - \beta), \quad \hat{a}_2 = \frac{1}{\sqrt{2}} (\hat{a} + \beta). $$

Since $I_{12}$ is proportional to the difference in photon number, we get

$$ \hat{n}_{12} = \hat{n}_2 - \hat{n}_1 = \beta^* \hat{a} + \beta \hat{a}^\dagger. $$

Substituting $\beta$ with $|\beta| e^{i\theta}$ yields

$$ \hat{n}_{12} = 2|\beta| \hat{q}_\theta, $$

where $\hat{q}_\theta = \hat{q} \cos \theta + \hat{p} \sin \theta$ is the generalized quadrature operator. Thus, a balanced homodyne detector apparatus measures the quadrature $\hat{q}_\theta$ of the signal field, where the reference phase $\theta$ is provided by the local oscillator.

A more detailed review on measuring quantum states of light is given by Leonhardt (1997) [34].

![Homodyne detection apparatus](image-url)

Figure 2.4: Homodyne detection apparatus. The ingoing signal is mixed with a local oscillator in a 50:50 beam splitter. Each outgoing signal strikes a photodetector that measures the photocurrent, and the two currents are then subtracted from one another.
Wigner negativity

As mentioned in the introduction, negativity of the Wigner function is related to non classical simulatability according to a theorem proven by Eisert and Mari [17]. In their work they show that a negative Wigner function can be identified as a quantum computational resource (similar to superposition), and that a quantum circuit where the initial state and all the following quantum operations are represented by positive Wigner functions can be classically efficiently simulated.

Lastly, some examples of how the Wigner function can look like is shown in Fig. 2.5, where the surface plot with the corresponding contour plot for the vacuum state, a coherent state and the single-photon Fock state is displayed. Note that both the vacuum and the coherent state is shaped like a Gaussian which does not display any negativity, while the Wigner function for the single-photon Fock state shows clear indication of negativity.

![Figure 2.5: Example of Wigner functions.](image)

2.7 Open quantum systems

An open quantum system is a quantum system that interacts with the environment. Open quantum systems play an essential part in both theoretical and experimental quantum physics, since to characterize a quantum system completely its interaction with its surroundings have to be included. In this section we will introduce the Lindblad master equation which describes the time-evolution of a quantum system that is interacting with an environment.
Closed quantum systems

The complete opposite of an open quantum system is a closed quantum system which is completely isolated from its surroundings. In the Schrödinger picture we have that the states $|\psi(t)\rangle$ are time-dependent while the operators are time-independent. The time-evolution for a closed quantum system in the Schrödinger picture is governed by the Schrödinger equation

$$\frac{i\hbar}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle.$$ 

More generally, if a closed quantum system consists of more than one state vector $|\psi\rangle$, the system is said to be in a mixed state. To describe the dynamics of mixed states the density operator formalism is used. A density operator is constructed as a statistical weighted sum of state vectors

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k|,$$

where $p_k$ is the probability of finding the system in the state $|\psi_k\rangle$ upon measurement. The density operator $\hat{\rho}$ is also subject to the normalization condition

$$\text{Tr}[\hat{\rho}] = 1.$$ 

For a closed quantum system in a mixed state, the time-evolution of the density operator can be shown to be given by (see appendix A.3)

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)].$$

This equation is formally known as the Liouville-von Neumann equation. Notice the subtle minus sign in front of the commutator compared to the Heisenberg equation of motion cf. Eq. (2.20). The minus sign is an indication that we are still “working” in the Schrödinger picture.

Lindblad master equation

In particular cases the closed quantum system description might be valid to a good approximation, but in some cases it will not [36]. Especially if we want do quantum computation, then we will need to have a good description of how noise from the environment shows up in our system and leads to decoherence. Typically the environment is assumed to be much larger i.e. containing many more degrees of freedom than the system [37]. For example, the environment can be thought of as many harmonic oscillators, like the free modes of the electromagnetic field [30]. In general the total Hamiltonian for a quantum system interacting with an environment can be written as

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{sys}} + \hat{H}_{\text{env}} + \hat{H}_{\text{int}},$$

where $\hat{H}_{\text{sys}}$ is the Hamiltonian for the system, $\hat{H}_{\text{env}}$ is the Hamiltonian describing the environment and $\hat{H}_{\text{int}}$ is the interaction Hamiltonian that couples the system to the environment. The equation of motion for the system as a whole can be considered closed and thus it follows the Liouville-von Neumann equation

$$\frac{d\hat{\rho}_{\text{tot}}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{tot}}, \hat{\rho}_{\text{tot}}(t)].$$

The goal of a master equation is to find the equation of motion of the system, not the combined system + environment. By taking the partial trace of the total density operator over the environment variables a reduced density operator for the system can be obtained

$$\hat{\rho}_{\text{sys}} = \text{Tr}_{\text{env}}[\hat{\rho}_{\text{tot}}].$$
The equation of motion describing the reduced density operator in presence of dissipation processes between the system and its environment in the interaction picture is given by the Lindblad master equation

\[
\frac{d\hat{\rho}_\text{sys}(t)}{dt} = -i\frac{1}{\hbar}\left[\hat{H}_\text{sys}(t), \hat{\rho}_\text{sys}(t)\right] + \sum_n \frac{1}{2}\left[2\hat{C}_n\hat{\rho}_\text{sys}(t)\hat{C}_n^\dagger - \hat{\rho}_\text{sys}(t)\hat{C}_n^\dagger\hat{C}_n - \hat{C}_n^\dagger\hat{C}_n\hat{\rho}_\text{sys}(t)\right],
\]

(2.32)

where \(\hat{C}_n = \hat{A}_n/\sqrt{\gamma_n}\) are known as collapse operators, where \(\hat{A}_n\) are the operators through which the environment couples to the system and \(\gamma_n\) are the corresponding decay rates. Equation (2.32) is a generalization of the Liouville-von Neumann equation to a system that include dissipation processes with its environment. The derivation of the Lindblad master equation will not be reproduced in this thesis, but can be found in several other sources [37], [38]. We will merely point out the different approximations made to arrive at the Lindblad master equation starting from the interaction picture of Eq. (2.31).

- **Separability**: At the initial time \(t = 0\) we assume that the system and the environment are uncorrelated, such that the total density matrix can be written as a tensor product \(\hat{\rho}_\text{tot}(0) = \hat{\rho}_\text{sys}(0) \otimes \hat{\rho}_\text{env}(0)\);

- **Born approximation**: This approximation assumes that the coupling between the system and environment is weak so that the influence of the system on the environment is small and that the time evolution of the environment density operator can therefore be neglected, in summary \(\hat{\rho}_\text{tot}(t) = \hat{\rho}_\text{sys}(t) \otimes \hat{\rho}_\text{env}\);

- **Markov approximation**: This approximation assumes that the rate of change of the system only depends on the state at that time. There is no explicit dependence on the state at previous times. The basic condition for this approximation is that the environment correlation functions decay sufficiently fast over a time \(\tau_\text{env}\) which is small compared to the time-scale \(\tau_\text{sys}\) over which the system varies, i.e. \(\tau_\text{sys} \gg \tau_\text{env}\).

- **Rotating wave approximation**: It involves discarding all the fast oscillating terms in the interaction picture proportional to \(\exp[i(\omega' - \omega)t]\) for \(\omega' \neq \omega\). For this approximation to be valid the inverse frequency differences should be small compared to the time-scale \(\tau_\text{sys}\) for which the system \(\hat{\rho}_\text{sys}\) varies appreciably, that is \(|\omega' - \omega|^{-1} \ll \tau_\text{sys}\).

These approximations generally apply to the interaction between matter and electromagnetic radiation because the Markov approximation is usually very well satisfied. E.g. consider an atom interacting with the free quantized electromagnetic field, in this case the typical radiative inverse atomic lifetimes are of the order \(\approx 10^8\) s\(^{-1}\), whereas optical frequencies are of the order \(10^{15}\) s\(^{-1}\) [37].

### 2.8 Measure of closeness between quantum states

We end this chapter on theory by introducing the concept of **fidelity**. In quantum information the measure of “closeness” between two density matrices \(\hat{\rho}\) and \(\hat{\sigma}\) is defined to be [22]

\[
\mathcal{F}(\hat{\rho}, \hat{\sigma}) = \text{Tr}\left(\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}\right).
\]

This quantity is known as the fidelity and it is bounded \(0 \leq \mathcal{F}(\hat{\rho}, \hat{\sigma}) \leq 1\), with \(\mathcal{F}(\hat{\rho}, \hat{\sigma}) = 1\) only if \(\hat{\rho} = \hat{\sigma}\).

In the special case when \(\hat{\rho}\) is described by a pure state \(|\psi_\rho\rangle \langle \psi_\rho|\) the fidelity reduces to

\[
\mathcal{F}(\hat{\rho}, \hat{\sigma}) = \text{Tr}\left(\sqrt{\langle \psi_\rho| \hat{\sigma}|\psi_\rho\rangle} |\psi_\rho\rangle \langle \psi_\rho|\right) = \sqrt{\langle \psi_\rho| \hat{\sigma}|\psi_\rho\rangle},
\]

which is the square root of the overlap between \(\hat{\sigma}\) and \(|\psi_\rho\rangle\).
The heart of our continuous variable quantum annealer device is the two-photon pumped Kerr-Nonlinear Resonator (KNR). It consists of a superconducting resonator made nonlinear with a Kerr-type nonlinearity, that causes the energy levels of the resonator to be non-equidistant [39]. In this chapter we will begin by providing the reader with a brief overview of circuit QED and superconductivity. After that we will introduce the two-photon pumped KNR in more detail. This device will allow encoding a combinatorial optimization problem in a network of two-photon pumped KNR’s where a single Ising spin is mapped to two quasi-orthogonal coherent states with opposite phases. The latter constitute a two-fold degenerate eigenspace for the two-photon pumped KNR in the rotating frame. Furthermore, we will demonstrate how the AQC Hamiltonian Eq. (1.1) is realized by adiabatically controlling the frequency and amplitude of the pumps, and how $N$ linearly coupled KNRs that are initialized to vacuum $|0\rangle$ at $t = 0$, will evolve to the ground state of an Ising Hamiltonian at $t = \tau$.

3.1 Circuit QED

Superconductivity is a quantum phenomenon that appears when certain metals are cooled down below a critical temperature. There, the metals become superconducting and a current can flow without electrical resistance [40]. Circuit quantum electrodynamics, or simply circuit QED, is based on superconducting electrical circuits. Similar to cavity QED, which studies atoms coupled to a single mode of the electromagnetic field inside a cavity, circuit QED studies on-chip implementation of QED using microwave photons [41]. In circuit QED a superconducting coplanar waveguide between two capacitors plays the role of the cavity and the atom is replaced with an artificial one [42]. Fig. 3.1 to the left illustrates a simple cavity that uses two mirrors facing each to confine photons. The illustration to the right depicts the corresponding circuit QED implementation.

While photons do not interact with each other in vacuum an effective photon-photon interaction can be engineered by the use of non-linear materials. One such effective interaction is called the Kerr effect, which is referred to when a material can be regarded of having a refractive index that is proportional to the intensity of the electromagnetic field [43]. A resonator can be made nonlinear with the use of a Josephson junction that introduces a Kerr-type nonlinearity [36]. A Josephson junction consists of two superconducting metals separated by
3.2 Two-photon pumped Kerr-nonlinear resonator

Two-photon pumped Kerr-nonlinear resonator

We now consider the proposed physical implementation of a continuous variable quantum annealer that was put forward by Puri et al. [15]. As already mentioned, the main component of the quantum annealer consists of a two-photon pumped KNR. A two-photon pumped KNR can be engineered by embedding a SQUID in the middle of a half-wavelength (\(\lambda/2\)) resonator, see Fig. 3.2. By modulating the magnetic flux \(\Phi_{\text{ext}}\) through the SQUID at twice the resonator frequency the SQUID can be used as a parametric two-photon pump. The Hamiltonian of a two-photon pumped KNR can be derived by expanding the Josephson potential of the SQUID up to fourth order and doing the rotating wave approximation, see appendix A.4. It results in

\[
\mathcal{H}(t) = \omega_r \hat{a}^\dagger \hat{a} - K \hat{a}^2 \hat{a}^\dagger + G \left( \hat{a}^{12} e^{-2i\omega_r t} + \hat{a}^2 \hat{a}^{12} e^{2i\omega_r t} \right)
\]

where \(\omega_r\) denotes the mode frequency of the resonator, \(K\) denotes the amplitude of the Kerr-nonlinearity, \(G\) denotes the amplitude of the two-photon pump, and the two-photon pump frequency is set to twice the resonator frequency. The time-dependence of the Hamiltonian can be removed by transforming to a frame rotating at the resonator frequency. Indeed, by

\[
\Phi_{\text{ext}} /4 /4 /4 /4 /4
\]

From here forth we will use natural units where \(\hbar = c = 1\).
doing the following unitary transformation $\hat{U}(t) = e^{i\omega t, \hat{a}^\dagger \hat{a}}$, we get

$$\hat{H} = -K\hat{a}^2 + G \left( \hat{a}^2 + \hat{a}^\dagger \right) = -K \left( \hat{a}^2 - \frac{G}{K} \right) \left( \hat{a}^2 - \frac{G}{K} \right) + \frac{G^2}{K}. \quad (3.1)$$

From the last expression it is evident that the two coherent states $|\pm \alpha\rangle$ with $\alpha = \sqrt{G/K}$ are are two degenerate eigenstates of this Hamiltonian$^2$ with eigenenergy

$$\hat{H} \pm \alpha \rangle = \frac{G^2}{K} |\pm \alpha\rangle.$$

Following Puri et al. [15] we now take advantage of this well-defined two state subspace and choose to encode the logical spins $|0\rangle$ and $|1\rangle$ onto these coherent states, i.e. we do the mapping $\{|0\rangle, |1\rangle\} \rightarrow \{|\mp \sqrt{G/K}\rangle, |\sqrt{G/K}\rangle\}$, where the bar is used to distinguish the logical spin states from the vacuum and single-photon Fock state. For sufficiently large $|\alpha|$ the states can be considered orthogonal, indeed following Eq. (2.30) we have that

$$|\langle 1|0\rangle|^2 = e^{-4|\alpha|^2}.$$

For instance if $|\alpha| = \sqrt{3}$, then $|\langle 1|0\rangle|^2 \approx 10^{-6}$. In what follows we will demonstrate that this mapping is robust against single-photon loss from the resonator (which we will consider as the main loss mechanism). In order to corroborate this fact, we start by showing that single-photon loss does not lead to spin-flip error, which is when a qubit flips e.g. $|0\rangle \rightarrow |1\rangle$. Consider the quantity $|\langle 1|\hat{a}|0\rangle|^2$ that describes the overlap between the two logical spin states $|0\rangle$ and $|1\rangle$ after a single-photon has been lost from the resonator. Using Eq. (2.28) and (2.30) we have that

$$|\langle 1|\hat{a}|0\rangle|^2 = |\alpha|^2 |\langle 1|0\rangle|^2 = |\alpha|^2 e^{-4|\alpha|^2} \approx 10^{-5},$$

for $|\alpha| = \sqrt{3}$. Therefore we propose to use a two-photon pump amplitude set to $G = 3K$ so that $|\alpha| = \sqrt{3K/K} = \sqrt{3}$, as such we will use $G = 3K$ throughout this thesis.

Next, we compute the semi-classical steady-state solution of the KNR. In the presence of single-photon loss the collapse operator in the Lindblad-master equation (2.32) takes the form $\hat{C} = \sqrt{\gamma} \hat{a}$ and the time-evolution of the density matrix is then given by

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}, \hat{\rho}] + \gamma \left( 2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a}\hat{\rho} \right).$$

This furthermore assumes that the system is in thermal equilibrium with a zero-temperature environment, so that the number of thermal photons in the system is negligible. This is a good approximation since the typical microwave cavity is at mK [45]. The described process is schematically sketched in Fig. 3.3. While it is possible to obtain the steady-state solution analytically [46], [47], it can be found quite easily (see appendix A.6) from the solution of the semi-classical equations of motion that the expected position of the states $|\pm \alpha\rangle$ in presence of single-photon loss are

$$\alpha = \frac{1}{2} \left( \frac{16G^2 - \gamma^2}{K^2} \right)^{1/4} \exp\left(-\frac{i}{2} \arctan\left( \frac{\gamma}{\sqrt{16G^2 - \gamma^2}} \right) \right).$$

Thus, if the two-photon pump amplitude is much larger than the single-photon loss rate, namely if $4G \gg \gamma$ or equivalently $12K \gg \gamma$, since $G = 3K$, then $\alpha \approx \sqrt{G/K}$ and the mapping is confined to the spin subspace spanned by $|0\rangle$ and $|1\rangle$. Furthermore $K/30 > \gamma$ is already achievable today in superconducting resonators [48]. When this condition is satisfied the steady-state density matrix of the system, $d\hat{\rho}/dt = 0$, takes the form

$$\hat{\rho}_{ss} = \frac{1}{2} \left( |\alpha\rangle \langle \alpha| + |\mp \alpha\rangle \langle \mp \alpha| \right),$$

$^2$Recall from section 2.5 that a coherent state is defined as an eigenstate to the annihilation operator $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$. 

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3.3. Two- & one-photon pumped Kerr-nonlinear resonator

We now turn our attention to the realization of the Ising Hamiltonian Eq. (2.1). We will demonstrate that a sufficiently weak single-photon pump can act like an effective magnetic field [16]. The single-photon pump can be physically implemented by capacitively coupling one end of the two-photon pumped KNR to a transmission line through which the single-photons are added to the resonator [39], see Figure 3.5. By considering the addition of a weak single-photon pump with a frequency that is in resonance with the resonator we get in a frame...
rotating at the resonator frequency that the the Hamiltonian is

$$\mathcal{H}_1 = -K\hat{a}^\dagger \hat{a}^2 + G(\hat{a}^\dagger^2 + \hat{a}^2) + F(\hat{a}^\dagger + \hat{a}), \quad (3.2)$$

where $F$ denotes the amplitude of the single-photon pump. With this additional pump it can be once again found (see appendix A.7) from the solution to the semi-classical equations of motion (neglecting losses) that the expected position of the coherent states $|\pm \alpha \rangle$ of the resonator are

$$\pm \alpha = \pm \sqrt{\frac{G}{K} + \frac{F}{4G}},$$

i.e. they are slightly displaced by a factor $F/4G$, but if $4G \gg F$ or equivalently $12K \gg F$, then $\pm \alpha \approx \pm \sqrt{G/K}$, and the states are once again confined to the spin subspace spanned by $|0\rangle$ and $|1\rangle$. Most importantly, this single-photon pump lifts the degeneracy between the states $|0\rangle$ and $|1\rangle$ by an amount

$$\Delta E = \langle \alpha | \hat{\mathcal{H}}_1 | \alpha \rangle - \langle -\alpha | \hat{\mathcal{H}}_1 | -\alpha \rangle = 4F\alpha.$$

In the spin subspace spanned by our two logical spins $|0\rangle$ and $|1\rangle$ the Hamiltonian of Eq. (3.2) can therefore be expressed as

$$\tilde{\mathcal{H}}_1 = (|0\rangle \langle 0| + |1\rangle \langle 1|) (-K\hat{a}^\dagger \hat{a}^2 + G(\hat{a}^\dagger^2 + \hat{a}^2) + F(\hat{a}^\dagger + \hat{a})) (|0\rangle \langle 0| + |1\rangle \langle 1|)$$

using the definition of a coherent state $\hat{a} |0\rangle = \alpha |0\rangle$ and the quasi-orthogonality condition $\langle 0|1\rangle = 0$, we get

$$\tilde{\mathcal{H}}_1 = (-K\alpha^4 + 2G\alpha^2) \hat{I} + 2F\alpha \hat{\sigma}^z$$

$$= 2F\alpha \hat{\sigma}^z + \text{const}$$

where $\hat{\sigma}^z = |1\rangle \langle 1| - |0\rangle \langle 0|$ is the Pauli-z matrix. This is the Ising Hamiltonian for a single Ising spin in a magnetic field of strength $|2F\alpha|$. Therefore the application of a weak single-photon drive induces an effective magnetic field.

Now, recall from the introduction that we require an initial Hamiltonian that does not commute with the final Hamiltonian and furthermore should have a simple non-degenerate ground state. We consider the case when the resonators are initialized to vacuum, since it is relative simple to prepare the resonators in such a state. We choose an initial Hamiltonian in the rotating frame of the following form

$$\mathcal{H}_0 = -\delta \hat{a}^\dagger \hat{a} - K\hat{a}^\dagger^2 \hat{a}^2 \quad (3.3)$$

where $\delta$ is a finite positive detuning to separate the vacuum state $|0\rangle$ from the single-photon Fock state $|1\rangle$ by an energy gap $\delta$. At first glance Eq. (3.3) might fill us with fear and trepidation since the Hamiltonian isn’t bounded from below. Fear not! One can show that this is a consequence of the approximations used in the derivation, namely that we truncated the expansion of the Josephson cosine potential to fourth order, see appendix A.4, and that the microscopic Hamiltonian from which it was derived from Eq. (A.7), does not exhibit this
property. We also have to remember that photon loss also naturally leads to lower photon number states. Therefore as pointed out by Nigg et al. (2016) [51] we have to think of the zero photon state in the rotating frame as the state with highest energy instead of the one with lowest energy. We also adopt the terminology by Nigg et al. and refer to the ground state, the state with highest energy, as the roof state.

The time-dependent Hamiltonian for the AQC algorithm is obtained by adiabatically varying the two- and single-photon pump amplitudes and frequencies (see appendix A.5)

\[
H(t) = \left(1 - \frac{t}{\tau}\right) H_0 + \frac{t}{\tau} H_1
\]

\[
= \left(1 - \frac{t}{\tau}\right) \left(-\delta a_1^\dagger a_1 - K a_1^{12} a_2^2\right) + \frac{t}{\tau} \left(-K a_1^{12} a_2^2 + G \left(a_1^{12} + a_2^2\right) + F \left(a_1^\dagger + a_1\right)\right),
\]

yielding Eq. (1.1) for the single Ising spin in a magnetic field problem. Notice that the Kerr term in the Hamiltonian above is actually time independent so that one only needs to vary the frequency and amplitude of the pumps.

### 3.4 Coupled two-photon pumped Kerr-nonlinear resonators

After having demonstrated that a weak single-photon pump induces an effective magnetic field we will now explain how the coupling between spins is realized.

The coupling between two resonators can in the simplest form be expressed as [52]

\[
g(a_1^\dagger a_2^\dagger + a_1^\dagger a_2),
\]

where \(g\) describes single-photon exchange rate between two capacitively coupled resonators. We will begin by considering the case of two capacitively coupled two-photon pumped KNRs as shown in Fig 3.6 and assume that the parameters \(K\) and \(G\) are identical for both resonators.

![Diagram of two capacitively coupled KNRs](image)

**Figure 3.6:** Two capacitively coupled KNRs.

We furthermore neglect the single-photon pump for the moment. The Hamiltonian for two coupled KNRs in the rotating frame is given by

\[
\hat{H}_1 = \sum_{i=1}^2 \left(-K a_i^{12} a_i^2\right) + G \left(a_i^{12} + a_i^2\right) + g \left(a_1^\dagger a_2 + a_1 a_2^\dagger\right).
\]

A steady state analysis shows that for sufficiently small coupling strengths \(g \ll 6K\) the two-photon pumped KNR is approximately kept in the subspace spanned by \(|0, 0\rangle, |0, 1\rangle, |1, 0\rangle\) and \(|1, 1\rangle\) (see appendix A.8). Similar to the previous section it can be shown that in the spin subspace spanned by the logical spins \(|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle\) this Hamiltonian takes the form

\[
\hat{H}_1 = 2ga^\dagger a^\dagger a^\dagger a + \text{const}.
\]

This is the Hamiltonian for two magnetically coupled spins with \(g > 0\) \((g < 0)\) corresponding to the ferromagnetic (antiferromagnetic) coupling between the spins in the rotating frame. The initial Hamiltonian for two coupled KNRs is chosen to be:

\[
\hat{H}_0 = \sum_{i=1}^2 \left(-\delta a_i^\dagger a_i - K a_i^{12} a_i^2\right) + g \left(a_i^\dagger a_2 + a_i a_2^\dagger\right).
\]
The two-spin Ising problem is then realized with the time-dependent Hamiltonian
\[
\mathcal{H}(t) = \left( 1 - \frac{t}{\tau} \right) \mathcal{H}_0 + \frac{t}{\tau} \mathcal{H}_1.
\]

Furthermore, in order to ensure that the vacuum state \(|0,0\rangle\) is the roof state of the initial Hamiltonian \(\mathcal{H}_0\) the detuning has to be greater than the single-photon exchange rate \(\delta > g\), which can be seen by looking at the eigenstate following the roof state
\[
\mathcal{H}_0 \left( \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}} \right) = (g - \delta) \left( \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}} \right).
\]

What is interesting about the two-spin Ising problem is that the ground state exhibits frustration. This implies that the ground state for the two-spin problem is degenerate. At \(t = \tau\) the two degenerate roof states for ferromagnetic and antiferromagnetic coupling are \(\{|0,0\rangle,|1,1\rangle\}\) and \(\{|0,1\rangle,|1,0\rangle\}\), respectively. It might seem like this can pose a problem since the gap closes between the two "highest" energy states, and therefore violates the adiabatic theorem. However, the theorem is not violated for the following reason. Recall from your introductory course in quantum mechanics that an observable \(A\) commutes with the Hamiltonian, i.e. \([\hat{A},\mathcal{H}] = 0\). It can be shown that the parity operator which is defined by \(\hat{P}_2 = \exp \left( i \pi \sum_{i=1}^{2} \hat{a}_i \hat{a}_i^\dagger \right)\) commutes with both the initial and final Hamiltonian, so that \([\hat{P}_2,\mathcal{H}(t)] = 0\). This means that parity is conserved and that transitions only occur between even-even and odd-odd parity eigenstates. So since the zero-photon Fock state \(|0\rangle\) has even parity and the single-photon Fock states \(|1\rangle\) has odd parity, degeneracy is not a problem.

We now possess the tools to write down the final Hamiltonian \(\mathcal{H}_f\) for \(N\) linearly coupled KNRs, which reads
\[
\mathcal{H}_f = \sum_{i=1}^{N} \left( -K \hat{a}_i^\dagger \hat{a}_i^2 + G \left( \hat{a}_i^{1\dagger} + \hat{a}_i^2 \right) + F_i \left( \hat{a}_i^\dagger + \hat{a}_i \right) \right) + \sum_{1 \leq i < j \leq N} g_{ij} \left( \hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i \right),
\]
where \(g_{ij}\) is the single-photon coupling strength between resonator \(i\) and \(j\). In the computational basis the final Hamiltonian becomes
\[
\mathcal{H}_f = 2 \alpha^2 \sum_{1 \leq i < j \leq N} g_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + 2 \alpha \sum_{i=1}^{N} F_i \hat{\sigma}_i^z + \text{const},
\]
which corresponds to the Ising Hamiltonian for \(N\) spins. The corresponding initial Hamiltonian can just as readily be written down
\[
\mathcal{H}_0 = \sum_{i=1}^{N} \left( -\delta \hat{a}_i^\dagger \hat{a}_i^2 - K \hat{a}_i^{1\dagger} \hat{a}_i^2 \right) + \sum_{1 \leq i < j \leq N} g_{ij} \left( \hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i \right).
\]
Lastly, the final component for this continuous variable quantum annealing architecture is the readout of the state of the computational states. This can be implemented through balanced homodyne detection that is enabled via capacitively coupled transmission lines [15] as described in section 2.6.
In this chapter we present the results of our numerical simulations of the Wigner negativity for the single Ising spin in a magnetic field problem and the success probability from running a small instance of the subset sum problem and the number partitioning problem. We begin with the former where we study the Wigner negativity during the time evolution of the single Ising spin in a magnetic field problem that is mapped to a single KNR subject to two- and one-photon pumping. We end with the numerical results for the subset sum problem and the number partitioning problem, where we show how these problems are mapped onto the final state of the two-photon pumped KNR, and we compute the success probability of obtaining the solution to the problems. At the end of each section we will have a discussion where we will comment on the results.

The detuning was set to $\delta = 1K$ and two-photon pump was set to $G = 3K$ in all the simulations, which corresponds to experimentally relevant parameters. All numerical results that are presented in this Chapter where written in Python using the QuTiP package [49], [50]. The dimension of the Hilbert space was truncated to $n = 16$ for each resonator in order to ensure a relative small numerical error (see appendix A.9). The Python code that was used to generate these results can be found in appendix B.

4.1 Wigner negativity for a single Kerr-nonlinear resonator

We have discussed that the Wigner function negativity is a necessary resource for computational speed-up of quantum algorithms over classical ones. Losses and experimental imperfections degrade this resource. Therefore, it is important to determine e.g. which rate of photon loss is tolerated before the generated state becomes Wigner positive during the full evolution of the quantum annealer, which prevents from any possible computational advantage. Therefore, we have varied the single-photon loss rate parameter for the trivial problem of finding the ground state of a single Ising spin in a magnetic field using the KNR architecture described in section 3.3 and studied whether the negativity of the Wigner function persists in the presence of losses. Furthermore, we have determined the loss parameter for when the Wigner function becomes positive for all times during the anneal.
4.1. Wigner negativity for a single Kerr-nonlinear resonator

Numerical simulation

The single-photon pump amplitude was set to \( F = 4G/100 = 0.12K \) to ensure that the condition \( 4G \gg F \) is satisfied. The time-dependent Hamiltonian for finding the ground state of the single Ising spin in a magnetic field problem is given by Eq. (3.4). We diagonalized the instantaneous Hamiltonian in Eq. (3.4) to obtain the eigenvalues and eigenstates in order to determine the adiabatic condition using Eq. (1.2). The time evolution of the eigenspectrum for the two “highest” energy levels is shown in Fig. 4.1 where the minimum energy gap is indicated with \( \Delta_{\text{min}} = 0.34K \). The adiabatic condition was found to be \( \tau \gg 6.86/K \).

Figure 4.1: Time evolution of eigenenergies for the two “highest” energy states for the KNR. The detuning is set to \( \delta = 1K \) and the two-photon pump amplitude is \( G = 3K \) while the single-photon pump amplitude is \( F = 0.12K \). The minimum energy gap indicated at \( t/\tau = 0.5 \) where it is \( \Delta_{\text{min}} = 0.34K \).

The adiabatic condition Eq. (1.2) is not given by an inequality but is instead determined by a strong inequality (characterized by a “\( \gg \)” sign), it can therefore be difficult to determine a precise value to exactly how much greater \( \tau \) must be in order for the adiabatic condition to be fulfilled. In this thesis when determining \( \tau \) we multiply as a rule of thumb the right hand side of Eq. (1.2) by a factor of one hundred, so that \( \tau \) is equal to 686/K in this case. With realistic experimental values \( K/2\pi = 750 \) KHz this would correspond to an evolution time of 146 \( \mu s \).

The resonator was initialized from vacuum \( |0\rangle \) and the Lindblad master equation was numerically solved the for the system with \( \hat{C} = \hat{a}\sqrt{\gamma} \) and \( \hat{H}(t) \) given by Eq. (3.4). Fig. 4.2 shows the minimum value of the Wigner function labelled \( W_{\text{min}}(\alpha) \), during the evolution of the KNR for the three different single-photon loss rates \( \gamma = 0, \ 0.012K \) and \( 0.12K \). In the absence of losses the Wigner negativity starts from zero and then peaks around \( t/\tau \approx 0.45 \) where \( W_{\text{min}}(\alpha) \approx -0.11 \), before returning to zero. When \( \gamma = 0.012K \) the minimum value of the Wigner function is \( W_{\text{min}}(\alpha) \approx -0.02 \) and happens around \( t/\tau \approx 0.4 \), and finally for \( \gamma = 0.12K \) the Wigner function is positive for all times.

Figure 4.2: The minimum value of the Wigner function during the evolution of the KNR when initialized to vacuum for three different loss parameters \( \gamma = 0, \ 0.012K \) and \( 0.12K \).
4.1. Wigner negativity for a single Kerr-nonlinear resonator

Fig. 4.3 and Fig. 4.4 show the Wigner function for the three different time steps $t/\tau = 0$, 0.45 and 1 of the KNR during the evolution for a positive and negative single-photon pump amplitude respectively. The success probability of reaching the logical spin down state $|1\rangle$ with $\gamma = 0$ and $F = 0.12K$ is calculated from the diagonal element $\langle 1|\rho(\tau)|1\rangle$ and was found to be 99.99%.

![Figure 4.3: The Wigner function of the KNR at three different times when the single-photon pump amplitude is positive $F = 0.12K$.](image)

![Figure 4.4: The Wigner function of the KNR at three different times when the single-photon pump amplitude is negative $F = -0.12K$.](image)

Discussion

The simulation of the quantum state evolution for the single-Ising spin in a magnetic field problem that is mapped to a single two- and one-photon pumped KNR in the absence of losses showed clear indication of Wigner negativity. For all loss rates the Wigner function has zero negativity at $t = 0$ and $t = \tau$. This is because the resonator state at $t = 0$ is the zero-photon Fock state and at $t = \tau$ is in a coherent state; as can be seen from Fig. 2.5, the Wigner function for these states does not exhibit any negativity.

A fair question is to ask why we observe Wigner negativity in the transition between the vacuum state $|0\rangle$ and the coherent state $|-\alpha\rangle$ or $|\alpha\rangle$. The negativity of the Wigner function that we observe is associated to the interference fringes in the Wigner function of an even cat state (see appendix A.10). Indeed, if the single-photon pump would not be present then the zero-photon state in which the resonator is initialized at the beginning of the evolution would evolve into an even cat-state, which is a super-position of two coherent states. Since the single-photon pump lifts the degeneracy between the coherent states of the superposition the system is slowly driven into the most energetically favourable state and not into an actual superposition. However at intermediate times the effect of the formation of a small even cat-state (often called kitten-state) can still be seen, in that negativity appears in the Wigner function, associated to the interference fringes of the cat, see the middle figure of Fig. 4.3 and 4.4. Furthermore by increasing the single-photon loss rate the coherence of the cat-state decreases, which reduces the size of the fringes. This is a reason to why the negativity decreases in the presence of dissipation and eventually vanishes.
From the results we found that around the single-photon loss parameter $\gamma = 0.12K$ the Wigner function becomes positive for all times. In experiments a single-photon loss rate below $\gamma < 0.012K$ is not unrealistic [53] and because we observe Wigner negativity for $\gamma = 0.012K$ we expect to observe Wigner negativity in the experiments as well. Therefore, we conclude that intermediate loss values less than $\gamma < 0.012K$ are tolerated, without that the Wigner function negativity vanishes completely. These simulations therefore point towards possible computational advantage.

4.2 Simulation of relevant combinatorial optimization problems

Subset sum problem

To demonstrate the capabilities of the two-photon pumped KNR architecture described in chapter 3, we have simulated a small instance of the subset sum problem that was introduced in section 2.3.

In order to map the subset sum problem onto a network of two-photon pumped KNRs we compare the couplings and the single-photon pumping strengths of the KNR Ising Hamiltonian Eq. (3.6) with the couplings and magnetic field strengths of the subset sum Ising Hamiltonian Eq. (2.3). By comparing these two equations we make the following correspondence for the couplings:

$$-2\alpha^2 g_{ij} \leftrightarrow J_{ij}. \quad (4.1)$$

The minus sign is introduced because $g_{ij} > 0$ ($g_{ij} < 0$) corresponds to ferromagnetic (antiferromagnetic) coupling while $J_{ij} > 0$ ($J_{ij} < 0$) corresponds to antiferromagnetic (ferromagnetic) coupling. Similar for the single-photon pumps we make the following correspondence:

$$2F_i \leftrightarrow h_i. \quad (4.2)$$

The subset sum problem we considered was defined by the set $n = \{-2, 1, 2\}$ and $m = 3$. This problem requires three coupled KNRs and the Ising spin configuration that satisfy this problem is $s_1 = -1$ and $s_2 = s_3 = +1$. By dividing Eq. (4.1) with $-2\alpha^2$ and replacing $J_{ij}$ with $n_in_j/2$ we get that the coupling between the KNR’s for the subset sum problem should be chosen as

$$g_{ij} = -A \frac{n_in_j}{4\alpha^2}; \quad g_{ii} = 0.$$  

Here $A$ is a positive scale factor which will leave the Ising problem invariant and is convenient to introduce in order to satisfy the physical constraints on the coupling strengths. Next we divide Eq. (4.2) by $2\alpha$ and replace $h_i$ with Eq. (2.2), to find that the single-photon pump amplitudes should be chosen as

$$F_i = A \frac{1}{2\alpha} \left( \frac{1}{2} \sum_{j=1}^{N} n_j - m \right) n_i,$$

where $A$ is the same scale factor. To make sure that the conditions $6K \gg g_{ij}$ and $12K \gg F_i$ are satisfied, we chose $A$ to be

$$A = \frac{K}{2 \max|h_i|} = \frac{K}{10}.$$  

It should be noted that the scale factor may vary depending on the given problem and the set of numbers in the problem. After having defined all the couplings and single-photon pump strengths we have diagonalized the time-dependent AQC Hamiltonian

$$\mathcal{H}(t) = \left( 1 - \frac{t}{\tau} \right) \mathcal{H}_0 + \frac{t}{\tau} \mathcal{H}_1, \quad (4.3)$$

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where the initial and final Hamiltonian are

\[
\mathcal{H}_1 = \sum_{i=1}^{3} \left( -K \hat{a}_{i}^\dagger \hat{a}_{i}^2 + G \left( \hat{a}_{i}^\dagger + \hat{a}_{i} \right) + F_i \left( \hat{a}_{i}^\dagger + \hat{a}_{i} \right) \right) + \sum_{1 \leq i < j \leq 3} g_{ij} \left( \hat{a}_{i}^\dagger \hat{a}_{j} + \hat{a}_{j}^\dagger \hat{a}_{i} \right),
\]

\[
\mathcal{H}_2 = \sum_{i=1}^{3} \left( -\delta \hat{a}_{i}^\dagger \hat{a}_{i} - K \hat{a}_{i}^2 \right) + \sum_{1 \leq i < j \leq 3} g_{ij} \left( \hat{a}_{i}^\dagger \hat{a}_{j} + \hat{a}_{j}^\dagger \hat{a}_{i} \right).
\]

In the spin subspace spanned by the computational basis the final Hamiltonian takes the form (dropping the constant)

\[
\mathcal{H}_1 = A \left( \frac{1}{2} \sum_{1 \leq i < j \leq 3} n_i n_j \tilde{\sigma}_i^z \tilde{\sigma}_j^z + \sum_{i=1}^{3} \left( \frac{3}{2} \sum_{j=1}^{3} n_j - m \right) n_i \tilde{\sigma}_i^z \right),
\]

realizing the Ising Hamiltonian for the subset sum problem. From diagonalizing the instantaneous AQC Hamiltonian Eq. (4.3) we found that the minimum energy gap was \( \Delta_{\text{min}} = 0.046 K \) and that the adiabatic condition was \( \tau \gg 7900 / K \) which requires an evolution time that is approximately \( \tau = 7900 / K \). Next we initialized each individual resonator to the zero-photon Fock state \(|0\rangle\), and numerically solved the Schrödinger equation with \( \mathcal{H}(t) \) given by Eq. (4.3). The success probability was then computed as the probability of occupation of the correct roof state at \( t = \tau \), that is, \( \langle 1,0,0|\rho(\tau)|1,0,0\rangle \), which corresponds to the solution \( s_1 = -1 \) and \( s_2 = s_3 = +1 \). The calculated success probability was found to be 99.999\% which shows that the evolution is to a very good approximation restricted to the computational subspace and that the KNR annealer correctly computes the solution of the subset sum problem. Fig. 4.5 shows the Wigner function at the end of the evolution for each individual resonator.

![Wigner function of each individual KNR at t = \( \tau \).](image)

**Figure 4.5:** The Wigner function for each individual KNR at \( t = \tau \).

### Number partitioning problem

To map the number partitioning problem onto a network of two-photon pumped KNRs we begin by comparing the KNR Ising Hamiltonian Eq. (3.6) with the number partitioning problem Ising Hamiltonian Eq. (2.5). The comparison shows that the single-photon exchange rate coupling between \( i \):th and \( j \):th resonator should be chosen as

\[
g_{ij} = -A \frac{n_i n_j}{\alpha^2}, \quad g_{ii} = 0.
\]

Since this problem does not require any additional single-photon pumps, we can safely include \( \alpha^2 \) that is in the denominator inside the scale factor \( A \), so that \( g_{ij} = -A n_i n_j \). We considered the number partitioning problem defined by the set \( n = \{ 3, 5, 8 \} \). This problem also requires three resonators coupled together. The fair partitioning to this problem is \( S_1 = \{ 3, 5 \} \) and \( S_2 = \{ 8 \} \) for which the Ising spin configurations that satisfy this problem are \( s_1 = s_2 = +1 \) and \( s_3 = -1 \) and the configuration with all spins flipped. We chose \( A = 0.0015 K \), to satisfy
6K ∝ g_{ij}. The initial and final Hamiltonian is given by

\begin{align*}
\mathcal{H}_1 &= \sum_{i=1}^{3} \left(-K\hat{a}_i^\dagger \hat{a}_i^2 + G \left(\hat{a}_i^{12} + \hat{a}_i^{21}\right)\right) + \sum_{1\leq i<j\leq 3} g_{ij} \left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger\right), \\
\mathcal{H}_0 &= \sum_{i=1}^{3} \left(-\hat{a}_i^\dagger \hat{a}_i - K\hat{a}_i^{12} \hat{a}_i^{21}\right) + \sum_{1\leq i<j\leq 3} g_{ij} \left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger\right).
\end{align*}

In the spin subspace spanned by the coherent eigenstates of the two-photon pumped KNR in the rotating frame the final Hamiltonian takes the form

\[ \mathcal{H}_1 = -A \sum_{1\leq i<j\leq 3} n_i n_j \hat{\sigma}_i^z \hat{\sigma}_j^z, \]

realizing the Ising Hamiltonian for the number partitioning problem. From diagonalizing the time-dependent AQC Hamiltonian we found that the minimum energy gap is \( \Delta_{\text{min}} = 0.3K \) and that the adiabatic condition was \( \tau \gg 3.92/K \), which requires an evolution time that is approximately \( \tau = 392/K \). Each resonator was then initialized to the vacuum \( |0\rangle \) and we numerically solved the Schrödinger equation. From this the success probability of reaching the desired state \( |0,0,1\rangle \) or \( |1,1,0\rangle \) was found to be 99.97%. Furthermore a calculation of the fidelity showed that the system reaches the entangled state \( \mathcal{N} \langle 0,0,1 | + |1,1,0\rangle \), where \( \mathcal{N} = 1/\sqrt{2 \left(1 + \exp(-6|\alpha|^2)\right)} \) is a normalization constant, with 100% fidelity.

Discussion

The numerical results that we have obtained show that in the ideal case with no losses the success probability was 99.98% for the subset sum problem and 99.97% for the number partitioning problem. The 0.02% - 0.03% error mostly comes from deviations from the computational subspace. A higher success probability could have been achieved by choosing a smaller scale factor since then the conditions \( 6K \gg g_{ij} \) and \( 12K \gg F_i \) would have been better satisfied. However, choosing a smaller scale factor decreases the minimum gap energy which leads to an increase in the evolution time of the algorithm. For example choosing \( A = 0.00025K \) for the number partitioning problem we get a success probability that is 100%, but the evolution time increases approximately by a factor of 6.

4.3 Remarks on scalability & the model

The results that we have presented confirm that the KNR architecture is a promising way of implementing quantum annealing. An important question that we didn’t address though, is the scalability of this model. The two-photon pumped KNR architecture could be scaled up using pairwise couplings between resonators only [15]. However, a combinatorial optimization requires in general long-range interactions between spins. Indeed, the coupling terms that appear in the problems are \( \propto n_i n_j \), and does so that the we require an all-to-all connectivity. It is usually difficult to implement experimentally such long-range interactions. One possible solution would be to embed the Ising spins in a so-called Chimera graph, as it is done in the D-wave system [54], [55]. Another possibility is to embed the Ising spins in a LHZ scheme which maps the Ising problem on a graph with local interactions only [56]. In the LHZ scheme \( N \) logical spins (the spins that define the optimization problem) are mapped to \( M = N(N-1)/2 \) physical spins (the spins available the lab). This was furthermore investigated by Puri et al. (2016) [15] where they showed how the LHZ scheme for two-photon pumped KNRs could be physically realized. A third connectivity solution was presented by Nigg et al. (2016) [51] where they showed that by leveraging the phenomenon of flux quantization for a network of Kerr-parametric oscillators they could achieve an all-to-all connected architecture. The investigation of any of these possible implementations was beyond the scope of this thesis and will be the object of further study.
As a final comment, we stress that in addition to non-rotating wave approximation corrections, the Hamiltonian of the SQUID will have higher order non-linearities than Eq. (3.1). Although, when Puri et al. [16] studied how higher order non-linearities affected the preparation of cat-states using a two-photon pumped KNR they found that the effect of higher-order terms appeared to be minimal. Therefore, in this thesis we have safely assumed that the first-order Kerr-nonlinearity Hamiltonian Eq. (3.1) is a meaningful physical model to work with.
5 Conclusion

The aim of this work was to investigate the continuous variable quantum annealing architecture proposed by Puri et al. [15], in connection to the possibility of using it to solve combinatorial optimization problems using superconducting quantum circuits. In contrast to qubit based quantum annealers the Ising spins of the continuous variable quantum annealer are encoded on the two stable eigenstates of the two-photon pumped Kerr-nonlinear resonator in the rotating frame. We have seen how the application of an additional weak single-photon pump effectively acts like a magnetic field, and how the coupling between resonators introduced an effective magnetic coupling between the Ising spins.

In this thesis we have studied the Wigner negativity for the single-Ising spin in a magnetic field problem that is encoded onto the two- and one-photon pumped KNR Hamiltonian in the rotating frame. Wigner negativity is a necessary albeit non sufficient resource for non-classical simulatability.

During the anneal in an ideal KNR, the Wigner function takes on negative values, which points towards possible computational advantage of the quantum annealer. However, the unavoidable presence of losses degrades the Wigner negativity. We have shown that for values of the loss parameter that are realistic experimentally, some negativity in the Wigner function during the evolution persists, which implies that quantum advantage is not ruled out. We have further determined for which threshold value of the loss parameters the Wigner negativity is eventually washed out, which identifies the onset of efficient classical simulatability by the theorem of Eisert and Mari [17].

In this thesis we have also shown how the subset sum problem and the number partitioning problem can be mapped onto this architecture and simulated a small instance of these problems. We found that the correct solution is obtained with high probability.

5.1 Future work

Recent studies have shown that stoquastic quantum annealers, i.e. annealers based on Hamiltonians with negative off diagonal elements in the “standard basis”[57], exhibit the same scaling of computational time with problem size as some classical simulation algorithms, such as quantum Monte Carlo (QMC) [58], [59]. This implies that for a physical quantum annealing device to have any chance of out-performing classical algorithms (such as QMC), it must also take advantage of nonstoquastic Hamiltonians i.e. Hamiltonians with positive off diagonal
elements, for which efficient QMC cannot be performed. E.g. the Hamiltonian for the qubit-based quantum annealer is stoquastic since the initial Hamiltonian has negative off-diagonal terms $-\sum_i \sigma_i^z$ in the $z$-basis. On the other hand, the Hamiltonian Eq. (3.5) and (3.4) has problem dependent sign off-diagonal elements in the Fock-basis $\sum_{ij} g_{ij} \left( \hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right)$. A future research project would be to understand whether a simple connection can be established between the stoquasticity of the Hamiltonian associated with the annealer and the positiveness of the Wigner function of the state that the annealer explores during the quantum evolution.

As a further perspective for future work, it would be of interest to study larger problems, as well as the performances of the continuous variable quantum annealer in comparison to a qubit based quantum annealer solving the same problem in the presence of dissipation. Unfortunately the computational hardware available limited us to simulate any larger problems.

Finally, we hope that this thesis can serve as a comprehensive background for understanding quantum annealing in continuous variable.
Appendix

A.1 The adiabatic theorem

In this section we will provide a simple and straightforward proof of the famous adiabatic theorem, first published by Max Born and Vladimir Fock (1928) [60].

**Theorem A.1.1** A particle that begin from the n-th eigenstate of a Hamiltonian that is gradually (adiabatically) changing from an initial form $\mathcal{H}_i$ into a final form $\mathcal{H}_f$, will remain in the n-th eigenstate [61].

Consider an arbitrary time-independent Hamiltonian $\mathcal{H}$, for which the Schrödinger equation reads

$$i\hbar \frac{d\Psi(x,t)}{dt} = \mathcal{H}\Psi(x,t).$$

Through separation of variables the time-independent Schrödinger equation can be obtained

$$\mathcal{H}\psi_n(x) = E_n\psi_n(x).$$

The general solution to the Schrödinger equation is given by a superposition of the separable solutions

$$\Psi(x,t) = \sum_n c_n\psi_n(x,t) = \sum_n c_n\psi_n(x)e^{-iE_n t/\hbar},$$

or by simply considering a specific eigenfunction

$$\Psi_n(x,t) = \psi_n(x)e^{-iE_n t/\hbar}.$$ 

Hence, the n-th eigenstate for a time-independent Hamiltonian remain in the n-th eigenstate, simply picking up a phase factor $-E_n t/\hbar$. On the contrary for a time-dependent Hamiltonian, the eigenenergies and eigenfunctions are themselves time-dependent. The instantaneous eigenstates and eigenenergies are defined as

$$\mathcal{H}(t)\psi_n(x,t) = E_n(t)\psi_n(x,t).$$

At any instant of time the eigenfunctions form a complete orthogonal set

$$\langle \psi_m(t)|\psi_n(t) \rangle = \delta_{mn}, \quad (A.1)$$
where the dependence of position is implicitly implied. We have also introduced the bra-ket notation $\psi_n(x) = \langle x | \psi_n \rangle$. The general solution to the Schrödinger equation is now given by
\[ \Psi(x, t) = \sum_n c_n(t) \psi_n(x, t) = \sum_n c_n(t) \psi_n(x, t) e^{i\theta_n(t)}, \tag{A.2} \]
where $\theta_n(t)$ is known as the dynamical phase factor
\[ \theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(s) ds. \]

Our task is to determine the coefficients $c_n(t)$, by substituting (A.2) into the Schrödinger equation we obtain
\[ i\hbar \sum_n (\dot{c}_n \psi_n + c_n \dot{\psi}_n + i c_n \psi_n \dot{\theta}_n) e^{i\theta_n} = \sum_n c_n E_n \psi_n e^{i\theta_n}. \]
The third term on the left cancel with the term on the right, since $\dot{\theta}_n = E_n$, thus we are left with
\[ i\hbar \sum_n (\dot{c}_n \psi_n + c_n \dot{\psi}_n) e^{i\theta_n} = 0. \]

Multiplying with an arbitrary eigenfunction $\langle \psi_m |$ from the left and using the orthogonality condition (A.1) yields
\[ \dot{c}_m = -\sum_n c_n \langle \psi_m | \psi_n \rangle e^{i(\theta_n - \theta_m)}. \]

To calculate the quantity $\langle \psi_m | \dot{\psi}_n \rangle$, we first observe that for $m \neq n$
\[ \frac{d}{dt} \langle \psi_m | \hat{\mathcal{H}} | \psi_n \rangle = 0 = \langle \dot{\psi}_m | \hat{\mathcal{H}} | \psi_n \rangle + \langle \psi_m | \hat{\mathcal{H}} | \dot{\psi}_n \rangle = E_n \langle \psi_m | \psi_n \rangle + E_m \langle \psi_m | \dot{\psi}_n \rangle + E_n \langle \dot{\psi}_m | \psi_n \rangle + E_m \langle \dot{\psi}_m | \psi_n \rangle. \]

Second note that
\[ \frac{d}{dt} \langle \psi_m | \psi_n \rangle = 0 = \langle \dot{\psi}_m | \psi_n \rangle + \langle \psi_m | \dot{\psi}_n \rangle, \]
which implies the relation $\langle \dot{\psi}_m | \psi_n \rangle = -\langle \psi_m | \dot{\psi}_n \rangle$, so
\[ \langle \dot{\psi}_m | \psi_n \rangle = \frac{\langle \psi_m | \hat{\mathcal{H}} | \psi_n \rangle}{E_n - E_m}, \quad (m \neq n). \]

This holds as long as no transitions between eigenstates occur
\[ \dot{c}_m = -c_m \langle \dot{\psi}_m | \psi_n \rangle = -c_m \langle \psi_m | \hat{\mathcal{H}} \psi_n \rangle \sum_{m \neq n} \frac{\langle \psi_m | \hat{\mathcal{H}} \psi_n \rangle}{E_n - E_m}. \]

Now, if the Hamiltonian is slowly changing, so that its time-derivative can be considered to be very small and that the energy difference $(E_n - E_m)^2$ is large compared to $|\langle \psi_m | \hat{\mathcal{H}} \psi_n \rangle|^2$, the second term becomes negligible. This approximation is known as the \textit{the adiabatic approximation}, and we conclude that
\[ \dot{c}_m \approx -c_m \langle \dot{\psi}_m | \psi_n \rangle. \]

By solving this equation one finds
\[ c_m(t) = c_m(0) \exp \left( -\int_0^t \langle \psi_m(s) | \dot{\psi}_m(s) \rangle ds \right) = c_m(0) e^{i\gamma_m(t)}, \]

A.1. The adiabatic theorem
where
\[ \gamma_m(t) = i \int_0^t \langle \dot{\psi}_m(s) | \dot{\psi}_m(s) \rangle \, ds, \]
is the geometrical (Berry) phase factor. Putting the obtained expression for the coefficients \( c_m(t) \) back into (A.2), we get that the \( n \)-th eigenstate is given by
\[ |\Psi_n(t)\rangle = e^{i\gamma_n(t)} e^{i\gamma_n(t)} |\psi_n(t)\rangle. \]
Hence, a system that starts out in the \( n \)-th eigenstate, will remain in the \( n \)-th eigenstate, simply picking up a couple of phase factors.

### A.2 Argument for noncommuting Hamiltonians in AQC

In AQC it is important that the initial \( \hat{H}_0 \) and final Hamiltonian \( \hat{H}_1 \) are two noncommuting Hamiltonians, i.e. \([\hat{H}_0, \hat{H}_1] \neq 0\). This can be understood by considering the following trivial example. Suppose that the initial and final Hamiltonian in AQC are given by
\[ \hat{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \hat{H}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \]
which is clearly two commuting Hamiltonians. Since they are both diagonal in the \( z \)-basis we’ll label the corresponding eigenvectors as
\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

It is easy to see that the ground state of \( \hat{H}_0 \) is \( |1\rangle \) and that the ground state of \( \hat{H}_1 \) is \( |0\rangle \). We mentioned in the introduction that AQC is based on the adiabatic theorem. For the theorem to hold there must always exist an energy gap between the eigenstates, see appendix A.1. Following the AQC algorithm Eq. (1.1) it can be seen that the energy gap between the eigenstates \( |1\rangle \) and \( |0\rangle \) closes at some point. In this example the energies becomes equal at the point \( t/\tau = 4/5 \), and so the gap between them closes. This is enough to see that \([\hat{H}_0, \hat{H}_1] \neq 0\) is a necessary condition for keeping the gap open.

### A.3 Liouville von Neumann equation

In this section we will derive the Liouville Von Neumann equation that describes the time-evolution of a density operator for a closed quantum system.

The density operator is defined by
\[ \hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k| \]
where \( p_k \) is a statistical weight. By taking the time derivative of the density operator we get
\[ \frac{\partial \hat{\rho}}{\partial t} = \frac{\partial}{\partial t} \sum_k p_k |\psi_k\rangle \langle \psi_k| \]
\[ = \sum_k p_k \left( -i \hbar \frac{\partial}{\partial t} |\psi_k\rangle \langle \psi_k| + \sum_k p_k |\psi_k\rangle \frac{\partial |\psi_k\rangle}{\partial t} \right) \]
Invoking Schrödinger’s equation, we are able to write the time-derivative of the state vector in terms of the system Hamiltonian acting on the state vector
\[ \frac{\partial |\psi\rangle}{\partial t} = -\frac{i}{\hbar} \hat{\mathcal{H}} |\psi\rangle \]
and thus we obtain
\[ \frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left( \sum_k p_k \hat{\mathcal{H}} |\psi_k\rangle \langle \psi_k| - \sum_k p_k |\psi_k\rangle \langle \psi_k| \hat{\mathcal{H}} \right) = -\frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\rho}] \]
which is the celebrated Liouville-von Neumann equation.
A.4 Circuit Lagrangian

\[
\Phi_{\text{ext}} \quad \phi_1 \quad \phi_2 \quad C_{J} \quad E_{J} \quad C_{J}
\]

Figure A.1: Left: Schematic illustration of a SQUID. Right: Circuit symbol for a Josephson junction with Josephson energy \( E_J \) and capacitance \( C_J \).

In this section of the appendix we aim to derive the two-photon pumped Kerr Hamiltonian starting from the circuit Lagrangian of a SQUID and follow the derivation presented by Nigg et al. [51]. The SQUID consists of two Josephson junctions connected in a superconducting loop with a magnetic flux \( \Phi_{\text{ext}}(t) \) penetrating the loop. Fig A.1 is meant to illustrate the SQUID and the corresponding circuit symbol of the Josephson junction. The circuit Lagrangian for a SQUID is given by [36]

\[
\mathcal{L} = \frac{1}{2} C_J \left( \dot{\phi}_1^2 + \dot{\phi}_2^2 \right) + E_J \left( \cos(2\pi \phi_1/\Phi_0) + \cos(2\pi \phi_2/\Phi_0) \right) \tag{A.3}
\]

where \( \phi_i \) is the phase difference across the junction and \( \Phi_0 = h/(2e) \) is the magnetic flux quantum. For convenience we chose units such that the flux quantum is \( \Phi_0 = h/(2e) = 2\pi \). The parameters \( C_J \) and \( E_J \) in Eq. (A.3) denote the capacitance and Josephson energy of each Josephson junction. Due to the fluxoid quantization condition, \( \phi_1 \) and \( \phi_2 \) are not independent variables. If a flux \( \Phi_{\text{ext}} \) is penetrating the loop, then the flux quantization requires that

\[
\phi_2 - \phi_1 = \Phi_{\text{ext}} ; \tag{A.4}
\]

where we have set the geometrical phase term to zero and neglected the induced flux term which is reasonable for small loops. We introduce the new variable

\[
\theta = \frac{1}{2}(\phi_1 + \phi_2) ;
\]

which together with Eq. (A.4) implies

\[
\phi_1 = \theta - \frac{1}{2} \Phi_{\text{ext}} ; \tag{A.5}
\]

\[
\phi_2 = \theta + \frac{1}{2} \Phi_{\text{ext}} . \tag{A.6}
\]

Inserting Eq. (A.5) and (A.6) into (A.3) and simplifying we get

\[
\mathcal{L} = \frac{1}{2} C \dot{\theta}^2 + 2E_J \cos \left( \frac{\Phi_{\text{ext}}}{2} \right) \cos(\theta) \]

with \( C = 2C_J \). We have also neglected the \( \frac{1}{4} C_J \dot{\phi}_{\text{ext}}^2 \) term which is independent of coordinates and does not affect the dynamics of the system. The Hamiltonian function is obtained as usual through the Legendre transformation

\[
\mathcal{H} = \sum_i Q_i \dot{\theta}_i - \mathcal{L} ,
\]

with \( Q_i = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} = C \dot{\theta}_i \) being the generalized momenta. From the Legendre transformation we obtain the Hamiltonian

\[
\mathcal{H} = E_C \frac{Q^2}{2e^2} - 2E_J \cos \left( \frac{\Phi_{\text{ext}}}{2} \right) \cos(\theta) . \tag{A.7}
\]
where $E_C \equiv e^2/C$ is the charging energy. Next, we consider a monochromatic magnetic flux modulation threaded through the loop that is of the form

$$\frac{\Phi_{\text{ext}}(t)}{2} = \Phi_{\text{dc}} + \delta \Phi(t)$$

where $\Phi_{\text{dc}}$ is the static magnetic flux and $\delta \Phi(t)$ is a time-dependent perturbation that we choose of the form

$$\delta \Phi(t) = \delta \Phi_{\text{ac}} \cos(\Omega t),$$

where $\delta \Phi_{\text{ac}}$ is the ac flux. The external flux cosine term in Eq. (A.7) can now be written as

$$\cos\left(\frac{\Phi_{\text{ext}}(t)}{2}\right) = \cos(\Phi_{\text{dc}}) \cos(\delta \Phi_{\text{ac}} \cos(\Omega t)) - \sin(\Phi_{\text{dc}}) \sin(\delta \Phi_{\text{ac}} \cos(\Omega t)).$$

Using the following mathematical tricks [62]

$$\cos(z \cos(\theta)) = J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(z) \cos(2n\theta)$$

$$\sin(z \cos(\theta)) = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}(z) \cos[(2n-1)\theta],$$

where $J_n(z)$ is the Bessel function of the first kind, we get

$$\cos\left(\frac{\Phi_{\text{ext}}(t)}{2}\right) = \cos(\Phi_{\text{dc}}) \left( J_0(\delta \Phi_{\text{ac}}) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(\delta \Phi_{\text{ac}}) \cos(2n\Omega t) \right)$$

$$- \sin(\Phi_{\text{dc}}) \left( 2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}(\delta \Phi_{\text{ac}}) \cos[(2n-1)\Omega t] \right).$$

If the ac-flux is sufficiently small, $|\delta \Phi_{\text{ac}}| \ll 1$, then we can keep only the two leading terms of the sum, that’s $J_0$ and $J_1$ and use the approximations $J_0(z) \approx 1$ and $J_1(z) \approx z/2$ yielding

$$\cos\left(\frac{\Phi_{\text{ext}}(t)}{2}\right) \approx \cos(\Phi_{\text{dc}}) + \sin(\Phi_{\text{dc}}) \delta \Phi_{\text{ac}} \cos(\Omega t).$$

For convenience we now separate the Hamiltonian into a time independent and time dependent part as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2(t)$, where

$$\mathcal{H}_0 = E_C \frac{Q^2}{2e^2} - 2E_J \cos(\Phi_{\text{dc}}) \cos(\theta)$$

$$\mathcal{H}_2(t) = -2E_J \sin(\Phi_{\text{dc}}) \delta \Phi_{\text{ac}} \cos(\Omega t) \cos(\theta).$$

We now expand the $\cos(\theta)$ to fourth order, keeping only the leading nonlinearity. Furthermore since we focus on the weak ac modulation $|\delta \Phi_{\text{ac}}| \ll 1$, we can neglect the nonlinear part of the time dependent term, then we have

$$\mathcal{H}_1 = E_C \frac{Q^2}{2e^2} + E_J \cos(\Phi_{\text{dc}}) \theta^2 - \frac{E_J}{12} \cos(\Phi_{\text{dc}}) \theta^4,$$

$$\mathcal{H}_2(t) = E_J \sin(\Phi_{\text{dc}}) \delta \Phi_{\text{ac}} \cos(\Omega t) \theta^2,$$

where we have dropped the coordinate independent terms that does not contribute to the dynamics. As the next step we promote $\theta$ and $Q$ to operators with the commutation relation

$$[\hat{\theta}, \hat{Q}] = i.$$

By introducing the creation and annihilation operator $\hat{a}^\dagger$ and $\hat{a}$ that obey the bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we have that

$$\hat{\theta} = \sqrt{\frac{1}{2Z}} (\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{Q} = -i \sqrt{\frac{Z}{2}} (\hat{a} - \hat{a}^\dagger),$$
Putting it all together we have

\[ \cos \Delta \] by an amount \( a \) two-photon pump frequency that is close to twice the resonator frequency, i.e. \( \Omega \) of the resonator frequency. For the third and last term in Eq. (A.3)

Next normal ordering the terms yields

\[ H = \sqrt{2E_C E_J \cos(\Phi_{dc})} \]

where we have defined the resonator frequency

\[ \omega_r = \sqrt{2E_C E_J \cos(\Phi_{dc})/\epsilon^2} \]

and have furthermore dropped the \( \omega_r/2 \) which simply corresponds to a constant energy shift. Next we move to the interaction picture. To switch into the interaction picture we divide the Schrödinger picture Hamiltonian into two parts:

\[ \hat{H}_S = \hat{H}_{S,0} + \hat{H}_{S,1} \]

where

\[ \hat{H}_{S,0} = \omega_r \hat{a}^\dagger \hat{a} \]

and

\[ \hat{H}_{S,1} = -\frac{E_C}{96\epsilon^2} (\hat{a} + \hat{a}^\dagger)^4 + \frac{E_J}{2Z} \sin(\Phi_{dc}) \delta \Phi_{ac} \cos (\Omega t) (\hat{a} + \hat{a}^\dagger)^2. \]

The Hamiltonian in the interaction picture is defined as

\[ \hat{H}_{\text{int}} = e^{i\hat{H}_{S,0} t} \hat{H}_{S,1} e^{-i\hat{H}_{S,0} t}, \]

and we thus get

\[ \hat{H}_{\text{int}} = \omega_r \hat{a}^\dagger \hat{a} - \frac{\epsilon_{\omega_r} \hat{a}^\dagger \hat{a}}{96\epsilon^2} \frac{E_C e^{-i\omega_r \hat{a}^\dagger \hat{a} t}}{2Z} \sin(\Phi_{dc}) \delta \Phi_{ac} \cos (\Omega t) e^{i\omega_r \hat{a}^\dagger \hat{a} t} e^{-i\omega_r \hat{a}^\dagger \hat{a} t}. \] (A.8)

Next normal ordering the terms yields

\[ : (\hat{a} + \hat{a}^\dagger)^4 : = \hat{a}^4 + 6\hat{a}^3 \hat{a} + 6\hat{a}^2 \hat{a}^2 + 4\hat{a}^2 \hat{a}^3 + 2\hat{a}^3 \hat{a} + 4\hat{a}^2 + 3, \]

\[ : (\hat{a} + \hat{a}^\dagger)^2 : = \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + \hat{a}^2 + 1. \]

Under the unitary transformation \( \hat{U}(t) = e^{-i\omega_r \hat{a}^\dagger \hat{a}} \) each \( \hat{a} \) and \( \hat{a}^\dagger \) transform according to

\[ e^{i\omega_r \hat{a}^\dagger \hat{a}} e^{-i\omega_r \hat{a}^\dagger \hat{a}} = \hat{a} e^{i\omega_r t}, \]

\[ e^{i\omega_r \hat{a}^\dagger \hat{a}} e^{-i\omega_r \hat{a}^\dagger \hat{a}} = \hat{a}^\dagger e^{-i\omega_r t}. \]

Now we use the rotating wave approximation (RWA) which corresponds to neglecting all rotating terms proportional to \( \exp(i\omega_r t) [39] \). In the second term of Eq. (A.8) the only non-rotating terms are \( 6\hat{a}^2 \hat{a}^2 \) and \( 12\hat{a} \hat{a}^\dagger \hat{a} \). The latter leads to a renormalization of the oscillator frequency by an amount \( \Delta \omega_r = -E_C/(8\epsilon^2) \) which we absorb into \( \omega_r \) and make an implicit redefinition of the resonator frequency. For the third and last term in Eq. (A.8) we specifically consider a two-photon pump frequency that is close to twice the resonator frequency, i.e. \( \Omega \approx 2\omega_r \). By writing \( \cos (\Omega t) \) in Euler form we find that the only non-rotating terms are \( \hat{a}^1 \hat{a}^2 \) and \( \hat{a}^2 \). Putting it all together we have

\[ \hat{H}_{\text{int}} = \omega_r \hat{a}^\dagger \hat{a} - \frac{E_C}{16\epsilon^2} \hat{a}^1 \hat{a}^2 + \frac{E_J}{4Z} \sin(\Phi_{dc}) \delta \Phi_{ac} \hat{a}^2. \]
Switching back to the Schrödinger picture we get
\[
\mathcal{\hat{H}}(t) = \omega_r \hat{a}^\dagger \hat{a} - K \hat{a}^{12} \hat{a}^2 + G(t) \left( e^{i \omega_p(t) t} \hat{a}^\dagger + e^{i \omega_p(t) t} \hat{a}^2 \right)
\]
(A.9)
which is the two-photon pumped Kerr-nonlinear resonator Hamiltonian, where we have defined the Kerr-nonlinear amplitude as
\[
K \equiv \frac{E_C}{16\epsilon^2},
\]
and the two-photon pump amplitude as
\[
G \equiv \frac{E_j}{4Z} \sin(\Phi_{DC}) \delta\Phi_{ac}.
\]

A.5 Transformation to the rotating frame

In this part of the appendix we will demonstrate how to derive the time-dependent AQC Hamiltonian for the two- and one-photon pumped KNR. Let’s consider the two- and one-photon parametrically driven Kerr-nonlinear resonator Hamiltonian that is given by
\[
\mathcal{\hat{H}}(t) = \omega_r \hat{a}^\dagger \hat{a} - K \hat{a}^{12} \hat{a}^2 + G(t) \left( e^{i \omega_p(t) t} \hat{a}^\dagger + e^{i \omega_p(t) t} \hat{a}^2 \right)
\]
where \(\omega_r\) is the frequency of the resonator, \(\omega_p(t)\) is the two-photon pump frequency and the one-photon pump frequency is chosen half times the two-photon pump frequency. We transform to a frame rotating at the single-photon pump frequency by performing the following unitary transformation \(\hat{U}(t) = e^{i \omega_p(t) t} \hat{a}^\dagger / \sqrt{2}\). Under a unitary transformation the state vector \(|\psi(t)\rangle\) transforms according to
\[
|\tilde{\psi}(t)\rangle = \hat{U}(t) |\psi(t)\rangle \rightarrow |\tilde{\psi}(t)\rangle = \hat{U}(t) |\psi(t)\rangle.
\]
(A.10)
Substituting the transformed state vector into the Schrödinger equation we obtain
\[
i \frac{\partial |\tilde{\psi}(t)\rangle}{\partial t} = i \frac{\partial}{\partial t} |\psi(t)\rangle + \hat{U}(t) i \frac{\partial |\psi(t)\rangle}{\partial t}
\]
\[
= i \frac{\partial \hat{U}(t)}{\partial t} |\psi(t)\rangle + \hat{U}(t) \mathcal{\hat{H}}(t) |\psi(t)\rangle
\]
by using Eq. (A.10) we replace \(|\psi(t)\rangle\) with \(\hat{U}^\dagger(t) |\tilde{\psi}(t)\rangle\) and get
\[
\frac{\partial |\tilde{\psi}(t)\rangle}{\partial t} = \left( i \frac{\partial \hat{U}(t)}{\partial t} \hat{U}^\dagger(t) + \hat{U}(t) \mathcal{\hat{H}}(t) \hat{U}^\dagger(t) \right) |\tilde{\psi}(t)\rangle = \mathcal{\hat{H}}(t) |\tilde{\psi}(t)\rangle
\]
where
\[
\mathcal{\hat{H}}(t) \equiv i \frac{\partial \hat{U}(t)}{\partial t} \hat{U}^\dagger(t) + \hat{U}(t) \mathcal{\hat{H}}(t) \hat{U}^\dagger(t)
\]
(A.11)
is the transformed Hamiltonian. We proceed by evaluating the first term in this expression
\[
i \frac{\partial \hat{U}(t)}{\partial t} \hat{U}^\dagger(t) = -\frac{1}{2} (\omega_p(t) + \dot{\omega_p}(t) t) \hat{a}^\dagger \hat{a} \hat{U}(t) \hat{U}^\dagger(t) = \frac{1}{2} (\omega_p(t) + \dot{\omega_p}(t) t) \hat{a}^\dagger \hat{a},
\]
where the dot \(\dot{\omega_p}(t)\) denotes the time-derivative. We then calculate the second term of Eq. (A.11) by invoking the Baker-Hausdorff lemma,
\[
e^A e^{-A} = a + [A,a] + \frac{1}{2!} [A,[A,a]] + \ldots
\]
where $A$ is a Hermitian operator. Applying this formula we get that each operator $\hat{a}$ and $\hat{a}^\dagger$ transform according to

$$\hat{U}^\dagger(t) \hat{a} \hat{U}(t) = \hat{a} e^{-i\omega_p(t)t/2 + \frac{(-i\omega_p(t)t/2)^2}{2!} + \ldots},$$

$$\hat{U}^\dagger(t) \hat{a}^\dagger \hat{U}(t) = \hat{a}^\dagger e^{i\omega_p(t)t/2 + \frac{(i\omega_p(t)t/2)^2}{2!} + \ldots},$$

and thus the Hamiltonian in the rotating frame is given by (we can now drop the tilde)

$$\hat{H}(t) = \left(\omega_r - \frac{1}{2}(\omega_p(t) + \dot{\omega}_p(t)t)\right)\hat{a}^\dagger \hat{a} - K\hat{a}^\dagger \hat{a}^2 + G(t)\left(\hat{a}^\dagger^2 + \hat{a}^2\right) + F(t)\left(\hat{a}^\dagger + \hat{a}\right).$$

By cleverly choosing the pump frequency

$$\omega_p(t) = 2\omega_r + 2\delta \left(1 - \frac{t}{2\tau}\right) \Rightarrow \dot{\omega}_p(t) = -\delta \frac{1}{\tau}$$

and the drive strengths

$$G(t) = \frac{t}{\tau} G \quad \text{and} \quad F(t) = \frac{t}{\tau} F,$$

the Hamiltonian in the rotating frame now reads

$$\hat{H}(t) = -\left(1 - \frac{t}{\tau}\right) \delta \hat{a}^\dagger \hat{a} - K\hat{a}^\dagger^2 \hat{a}^2 + \frac{t}{\tau} G\left(\hat{a}^\dagger^2 + \hat{a}^2\right) + \frac{t}{\tau} F\left(\hat{a}^\dagger + \hat{a}\right)$$

$$= \left(1 - \frac{t}{\tau}\right) \left(-\delta \hat{a}^\dagger \hat{a} - K\hat{a}^\dagger^2 \hat{a}^2\right) + \frac{t}{\tau} \left(-K\hat{a}^\dagger^2 \hat{a}^2 + G\left(\hat{a}^\dagger^2 + \hat{a}^2\right) + F\left(\hat{a}^\dagger + \hat{a}\right)\right)$$

(A.12)

where we have defined

$$\hat{H}_0 \equiv \left(-\delta \hat{a}^\dagger \hat{a} - K\hat{a}^\dagger^2 \hat{a}^2\right) \quad \text{and} \quad \hat{H}_1 \equiv -K\hat{a}^\dagger^2 \hat{a}^2 + G\left(\hat{a}^\dagger^2 + \hat{a}^2\right) + F\left(\hat{a}^\dagger + \hat{a}\right).$$

### A.6 Steady state & stability

In this section of the appendix we will demonstrate one way to obtain the steady state fixed points for the two-photon pumped KNR in presence of single-photon loss.

The Lindblad master equation, which is an operator valued equation, can be turned into a differential equation (Fokker-Planck like equation) by first writing the density operator $\hat{\rho}$ describing the state of the system, in terms of the coherent state projectors $|\alpha\rangle \langle \alpha|$. The coherent state representation of the density matrix is defined by $[29], [63]$

$$\hat{\rho}(t) = \int d^2\alpha P(\alpha, \alpha^*, t) |\alpha\rangle \langle \alpha|,$$

(A.13)

where $P(\alpha, \alpha^*, t)$ is some weight function that is normalized such that $\text{Tr}[\hat{\rho}(t)] = 1$. The integral of Eq. (A.13) is formally known as the Glauber Sudarshan P-representation of the state $\hat{\rho}(t)$. To derive an equation for $P(\alpha, \alpha^*, t)$ we substitute the P-representation into the Lindblad master equation (3.2) and use the properties

$$\hat{a} |\alpha\rangle \langle \alpha| = \alpha |\alpha\rangle \langle \alpha|,$$

$$|\alpha\rangle \langle \alpha| \hat{a}^\dagger = \alpha^* |\alpha\rangle \langle \alpha|,$$

$$|\alpha\rangle \langle \alpha| \hat{a} = \left(\frac{\partial}{\partial \alpha} + \alpha^*\right) |\alpha\rangle \langle \alpha|,$$

$$\hat{a}^\dagger |\alpha\rangle \langle \alpha| = \left(\frac{\partial}{\partial \alpha^*} + \alpha\right) |\alpha\rangle \langle \alpha|,$$

A fixed point is when a solution does not change in time.
which can be derived from the definition of the coherent state. Next we perform integration by parts and assume that the boundary conditions at infinity are zero. This introduces a minus sign in front of each differential operator. The above properties show that we have the following correspondences

\[ \hat{a} \hat{\rho}(t) \leftrightarrow \alpha P(\alpha, \alpha^*, t), \]
\[ \hat{a}^\dagger \hat{\rho}(t) \leftrightarrow \left( \alpha^* - \frac{\partial}{\partial \alpha^*} \right) P(\alpha, \alpha^*, t), \]
\[ \hat{\rho}(t) \hat{a} \leftrightarrow \left( \alpha - \frac{\partial}{\partial \alpha^*} \right) P(\alpha, \alpha^*, t), \]
\[ \hat{\rho}(t) \hat{a}^\dagger \leftrightarrow \alpha^* P(\alpha, \alpha^*, t). \]  

(A.14)

Substituting Eq. (A.13) into the master equation Eq. (3.2) and using the identities given by Eq. (A.14) we get

\[
\frac{\partial P(\alpha, \alpha^*, t)}{\partial t} = \left( -\frac{\partial}{\partial \alpha}(i2K\alpha^2 \alpha^* - i2G\alpha^* - \frac{\gamma}{2}\alpha) - \frac{\partial}{\partial \alpha^*}(\alpha - i2K\alpha^2 \alpha + i2G\alpha - \frac{\gamma}{2}\alpha^*), \right. \\
\left. + \frac{\partial^2}{\partial \alpha^2}(iK\alpha^2 - iG) + \frac{\partial^2}{\partial \alpha^2}(iK\alpha^2 + iG) \right) P(\alpha, \alpha^*, t). 
\]

(A.15)

This equation is on the form of a Fokker-Planck equation and can moreover be written as a stochastic differential equation [38]

\[
\frac{d\alpha}{dt} = i2 \left( K\alpha^2 \alpha^* - G\alpha^* + i\frac{\gamma}{4}\alpha \right) + \sqrt{2(2K\alpha^2 - G)}\eta(t), \\
\frac{d\alpha^*}{dt} = i2 \left( -K\alpha^2 \alpha + G\alpha + i\frac{\gamma}{4}\alpha^* \right) + \sqrt{2(2K\alpha^2 - G)}\eta^*(t). 
\]

(A.16)

It can be shown that this stochastic differential equation is indeed equivalent to the Fokker-Planck equation (A.15) [37]. Here \( \eta(t), \eta^*(t) \) are stochastic forces (Langevin terms) with zero mean. The semi-classical or mean value equation is obtained by taking the average, the Langevin terms thus disappear since \( \langle \eta(t) \rangle = \langle \eta(t)^* \rangle = 0 \) [38]. The semi-classical equation of motion thus are

\[
\frac{d\alpha}{dt} = i2 \left( K\alpha^2 \alpha^* - G\alpha^* + i\frac{\gamma}{4}\alpha \right), \\
\frac{d\alpha^*}{dt} = i2 \left( -K\alpha^2 \alpha + G\alpha + i\frac{\gamma}{4}\alpha^* \right). 
\]

(A.17)

The steady-state to the semi-classical equation of motion can now readily be obtained by setting \( \frac{d\alpha}{dt} = 0 \), and solving this equation. The equation \( K\alpha^2 \alpha^* - G\alpha^* + i\frac{\gamma}{4}\alpha = 0 \) has three solutions which are given by \( \alpha_i^* = 0 \) and \( \alpha_{2,3}^* = \pm re^{i\theta} \), where

\[
r = \frac{1}{2} \left( \frac{16G^2 - \gamma^2}{K^2} \right)^{1/4} \quad \text{and} \quad \theta = -\frac{1}{2} \arctan \left( \frac{\gamma}{\sqrt{16G^2 - \gamma^2}} \right),
\]

which reduces to \( \alpha_{2,3}^* = \sqrt{G/K} \) in the case of \( 4G \gg \gamma \).

To investigate the stability of these solutions we linearise Eq. (A.17) around the steady states

\[ \alpha_i(t) = \alpha_i^* + \delta\alpha_i(t), \quad (i = 1, 2, 3), \]

where \( \alpha_i^* \) is the \( i:th \) steady-state solution to Eq. (A.17) and \( \delta\alpha_i \) is a small perturbation. The linearized equations written in matrix form are

\[
\frac{d}{dt} \begin{pmatrix} \delta\alpha_1(t) \\ \delta\alpha_2(t) \\ \delta\alpha_3(t) \end{pmatrix} = i2 \begin{pmatrix} 2|\alpha_i^*|^2 + i\gamma/4 & K(\alpha_i^*)^2 - G & K(\alpha_i^*)^2 \end{pmatrix} \begin{pmatrix} \delta\alpha_1(t) \\ \delta\alpha_2(t) \\ \delta\alpha_3(t) \end{pmatrix}.
\]
and the eigenvalues to the matrix equation are
\[
\lambda_{\pm} = -\frac{\gamma}{2} \pm 2 \text{Im} \left( \sqrt{3K^2|\alpha_{\epsilon}|^4 + G \left( K(\alpha_{\epsilon})^2 - G + (\alpha_{\epsilon})^2K \right)} \right).
\]

Stability of the fixed points require all eigenvalues to have a positive real part less than or equal to zero [38]. We begin by examining the first steady state solution which is the origin \(\alpha_{\epsilon}^* = 0\), it has eigenvalues \(\lambda_{s} = -\gamma/2 \pm 2G\), so if \(4G > \gamma\), then the origin is an unstable solution. The eigenvalues to the second and third steady state solution \(\alpha_{2,3}^* = \pm r e^{i\theta}\) is most easily analyzed in the case when \(4G \gg \gamma\), where both steady state solutions has the degenerate eigenvalue \(-\gamma/2\), and are thus stable solutions.

### A.7 Effect of single-photon pump

In this section we will study the effect that is obtained when a single-photon pump is added to the two-photon pumped KNR. For the moment we will neglect the effect of single-photon loss, since it has already been treated in appendix A.6. The Hamiltonian for the two- and one-photon KNR written in a frame rotating at the resonator frequency is given by
\[
\mathcal{H} = -K\hat{a}^\dagger \hat{a}^2 + G(\hat{a}^\dagger \hat{a}^2) + F(\hat{a}^\dagger + \hat{a}).
\]

In the Heisenberg picture the time evolution of the annihilation operator is given by
\[
\frac{d\hat{a}}{dt} = i[\mathcal{H}, \hat{a}] = i \left( 2K\hat{a}^\dagger \hat{a}^2 - 2G\hat{a}^\dagger - F \right).
\]

To obtain the semi-classical equations of motion we take the average \(\langle \hat{a} \rangle = \alpha\), which is obtained by simply replacing \(\hat{a}\) with the complex variable \(\alpha\),
\[
\frac{d\alpha}{dt} = i \left( 2K\alpha^* \alpha^2 - 2G\alpha^* - F \right).
\]

The steady state to this equation has three solutions which are of the form \(\alpha_1 = (-2\epsilon, 0)\) and \(\alpha_{2,3} = (\pm \alpha + \epsilon, 0)\) where \(\alpha = \sqrt{G/K}\) and \(\epsilon = F/4G\). Hence if \(4G \gg F\) so that \(\epsilon \to 0\), then \(\alpha_{2,3} \approx \pm \sqrt{G/K}\). To investigate the stability of the steady state solutions we do a small perturbation around the fixed point. The linearized equation on motion are
\[
\begin{pmatrix}
\frac{d(\delta \alpha)}{dt} \\
\frac{d(\delta \alpha^*)}{dt}
\end{pmatrix} = i 2 \begin{pmatrix}
2K|\alpha_1|^2 & K(\alpha_1^*)^2 - G \\
-K(\alpha_1^*)^2 - G & -2K|\alpha_1|^2
\end{pmatrix} \begin{pmatrix}
\delta \alpha \\
\delta \alpha^*
\end{pmatrix},
\]

which has the eigenvalues
\[
\lambda_{\pm} = \pm 2 \text{Im} \left( \sqrt{4K^2|\alpha_1|^4 - |G - K(\alpha_1^*)|^2} \right).
\]

We begin by examining the eigenvalues for the first steady state solution \(\alpha_{1}^* = -2\epsilon\). It has eigenvalues of the form \(\lambda_{\pm} = \pm 2 \text{Im} \left( \sqrt{K^2F^2(2G - G^2)} \right)\). If the expression inside the square root is negative then the \(\alpha_{1}^* \) will have an eigenvalue with a positive real part. This happens if \(F < \sqrt{2}K\alpha^3\). The eigenvalue of the two other solutions \(\alpha_{2}^* \) and \(\alpha_{3}^* \) both has \(\lambda_{\pm} = 0\) as eigenvalue and are therefore stable.

### A.8 Coupling between two Kerr-nonlinear resonators

In this section of the appendix we will obtain the fixed points for two linearly coupled KNR’s. The Hamiltonian for two linearly coupled two-photon pumped KNRs in the rotating frame is given by
\[
\mathcal{H} = \sum_{i=1}^{2} \left( -K\hat{a}_i^\dagger \hat{a}_i^2 \right) + G(\hat{a}_i^\dagger \hat{a}_i^2) + g(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger).\]
The Heisenberg equations of motion tell us how the operators evolve in time

\[
\frac{d a_1}{dt} = i \left[ H, a_1 \right] = i \left( 2K a_1^\dagger a_2^\dagger - 2G a_1^\dagger - g a_2 \right),
\]

\[
\frac{d a_2}{dt} = i \left[ H, a_2 \right] = i \left( 2K a_2^\dagger a_1^\dagger - 2G a_2^\dagger - g a_1 \right).
\]

Once again the mean field or semi-classical equations are obtained by replacing \( a_1 \to \alpha \) and \( a_2 \to \beta \), where \( \alpha \) and \( \beta \) are two complex variables. We thus get that the semi-classical equations of motion for two coupled KNRs are

\[
\frac{d \alpha}{dt} = i \left( 2K^* \alpha^2 - 2G^* \alpha - g \beta \right),
\]

\[
\frac{d \beta}{dt} = i \left( 2K^* \beta^2 - 2G^* \beta - g \alpha \right).
\]

The solutions to the steady state equations are

\[
\{ \alpha_1^s, \beta_1^s \} = \{ 0, 0 \},
\]

\[
\{ \alpha_2^s, \beta_2^s \} = \{ \pm \sqrt{\frac{2G + g}{2K}}, \pm \sqrt{\frac{2G + g}{2K}} \}
\]

\[
\{ \alpha_3^s, \beta_3^s \} = \{ \mp \sqrt{\frac{2G + g}{2K}}, \mp \sqrt{\frac{2G + g}{2K}} \}
\]

which reduces to \( \alpha_2^s \sim \sqrt{G/K} \) when \( 2G \gg g \), and the states are kept in the subspace spanned by \( |0, 0\rangle \), \( |0, 1\rangle \), \( |1, 0\rangle \) and \( |1, 1\rangle \). Following the same kind of stability analysing procedure that we did in section A.6 we find likewise that the origin is an unstable solution while \( \alpha_2^s \) are stable solutions.

### A.9 Error estimation

In this part of the appendix we aim at providing an order of magnitude of the error that is inherent in the truncation of the Hilbert space dimension corresponding to \( n = 16 \) photons, as we have used in the numerical simulations of Chapter 5. In order to do so, we evaluate the error that this truncation entails on the normalization condition for a coherent state. From the normalization condition of a coherent we have that

\[
\langle \alpha | \alpha \rangle = 1 = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!},
\]

which can be written as

\[
1 - e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = 0.
\]

If we truncate the Fock space at the number \( x \) then the numerical error \( \epsilon \) is given by

\[
\epsilon = 1 - e^{-|\alpha|^2} \sum_{n=0}^{x} \frac{|\alpha|^{2n}}{n!}.
\]

and the error is bounded between \( 0 \leq \epsilon \leq 1 - e^{-|\alpha|^2} \). To have an error that’s bounded between zero and one we have to divide by \( 1 - e^{-|\alpha|^2} \), such that

\[
\epsilon = \left( 1 - e^{-|\alpha|^2} \sum_{n=0}^{x} \frac{|\alpha|^{2n}}{n!} \right) \frac{1}{1 - e^{-|\alpha|^2}}.
\]

Now if \( x = 16 \) and \( |\alpha| = \sqrt{3} \) we find that the numerical error is approximately \( \epsilon \approx 10^{-8} \), which is a considerably small numerical error.
A.10 Generation of cat states using a two-photon pumped KNR

The two-photon pumped KNR can be used as a mean to generate Schrödinger cat states. Cat states are a superposition of two coherent states with opposite phase, defined by

$$|C^\pm_o\rangle = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}},$$

where the term in the denominator is a normalization factor. $|C^+_o\rangle$ is called an even-cat state because when written in the Fock basis it only contain Fock states with even numbers and $|C^-_o\rangle$ is called an odd-cat state because when written in the Fock-basis it only contains Fock states with odd numbers. Cat states are of interest because they have the potential to be used as logical states for universal quantum computation [16], [64]. To generate a cat state it can first be noted that since the two-coherent states $|\pm \alpha\rangle = \pm \sqrt{G/K}$ are two-degenerate eigenstates of the two-photon pumped KNR Eq. (3.1) so is any superposition of these states. It can be easily checked that the even and odd cat state with $\alpha = \sqrt{G/K}$ are also two degenerate eigenstates of the two-photon pumped KNR Eq. (3.1) with eigenenergy $G^2/K$. Second, to generate a cat state the detuning and single-photon pump is set to zero in the time-dependent Hamiltonian (3.4). The vacuum and single-photon Fock state are then degenerate eigenstates of the initial Hamiltonian. Under adiabatic evolution the zero-photon Fock state will evolve into the even cat-state since parity is conserved, and the single-photon Fock state will evolve into the odd-cat state for the same reason.

Numerical simulations show that, for $G = 3K$ and $\tau = 158$ as given by the adiabatic condition, if the resonator is initialized to $|0\rangle$ ($|1\rangle$) it will evolve into the even cat state (odd cat state) with 100% fidelity, within the numerical error. Figure A.2 (a) and (b) show the Wigner function of the even and odd cat. The Wigner negativity that appears between the two Gaussian peaks are interference fringes and are sometimes referred to as the cat’s whiskers. These interference fringes are very characteristic for a cat state. When the amplitude of $|\alpha|$ is small the states are often referred to as “kitten” states. For example an even cat (“kitten”) state with $|\alpha| = \sqrt{1/2}$ is shown in Fig. A.2 (c).

![Figure A.2: Cat states.](image-url)
B Code

B.1 Single Ising spin in a magnetic field: eigenspectrum & adiabatic condition

# import packages
import numpy as np
import matplotlib.pyplot as plt
from qutip import *

# define parameters
N = 16  # number of levels in Hilbert space
a = destroy(N)  # create lowering operator corresponding to N number states
K = 1  # amplitude of Kerr non-linearities
G = 3  # amplitude of two-photon drive
F = 4*G/100  # amplitude of single-photon drive
delta = 1  # amplitude of dephasing

# define time vector s = t / tau
s = np.linspace(0.0, 1.0, 101); r = range(len(s))

# initial Hamiltonian
H_0 = - delta * a.dag() @ a - K * pow(a.dag(), 2) @ pow(a, 2)

# final Hamiltonian
H_1 = - K * pow(a.dag(), 2) @ pow(a, 2) + G * (pow(a.dag(), 2) + pow(a, 2))
+ F * (a.dag() + a)

# total Hamiltonian
def H(s):
    return (1 - s) @ H_0 + s @ H_1

# eigenstates and eigenenergies
E = [H(i).eigenstates(sparse=False, sort='high', eigvals=2) for i in s]
B.2. Single Ising spin in a magnetic field: Wigner function

```python
# find \( \min (E_e - E_g)^2 \)
minval = \( \min(\text{pow}(\text{np.subtract}\([E[i][0][1] \text{ for i in r]},[E[i][0][0] \text{ for i in r}]),2))\)
print(‘Minimum energy gap: %.2f’ % minval)

# find \( \max |<\psi_e(s)|\frac{dH(s)}{ds}|\psi_g(s)>| \)
maxstate = \( \max(\text{abs}((H_1 - H_0).\text{matrix_element}(E[i][1][1],E[i][1][0])) \text{ for i in r})\)

# adiabatic condition
print(‘Adiabatic condition: \(\tau \gg \%s\)” % (maxstate/minval))

# plot energy eigenspectrum
fig, ax = plt.subplots()
for i in range(2):
    ax.plot(s, [E[j][0][i] \text{ for j in r}], lw=2)
plt.xticks(np.linspace(0,1,11), fontsize=12)
plt.yticks(fontsize=12)
plt.xlabel(r’$t/\tau$’, fontsize=14)
plt.ylabel(r’$E/K$’, fontsize=14)
plt.show()

B.2 Single Ising spin in a magnetic field: Wigner function

""
Continuous Variable Quantum Annealer
Single Ising Spin Density Matrix
""

# import packages
import numpy as np
import matplotlib.pyplot as plt
from qutip import *

# define parameters
N = 16 \# number of levels in Hilbert space
a = destroy(N) \# create lowering operator corresponding to N number states
K = 1 \# amplitude of Kerr nonlinearities
G = 3 \# amplitude of two-photon drive
F = 4*G/100 \# amplitude of single-photon drive
delta = 1 \# amplitude of dephasing

# computational basis
alpha = \text{pow}(G,1/2) \# amplitude of coherent state
up = \text{coherent}(N,-alpha) \# \text{|uparrow}\rightarrow |0>
down = \text{coherent}(N,alpha) \# \text{|downarrow}\rightarrow |1>

# define time vector
tau = 686; t = \text{np.linspace}(0, tau, 20*tau)

# initial Hamiltonian
H_0 = \text{delta} \times \text{a.dag()} \times \text{a} \text{ - K} \times \text{pow(a.dag(),2)} \text{ * pow(a,2)

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B.2. Single Ising spin in a magnetic field: Wigner function

# final Hamiltonian
H_1 = - K * pow(a.dag(), 2) * pow(a, 2) + G * (pow(a.dag(), 2) + pow(a, 2)) \ 
+ F * (a.dag() + a)

# time dependent factor for the initial Hamiltonian
def H_0_coeff(t, args):
    return (1 - t/tau)

# time dependent factor for the final Hamiltonian
def H_1_coeff(t, args):
    return t/tau

# total Hamiltonian
H = [[H_0, H_0_coeff], [H_1, H_1_coeff]]

# single-photon loss rate
gamma = 0
# collapse operators
if gamma == 0:
    c_op = []
else:
    c_op = [np.sqrt(gamma) * a]

# initial state; create a Fock vacuum state vector |0> in a Hilbert space
# with N number states
psi0 = fock(N, 0)
# find solution to master equation
rho = mesolve(H, psi0, t, c_op, progress_bar=True);

# selects rho(t_final)
if not c_op:
    rho_final = rho.states[-1] * rho.states[-1].dag()
else:
    rho_final = rho.states[-1]

# find Wigner function for rho(t_final)
xvec = np.linspace(-4, 4, 400)
yvec = np.linspace(-2, 2, 200)
W = wigner(rho_final, xvec, yvec, 'iterative', 2)
wmap = wigner_cmap(W)  # Generate Wigner colormap

# find negative Wigner function
l = np.int_(np.linspace(0, len(t)-1, 101))
W_min = [wigner(rho.states[i], xvec, yvec, 'iterative', 2).min() for i in l]

# calculate the fidelity
rho_true = coherent_dm(N, alpha)
fid = fidelity(rho_true, rho_final)
print('Fidelity = %s %%' % (round(fid*100, 2)))

# calculate the success probability from the matrix element <1|rho|1>
success = rho_final.matrix_element(down, down)
print('Success probability = %s %%' % (round(abs(success)*100, 2)))

# plot Wigner function
plt.figure()
plt.subplot(111, aspect='equal')
im = plt.contourf(xvec, yvec, W, 100, cmap = wmap)
plt.xlabel(r'$Re$(\textit{\textalpha})', fontsize = 14)
plt.ylabel(r'$Im$(\textit{\textalpha})', fontsize = 14)
plt.xticks(np.arange(min(xvec), max(xvec)+1, 2), fontsize = 12)
plt.yticks(np.arange(min(yvec), max(yvec)+1, 2), fontsize = 12)

# colorbar
cax = plt.axes([0.125, 0.8, 0.775, 0.05]) # left, bottom, width, height
plt.colorbar(im, cax = cax, orientation = 'horizontal', ticklocation = 'top',
             ticks = np.arange(round(W.min()),1), round(W.max(),1)+0.1,0.1))

# plot negativity of Wigner function
plt.figure()
plt.subplot(111)
plt.plot((1/len(t)), W_min, color='#192B53', lw=3)
plt.xlabel(r'$t$/$t\_\tau$', fontsize = 14)
plt.ylabel(r'W$_{\text{min}}$(\textit{\textalpha})', fontsize = 14)
plt.xticks(np.linspace(0,1,11), fontsize = 12)
plt.yticks(fontsize = 12)
plt.ylim([-1.2,0])
plt.margins(0.005, 0.01)
plt.tight_layout()

plt.show()

B.3 Subset sum problem

# import packages
import numpy as np
import matplotlib.pyplot as plt
from qutip import *

# numbers
n = [-2,1,2]
m = 3

c = 1/2 * sum(n[i] for i in range(len(n))) - m

# magnetic field
h = [c*n[i] for i in range(len(n))]

# couplings
J = [[-n[i]*n[j] for i in range(len(n))] for j in range(len(n))]

# Define system parameters
N = 16 # number of levels in Hilbert space
K = 1 # amplitude of Kerr nonlinearities
G = 3 # amplitude of two-photon drive
delta = 1 # amplitude of dephasing
alpha = pow(G,1/2) # amplitude of coherent state

# scale factor
A = 1/(max(list(map(lambda x: abs(x), h)))+2)
B.3. Subset sum problem

# single photon pumps
F1 = A*h[0]/(2*alpha)  # amplitude of single-photon drive 1
F2 = A*h[1]/(2*alpha)  # amplitude of single-photon drive 2
F3 = A*h[2]/(2*alpha)  # amplitude of single-photon drive 3

# single-photon exchange rate between resonators
g12 = A*J[0][1]/(4*alpha**2)  # coupling strength between resonator 1 and 2
g13 = A*J[0][2]/(4*alpha**2)  # coupling strength between resonator 1 and 3
g23 = A*J[1][2]/(4*alpha**2)  # coupling strength between resonator 2 and 3

# computational basis
up = coherent(N,-alpha)  # |uparrow> -> |0>
down = coherent(N,alpha)  # |downarrow> -> |1>

# create lowering operators
a = destroy(N)
a1 = tensor(a, qeye(N), qeye(N))  # lowering operator for first resonator
a2 = tensor(qeye(N), a, qeye(N))  # lowering operator for second resonator
a3 = tensor(qeye(N), qeye(N), a)  # lowering operator for third resonator

# define time vector
tau = 79*100; t = np.linspace(0, tau, 10*tau)

# initial Hamiltonian
H_0 = -delta * a1.dag() * a1 - K * pow(a1.dag(),2) * pow(a1,2) \ 
- delta * a2.dag() * a2 - K * pow(a2.dag(),2) * pow(a2,2) \ 
- delta * a3.dag() * a3 - K * pow(a3.dag(),2) * pow(a3,2) \ 
+ g12 * (a1.dag()) * a2 + a2.dag() * a1) \ 
+ g13 * (a1.dag()) * a3 + a3.dag() * a1) \ 
+ g23 * (a2.dag()) * a3 + a3.dag() * a2)

# final Hamiltonian
H_1 = -K * pow(a1.dag(),2) * pow(a1,2) + G * (pow(a1.dag(),2) + pow(a1,2)) \ 
+ F1 * (a1.dag()) + a1) \ 
- K * pow(a2.dag(),2) * pow(a2,2) + G * (pow(a2.dag(),2) + pow(a2,2)) \ 
+ F2 * (a2.dag()) + a2) \ 
- K * pow(a3.dag(),2) * pow(a3,2) + G * (pow(a3.dag(),2) + pow(a3,2)) \ 
+ F3 * (a3.dag()) + a3) \ 
+ g12 * (a1.dag()) * a2 + a2.dag() * a1) \ 
+ g13 * (a1.dag()) * a3 + a3.dag() * a1) \ 
+ g23 * (a2.dag()) * a3 + a3.dag() * a2)

# time dependent factor for the initial Hamiltonian
def H_0_coeff(t, args):
    return (1 - t/tau)

# time dependent factor for the final Hamiltonian
def H_1_coeff(t, args):
    return t/tau

# total Hamiltonian
H = [[H_0, H_0_coeff], [H_1, H_1_coeff]]
B.4 Number partitioning problem

# find solution to master equation
psi0 = tensor(fock(N, 0), fock(N, 0), fock(N, 0)) # initial state
rho = mesolve(H, psi0, t, [], progress_bar=True)

# selects rho(t_final)
rho_final = ket2dm(rho.states[-1])

# answer
psi_ans = tensor(down, up, up)

# compute the success probability
success = rho_final.matrix_element(psi_ans, psi_ans)
print(f'Success Probability = %s%%
(round(abs(success)*100, 2))')

Wigner function for each individual resonator
---
rho = [rho_final.ptrace(i) for i in range(3)]

# find Wigner function
xvec = np.linspace(-4, 4, 400); yvec = np.linspace(-2, 2, 200)
W = [wigner(rho[i], xvec, yvec, 'iterative', 2) for i in range(3)]
wmap = [wigner_cmap(W[i]) for i in range(3)]

for i in range(3):
    # plot Wigner function
    plt.figure()
    plt.subplot(111, aspect='equal')
    im = plt.contourf(xvec, yvec, W[i], 100, cmap=wmap[i])
    plt.xlabel(r'Re$(\alpha)$', fontsize=14)
    plt.ylabel(r'Im$(\alpha)$', fontsize=14)
    plt.xticks(np.arange(min(xvec), max(xvec)+1, 2), fontsize=12)
    plt.yticks(np.arange(min(yvec), max(yvec)+1, 2), fontsize=12)
    # colorbar
cax = plt.axes([0.125, 0.8, 0.775, 0.05]) # left, bottom, width, height
    plt.colorbar(im, cax=cax, orientation='horizontal',
    ticklocation='top',
    ticks=np.arange(round(W[i].min(),1),round(W[i].max(),1)+0.1,0.1)
    plt.show()

B.4 Number partitioning problem

Continuous Variable Quantum Annealing
Number Partitioning Problem
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# import packages
import numpy as np
from qutip import *

# numbers
n = [3, 5, 8]
# couplings
B.4. Number partitioning problem

\[ J = \left[ -n[i] \cdot n[j] \text{ for } i \text{ in range(len(n))} \right] \text{ for } j \text{ in range(len(n))} \]

# Define system parameters
N = 16 # number of levels in Hilbert space
K = 1 # amplitude of Kerr nonlinearities
G = 3 # amplitude of two-photon drive
alpha = pow(G, 1/2) # amplitude of coherent state
delta = 1 # amplitude of dephasing

# scale factor
A = 0.015

# Couplings
g12 = A*J[0][1] # coupling strength between resonator 1 and 2
g13 = A*J[0][2] # coupling strength between resonator 1 and 3
g23 = A*J[1][2] # coupling strength between resonator 2 and 3

# computational basis
up = coherent(N, -alpha) # /uparrow> -> |0>
down = coherent(N, alpha) # /downarrow> -> |1>

# tensor
a = destroy(N)
a1 = tensor(a, qeye(N), qeye(N)) # lowering operator for first resonator
a2 = tensor(qeye(N), a, qeye(N)) # lowering operator for second resonator
a3 = tensor(qeye(N), qeye(N), a) # lowering operator for third resonator

# define time vector
tau = 3.92 * 100; t = np.linspace(0, tau, 10*tau)

# initial Hamiltonian
H_0 = -delta * a1.dag() * a1 - K * pow(a1.dag(), 2) * pow(a1, 2) \ 
- delta * a2.dag() * a2 - K * pow(a2.dag(), 2) * pow(a2, 2) \ 
- delta * a3.dag() * a3 - K * pow(a3.dag(), 2) * pow(a3, 2) \ 
+ g12 * (a1.dag() * a2 + a2.dag() * a1) \ 
+ g13 * (a1.dag() * a3 + a3.dag() * a1) \ 
+ g23 * (a2.dag() * a3 + a3.dag() * a2)

# final Hamiltonian
H_1 = -K * pow(a1.dag(), 2) * pow(a1, 2) + G * (pow(a1.dag(), 2) + pow(a1, 2)) \ 
- K * pow(a2.dag(), 2) * pow(a2, 2) + G * (pow(a2.dag(), 2) + pow(a2, 2)) \ 
- K * pow(a3.dag(), 2) * pow(a3, 2) + G * (pow(a3.dag(), 2) + pow(a3, 2)) \ 
+ g12 * (a1.dag() * a2 + a2.dag() * a1) \ 
+ g13 * (a1.dag() * a3 + a3.dag() * a1) \ 
+ g23 * (a2.dag() * a3 + a3.dag() * a2)

# time dependent factor for the initial Hamiltonian
def H_0_coeff(t, args):
    return (1 - t/tau)

# time dependent factor for the final Hamiltonian
def H_1_coeff(t, args):
    return t/tau
# total Hamiltonian
\[ H = \begin{bmatrix} [H_0, H_0\text{\_coeff}] & [H_1, H_1\text{\_coeff}] \end{bmatrix} \]

# find solution to master equation
psi0 = tensor(fock(N, 0), fock(N, 0), fock(N, 0)) # initial state
rho = mesolve(H, psi0, t, [], progress_bar = True)

# selects rho(t\_final)
rho\_final = ket2dm(rho\_states[-1])

# answer
psi0 = tensor(down, down, up)
psi1 = tensor(up, up, down)

# compute the success probability
success = rho\_final.matrix\_element(psi0, psi0) + rho\_final.matrix\_element(psi1, psi1)
print('Success probability = %s%% (%s)' % (round(abs(success)*100, 2)))

# compute the fidelity
nrm = 1/pow(2*(1+np.exp(-6*pow(G, 2)))*1/2) # normalization
psi\_ans = nrm * (psi0+psi1)
rho\_ans = ket2dm(psi\_ans)
fidelity = fidelity(rho\_ans, rho\_final)
print('Fidelity = %s%% (%s)' % (round(fidelity*100, 2)))
Bibliography


