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# A dual consistent summation-by-parts formulation for the linearized incompressible Navier-Stokes equations posed on deforming domains 

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#### Abstract

In this article, well-posedness and dual consistency of the linearized constant coefficient incompressible Navier-Stokes equations posed on time-dependent spatial domains are studied. To simplify the derivation of the dual problem and improve the accuracy of gradients, the second order formulation is transformed to first order form. Boundary conditions that simultaneously lead to boundedness of the primal and dual problems are derived.

Fully discrete finite difference schemes on summation-by-parts form, in combination with the simultaneous approximation technique, are constructed. We prove energy stability and discrete dual consistency and show how to construct the penalty operators such that the scheme automatically adjusts to the variations of the spatial domain. As a result of the aforementioned formulations, stability and discrete dual consistency follow simultaneously.

The method is illustrated by considering a deforming time-dependent spatial domain in two dimensions. The numerical calculations are performed using high order operators in space and time. The results corroborate the stability of the scheme and the accuracy of the solution. We also show that linear functionals are superconverging. Additionally, we investigate the convergence of non-linear functionals and the divergence of the solution.


Keywords: incompressible Navier-Stokes equations, deforming domain, stability, dual consistency, high order accuracy, superconvergence

## 1. Introduction

The incompressible Navier-Stokes equations are used in various disciplines such as meteorological studies [1], biomechanic developments in cardiovascular and capillary blood flow simulations [2], external aerodynamic analyses [3], [4], [5], acoustic modelings [6] and many other industrial applications. The equations have been formulated in several alternative ways. The most commonly used formulation is the so-called velocity-pressure formulation where the divergence relation is not explicitly satisfied. In such formulations, one has to employ special numerical techniques or boundary procedures to enforce the zero divergence condition on the velocity field. Examples of these techniques and procedures include the use of staggered grids [7], fractional steps or projection methods that satisfy the incompressibility condition [8] and new boundary conditions for open boundaries that damp the divergence [9]. Other forms of the incompressible NavierStokes equations are the velocity-divergence and vorticity-stream function formulations. In this article, we consider the velocity-divergence formulation and preserve zero divergence to the order of accuracy of the scheme, directly.

In many applications, functionals of the solution are more interesting than the solution itself. These functionals are weighted integrals of the solution over the spatial domain. Typical functionals, in aerodynamics applications for example, are the lift and drag coefficients. Dual consistent schemes on Summation-by-Parts (SBP) form in combination with the Simultaneous Approximation Term (SAT) technique, are investigated in [10], [11], [12], [13] and [14] for a variety of problems posed on fixed spatial domains. One very appealing result of having a dual consistent finite difference approximation of linear problems is that it leads to superconverging linear functionals [13], [14]. In [10], [11] and [12], it was found that the only requirement for having a dual consistent scheme on SBP-SAT form is that a specific subset of values for the SAT penalty coefficients/operators in the range of values for which stability is guaranteed, must be chosen. Consequently, superconverging linear functional approximations comes with no additional computational costs.

In this article, we extend the SBP-SAT dual consistency approach to the twodimensional linearized constant coefficient incompressible Navier-Stokes equations on time-dependent spatial domains, and start by reducing the problem to a first order system. The reduction to first order form simplifies the derivation of dual consistency and improves the accuracy of the computed first derivatives including the divergence [15], [16]. Next, a time-dependent coordinate transformation is used to map the problem from a moving spatial domain into a fixed one.

Additionally, we apply the techniques in [17] such that the Numerical Geometric Conservation Law (NGCL) is preserved under the coordinate transformation.

The development in this article is the second step in the development of solution procedures for the three-dimensional time-dependent nonlinear incompressible Navier-Stokes equations on moving meshes. The first step was taken in [19], where energy stable boundary conditions were derived, and fully discrete stability was proved for the nonlinear equations. The equations in [19] were on second order form and the geometry was Cartesian. In this paper we focus on i) investigating the potential advantages (such as higher accuracy of the derivatives and divergence) of posing the equations on first order form and ii) the additional complications (e.g. the requirement to satisfy the Numerical Geometric Conservation Law) with moving curvilinear meshes. This investigation is more easily done for the linearized constant coefficient problem. With the knowledge gained in i)+ii), we are prepared for the development of a provably stable high order accurate nonlinear solver on moving meshes.

The rest of this article proceeds as follows. In section 2, we analyze the continuous primal problem, choose bounded boundary conditions and derive the dual (or adjoint) problem together with bounded dual boundary conditions. In section 3, the discrete problem is constructed, stability of the primal and dual problems are investigated and the requirements for a dual consistent approximation are specified. Numerical experiments are performed in section 4, where we show the convergence of the solution, divergence and functionals. Finally, we summarize and draw conclusions in section 5.

## 2. The incompressible Navier-Stokes equations

Consider the two-dimensional incompressible Navier-Stokes equations

$$
\begin{align*}
u_{t}+u u_{x}+v u_{y}+p_{x} & =v\left(u_{x x}+u_{y y}\right), \\
v_{t}+u v_{x}+v v_{y}+p_{y} & =v\left(v_{x x}+v_{y y}\right),  \tag{1}\\
u_{x}+v_{y} & =0,
\end{align*}
$$

where $(x, y) \in \Omega(t)$ and $t \in[0, T]$. In (1), $x, y$ and $t$ are the spatial coordinates and time, respectively. The subscripts $t, x$ and $y$ denote partial derivatives in their respective directions, $p$ is the pressure (divided by the constant density), $v>0$ is the constant kinematic viscosity and $\Omega(t)$ is a moving and/or deforming spatial domain.

We rewrite (1) in matrix vector form as

$$
\begin{equation*}
S U_{t}+\hat{A}(U) U_{x}+\hat{B}(U) U_{y}=C\left(U_{x x}+U_{y y}\right) \tag{2}
\end{equation*}
$$

where $U=[u, v, p]^{T}$,

$$
S=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right], \hat{A}(U)=\left[\begin{array}{ccc}
u & 0 & 1 \\
0 & u & 0 \\
1 & 0 & 0
\end{array}\right], \hat{B}(U)=\left[\begin{array}{ccc}
v & 0 & 0 \\
0 & v & 1 \\
0 & 1 & 0
\end{array}\right]
$$

and $C=v S$. We linearize (2) and obtain the constant coefficient problem as

$$
\begin{equation*}
S U_{t}+A U_{x}+B U_{y}=C\left(U_{x x}+U_{y y}\right) \tag{3}
\end{equation*}
$$

In (3), $A=\hat{A}(\bar{U})$ and $B=\hat{B}(\bar{U})$ where the bar sign indicates the reference state around which we have linearized.

In order to simplify the derivation of the dual problem, the problem (3) is reduced to a first order system, through the transformations $V=U_{x}$ and $W=U_{y}$. The result is

$$
\begin{equation*}
\mathbf{S U}_{t}+\mathbf{A} \mathbf{U}_{x}+\mathbf{B} \mathbf{U}_{y}+\mathbf{C U}=0 \tag{4}
\end{equation*}
$$

where $\mathbf{U}=\left[U^{T}, V^{T}, W^{T}\right]^{T}$ and

$$
\mathbf{S}=\left[\begin{array}{lll}
S & &  \tag{5}\\
& 0 & \\
& & 0
\end{array}\right], \mathbf{A}=\left[\begin{array}{ccc}
A & -C & 0 \\
-C & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \mathbf{B}=\left[\begin{array}{ccc}
B & 0 & -C \\
0 & 0 & 0 \\
-C & 0 & 0
\end{array}\right], \mathbf{C}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & C & 0 \\
0 & 0 & C
\end{array}\right]
$$

Remark 1. Note that all the systems (2), (3) and (5) are symmetric.
Remark 2. The first order formulation makes it possible to obtain design order of accuracy of the divergence. This will be discussed later in this article.

### 2.1. Time-dependent coordinate transformation

An invertible time-dependent coordinate transformation is considered,

$$
\begin{aligned}
x & =x(\tau, \xi, \eta), \quad y=y(\tau, \xi, \eta), \quad t=\tau \\
\xi & =\xi(t, x, y), \quad \eta=\eta(x, y, t), \quad \tau=t
\end{aligned}
$$

The transformation satisfies

$$
\left[\begin{array}{l}
\partial / \partial \xi \\
\partial / \partial \eta \\
\partial / \partial \tau
\end{array}\right]=\underbrace{\left[\begin{array}{lll}
x_{\xi} & y_{\xi} & 0 \\
x_{\eta} & y_{\eta} & 0 \\
x_{\tau} & y_{\tau} & 1
\end{array}\right]}_{:=[J]}\left[\begin{array}{l}
\partial / \partial x \\
\partial / \partial y \\
\partial / \partial t
\end{array}\right]
$$




Figure 1: A schematic of the domain in Cartesian and curvilinear coordinates
where the subscripts $\tau, \xi$ and $\eta$ denote partial derivatives and $[J]$ is the Jacobian matrix of the transformation. The metric relations [18], [20] are

$$
\begin{aligned}
& J \xi_{t}=x_{\eta} y_{\tau}-x_{\tau} y_{\eta}, \quad J \xi_{x}=y_{\eta}, \quad J \xi_{y}=-x_{\eta}, \\
& J \eta_{t}=y_{\xi} x_{\tau}-x_{\xi} y_{\tau}, \quad J \eta_{x}=-y_{\xi}, \quad J \eta_{y}=x_{\xi},
\end{aligned}
$$

where $J=x_{\xi} y_{\eta}-x_{\eta} y_{\xi}>0$ is the determinant of $[J]$.
All non-singular transformations satisfy the Geometric Conservation Law (GCL)

$$
\begin{align*}
J_{\tau}+\left(J \xi_{t}\right) \xi+\left(J \eta_{t}\right)_{\eta} & =0 \\
\left(J \xi_{x}\right) \xi+\left(J \eta_{x}\right)_{\eta} & =0  \tag{6}\\
\left(J \xi_{y}\right) \xi+\left(J \eta_{y}\right)_{\eta} & =0
\end{align*}
$$

Now, the transformation is applied to the spatial domain $\Omega(t)$ and results in a fixed domain, $\Phi$. A schematic of $\Omega(t)$ and $\Phi$ is given in Figure 1 .

The chain rule applied to (4) and the result multiplied with $J$ yields

$$
\begin{equation*}
\mathscr{S} \mathbf{U}_{\tau}+\mathscr{A} \mathbf{U}_{\xi}+\mathscr{B} \mathbf{U}_{\eta}+\mathscr{C} \mathbf{U}=0, \quad(\xi, \eta) \in \Phi, \quad \tau \in[0, T] \tag{7}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{S} & =J \mathbf{S} \\
\mathscr{A} & =\left(J \xi_{t}\right) \mathbf{S}+\left(J \xi_{x}\right) \mathbf{A}+\left(J \xi_{y}\right) \mathbf{B}, \\
\mathscr{B} & =\left(J \eta_{t}\right) \mathbf{S}+\left(J \eta_{x}\right) \mathbf{A}+\left(J \eta_{y}\right) \mathbf{B},  \tag{8}\\
\mathscr{C} & =J \mathbf{C} .
\end{align*}
$$

Finally, the GCL given in (6) is applied to (8) and results in

$$
\begin{equation*}
\mathscr{S}_{\tau}+\mathscr{A}_{\xi}+\mathscr{B}_{\eta}=0, \tag{9}
\end{equation*}
$$

by which (7) can be rewritten in conservative form, as

$$
\begin{equation*}
(\mathscr{S} \mathbf{U})_{\tau}+(\mathscr{A} \mathbf{U})_{\xi}+(\mathscr{B} \mathbf{U})_{\eta}+\mathscr{C} \mathbf{U}=0, \quad(\xi, \eta) \in \Phi, \quad \tau \in[0, T] \tag{10}
\end{equation*}
$$

Remark 3. The equation (10) is denoted as conservative with a slight abuse of notation, since it has a lower order term that comes from the reduction to a first order system.

Remark 4. The formulations (7) and (10) are equivalent. We prefer the conservative form and build our scheme based on that.

### 2.2. Well-posedness of the primal problem

The energy method (multiplying (10) from the left with the transpose of the solution and integrating over the spatial and temporal domains) gives

$$
\begin{align*}
\|\mathbf{U}(T, \xi, \eta)\|_{\mathscr{S}(T)}^{2}-\|\mathbf{U}(0, \xi, \eta)\|_{\mathscr{S}(0)}^{2} & +2 \int_{0}^{T} \iint_{\Phi} \mathbf{U}^{T} \mathscr{C} \mathbf{U} d \Phi d \tau=  \tag{11}\\
& -\int_{0}^{T} \oint_{\delta \Phi} \mathbf{U}^{T} D \mathbf{U} d s d \tau
\end{align*}
$$

In (11), $D=(\mathscr{A}, \mathscr{B}) \cdot n=n_{1} \mathscr{A}+n_{2} \mathscr{B}$, where $n=\left(n_{1}, n_{2}\right)$ is the outward pointing unit normal from the boundary of $\Phi$, denoted by $\delta \Phi$. Additionally, the norms are defined by

$$
\begin{equation*}
\|\mathbf{U}\|_{\mathscr{S}}^{2}=\iint_{\Phi} \mathbf{U}^{T} \mathscr{S} \mathbf{U} d \xi d \eta=\iint_{\Phi} U^{T} J S U d \xi d \eta=\iint_{\Phi} J\left(u^{2}+v^{2}\right) d \xi d \eta . \tag{12}
\end{equation*}
$$

The volume term in (11) is dissipative, i.e., matrix $\mathscr{C} \geq 0$.
The matrix $D$ in (11) can be diagonalized as $D=X \Lambda X^{T}$, where the decomposition $\Lambda=\Lambda_{+}+\Lambda_{-}$splits up the eigenvalue matrix into non-negative and negative parts, respectively. Moreover, the eigenvector matrix $X$ can be rearranged as $X=\left[X_{+}, X_{-}\right]$where $X_{+}$and $X_{-}$are the eigenvectors corresponding to $\Lambda_{+}$and $\Lambda_{-}$, respectively. We can now rewrite (11) as

$$
\begin{align*}
& \|\mathbf{U}(T, \xi, \eta)\|_{\mathscr{S}(T)}^{2}-\|\mathbf{U}(0, \xi, \eta)\|_{\mathscr{S}(0)}^{2}+2 \int_{0}^{T} \iint_{\Phi} \mathbf{U}^{T} \mathscr{C} \mathbf{U} d \Phi d \tau=  \tag{13}\\
& -\int_{0}^{T} \oint_{\delta \Phi}\left(X_{+}^{T} \mathbf{U}\right)^{T} \Lambda_{+}\left(X_{+}^{T} \mathbf{U}\right) d s d \tau-\int_{0}^{T} \oint_{\delta \Phi}\left(X_{-}^{T} \mathbf{U}\right)^{T} \Lambda_{-}\left(X_{-}^{T} \mathbf{U}\right) d s d \tau
\end{align*}
$$

In order to bound the energy of the solution in (13), boundary conditions must be applied to the potentially growing terms, associated with the negative eigenvalues. For simplicity, we choose the boundary conditions

$$
\begin{equation*}
\left(X_{-}^{T} \mathbf{U}\right)_{j}=\left(X_{-}^{T} \mathbf{U}_{\infty}\right)_{j} \text { if } \Lambda_{j j}<0 \tag{14}
\end{equation*}
$$

where $j \in\{1,2, \ldots, 9\}$ and $\mathbf{U}_{\infty}$ is a reference solution at the boundary $\delta \Phi$. Moreover, we consider an initial condition of the form $\mathbf{U}(0, \xi, \eta)=f(\xi, \eta)$.

Remark 5. In (14), we choose one particular set of boundary conditions that lead to the boundedness of the primal problem. Other forms of boundary conditions that lead to an energy estimate, as addressed in [21], are possible to use.

Remark 6. If pressure data is used in (14), the pressure is determined uniquely, otherwise, it is determined up to a constant value.

Strong imposition of the initial and boundary conditions to (13) leads to the energy estimate,

$$
\begin{align*}
\|\mathbf{U}(T, \xi, \eta)\|_{\mathscr{S}(T)}^{2} & +\int_{0}^{T} \oint_{\delta \Phi}\left(X_{+}^{T} \mathbf{U}\right)^{T} \Lambda_{+}\left(X_{+}^{T} \mathbf{U}\right) d s d \tau+2 \int_{0}^{T} \int_{\Phi} \mathbf{U}^{T} \mathscr{C} \mathbf{U} d \Phi d \tau  \tag{15}\\
& =\|f(\xi, \eta)\|_{\mathscr{S}(0)}^{2}-\int_{0}^{T} \oint_{\delta \Phi}\left(X_{-}^{T} \mathbf{U}_{\infty}\right)^{T} \Lambda_{-}\left(X_{-}^{T} \mathbf{U}_{\infty}\right) d s d \tau
\end{align*}
$$

The estimate (15) bounds the velocity and its gradients and hence leads to uniqueness of the velocity field. However, there is no bound on the pressure, and hence it is not unique. Existence is given by the fact that we use the correct (minimal) number of boundary conditions equal to the number of elements in $\Lambda_{-}$[22] and [23]. We summarize the results of this section in the following proposition.

Proposition 1. Equation (10) augmented with the boundary condition (14) has a unique velocity field with the bound in (15).

Remark 7. In the incompressible Navier-Stokes equations, the pressure is not unique, but on the other hand it does not have to be. The pressure itself is not a thermodynamical property of the flow whereas its spatial derivatives, which represent forces in the momentum equations, are significant.

For later reference, consider the south boundary where $\eta=0$ (indicated by subscript $s$ ) and $n=(0,-1)$. Then, $D_{s}=-\mathscr{B}_{s}=-X_{\mathscr{B}_{s}} \Lambda_{\mathscr{B}_{s}} X_{\mathscr{B}_{s}}^{T}$, and the estimate in (15) becomes

$$
\begin{align*}
& \|\mathbf{U}(T, \xi, \eta)\|_{\mathscr{S}(T)}^{2}-\int_{0}^{T} \int\left[\left(X_{\mathscr{B}_{s}}\right)_{-}^{T} \mathbf{U}\right]^{T}\left(\Lambda_{\mathscr{B}_{s}}\right)_{-}\left[\left(X_{\mathscr{B}_{s}}\right)_{-}^{T} \mathbf{U}\right] d \xi d \tau+2 \int_{0}^{T} \int_{\Phi} \mathbf{U}^{T} \mathscr{C} \mathbf{U} d \Phi d \tau \\
& =\|f(\xi, \eta)\|_{\mathscr{S}(0)}^{2}+\int_{0}^{T} \int\left[\left(X_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]^{T}\left(\Lambda_{\mathscr{B}_{s}}\right)_{+}\left[\left(X_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right] d \xi d \tau . \tag{16}
\end{align*}
$$

### 2.3. The dual problem

To derive the dual problem, we consider (10) augmented with a functional, homogeneous initial and boundary conditions and a forcing function $F$, as

$$
\begin{array}{rlrlrl}
(\mathscr{S} \mathbf{U})_{\tau}+(\mathscr{A} \mathbf{U})_{\xi}+(\mathscr{B} \mathbf{U})_{\eta}+\mathscr{C} \mathbf{U} & =F, & (\xi, \eta) & \in \Phi \\
X_{-}^{T} \mathbf{U} & =0, & (\xi, \eta) \in \delta \Phi  \tag{17}\\
U(\xi, \eta) & =0, & \tau=0 \\
\mathscr{J}(\mathbf{U}) & =(\mathbf{U}, G) . & &
\end{array}
$$

In actual calculations, the right hand side $F$ may be identically zero. Here, we use it to derive the dual problem. In (17), $\mathscr{J}(\mathbf{U})$ is a linear functional of the solution with a weight function $G$ given by

$$
\begin{equation*}
\mathscr{J}(\mathbf{U})=(\mathbf{U}, G):=\iint_{\Phi} \mathbf{U}^{T} G d \xi d \eta . \tag{18}
\end{equation*}
$$

Remark 8. In the remainder of the analysis, it is convenient to switch between the integral and inner product notations in (18).

Our objective is to find the dual solution $\Theta=[\theta, \Psi, \Gamma]^{T}$, by searching for it in an appropriate function space [10], [14], such that

$$
\begin{equation*}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau=\int_{0}^{T}(\boldsymbol{\Theta}, F) d \tau \tag{19}
\end{equation*}
$$

As an initial step, we observe that

$$
\begin{equation*}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau=\int_{0}^{T}(\mathbf{U}, G) d \tau-\int_{0}^{T}\left(\boldsymbol{\Theta},(\mathscr{S} \mathbf{U})_{\tau}+(\mathscr{A} \mathbf{U})_{\xi}+(\mathscr{B} \mathbf{U})_{\eta}+\mathscr{C} \mathbf{U}-F\right) d \tau \tag{20}
\end{equation*}
$$

Next, we add and subtract terms to obtain

$$
\begin{align*}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau & =\int_{0}^{T}(\mathbf{U}, G) d \tau-\int_{0}^{T}\left(\boldsymbol{\Theta},(\mathscr{S} \mathbf{U})_{\tau}+(\mathscr{A} \mathbf{U})_{\xi}+(\mathscr{B} \mathbf{U})_{\eta}+\mathscr{C} \mathbf{U}-F\right) d \tau \\
& \pm \int_{0}^{T}\left[\left(\boldsymbol{\Theta}_{\tau}, \mathscr{S} \mathbf{U}\right)+\left(\boldsymbol{\Theta}_{\xi}, \mathscr{A} \mathbf{U}\right)+\left(\boldsymbol{\Theta}_{\eta}, \mathscr{B} \mathbf{U}\right)\right] d \tau \tag{21}
\end{align*}
$$

One can rearrange (21) and use the symmetry of the matrices to arrive at

$$
\begin{aligned}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau=\int_{0}^{T}(\boldsymbol{\Theta}, F) d \tau & -\int_{0}^{T} \iint_{\Phi}\left(\boldsymbol{\Theta}^{T} \mathscr{S} \mathbf{U}\right)_{\tau} d \xi d \eta d \tau \\
& -\int_{0}^{T} \iint_{\Phi}\left(\left(\boldsymbol{\Theta}^{T} \mathscr{A} \mathbf{U}\right)_{\xi}+\left(\boldsymbol{\Theta}^{T} \mathscr{B} \mathbf{U}\right)_{\eta}\right) d \xi d \eta d \tau \\
& +\int_{0}^{T}\left(\mathscr{S} \boldsymbol{\Theta}_{\tau}+\mathscr{A} \boldsymbol{\Theta}_{\xi}+\mathscr{B} \boldsymbol{\Theta}_{\eta}-\mathscr{C} \boldsymbol{\Theta}+G, \mathbf{U}\right) d \tau
\end{aligned}
$$

Integration by parts together with the use of Green-Gauss theorem leads to

$$
\begin{align*}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau & =\int_{0}^{T}(\boldsymbol{\Theta}, F) d \tau-\left.(\boldsymbol{\Theta}, \mathscr{S} \mathbf{U})\right|_{\tau=0} ^{\tau=T}-\int_{0}^{T} \oint_{\delta \Phi} \boldsymbol{\Theta}^{T} D \mathbf{U} d s d \tau  \tag{22}\\
& +\int_{0}^{T}\left(\mathscr{S} \boldsymbol{\Theta}_{\tau}+\mathscr{A} \boldsymbol{\Theta}_{\xi}+\mathscr{B} \boldsymbol{\Theta}_{\eta}-\mathscr{C} \boldsymbol{\Theta}+G, \mathbf{U}\right) d \tau
\end{align*}
$$

By applying the previous eigenvalue decomposition of $D$, and considering the homogeneous version of the initial and boundary conditions for the primal problem given in (17), we obtain

$$
\begin{align*}
\int_{0}^{T} \mathscr{J}(\mathbf{U}) d \tau & =\int_{0}^{T}(\boldsymbol{\Theta}, F) d \tau-(\boldsymbol{\Theta}, \mathscr{S} \mathbf{U})_{\tau=T^{-}} \int_{0}^{T} \oint_{\delta \Phi}\left(X_{+}^{T} \boldsymbol{\Theta}\right)^{T} \Lambda_{+}\left(X_{+}^{T} \mathbf{U}\right) d s d \tau \\
& +\int_{0}^{T}\left(\mathscr{S} \boldsymbol{\Theta}_{\tau}+\mathscr{A} \boldsymbol{\Theta}_{\xi}+\mathscr{B} \boldsymbol{\Theta}_{\eta}-\mathscr{C} \boldsymbol{\Theta}+G, \mathbf{U}\right) d \tau . \tag{23}
\end{align*}
$$

To arrive at (19), the last three terms in (23) must vanish. Therefore, the dual problem is given by

$$
\begin{array}{rlrl}
-\mathscr{S} \boldsymbol{\Theta}_{\tau}-\mathscr{A} \boldsymbol{\Theta}_{\xi}-\mathscr{B} \boldsymbol{\Theta}_{\eta}+\mathscr{C} \boldsymbol{\Theta} & =G,(\xi, \eta) & \in \Phi \\
X_{+}^{T} \boldsymbol{\Theta} & =0, \quad(\xi, \eta) \in \delta  \tag{24}\\
\boldsymbol{\Theta}(\xi, \eta) & =0, & \tau & =T
\end{array}
$$

We can prove
Proposition 2. The dual problem in (24) is bounded.
Proof. By the use of (9) in (24), we can rewrite the dual equation in conservative form as

$$
\begin{equation*}
-(\mathscr{S} \boldsymbol{\Theta})_{\tau}-(\mathscr{A} \boldsymbol{\Theta})_{\xi}-(\mathscr{B} \boldsymbol{\Theta})_{\eta}+\mathscr{C} \boldsymbol{\Theta}=0,(\xi, \eta) \in \Phi, \tau \in[0, T] \tag{25}
\end{equation*}
$$

where we ignored the forcing function. The energy method applied to (25) and the matrix $D$ decomposed as before, lead to

$$
\begin{align*}
& -\|\boldsymbol{\Theta}(T, \xi, \eta)\|_{\mathscr{S}(T,)}^{2}+\|\boldsymbol{\Theta}(0, \xi, \eta)\|_{\mathscr{S}(0)}^{2}+2 \int_{0}^{T} \iint_{\Phi} \boldsymbol{\Theta}^{T} \mathscr{C} \boldsymbol{\Theta} d \Phi d \tau=  \tag{26}\\
+ & \int_{0}^{T} \oint_{\delta \Phi}\left(X_{+}^{T} \boldsymbol{\Theta}\right)^{T} \Lambda_{+}\left(X_{+}^{T} \boldsymbol{\Theta}\right) d s d \tau+\int_{0}^{T} \oint_{\delta \Phi}\left(X_{-}^{T} \boldsymbol{\Theta}\right)^{T} \Lambda_{-}\left(X_{-}^{T} \boldsymbol{\Theta}\right) d s d \tau
\end{align*}
$$

With zero initial data and the dual boundary conditions in (24), we obtain

$$
\begin{equation*}
\|\boldsymbol{\Theta}(0, \xi, \eta)\|_{\mathscr{S}(0)}^{2}+2 \int_{0}^{T} \iint_{\Phi} \boldsymbol{\Theta}^{T} \mathscr{C} \boldsymbol{\Theta} d \xi d \eta d \tau-\int_{0}^{T} \oint_{\delta \Phi}\left(X_{-}^{T} \boldsymbol{\Theta}\right)^{T} \Lambda_{-}\left(X_{-}^{T} \boldsymbol{\Theta}\right) d s d \tau=0 \tag{27}
\end{equation*}
$$

By (27), we conclude that the dual problem (24) is bounded.
For later reference, with only the south boundary term considered, the dual energy estimate becomes

$$
\begin{align*}
\|\boldsymbol{\Theta}(0, \xi, \eta)\|_{\mathscr{S}(0)}^{2} & +2 \int_{0}^{T} \iint_{\Phi} \boldsymbol{\Theta}^{T} \mathscr{C} \boldsymbol{\Theta} d \xi d \eta d \tau  \tag{28}\\
& +\int_{0}^{T} \int\left[\left(X_{\mathscr{B}_{s}}\right)_{+}^{T} \boldsymbol{\Theta}\right]^{T}\left(\Lambda_{\mathscr{B}_{s}}\right)_{+}\left[\left(X_{\mathscr{B}_{s}}\right)_{+}^{T} \boldsymbol{\Theta}\right] d \xi d \tau=0 .
\end{align*}
$$

## 3. The discrete problem

We discretize $\Phi=[0,1] \times[0,1]$, with a spatial mesh of $N+1$ and $M+1$ grid points in $\xi$ and $\eta$ directions, respectively. In time, we use a mesh of size $K+1$ from $t=0$ to $t=T$. The fully discrete numerical solution is a vector of size
$9(K+1)(N+1)(M+1)$, arranged as

$$
\tilde{\mathbf{U}}=\left[\begin{array}{c}
\tilde{\mathbf{U}}_{0}  \tag{29}\\
\vdots \\
{\left[\tilde{\mathbf{U}}_{k}\right]} \\
\vdots \\
\tilde{\mathbf{U}}_{K}
\end{array}\right] ;\left[\tilde{\mathbf{U}}_{k}\right]=\left[\begin{array}{c}
\tilde{\mathbf{U}}_{0} \\
\vdots \\
{\left[\tilde{\mathbf{U}}_{n}\right]} \\
\vdots \\
\tilde{\mathbf{U}}_{N}
\end{array}\right]_{k} ;\left[\tilde{\mathbf{U}}_{n}\right]_{k}=\left[\begin{array}{c}
\tilde{\mathbf{U}}_{0} \\
\vdots \\
{\left[\tilde{\mathbf{U}}_{m}\right]} \\
\vdots \\
\tilde{\mathbf{U}}_{M}
\end{array}\right]_{k n} ;\left[\tilde{\mathbf{U}}_{m}\right]_{k n}=\left[\begin{array}{c}
\tilde{U} \\
\tilde{V} \\
\tilde{W}
\end{array}\right]_{k n m}=\tilde{\mathbf{U}}_{k n m}
$$

where $\tilde{\mathbf{U}}_{k n m} \approx \mathbf{U}\left(\tau_{k}, \xi_{n}, \eta_{m}\right)$. A schematic of the spatial mesh at $\tau=\tau_{k}$ and the indexing used is shown in Figure 2.


Figure 2: A schematic of the mesh and the indexing used in the arrangement of the numerical solution at $\tau=\tau_{k}$.

The first derivative $\phi_{\xi}$ is approximated by $D_{\xi} \phi$, where $D_{\xi}$ is a so-called SBP operator of the form

$$
D_{\xi}=P_{\xi}^{-1} Q_{\xi}
$$

and $\phi=\left[\phi_{0}, \phi_{1}, \cdots, \phi_{N}\right]^{T}$ is a smooth function injected in each grid point in the $\xi$
direction. The matrix $P_{\xi}$ is symmetric positive definite, and $Q_{\xi}$ is almost skewsymmetric satisfying

$$
\begin{equation*}
Q_{\xi}+Q_{\xi}^{T}=B_{\xi}=E_{1}-E_{0}=\operatorname{diag}(-1,0, \ldots, 0,1) \tag{30}
\end{equation*}
$$

In (30), $E_{0}=\operatorname{diag}(1,0, \ldots, 0)$ and $E_{1}=\operatorname{diag}(0, \ldots, 0,1)$, both of size $(N+1) \times(N+$ 1). The difference operators in $\eta$ and $\tau$ directions, i.e., $D_{\eta}=P_{\eta}^{-1} Q_{\eta}$ of size $(M+$ 1) $\times(M+1)$ and $D_{\tau}=P_{\tau}^{-1} Q_{\tau}$ of size $(K+1) \times(K+1)$, are defined in the same way.

A first derivative SBP operator is a $2 s$-order accurate central difference operator which is modified close to the boundaries such that it becomes one-sided. Together with a diagonal norm $P_{\xi, \eta, \tau}$, the boundary closure is $s$-order accurate, making a point-wise stable first order approximation $s+1$ order accurate globally [35], [24], [25], [26]. For more details on non-standard SBP operators see [27], [28], [29] and [30].

A finite difference operator in multiple space dimensions including the time discretization [31], [32], on SBP form, is constructed by extending the one-dimensional SBP operators in a tensor product fashion as

$$
\begin{align*}
& \mathscr{D}_{\tau}=D_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes I, \\
& \mathscr{D}_{\xi}=I_{\tau} \otimes D_{\xi} \otimes I_{\eta} \otimes I,  \tag{31}\\
& \mathscr{D}_{\eta}=I_{\tau} \otimes I_{\xi} \otimes D_{\eta} \otimes I .
\end{align*}
$$

In (31), $\otimes$ represents the Kronecker product [33] which is defined as

$$
A \otimes B=\left[\begin{array}{ccc}
a_{00} B & \ldots & a_{0 m} B \\
\vdots & \ddots & \vdots \\
a_{n 0} B & \ldots & a_{n m} B
\end{array}\right]
$$

for the $(n+1) \times(m+1)$ matrix $A=\left\{a_{i j}\right\}$, and $B$ of arbitrary size. In (31), and in the remainder of this article, all matrices in the first position of the Kronecker products are of size $(K+1) \times(K+1)$, the second position $(N+1) \times(N+1)$, the third position $(M+1) \times(M+1)$ and the fourth position $9 \times 9$. Additionally, $I_{\tau}, I_{\xi}$, $I_{\eta}$ and I denote the identity matrices with a size consistent with their positions in the Kronecker product.

In (17), the coefficient matrices on the left hand side are variable and symmetric, therefore, prior to constructing the scheme, we apply the splitting technique in [34]. The forcing function on the right hand side of (17) is ignored, since it does
not affect the stability analysis. The result is

$$
\begin{align*}
\frac{1}{2}\left((\mathscr{S} \mathbf{U})_{\tau}+\mathscr{S} \mathbf{U}_{\tau}+\mathscr{S}_{\tau} \mathbf{U}\right) & +\frac{1}{2}\left((\mathscr{A} \mathbf{U})_{\xi}+\mathscr{A} \mathbf{U}_{\xi}+\mathscr{A}_{\xi} \mathbf{U}\right) \\
& +\frac{1}{2}\left((\mathscr{B} \mathbf{U})_{\eta}+\mathscr{B} \mathbf{U}_{\eta}+\mathscr{B}_{\eta} \mathbf{U}\right)+\mathscr{C} \mathbf{U}=0 \tag{32}
\end{align*}
$$

The fully discrete version of (32) on SBP-SAT form, including weakly imposed initial and boundary conditions, is

$$
\begin{align*}
\frac{1}{2}\left[\mathscr{D}_{\tau} \tilde{\mathscr{S}}+\tilde{\mathscr{S}} \mathscr{D}_{\tau}+\tilde{\mathscr{S}}_{\tau}\right] \tilde{\mathbf{U}} & +\frac{1}{2}\left[\mathscr{D}_{\xi} \tilde{\mathscr{A}}+\tilde{\mathscr{A}} \mathscr{D}_{\xi}+\tilde{\mathscr{A}}_{\xi}\right] \tilde{\mathbf{U}}+\frac{1}{2}\left[\mathscr{D}_{\eta} \tilde{\mathscr{B}}+\tilde{\mathscr{B}} \mathscr{D}_{\eta}+\tilde{\mathscr{B}}_{\eta}\right] \tilde{\mathbf{U}}+\tilde{\mathscr{C}} \tilde{\mathbf{U}} \\
& =\mathscr{P}_{i}^{-1} \Sigma_{i}(\tilde{\mathbf{U}}-\mathbf{f})+\mathscr{P}_{s}^{-1} \Sigma_{s}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right] . \tag{33}
\end{align*}
$$

In (33), $\mathscr{P}_{i}^{-1}=P_{\tau}^{-1} E_{0} \otimes I_{\xi} \otimes I_{\eta} \otimes I$ and $\mathscr{P}_{s}^{-1}=I_{\tau} \otimes I_{\xi} \otimes P_{\eta}^{-1} E_{0} \otimes I$ where the subscripts $i$ and $s$ indicate initial and south, respectively. We have only considered the south boundary procedure in (33), since the treatment of other boundaries is similar. Additionally in (33), $\mathbf{U}_{\infty}$ and $\mathbf{f}$, are vectors containing boundary and initial data at the relevant positions.

In (33), $\tilde{\mathscr{S}}, \tilde{\mathscr{S}}_{\tau}, \tilde{\mathscr{A}}, \tilde{\mathscr{A}_{\xi}}, \tilde{\mathscr{B}}, \tilde{\mathscr{B}} \eta, \tilde{\mathscr{C}}$ and $\tilde{X}_{\mathscr{B}_{s}}$ are block diagonal matrices that approximate $\mathscr{S}, \mathscr{S}_{\tau}, \mathscr{A}, \mathscr{A}_{\xi}, \mathscr{B}, \mathscr{B}_{\eta}, \mathscr{C}$ and $X_{\mathscr{B}_{s}}$ pointwise in the following way

$$
\begin{aligned}
& \tilde{\mathscr{A}}_{\xi}=\left[\begin{array}{lllll}
{\left[\tilde{\mathscr{A}}_{\xi}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\tilde{\mathscr{A}}_{\xi}\right]_{k}} & & \\
& & & \ddots & \\
& & & & {\left[\tilde{\mathscr{A}}_{\xi}\right]_{K}}
\end{array}\right],\left[\tilde{\mathscr{A}}_{\xi}\right]_{k}=\left[\begin{array}{lllll}
{\left[\tilde{\mathscr{A}}_{k}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\tilde{\mathscr{A}}_{\xi}\right]_{n}} & & \\
& & & \ddots & \\
& & & & {\left[\tilde{\mathscr{A}}_{\xi}\right]_{N}}
\end{array}\right] \text {, } \\
& {\left[\tilde{\mathscr{A}}_{\xi}\right]_{k n}=\left[\begin{array}{llllll}
{\left[\tilde{\mathscr{A}}_{\xi}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {[\tilde{\mathscr{A}}]_{m}} & & \\
& & & \ddots & \\
& & & & {\left[\tilde{\mathscr{A}}_{\xi}\right]_{M}}
\end{array}\right]_{k n},\left[\tilde{\mathscr{A}}_{\xi}\right]_{k n m} \approx \mathscr{A} \mathcal{A}_{\xi}\left(\tau_{k}, \xi_{n}, \eta_{m}\right) .}
\end{aligned}
$$

The matrices $\tilde{\mathscr{S}}, \tilde{\mathscr{S}}_{\tau}, \tilde{\mathscr{A}}, \tilde{\mathscr{A}}_{\xi}, \tilde{\mathscr{B}}, \tilde{\mathscr{B}}_{\eta}$ and $\tilde{\mathscr{C}}$ include numerical approximations of the metric terms [18, 20].

We use

$$
\begin{aligned}
\tilde{\mathscr{S}} & =\bar{J}\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{S}\right), \\
\tilde{\mathscr{A}} & =\overline{J \xi_{t}}+\overline{J \xi_{x}}\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{A}\right)+\overline{J \xi_{y}}\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{B}\right), \\
\tilde{\mathscr{B}} & =\overline{J \eta_{t}}+\overline{J \eta_{x}}\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{A}\right)+\overline{J \eta_{y}}\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{B}\right),
\end{aligned}
$$

where $\bar{J}, \overline{J \xi_{t}}, \overline{J \xi_{x}}, \overline{J \xi_{y}}, \overline{J \eta_{t}}, \overline{J \eta_{x}}$ and $\overline{J \eta_{y}}$ are diagonal matrices, containing pointwise approximations of $J, J \xi_{t}, J \xi_{x}, J \xi_{y}, J \eta_{t}, J \eta_{x}$ and $J \eta_{y}$, respectively. The numerical metric terms are defined as

$$
\begin{align*}
\bar{J} & =\operatorname{diag}\left[\mathscr{D}_{\eta} M^{(1)}-\mathscr{D}_{\xi} M^{(2)}\right], & & \\
\overline{J \xi_{t}} & =\operatorname{diag}\left[\mathscr{D}_{\tau} M^{(2)}-\mathscr{D}_{\eta} M^{(3)}\right], & \overline{J \xi_{x}}=\operatorname{diag}\left[\mathscr{D}_{\eta} \mathbf{y}\right], & \overline{J \xi_{y}}=-\operatorname{diag}\left[\mathscr{D}_{\eta} \mathbf{x}\right], \\
\overline{J \eta_{t}} & =\operatorname{diag}\left[\mathscr{D}_{\xi} M^{(3)}-\mathscr{D}_{\tau} M^{(1)}\right], & \overline{J \eta_{x}}=-\operatorname{diag}\left[\mathscr{D}_{\xi} \mathbf{y}\right], & \overline{J \eta_{y}}=\operatorname{diag}\left[\mathscr{D}_{\xi} \mathbf{x}\right] . \tag{34}
\end{align*}
$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors containing the x and y coordinates of the Cartesian mesh arranged similar to the numerical solution in (29). Moreover,

$$
\begin{equation*}
M^{(1)}=\operatorname{diag}(\mathbf{y}) \mathscr{D}_{\xi} \mathbf{x}, \quad M^{(2)}=\operatorname{diag}(\mathbf{y}) \mathscr{D}_{\eta} \mathbf{x}, \quad M^{(3)}=\operatorname{diag}(\mathbf{y}) \mathscr{D}_{\tau} \mathbf{x} \tag{35}
\end{equation*}
$$

and

$$
\begin{align*}
& \tilde{\mathscr{P}}_{\tau}=\left(\mathscr{D}_{\tau} \bar{J}\right)\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \mathbf{S}\right), \\
& \tilde{\mathscr{A}}_{\xi}=\left(\mathscr{D}_{\xi} \overline{J \xi_{t}}\right)+\left(\mathscr{D}_{\xi} \bar{J} \xi_{x}\right)\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \overline{\mathbf{A}}\right)+\left(\mathscr{D}_{\xi} \overline{J \xi_{y}}\right)\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \overline{\mathbf{B}}\right)  \tag{36}\\
& \tilde{\mathscr{B}}_{\eta}=\left(\mathscr{D}_{\eta} \bar{J} \eta_{t}\right)+\left(\mathscr{D}_{\eta} \bar{J} \eta_{x}\right)\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \overline{\mathbf{A}}\right)+\left(\mathscr{D}_{\eta} \bar{J} \eta_{y}\right)\left(I_{\tau} \otimes I_{\xi} \otimes I_{\eta} \otimes \overline{\mathbf{B}}\right) .
\end{align*}
$$

Furthermore, in (33), $\Sigma_{s}$ is the penalty matrix corresponding to the weak imposition of the south boundary condition, arranged as

$$
\begin{gathered}
\left.\Sigma_{s}=\left[\begin{array}{llllll}
{\left[\Sigma_{s}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\Sigma_{s}\right]_{k}} & & \\
& & & \ddots & \\
& & & & {\left[\Sigma_{s}\right]_{K}}
\end{array}\right] ; \text { 法 }\right]_{k}=\left[\begin{array}{lllll}
{\left[\Sigma_{s}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\Sigma_{s}\right]_{n}} & & \\
& & & \ddots & \\
& & & & \left.\left[\Sigma_{s}\right]_{N}\right]_{k}
\end{array}\right],\left[\Sigma_{s}\right]_{k n}=\left[\begin{array}{llll}
{\left[\Sigma_{s}\right]_{0}} & & \\
& 0 & & \\
& & \ddots & \\
& & & 0
\end{array}\right]_{k n}
\end{gathered}
$$

where $\left[\Sigma_{s}\right]_{k n 0}$ operates on the south boundary, i.e., $\left(\xi_{n}, \eta_{0}\right)$ for $n \in\{0, \ldots, N\}$, see Figure 2.

Finally, in (33), $\Sigma_{i}$ is the penalty operator for the weak imposition of the initial condition arranged as

$$
\begin{aligned}
& \Sigma_{i}=\left[\begin{array}{cccc}
{\left[\Sigma_{i}\right]_{0}} & & & \\
& 0 & & \\
& & \ddots & \\
& & & 0
\end{array}\right] ; \\
& {\left[\Sigma_{i}\right]_{0}=\left[\begin{array}{lllll}
{\left[\Sigma_{i}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\Sigma_{i}\right]_{n}} & & \\
& & & \ddots & \\
& & & & {\left[\Sigma_{i}\right]_{N}}
\end{array}\right]_{0} ;\left[\Sigma_{i}\right]_{0 n}=\left[\begin{array}{lllll}
{\left[\Sigma_{i}\right]_{0}} & & & & \\
& \ddots & & & \\
& & {\left[\Sigma_{i}\right]_{m}} & & \\
& & & \ddots & \\
& & & & {\left[\Sigma_{i}\right]_{M}}
\end{array}\right]_{0 n}}
\end{aligned}
$$

where $\left[\Sigma_{i}\right]_{0 n m}$ operates on the mesh point $\left(\xi_{n}, \eta_{m}\right)$, see Figure 2 , at $\tau=0$.
Lemma 1. By using the definitions given in (34), (35) and (36), the Numerical Geometric Conservation Law (NGCL) holds and results in

$$
\begin{equation*}
\tilde{\mathscr{S}}_{\tau}+\tilde{\mathscr{A}}_{\xi}+\tilde{\mathscr{B}}_{\eta}=0 . \tag{37}
\end{equation*}
$$

See [17] for the proof.
Remark 9. By using SBP in time we get a time-discretization operator that clearly commutes with the spatial operators and (37) follows directly, for details see [17].

### 3.1. Stability of the primal problem

The discrete energy method applied to (33) yields

$$
\begin{aligned}
& \tilde{\mathbf{U}}^{T}\left[B_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right] \tilde{\mathscr{S}} \tilde{\mathbf{U}}+\tilde{\mathbf{U}}^{T}\left[P_{\tau} \otimes B_{\xi} \otimes P_{\eta} \otimes I\right] \tilde{\mathscr{A}} \tilde{\mathbf{U}}+ \\
& \tilde{\mathbf{U}}^{T}\left[P_{\tau} \otimes P_{\xi} \otimes B_{\eta} \otimes I\right] \tilde{\mathscr{B}} \tilde{\mathbf{U}}+\tilde{\mathbf{U}}^{T} \mathscr{P}\left(\tilde{\mathscr{S}}_{\tau}+\tilde{\mathscr{A}}_{\xi}+\tilde{\mathscr{B}}_{\eta}\right) \tilde{\mathbf{U}}+2 \tilde{\mathbf{U}}^{T} \mathscr{P} \tilde{\mathscr{C}} \tilde{\mathbf{U}}= \\
& \tilde{\mathbf{U}}^{T}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i}(\tilde{\mathbf{U}}-\mathbf{f})+(\tilde{\mathbf{U}}-\mathbf{f})^{T} \Sigma_{i}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathbf{U}}+ \\
& \tilde{\mathbf{U}}^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]+ \\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]^{T} \Sigma_{s}^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \tilde{\mathbf{U}}}
\end{aligned}
$$

where $\mathscr{P}=P_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I$. The use of the NGCL (37) and only considering the south boundary terms lead to

$$
\begin{align*}
& \tilde{\mathbf{U}}^{T}\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}} \tilde{\mathbf{U}}+2 \tilde{\mathbf{U}}^{T} \mathscr{P} \tilde{\mathscr{C}} \tilde{\mathbf{U}}= \\
& \tilde{\mathbf{U}}^{T}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)\left(\tilde{\mathscr{S}}+2 \Sigma_{i}\right) \tilde{\mathbf{U}}-\tilde{\mathbf{U}}^{T}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i} \mathbf{f}- \\
& \mathbf{f}^{T} \Sigma_{i}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathbf{U}}+\tilde{\mathbf{U}}^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \tilde{\mathscr{B}}_{s} \tilde{\mathbf{U}}+  \tag{38}\\
& \tilde{\mathbf{U}}^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}\left[\left(\tilde{X}_{\mathscr{B}_{s}}{ }_{+}^{T} \tilde{\mathbf{U}}-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]+\right. \\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]^{T} \Sigma_{s}^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \tilde{\mathbf{U}} .}
\end{align*}
$$

The decomposition $\tilde{\mathscr{B}}_{s}=\tilde{X}_{\mathscr{B}_{s}} \tilde{\Lambda}_{\mathscr{B}_{s}} \tilde{X}_{\mathscr{B}_{s}}^{T}$ and $\Sigma_{s}=\left(\tilde{X}_{\mathscr{B}_{s}}\right)+\hat{\Sigma}_{s}$ inserted in (38) gives

$$
\begin{align*}
& \|\tilde{\mathbf{U}}\|_{\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)}^{2}+2 \tilde{\mathbf{U}}^{T} \mathscr{P} \tilde{\mathscr{C}} \tilde{\mathbf{U}}=\tilde{\mathbf{U}}^{T}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)\left(\tilde{\mathscr{S}}+2 \Sigma_{i}\right) \tilde{\mathbf{U}}- \\
& \tilde{\mathbf{U}}^{T}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i} \mathbf{f}-\mathbf{f}^{T} \Sigma_{i}\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathbf{U}}+ \\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}^{T} \tilde{\mathbf{U}}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{-}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}^{T} \tilde{\mathbf{U}}\right]+}  \tag{39}\\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left[\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{+}+2 \hat{\Sigma}_{s}\right]\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}\right]-} \\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}^{T}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \hat{\Sigma}_{s}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]-} \\
& {\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}_{\infty}\right]^{T} \hat{\Sigma}_{s}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\mathbf{U}}\right],}
\end{align*}
$$

where

$$
\|\tilde{\mathbf{U}}\|_{\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}}^{2}=\tilde{\mathbf{U}}^{T}\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}} \tilde{\mathbf{U}}^{T} \approx \iint_{\Phi} J\left(u^{2}+v^{2}\right) d \xi d \eta
$$

To find stability requirements, it is convenient to consider zero data in (39). Zero energy growth is obtained under the conditions

$$
\begin{align*}
\Sigma_{i} & \leq-\tilde{\mathscr{S}} / 2, \\
\left(\hat{\Sigma}_{s}\right)_{j j} & \leq-\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{j j} / 2  \tag{40}\\
\left(\hat{\Sigma}_{s}\right)_{j j} & \text { if } \quad\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{j j}>0 \\
& \text { if } \quad\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{j j} \leq 0
\end{align*}
$$

for $j \in\{0, \ldots, 9(K+1)(N+1)(M+1)\}$.

Remark 10. For other boundary conditions than the ones chosen in (14), the analysis above would remain the same if they lead to energy stability.

### 3.2. Stability of the dual problem

Similar to the primal case, the splitting technique is applied to the dual problem in (24), prior to constructing the scheme. The result is

$$
\begin{align*}
-\frac{1}{2}\left[(\mathscr{S} \boldsymbol{\Theta})_{\tau}+\mathscr{S} \boldsymbol{\Theta}_{\tau}+\mathscr{S}_{\tau} \boldsymbol{\Theta}\right] & -\frac{1}{2}\left[(\mathscr{A} \boldsymbol{\Theta})_{\xi}+\mathscr{A} \boldsymbol{\Theta}_{\xi}+\mathscr{A}_{\xi} \boldsymbol{\Theta}\right]  \tag{41}\\
& -\frac{1}{2}\left[(\mathscr{B} \boldsymbol{\Theta})_{\eta}+\mathscr{B}_{\eta}+\mathscr{B}_{\eta} \boldsymbol{\Theta}\right]+\mathscr{C} \boldsymbol{\Theta}=0
\end{align*}
$$

where we ignored the forcing function. Next, we discretize (41), as in the primal case, and weakly impose the homogeneous final time and boundary conditions given in (24). Again, we only consider the south boundary procedure. The discrete dual problem becomes

$$
\begin{align*}
-\frac{1}{2}\left[D_{\tau} \tilde{\mathscr{S}}+\tilde{\mathscr{S}} D_{\tau}+\tilde{\mathscr{S}}_{\tau}\right] \tilde{\boldsymbol{\Theta}} & -\frac{1}{2}\left[D_{\xi} \tilde{\mathscr{A}}+\tilde{\mathscr{A}} D_{\xi}+\tilde{\mathscr{A}}_{\xi}\right] \tilde{\boldsymbol{\Theta}}-\frac{1}{2}\left[D_{\eta} \tilde{\mathscr{B}}+\tilde{\mathscr{B}} D_{\eta}+\tilde{\mathscr{B}}_{\eta}\right] \tilde{\boldsymbol{\Theta}}+\tilde{\mathscr{C}} \tilde{\boldsymbol{\Theta}} \\
& =\mathscr{P}_{f}^{-1} \Sigma_{f} \tilde{\boldsymbol{\Theta}}+\mathscr{P}_{s}^{-1} \Sigma_{s}^{d}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\boldsymbol{\Theta}}\right] . \tag{42}
\end{align*}
$$

In (42), $\mathscr{P}_{f}^{-1}=P_{\tau}^{-1} E_{1} \otimes I_{\xi} \otimes I_{\eta} \otimes I, \Sigma_{f}$ and $\Sigma_{s}^{d}$ are the penalty operators corresponding to the weak final time and south boundary procedure of the dual problem, respectively. We consider $\Sigma_{s}^{d}=\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-} \hat{\Sigma}_{s}^{d}$, where $\hat{\Sigma}_{s}^{d}$ is diagonal.

We use the discrete energy method (multiplying (42) with $\tilde{\boldsymbol{\Theta}}^{T} \mathscr{P}$ ) to study the stability of the dual problem. This procedure is identical to the stability analysis for the primal problem, and therefore not repeated here. Stability of the dual problem requires
where $j \in\{0, \ldots, 9(K+1)(N+1)(M+1)\}$.

### 3.3. The dual consistent approximation

So far, we have determined conditions for the penalty operators ( $\Sigma_{i}, \Sigma_{f}, \Sigma_{s}$ and $\Sigma_{s}^{d}$ ), given in (40) and (43), that lead to stable primal (33) and dual (42) approximations. Now, we aim for the specific choices of the penalty operators, that make the approximations dual consistent. The procedure is as follows:

1. We consider zero initial and boundary data in (33) and add a non-zero forcing function, $\mathbf{F}$, to the right hand side. The result can be rewritten in form of

$$
\begin{equation*}
\mathscr{L} \tilde{\mathbf{U}}=\mathscr{P} \mathbf{F}, \tag{44}
\end{equation*}
$$

where $\mathscr{L}$ contains the SBP-SAT contributions.
2. We consider zero initial and boundary data in (42) and add a non-zero forcing function, $\mathbf{G}$, to the right hand side. The result can be rewritten as

$$
\begin{equation*}
\mathscr{L}^{d} \tilde{\boldsymbol{\Theta}}=\mathscr{P} \mathbf{G} \tag{45}
\end{equation*}
$$

where $\mathscr{L}^{d}$ contains the SBP-SAT contributions.
3. A dual consistent approximation is obtained if

$$
\begin{equation*}
\mathscr{L}=\left(\mathscr{L}^{d}\right)^{T}, \tag{46}
\end{equation*}
$$

by which we can find the specific values of the penalty operators.
The operator $\mathscr{L}$ in (44) is

$$
\begin{align*}
\mathscr{L}= & \frac{1}{2}\left[\left(Q_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}+\tilde{\mathscr{S}}\left(Q_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)+\tilde{\mathscr{S}}_{\tau}\right]+ \\
& \frac{1}{2}\left[\left(P_{\tau} \otimes Q_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{A}}+\tilde{\mathscr{A}}\left(P_{\tau} \otimes Q_{\xi} \otimes P_{\eta} \otimes I\right)+\tilde{\mathscr{A}}_{\xi}\right]+  \tag{47}\\
& \frac{1}{2}\left[\left(P_{\tau} \otimes P_{\xi} \otimes Q_{\eta} \otimes I\right) \tilde{\mathscr{B}}+\tilde{\mathscr{B}}\left(P_{\tau} \otimes P_{\xi} \otimes Q_{\eta} \otimes I\right)+\tilde{\mathscr{B}}_{\eta}\right]+ \\
& \mathscr{P} \tilde{\mathscr{C}}-\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i}-\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T},
\end{align*}
$$

and the operator $\mathscr{L}^{d}$ in (45) is

$$
\begin{align*}
\mathscr{L}^{d}= & -\frac{1}{2}\left[\left(Q_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}+\tilde{\mathscr{S}}\left(Q_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)+\tilde{\mathscr{S}}_{\tau}\right] \\
& -\frac{1}{2}\left[\left(P_{\tau} \otimes Q_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{A}}+\tilde{\mathscr{A}}\left(P_{\tau} \otimes Q_{\xi} \otimes P_{\eta} \otimes I\right)+\tilde{\mathscr{A}}_{\xi}\right]  \tag{48}\\
& -\frac{1}{2}\left[\left(P_{\tau} \otimes P_{\xi} \otimes Q_{\eta} \otimes I\right) \tilde{\mathscr{B}}+\tilde{\mathscr{B}}\left(P_{\tau} \otimes P_{\xi} \otimes Q_{\eta} \otimes I\right)+\tilde{\mathscr{B}}_{\eta}\right] \\
& +\mathscr{P} \tilde{\mathscr{C}}-\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{f}-\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}^{d}\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}^{T} .
\end{align*}
$$

Subtracting (47) from the transpose of (48), using the NGCL given in Lemma (1) and the SBP property (30) give

$$
\begin{align*}
\left(\mathscr{L}^{d}\right)^{T}-\mathscr{L}= & -\frac{1}{2}\left[\left(B_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}+\tilde{\mathscr{S}}\left(B_{\tau} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)\right] \\
& -\frac{1}{2}\left[\left(P_{\tau} \otimes B_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{A}}+\tilde{\mathscr{A}}\left(P_{\tau} \otimes B_{\xi} \otimes P_{\eta} \otimes I\right)\right] \\
& -\frac{1}{2}\left[\left(P_{\tau} \otimes P_{\xi} \otimes B_{\eta} \otimes I\right) \tilde{\mathscr{B}}+\tilde{\mathscr{B}}\left(P_{\tau} \otimes P_{\xi} \otimes B_{\eta} \otimes I\right)\right]  \tag{49}\\
& -\Sigma_{f}\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}\left(\Sigma_{s}^{d}\right)^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \\
& +\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i}+\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T}
\end{align*}
$$

By considering only the terms corresponding to the initial and final time and the south boundary in (49), we obtain

$$
\begin{align*}
\left(\mathscr{L}^{d}\right)^{T}-\mathscr{L}=- & {\left[\left(E_{1}-E_{0}\right) \otimes P_{\xi} \otimes P_{\eta} \otimes I\right] \tilde{\mathscr{S}}+\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \tilde{\mathscr{B}} } \\
& -\Sigma_{f}\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)-\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}\left(\Sigma_{s}^{d}\right)^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)  \tag{50}\\
& +\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \Sigma_{i}+\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right) \Sigma_{s}\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T}+R .
\end{align*}
$$

In (50), $R$ includes the contribution of the other boundaries than the south. In order to arrive at (50), we have used the fact that $\tilde{\mathscr{B}}$ commutes with $\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)$. Next, we use the decompositions $\Sigma_{s}=\left(\tilde{X}_{B_{s}}\right)_{+} \hat{\Sigma}_{s}$ and $\Sigma_{s}^{d}=\left(\tilde{X}_{B_{s}}\right)_{-} \hat{\Sigma}_{s}^{d}$, as introduced in sections 3.1 and 3.2 and arrive at

$$
\begin{align*}
\left(\mathscr{L}^{d}\right)^{T}-\mathscr{L}= & -\left[E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right]\left(\tilde{\mathscr{S}}+\Sigma_{f}\right)+\left[E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right]\left(\tilde{\mathscr{S}}+\Sigma_{i}\right) \\
& +\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left[\left(\tilde{X}_{B_{s}}\right)_{+}\left(\hat{\Sigma}_{s}+\left(\tilde{\Lambda}_{B_{s}}\right)_{+}\right)\left(\tilde{X}_{B_{s}}\right)_{+}^{T}\right] \\
& +\left[\left(\tilde{X}_{B_{s}}\right)_{-}\left(\left(\tilde{\Lambda}_{B_{s}}\right)_{-}-\hat{\Sigma}_{s}^{d}\right)\left(\tilde{X}_{B_{s}}\right)_{-}^{T}\right]\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)+R . \tag{51}
\end{align*}
$$

To secure (46) we need

$$
\begin{align*}
& \Sigma_{i}=-\tilde{\mathscr{S}} \\
& \hat{\Sigma}_{s}=-\frac{\tilde{\Lambda}_{\tilde{B}_{s}}+\left|\tilde{\Lambda}_{\tilde{\mathscr{B}}_{s}}\right|}{2}=-\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{+} \\
& \Sigma_{f}=-\tilde{\mathscr{S}}  \tag{52}\\
& \hat{\Sigma}_{s}^{d}=\frac{\tilde{\Lambda}_{\mathscr{B}_{s}}-\left|\tilde{\Lambda}_{\tilde{B}_{s}}\right|}{2}=\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{-} .
\end{align*}
$$

Note that the results in (52) satisfy the stability conditions in (40) and (43).
Finally, we substitute the primal penalty operators in (52) into (39) and consider non-zero initial and boundary data. The result is

$$
\begin{align*}
\|\tilde{\mathbf{U}}\|_{\left(E_{1} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}}^{2} & -\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}^{T} \tilde{\mathbf{U}}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{-}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{-}^{T} \tilde{\mathbf{U}}\right]+2 \tilde{\mathbf{U}}^{T} \mathscr{P} \tilde{\mathscr{C}} \tilde{\mathbf{U}} \\
& =\left\|\left.\mathbf{f}\right|_{\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{T}}} ^{2}-\right\| \tilde{\mathbf{U}}-\mathbf{f} \|_{\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right)}^{2} \tilde{\mathscr{S}} \\
& +\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left(\tilde{\Lambda}_{\mathscr{R}_{s}}\right)_{+}\left[\left(\tilde{X}_{\mathscr{R}_{s}}\right)_{+}^{T} \mathbf{U}_{\infty}\right] \\
& -\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T}\left(\tilde{\mathbf{U}}-\mathbf{U}_{\infty}\right)\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{+}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T}\left(\tilde{\mathbf{U}}-\mathbf{U}_{\infty}\right)\right] . \tag{53}
\end{align*}
$$



Figure 3: A schematic of the deforming domain

Note that the estimate in (53) mimics the continuous counterpart in (16).
Similarly, substituting the dual penalty operators in (52) to (42) leads to

$$
\begin{equation*}
\|\tilde{\boldsymbol{\Theta}}\|_{\left(E_{0} \otimes P_{\xi} \otimes P_{\eta} \otimes I\right) \tilde{\mathscr{S}}}^{2}-\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\boldsymbol{\Theta}}\right]^{T}\left(P_{\tau} \otimes P_{\xi} \otimes E_{0} \otimes I\right)\left(\tilde{\Lambda}_{\mathscr{B}_{s}}\right)_{+}\left[\left(\tilde{X}_{\mathscr{B}_{s}}\right)_{+}^{T} \tilde{\boldsymbol{\Theta}}\right]^{T}+2 \tilde{\boldsymbol{\Theta}}^{T} \mathscr{P} \tilde{\mathscr{C}} \tilde{\boldsymbol{\Theta}}=0 . \tag{54}
\end{equation*}
$$

The dual estimate in (54) mimics the continuous counterpart given in (28).
Remark 11. Note that dual consistency is achieved by the specific choice (52). This is merely a choice of the specific penalty parameters for the primal problem. It does not require the solution of the dual problem. Hence, dual consistency comes at no extra cost.

## 4. Numerical experiments

We consider (4) with $(\bar{u}, \bar{v})=(1,1)$ and $v=0.1$ posed on a deforming domain where the boundaries coincide with segments of constant polar coordinates. The coordinate transformation is shown schematically in Figure 3. The polar coordi-


Figure 4: A schematic of the deforming domain at different times
nates are

$$
r(x, y, t)=\sqrt{x(t)^{2}+y(t)^{2}}, \quad \phi(x, y, t)=\tan ^{-1}(y(t) / x(t))
$$

and the exact description of the moving boundaries is given by

$$
\begin{aligned}
& r_{d}(t)=1+0.1 \sin (2 \pi t), r_{b}(t)=2-0.2 \sin (2 \pi t), \\
& \phi_{a}(t)=0+0.1 \sin (2 \pi t), \phi_{c}(t)=(\pi / 2)-0.2 \sin (2 \pi t),
\end{aligned}
$$

where $a, b, c$ and $d$ are shown in Figure 3. Next, we scale the polar coordinates such that the fixed domain becomes the unit square, as

$$
\xi(x, y, t)=\frac{r(x, y, t)-r_{d}(t)}{r_{b}(t)-r_{d}(t)}, \eta(x, y, t)=\frac{\phi(x, y, t)-\phi_{a}(t)}{\phi_{c}(t)-\phi_{a}(t)} .
$$

A schematic of the deforming domain at different times is given in Figure 4.
Remark 12. The transformation chosen here is only for illustration purposes. In realistic applications any non-orthogonal mesh satisfying the GCL can be used.

### 4.1. Remarks on the implementation

The time interval $[0, T]$ is divided into an arbitrary number of smaller intervals. The final solution in each interval is weakly imposed as initial data for the next interval. By using this technique, the computation becomes faster and more memory efficient. The reason is that instead of constructing a matrix of size $K$, associated with the time integration we construct smaller blocks each of size $K / N_{b}$ where $N_{b}$ denotes the number of blocks. More details on the SBP multi-block formulation in time is outlined in [32].

To compute $\tilde{U}$ for each computational block, we rewrite (33) as

$$
\begin{equation*}
\mathscr{L} \tilde{U}=\mathscr{R} \tag{55}
\end{equation*}
$$

In (55), $\mathscr{L}$ is a matrix of size $9\left(K_{b}+1\right)(N+1)(M+1)$ containing all SBP-SAT contributions excluding data. Additionally, $\mathscr{R}$ is a $9\left(K_{b}+1\right)(N+1)(M+1)$ vector which includes data. The Generalized Minimal Residual Method (GMRES) is used to solve (55).

### 4.2. Order of accuracy for the solution

In the theoretical section, we proved that we have a stable and consistent scheme. To verify that also our implementation is correct, we will check and make sure that we get the correct order of accuracy for the solution and functionals.

To determine the order of accuracy of the approximation, the method of manufactured solutions with

$$
\begin{equation*}
U=\left[\sin \left(x^{2}-t\right), \cos (x-t), \sin (y-t)\right]^{T} \tag{56}
\end{equation*}
$$

is used. By using (56), $\mathbf{U}=\left[U^{T}, V^{T}, W^{T}\right]^{T}$ where $V=\partial U / \partial x$ and $W=\partial U / \partial y$, we can quantify the error as $e=\mathbf{U}-\tilde{\mathbf{U}}$.

We examine the scheme for SBP operators of order $2 s$ in the interior and $s$ close to the boundaries, where $s \in\{1,2,3\}$. The fifth order accurate SBP operator SBP84, with a sufficiently large $K$, is used in time.

The rates of convergence are calculated as

$$
\begin{equation*}
p=\frac{\log \frac{\left\|e^{(1)}\right\|_{\tilde{J} \mathscr{P}}}{\left\|e^{(2)}\right\|_{\bar{J} \mathscr{P}}}}{\log \frac{N^{(1)} M^{(1)}}{N^{(2)} M^{(2)}}} . \tag{57}
\end{equation*}
$$

Table 1: Convergence rates for the solution at $\mathrm{T}=1$, for a sequence of mesh refinements, SBP84 in time with sufficiently small time steps is used

| $N, M$ | 21 | 31 | 41 | 51 |
| :---: | :---: | :---: | :---: | :---: |
| $S B P 21$ | 2.0152 | 1.9970 | 1.9937 | 1.9965 |
| SBP42 | 3.0737 | 3.0012 | 3.0063 | 3.0745 |
| SBP63 | 5.1925 | 3.7985 | 4.2519 | 4.4549 |

Table 2: Convergence rates for the divergence at $\mathrm{T}=1$, for a sequence of mesh refinements, SBP84 in time with sufficiently small time steps is used

| $N, M$ | 21 | 31 | 41 | 51 |
| :---: | :---: | :---: | :---: | :---: |
| $S B P 21$ | 1.8270 | 2.1751 | 1.9861 | 2.0334 |
| $S B P 42$ | 9.0553 | 4.3946 | 4.2267 | 4.3441 |
| $S B P 63$ | 15.6189 | 4.9515 | 5.1150 | 4.5048 |

where superscripts (1) and (2) denote two mesh levels with $\left(N^{(1)}+1\right) \times\left(M^{(1)}+1\right)$ and $\left(N^{(2)}+1\right) \times\left(M^{(2)}+1\right)$ grid points, respectively.

In Table 1 , the convergence rates of the solution are shown for a sequence of spatial mesh refinements. The results corroborate that the scheme is design order accurate [35], [26]. Table 2 shows, the convergence rates of the divergence. Due to the first order formulation of the problem, the divergence converges with the design order of accuracy of the scheme, i.e., with the same order as the variables themselves [15], [16].

The calculations in Table 1 and 2 are computationally demanding, since we solve a three dimensional system of equations with nine variables. The convergence rates for SBP21, 42 and 63 operators are presented in Table 1. The results in Table 2 for the divergence are a bit erratic, but still reasonable, especially since the rates are slightly higher than expected.

### 4.3. Order of accuracy for functionals

Based on the theory, a superconverging linear functional should be obtained for linear problems and dual consistent approximations [10], [11], [12], [13], [14].

Table 3: Convergence rates for $J_{1}$ at $\mathrm{T}=1$, for a sequence of mesh refinements, SBP84 in time with sufficiently small time steps is used

| $N, M$ | 21 | 31 | 41 | 51 |
| :---: | :---: | :---: | :---: | :---: |
| $S B P 21$ | 1.9981 | 1.9604 | 1.9756 | 1.9647 |
| SBP42 | 4.0287 | 4.3275 | 4.6774 | 6.0737 |
| SBP63 | 9.6653 | 6.8172 | 8.8344 | 6.2045 |

Table 4: Convergence rates for $J_{2}$ at $\mathrm{T}=1$, for a sequence of mesh refinements, SBP84 in time with sufficiently small time steps is used

| $N, M$ | 21 | 31 | 41 | 51 |
| :---: | :---: | :---: | :---: | :---: |
| $S B P 21$ | 1.9933 | 1.9759 | 1.9786 | 1.9815 |
| $S B P 42$ | 3.5550 | 4.0678 | 4.6536 | 5.6233 |
| $S B P 63$ | 9.5206 | 6.7957 | 2.5920 | 7.4805 |

Here, we will compute both linear and non-linear functionals to see if superconvergence is obtained also in the nonlinear case.

The linear and non-linear functionals that we consider are

$$
J_{1}(U)=\int_{\Phi} u d \Phi \text { and } J_{2}(U)=\int_{\Phi} \frac{1}{2}\left(u^{2}+v^{2}\right) d \Phi
$$

respectively. Additionally, we calculate the integral of the divergence as

$$
J_{3}(U)=\int_{\Phi} u_{x}+v_{y} d \Phi
$$

The exact functionals are computed using (56) and the rates of convergence of the numerical functionals toward the exact ones (evaluated at the final time, $\mathrm{T}=1$ ) are calculated by using (57). The rates of convergence for SBP21, SBP42 and SBP63 are given in Tables 3-5.

As shown in Tables 3 and 4, superconvergence is achieved for both $J_{1}$ and $J_{2}$. Superconvergence is also achieved for $J_{3}$ when using $S B P 21$ and $S B P 42$, but not quite for $S B P 63$, as seen in Table 5. The reason for the erratic behaviors for SBP63 in the $J_{2}$ case could be lack of sufficient resolution.

Table 5: Convergence rates for $J_{3}$ at $\mathrm{T}=1$, for a sequence of mesh refinements, SBP84 in time with sufficiently small time steps is used

| $N, M$ | 21 | 31 | 41 | 51 |
| :---: | :---: | :---: | :---: | :---: |
| $S B P 21$ | 2.1527 | 2.5231 | 2.5231 | 2.4508 |
| $S B P 42$ | 9.5427 | 4.8729 | 4.9226 | 5.2762 |
| $S B P 63$ | 21.1193 | 1.4537 | 4.3645 | 2.7469 |

## 5. Summary and conclusions

A high order, fully discrete, stable and dual consistent approximation of the linearized constant coefficient incompressible Navier-Stokes on first order form was developed. The derivations for the continuous problem were done by reducing the second order system to first order form. Boundary conditions that simultaneously lead to boundedness of the primal and dual problems posed on time-dependent spatial domains were derived.

Stability and dual consistency were obtained by using summation-by-parts operators in combination with the simultaneous approximation term technique. Penalty formulations that adjust to the time variations of the spatial geometry such that stability and dual consistency follow automatically, were derived.

The order of accuracy of the solution, the divergence and linear and non-linear functionals were examined numerically. Design order of accuracy was obtained for the solution and the divergence. Both the linear and nonlinear functionals considered superconverged.

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