On equivalence of inverse and forward IV estimators with application to quadcopter modeling

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Abstract
This paper concerns the estimation of a dynamic model from two measured signals when it is not clear which signal should be used as input to the model. In this case, both a forward and an inverse model can be estimated. Here, a basic instrumental variable approach is used and it is shown that the forward and inverse model estimators give identical parameter estimates provided that corresponding model structures have been used. Furthermore, it is shown that this scenario occurs when properties of a quadcopter are estimated from accelerometer and gyro signals and, hence, that it does not matter which signal is used as input.

Keywords: system identification, instrumental variable, inverse model, quadcopter

1. INTRODUCTION

The standard framework for system identification includes the notion of an input signal that enters a well-defined system and produces an output signal. The system and thus also the output signal are usually affected by noise but the input signal is typically known exactly. Under these assumptions, it is most common to estimate a model from the input to the output of the system and most system identification methods are designed like this.

However, the standard system identification assumptions are not always satisfied. For example, the exact input signal might be unknown and replacing it with a noisy measurement leads to an errors-in-variables (EIV) problem (Söderström, 2007). Furthermore, the system might be more complex than in the standard framework such that there is still a dynamical relation between the available signals but no clear distinction between input and output. A typical example is a mechanical system where two sensors measure the movements at different places or in different directions but where the external force or torque that generates the movements is unknown. For example, this setup is common in vibration analysis (Maia et al., 2001; Devriendt and Guillaume, 2008) and the model estimation problem is sometimes called sensor-only blind system identification (D'Amato et al., 2009).

Similar examples can be found in electrical power grids, communication systems, process industry applications, and biochemical reactions. Many of these complex systems can be modeled as dynamic networks where the system identification problem is to estimate a particular part of the network, e.g., the dynamical subsystem that connects two nodes in the network (Chiuso and Pillonetto, 2012; Van den Hof et al., 2013; Dankers et al., 2015; Weerts et al., 2015; Linder and Enqvist, 2017b,a). Instrumental variable (IV) methods are commonly used in this setting since they provide a way to handle challenges concerning confounding variables and EIV settings, provided that instruments with particular correlation properties can be constructed.

The focus of this paper is on the use of IV methods in complex settings where two measured signals, u_t and y_t, are available but where it is not obvious which signal should be viewed as input. The performance of some system identification methods depends heavily on the input-output choice (Jung and Enqvist, 2013), but it will be shown here that the basic IV estimators with u_t or y_t as input give identical results provided that corresponding model structures are used in both cases. Furthermore, estimation of some properties of a quadcopter based on signals from an inertial measurement unit (IMU) will be discussed and some previous results will be generalized (Ho et al., 2017a,b). In this application, it is not obvious whether the lateral acceleration or the roll rate should be used as input, but the previous analysis shows that both choices are equivalent despite quite different levels of disturbances in the two signals.

The paper outline is as follows. In Sec. 2, the problem of estimating the forward and inverse models are formulated, and the residuals obtained with the model estimates are also derived. The IV method is described in Sec. 3. The equivalence of the forward and inverse IV model estimators is shown in Sec. 4 and verified using several simulation studies in Sec. 5. The experimental results for estimation of the center of gravity/center of rotation of a quadcopter are given in Sec. 6, and Sec. 7 concludes the paper.

2. PROBLEM FORMULATION

2.1 Forward model

The considered system is a single input-single output (SISO) system that has a noise-free scalar input u^o_t and a noise-free scalar output y^o_t. Noisy measurements u_t and y_t of the two signals are available and the relationships between all signals can be written

$$y_t^o = G^o(q)u_t^o, \tag{1}$$
$$u_t = u_t^o + T^o(q)v_t, \tag{2}$$
$$y_t = y_t^o + H^o(q)e_t, \tag{3}$$

where v_t and e_t are the zero mean white noises affecting the input and output, respectively. $G^o(q)$, $H^o(q)$ and $T^o(q)$ are rational functions in the time shift operator q ($qy_t = y_{t+1}$) according to

$$G^o(q) = \frac{B^o(q)}{A^o(q)} = \frac{b_0^o q^{-n_k} + \cdots + b_{n_b-1}^o q^{-n_b-n_k+1}}{1 + a_1^o q^{-1} + \cdots + a_{n_a}^o q^{-n_a}}, \tag{4}$$

$$H^o(q) = \frac{C^o(q)}{D^o(q)} = \frac{1 + c_1^o q^{-1} + \cdots + c_{n_c}^o q^{-n_c}}{1 + d_1^o q^{-1} + \cdots + d_{n_d}^o q^{-n_d}}, \tag{5}$$

$$T^o(q) = \frac{S^o(q)}{F^o(q)} = \frac{1 + s_1^o q^{-1} + \cdots + s_{n_s}^o q^{-n_s}}{1 + f_i^o q^{-1} + \cdots + f_{n_f}^o q^{-n_f}}. \tag{6}$$

The transfer functions of the noise models $H^o(q)$ and $T^o(q)$ are assumed to be stable and coprime.

The aim of this work is to derive estimates of the model $G^o(q)$ and its inverse form $(G^o)^{-1}(q)$. We rewrite the forward model in a regression form as

$$y_t = \varphi_t^T \theta_F + w_t, \tag{7}$$

where

$$\begin{aligned} \varphi_t^T &= [u_{t-n_k}, \ldots, u_{t-n_b-n_k+1}, -y_{t-n_a}, \ldots, -y_{t-1}], \\ &= [u_{t-n_k}, \tilde{\varphi}_t^T], \\ \theta_F^T &= [b_0^o, \ldots, b_{n_b-1}^o, a_{n_a}^o, \ldots, a_1^o], \end{aligned}$$

The residual w_t of the forward system is generated as

$$w_t = y_t - \varphi_t^T \theta_F, \tag{8a}$$
$$= A^o(q)\big[-G^o(q)T^o(q)v_t + H^o(q)e_t\big], \tag{8b}$$
$$= -B^o(q)T^o(q)v_t + A^o(q)H^o(q)e_t, \tag{8c}$$

which implies that w_t could be modeled as $w_t = L^o(q)^{-1}\eta_{F,t}$ where $L^o(q)^{-1}$ is stable function and the noise $\eta_{F,t}$ is white. Note that the transfer function $G^o(q)$ could be unstable and non-minimum phase.

2.2 Inverse model

We now consider the problem of rewriting the inverse model $G^o(q)^{-1}$ on regression form. First, we can write the inverse model relation as

$$\begin{aligned} u_t &= G^o(q)^{-1}y_t + G^o(q)^{-1}\big[G^o(q)T^o(q)v_t - H^o(q)e_t\big], \\ &= G^o(q)^{-1}y_t + T^o(q)v_t - G^o(q)^{-1}H^o(q)e_t. \end{aligned}$$

Since the forward model $G^o(q)$ is rational proper, the inverse model will be non-causal. Under the assumption that $b_0^o \neq 0$, this inverse model can be written on regression form as

$$u_{t-n_k} = \psi_t^T \gamma_I + \varepsilon_t, \tag{9}$$

where

$$\begin{aligned} \psi_t^T &= [-u_{t-n_k-1}, \ldots, -u_{t-n_k-n_b+1}, y_{t-n_a}, \ldots, y_t], \\ &= [-\tilde{\varphi}_t^T, y_t], \\ \gamma_I &= [\frac{b_1^o}{b_0^o}, \ldots, \frac{b_{n_b-1}^o}{b_0^o}, \frac{a_{n_a}^o}{b_0^o}, \ldots, \frac{a_1^o}{b_0^o}, \frac{1}{b_0^o}]^T, \end{aligned}$$

and the residual ε_t is generated as

$$\varepsilon_t = u_{t-n_k} - \psi_t^T \gamma_I, \tag{10a}$$
$$= \frac{1}{b_0^o}B^o(q)\big[T^o(q)v_t - G^o(q)^{-1}H^o(q)e_t\big], \tag{10b}$$
$$= B^o(q)T^o(q)\frac{v_t}{b_0^o} - A^o(q)H^o(q)\frac{e_t}{b_0^o}. \tag{10c}$$

2.3 Estimation methods

Many estimation methods have been developed for system identification of linear dynamic systems with noise-corrupted input and output measurements (see, for example, Söderström, 2007). One approach is to use an instrument signal that is correlated with the regressors and uncorrelated with the noise signals. Such an IV method can provide consistent estimates of the parameters for linear dynamic systems in EIV and closed-loop settings (Ljung, 1999).

3. INSTRUMENTAL VARIABLE METHOD

In the EIV framework, it is well known that the conventional least-squares (LS) method might not be able to provide a consistent estimator (Ljung, 1999). To overcome this limitation, one alternative is to use the IV method, which is a correlation-based method. This method is based on the use of an instrument vector ζ_t to extract the interesting information from the input u_t and output y_t.

One important aspect of the IV method is to choose a pre-filter $L(q)$, which has a considerable effect on the covariance of the estimated parameters (Ljung, 1999). If the true noise model is assumed to be known, the estimated covariance of the parameters can be minimized by using the inverse of the noise model as the prefilter $L(q)$, which means that the filtered input $\bar{u}_t = L(q)u_t$, the filtered output $\bar{y}_t = L(q)y_t$ and the filtered instrument $\bar{\zeta}_t = L(q)\zeta_t$ are used in the estimator. In practice, an approximation of the true noise model is often used to define the prefilter.

With $p = \max\{n_a, n_b + n_k - 1\}$, a dataset with measurements from $t = 1$ to N can be used to construct the matrices

$$\begin{aligned} \bar{\tilde{\Phi}}_N &= \left[\bar{\tilde{\varphi}}_{p+1} \;\; \bar{\tilde{\varphi}}_{p+2} \;\; \ldots \;\; \bar{\tilde{\varphi}}_N\right]^T, \\ \bar{Z}_N &= \left[\bar{\zeta}_{p+1} \;\; \bar{\zeta}_{p+2} \;\; \ldots \;\; \bar{\zeta}_N\right]^T, \\ \bar{Y}_N &= \left[\bar{y}_{p+1} \;\; \bar{y}_{p+2} \;\; \ldots \;\; \bar{y}_N\right]^T, \\ \bar{U}_N &= \left[\bar{u}_{p-n_k+1} \;\; \bar{u}_{p-n_k+2} \;\; \ldots \;\; \bar{u}_{N-n_k}\right]^T. \end{aligned} \tag{11}$$

Using a basic IV method, an estimate of the parameter vector θ_F of the forward model can be obtained by solving

$$\hat{\theta}_F = \operatorname*{sol}_{\theta_F}\left[\frac{1}{N}\bar{Z}_N^T\bar{\Phi}_{F,N}\theta_F - \frac{1}{N}\bar{Z}_N^T\bar{Y}_N = 0\right], \tag{12}$$

where $\bar{\Phi}_{F,N} = [\bar{U}_N, \bar{\tilde{\Phi}}_N]$.

Similarly, an estimate of the parameter vector γ_I of the inverse model can be obtained by solving

$$\hat{\gamma}_I = \operatorname*{sol}_{\gamma_I}\left[\frac{1}{N}\bar{Z}_N^T\bar{\Phi}_{I,N}\gamma_I - \frac{1}{N}\bar{Z}_N^T\bar{U}_N = 0\right], \tag{13}$$

where $\bar{\Phi}_{I,N} = [-\bar{\tilde{\Phi}}_N, \bar{Y}_N]$. Let $\hat{\tilde{\gamma}}_I$ be the vector consisting of the first $n_a + n_b - 1$ elements of $\hat{\gamma}_I$ and let $\hat{\gamma}_{I,n_a+n_b}$ be the last element. If $\hat{\gamma}_{I,n_a+n_b} \neq 0$, an estimate $\hat{\theta}_I$ of the forward model parameters can be obtained from $\hat{\gamma}_I$ as

$$\hat{\theta}_I = \frac{1}{\hat{\gamma}_{I,n_a+n_b}}\left[1 \;\; \hat{\tilde{\gamma}}_I\right]^T \tag{14}$$

If the true input and output would have been known, one choice of filtered instrument $\bar{\zeta}_t$ could have been

$$\bar{\zeta}_t = \left[\bar{u}^o_{t-n_k}, \dots, \bar{u}^o_{t-n_b-n_k+1}, -\bar{y}^o_{t-n_a}, \dots, -\bar{y}^o_{t-1}\right]^T, \qquad (15)$$

in which $\bar{u}^o_t$ and $\bar{y}^o_t$ are the filtered noise-free input and output, respectively. However, this is of course not the case in practice and approximations of the instruments in (15) are often used instead. Such approximations can sometimes be created using a known external signal, such as the reference signal in a closed-loop system.

For the basic IV method, the estimated covariance matrix $\hat{P}_{\theta_F}$ for $\hat{\theta}_F$ is given by

$$\hat{P}_{\theta_F} = \hat{\sigma}^2 \left[\bar{Z}_N^T \bar{\Phi}_{F,N}\right]^{-1} \left[\bar{Z}_N^T \bar{Z}_N\right] \left[\bar{Z}_N^T \bar{\Phi}_{F,N}\right]^{-T}, \qquad (16)$$

where $\hat{\sigma}^2$ is the estimated variance of the model residual.

One way to improve the performance of the IV method when the true noise model and the instruments in (15) are unknown is to use an iterative approach in which a sequence of refined IV estimates are computed. One particular example of such a refined IV approach is outlined in the following algorithm:

- Compute the first estimate with the prefilter $\hat{L}(q) = 1$ and estimates of y^o_t and u^o_t as instruments. These estimates can be obtained using a known external signal r_t and estimated black-box models from r_t to y_t and u_t.
- For $k = 1, 2, \dots$
  + Compute the residual $w_t = y_t - \varphi^T_{F,t} \hat{\theta}^{(k)}_F$.
  + Use the IVARMA method to estimate a noise model $w_t = \hat{L}(q)^{-1} \eta_{F,t}$ where $\eta_{F,t}$ is assumed to be white (Young, 2015).
  + Simulate noise-free signals $\hat{y}^o_t$ and $\hat{u}^o_t$ and use the prefilter $\hat{L}(q)$ to filter y_t, u_t, $\hat{y}^o_t$ and $\hat{u}^o_t$.
  + Generate the matrices in (11) with the instruments in (15) using these filtered signals and estimate the parameter vector $\hat{\theta}^{(k+1)}_F$.
- Repeat until $\hat{\theta}^{(k+1)}_F$ appears to have converged according to a stop criterion $\|\hat{\theta}^{(k+1)}_F - \hat{\theta}^{(k)}_F\|_2 / \|\hat{\theta}^{(k)}_F\|_2 < \delta$ or if $k > k_{\max}$.

4. ESTIMATION OF FORWARD AND INVERSE MODELS

Since the available signals u_t and y_t could have very different signal-to-noise ratios, it seems relevant to consider whether the signal qualities should affect the choice of a forward or an inverse model estimator. However, it turns out that the basic IV estimators are equivalent.

Lemma 1. Assume that the collected dataset with N input and output measurements and the chosen instrument vector $\bar{\zeta}_t \in \mathbb{R}^{n_a+n_b}$ are such that the forward and inverse IV estimates in (12) and (13) are unique and that $\hat{b}_0 = \hat{\theta}_{F,1} \neq 0$ and $\hat{\gamma}_{I,n_a+n_b} \neq 0$. Then, it holds that

$$\hat{\theta}_F = \hat{\theta}_I. \qquad (17)$$

Proof. From (12), the estimate of θ_F of the forward model is given as

$$\hat{\theta}_F = \underset{\theta_F}{\text{sol}} \left[\frac{1}{N} \bar{Z}_N^T \bar{\Phi}_{F,N} \theta_F - \frac{1}{N} \bar{Z}_N^T \bar{Y}_N = 0\right], \qquad (18)$$

where $\bar{\Phi}_{F,N} = [\bar{U}_N, \widetilde{\bar{\Phi}}_N]$, which implies

$$\begin{bmatrix} \hat{\theta}_F \\ 1 \end{bmatrix} \in \text{Null}\{\bar{Z}_N^T [-\bar{\Phi}_{F,N}, \bar{Y}_N]\}. \qquad (19)$$

Dividing this vector with $\hat{b}_0$ gives

$$\begin{bmatrix} \dfrac{\hat{\theta}_F}{\hat{b}_0} \\ \dfrac{1}{\hat{b}_0} \end{bmatrix} \in \text{Null}\{\bar{Z}_N^T \underbrace{[-\bar{\Phi}_{F,N}, \bar{Y}_N]}_{[-\bar{U}_N, \bar{\Phi}_{I,N}]}\}. \qquad (20)$$

Defining $\frac{\hat{\theta}_F}{\hat{b}_0} = [1, \frac{\hat{b}_1}{\hat{b}_0}, \dots, \frac{\hat{b}_{n_b-1}}{\hat{b}_0}, \frac{\hat{a}_{n_a}}{\hat{b}_0}, \dots, \frac{\hat{a}_1}{\hat{b}_0}]^T = [1, \frac{\hat{\tilde{\theta}}^T_F}{\hat{b}_0}]^T$, the previous expression can be rewritten as

$$\begin{bmatrix} 1 \\ \dfrac{\hat{\tilde{\theta}}_F}{\hat{b}_0} \\ \dfrac{1}{\hat{b}_0} \end{bmatrix} \in \text{Null}\{\bar{Z}_N^T [-\bar{U}_N, \bar{\Phi}_{I,N}]\}, \qquad (21)$$

which implies that $\hat{\gamma}_F = [\frac{\hat{\tilde{\theta}}^T_F}{\hat{b}_0}, \frac{1}{\hat{b}_0}]^T$ is also the solution to the inverse IV problem. Since the solution to the inverse IV problem is unique, this implies that $\hat{\gamma}_I = \hat{\gamma}_F$. Hence, the result follows.

A straightforward consequence of this result is that the model residuals also are equal except for a scale factor.

Corollary 2. Under the same assumptions as in Lemma 1, it holds that

$$\hat{w}_t = -\frac{1}{\hat{b}_0} \hat{\varepsilon}_t, \qquad (22)$$

where $\hat{w}_t$ and $\hat{\varepsilon}_t$ are the forward and inverse model residuals, respectively.

Proof. The residual of the forward model is

$$\begin{aligned} \hat{w}_t &= y_t - \varphi^T_t \hat{\theta}_F = y_t - [u_{t-n_k}, \tilde{\varphi}^T_t] \hat{\theta}_F \\ &= y_t - [u_{t-n_k}, \tilde{\varphi}^T_t] \begin{bmatrix} \hat{b}_0 \\ \hat{\tilde{\theta}}_F \end{bmatrix}. \end{aligned}$$

On the other hand, the residual of the inverse model is

$$\begin{aligned} \hat{\varepsilon}_t &= u_{t-n_k} - \psi^T_t \hat{\gamma}_I = u_{t-n_k} - [-\tilde{\varphi}^T_t, y_t] \hat{\gamma}_I \\ &= u_{t-n_k} - [-\tilde{\varphi}^T_t, y_t] \begin{bmatrix} \dfrac{\hat{\tilde{\theta}}_F}{\hat{b}_0} \\ \dfrac{1}{\hat{b}_0} \end{bmatrix} = -\frac{1}{\hat{b}_0} y_t + u_{t-n_k} + \tilde{\varphi}^T_t \frac{\hat{\tilde{\theta}}_F}{\hat{b}_0} \\ &= -\frac{1}{\hat{b}_0} \hat{w}_t. \end{aligned}$$

Remark:

- The residuals of the forward and inverse models could be modeled as

$$\begin{aligned} w_t &= -B^o(q) T^o(q) v_t + A^o(q) H^o(q) e_t = \left(L^o(q)\right)^{-1} \eta_{F,t}, \\ \varepsilon_t &= B^o(q) T^o(q) \frac{v_t}{b^o_1} - A^o(q) H^o(q) \frac{e_t}{b^o_1} = \left(L^o(q)\right)^{-1} \eta_{I,t}, \end{aligned}$$

  where $\eta_{F,t}$ and $\eta_{I,t}$ are the driving white noises of the forward and inverse noise models, respectively. The difference between w_t and ε_t is a factor $-\frac{1}{b^o_0}$ which affects the variance of the noises $\eta_{F,t}$ and $\eta_{I,t}$. If an ARMA model estimator is used, the same estimated noise model $\hat{L}(q)^{-1}$ is obtained in the forward and inverse approaches. Hence, $\hat{L}(q)$ can be used to filter the residuals to achieve estimates of the driving white noises $\eta_{F,t}$ and $\eta_{I,t}$. The variances of these signals are related as $\text{Var}[\eta_{I,t}] = \left(\frac{1}{b^o_0}\right)^2 \text{Var}[\eta_{F,t}]$.

Table 1. The estimates of the parameters of the forward and inverse models obtained from the first simulation study when the basic IV method is used.

Parameter	Forward model	Inverse model
$a_1^o = -0.995$	-0.9943 ± 0.0036	-0.9943 ± 0.0036
$b_1^o = 0.01$	0.0100 ± 0.0006	0.0100 ± 0.0006

Table 2. The differences of the parameter estimates obtained from the first simulation study: $\hat{\theta}_F - \hat{\theta}_I$ and $\mathrm{std}(\theta_F) - \mathrm{std}(\theta_I)$

Parameter	Mean estimate	Standard deviation
$a_1^o = -0.995$	0.6178×10^{-7}	0.2504×10^{-7}
$b_1^o = 0.01$	-0.1812×10^{-7}	0.0217×10^{-7}

- Since $\hat{\theta}_F = \hat{\theta}_I$ for the basic IV method, it follows that these estimators also must have equal covariance matrices. Hence, the estimate of the covariance matrix for the forward IV approach can be used also for $\hat{\theta}_I$. Note that the estimated covariance matrix $\hat{P}_{\gamma_I}$ of $\hat{\gamma}_I$ could also be derived similarly compared to $\hat{P}_{\theta_F}$ in (16).
- The estimated parameter vector $\hat{\theta}_F$ using (12) is unique if the system is persistently exciting (the matrix $\bar{Z}_N^T \bar{\Phi}_{F,N}$ is invertible) with the instrument as an approximation of (15). Therefore, this instrument vector can also be used to obtain $\hat{\gamma}_I$ of the inverse model.

5. SIMULATION STUDY

In this section, we present the results of two Monte Carlo simulations where the estimation of forward and inverse models using the basic IV method with optimal instruments has been investigated.

5.1 Simulation 1

In the first simulation, the data is generated using

$$G^o(q) = \frac{0.01q^{-1}}{1 - 0.995q^{-1}},$$
$$H^o(q) = \frac{1 + 0.9q^{-1}}{1 - 0.9q^{-1}}.$$

and $T^o(q) = 1$. The Monte Carlo simulation is performed with 100 runs to create different realizations with data length 5000 of the noises $v \sim \mathcal{N}(0, 0.1^2)$ and $e \sim \mathcal{N}(0, 0.1^2)$. The input is generated as

$$u_t^o = \frac{1}{1 - 1.95q^{-1} + 0.975q^{-2}} \delta_t, \tag{23}$$

where $\delta_t \sim \mathcal{N}(0, 1.0^2)$ can be considered a known external signal.

The results are shown in Table 1 and Table 2 for the mean and standard deviation estimates of the parameters and the differences between these values, respectively. From Table 1, we can observe that the estimates of θ_F and θ_I are accurate and identical. Their differences shown in Table 2 are insignificant and probably caused by numerical errors.

5.2 Simulation 2

In order to verify our findings, we perform a second simulation with transfer functions

Table 3. The estimates of the parameters of the forward and inverse models obtained from the second simulation study when the basic IV method is used.

Parameter	Forward model	Inverse model
$a_1^o = -1.5$	-1.4995 ± 0.0058	-1.4995 ± 0.0058
$a_2^o = 0.8$	0.7999 ± 0.0058	0.7999 ± 0.0058
$b_1^o = 0.5$	0.5011 ± 0.0073	0.5011 ± 0.0073
$b_2^o = 0.4$	0.4002 ± 0.0085	0.4002 ± 0.0085

Table 4. The differences of the parameter estimates obtained from the second simulation study: $\hat{\theta}_F - \hat{\theta}_I$ and $\mathrm{std}(\theta_F) - \mathrm{std}(\theta_I)$

Parameter	Mean estimate	Standard deviation
$a_1^o = -1.5$	-0.0953×10^{-6}	0.4218×10^{-7}
$a_2^o = 0.8$	0.0889×10^{-6}	0.1319×10^{-7}
$b_1^o = 0.5$	-0.0092×10^{-6}	0.6001×10^{-7}
$b_2^o = 0.4$	-0.1036×10^{-6}	0.3552×10^{-7}

$$G^o(q) = \frac{0.5q^{-1} + 0.4q^{-2}}{1 - 1.5q^{-1} + 0.8q^{-2}},$$
$$H^o(q) = \frac{1 + 0.9q^{-1}}{1 - 0.9q^{-1}}$$

and $T^o(q) = 1$. 100 runs of Monte Carlo simulations are conducted with different realizations of $v \sim \mathcal{N}(0, 0.5^2)$ and $e \sim \mathcal{N}(0, 0.5^2)$. The input is generated as

$$u_t^o = \frac{1}{1 - 1.95q^{-1} + 0.975q^{-2}} \delta_t, \tag{24}$$

where $\delta_t \sim \mathcal{N}(0, 1.0^2)$. For each Monte Carlo run, 5000 samples are created.

The estimates of the parameters of the forward and inverse models are shown in Table 3 while Table 4 shows the differences between these estimates. Again, it can be seen that the mismatch between the two estimates of θ_F and θ_I is quite small.

6. ESTIMATING THE CENTER OF GRAVITY OF A QUADCOPTER

In this section, the problem of estimating the center of gravity (CoG) of a quadcopter is considered. Ideally, the CoG of a quadcopter is designed to coincide with the intersection of its frame arms and the onboard IMU is supposed to be placed close to the CoG. Therefore, during any aggressive maneuver of the quadcopter, the second derivative of the Euler angles will not contribute significantly to the IMU measurements, which is beneficial since such contributions could cause side effects in inertial-based navigation methods.

However, due to the maneuverability and capability of quadcopters, they are used in variety of applications, e.g., carrying various payloads such as cameras or other items. In these applications, the CoG is shifted into a new position and it can be useful to be able to estimate this shift.

We consider a quadcopter as in Fig. 1. The position of the quadcopter in the inertial frame is defined as $\boldsymbol{\xi} = [x \quad y \quad z]^T$. The roll, pitch and yaw angles ϕ, θ and ψ denote the orientation of the quadcopter. These Euler angles are collected in $\boldsymbol{\eta} = [\phi \quad \theta \quad \psi]^T$.

The distance from the IMU to the shifted CoG is d, which is the parameter to be estimated. The translational model of the

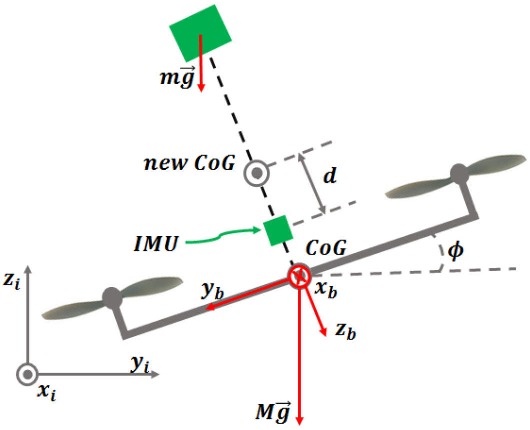

Figure 1. The inertial and the body coordinate frames of the quadcopter. The quadcopter is carrying a payload, which is modeled as a point mass.

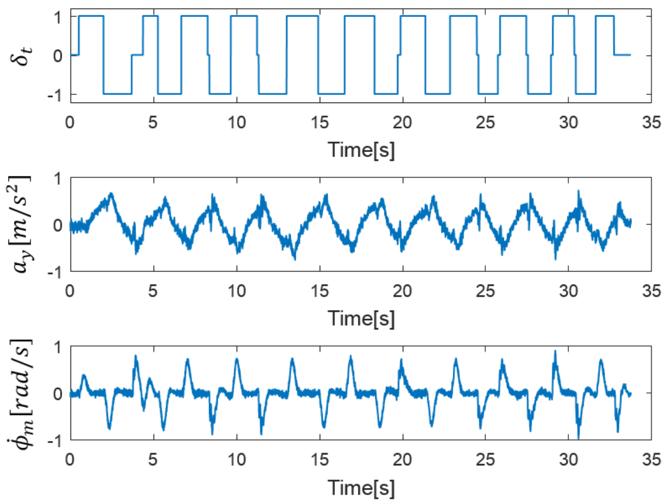

Figure 2. The experimented data. The reference signal $\delta(t)$ is found in the top sub-figure, the lateral acceleration $a_{y,s}$ in the middle one and the bottom sub-figure shows the roll rate $\dot{\phi}_s$.

quadcopter is projected to the x-y plane which gives

$$\dot{v} = g\sin\phi - \frac{\lambda}{\tilde{m}}v, \tag{25}$$

where v is the velocity in the y direction of the body frame, $\tilde{m} = M + m$ with M as the mass of the quadcopter and m as the mass of the load.

This model contains a drag force that is linearly dependent on the velocity of the quadcopter, and which has been used earlier in Ho et al. (2017a) for mass estimation purposes. The measurements from the IMU are

$$\begin{aligned} \dot{\phi}_m &= \dot{\phi} + e_{\dot{\phi}}, \\ a_y &= g\sin\phi - \dot{v} + d\ddot{\phi} + e_{a_y}. \end{aligned} \tag{26}$$

The first term in the acceleration measurement is due to the gravity, the second term is the contribution from the lateral acceleration and the third term is the angular acceleration around the x_b axis.

The lateral model (25) can be linearized under a small angle assumption ($\sin(\phi) \approx \phi$ and $\cos(\theta) \approx 1$) which gives

Table 5. The estimated $\hat{\theta}_F$ of the forward model and $\hat{\theta}_I$ of the inverse model with their standard deviation using (16).

Par	Forward ($\hat{\theta}_F$)	Inverse ($\hat{\theta}_I$)	std	Inverse ($\hat{\gamma}_I$)
α_1	0.1103	0.1104	0.1696×10^{-3}	$\widehat{\frac{1}{\alpha_1}} = 9.0546$
α_2	-0.2204	-0.2206	0.3349×10^{-3}	$\widehat{\frac{\alpha_2}{\alpha_1}} = -1.9975$
α_3	0.1101	0.1102	0.1657×10^{-3}	$\widehat{\frac{\alpha_3}{\alpha_1}} = 0.9975$
β_1	-2.9917	-2.9917	0.1739×10^{-3}	$\widehat{\frac{\beta_1}{\alpha_1}} = -27.0882$
β_2	2.9834	2.9834	0.3426×10^{-3}	$\widehat{\frac{\beta_2}{\alpha_1}} = 27.0132$
β_3	-0.9917	-0.9917	0.1691×10^{-3}	$\widehat{\frac{\beta_3}{\alpha_1}} = -8.9794$

$$\dot{v} = g\phi - \frac{\lambda_1}{\tilde{m}}v + \bar{\tau}, \tag{27}$$

where $\bar{\tau}$ represents both process noise and unmodeled dynamics.

Combining (26) and (27) yields a model

$$\dot{\phi}_m - e_{\dot{\phi}} = \frac{p^2 + \frac{\lambda}{\tilde{m}}p}{dp^3 + d\frac{\lambda}{\tilde{m}}p^2 + g\frac{\lambda}{\tilde{m}}}(a_y - e_{a_y}) + \bar{\tau}, \tag{28}$$

from the lateral acceleration measurement to the measured roll rate, where p is the differential operator. This model can also be written as

$$\dot{\phi}_m = \frac{p^2 + \frac{\lambda}{\tilde{m}}p}{dp^3 + d\frac{\lambda}{\tilde{m}}p^2 + g\frac{\lambda}{\tilde{m}}}a_y + e_F = G_F(p)a_y + e_F. \tag{29}$$

Moreover, it should be noted that the total noise term e_F typically is colored and that it might thus be beneficial to include a noise model or a prefilter in the estimator.

In fact, the measurements are taken in the discrete time domain and they need to be related to the model. Here, the transfer function $G_F(p)$ is discretized using $p = \frac{q-1}{T}$ which gives

$$G_T(q) = \frac{\alpha_1 q^{-1} + \alpha_2 q^{-2} + \alpha_3 q^{-3}}{1 + \beta_1 q^{-1} + \beta_2 q^{-2} + \beta_3 q^{-3}}, \tag{30}$$

where

$$\alpha_1 = \frac{T}{d}, \quad \alpha_2 = (-2 + T\frac{\lambda}{\tilde{m}})\frac{T}{d}, \quad \alpha_3 = (1 - T\frac{\lambda}{\tilde{m}})\frac{T}{d},$$

$$\beta_1 = -3 + T\frac{\lambda}{\tilde{m}}, \quad \beta_2 = 3 - 2T\frac{\lambda}{\tilde{m}}, \quad \beta_3 = -1 + T\frac{\lambda}{\tilde{m}} + g\frac{\lambda T^3}{\tilde{m}d}.$$

The corresponding inverse model has the roll rate measurement as input and lateral acceleration measurement as output and can be written

$$a_y = \frac{\frac{1}{\alpha_1} + \frac{\beta_1}{\alpha_1}q^{-1} + \frac{\beta_2}{\alpha_1}q^{-2} + \frac{\beta_3}{\alpha_1}q^{-3}}{1 + \frac{\alpha_2}{\alpha_1}q^{-1} + \frac{\alpha_3}{\alpha_1}q^{-2}} q\dot{\phi}_m + e_I. \tag{31}$$

Fig. 2 shows the signals collected from an experiment with an AR Drone quadcopter. The reference signal δ_t is created in such a way that the AR Drone moves left and right with a constant altitude and zero pitch angle. The measured signals have been collected with a sampling time $T = 0.005$ s

Table 5 shows the estimate of θ_F and θ_I with their standard deviations. As expected, the estimates of θ_F and θ_I are similar. Furthermore, an estimate of d can be obtained using the estimate of $\alpha_{1,F}$ or $\alpha_{1,I}$. With the estimate $\hat{\alpha}_{1,F}$ and its variance $\hat{P}_{\alpha_{1,F}}$, the estimate of d is $\hat{d}_F = \frac{T}{\hat{\alpha}_{1,F}}$ and its variance is computed as $\hat{P}_{d,F} = \frac{\hat{d}_F^4}{T^2}\hat{P}_{\alpha_{1,F}}$. A similar estimate can also be obtained from

Table 6. The estimates of the CoG d with standard deviations obtained from the forward model and the inverse model, respectively.

Forward model ($\hat{d}_F$)	Inverse model ($\hat{d}_I$)
4.5312 ± 0.00696 cm	4.5273 ± 0.00695 cm

$\hat{\alpha}_{1,I}$ and $\hat{P}_{\alpha_{1,I}}$ as $\hat{d}_I = \frac{T}{\hat{\alpha}_{1,I}}$ and $\hat{P}_{d,I} = \frac{\hat{d}_I^4}{T^2}\hat{P}_{\alpha_{1,I}}$. These estimates are shown in Table 6, and as can be seen there, the estimates of d obtained from the forward and inverse models are similar.

Finally, if the position and mass of the load are known, the CoG of the unloaded quadcopter can also be computed. This estimate will not depend on the choice of a forward or inverse model either.

7. CONCLUSION

In this work, we have considered the problem of estimating forward and inverse models of a system using an IV approach. The main observation is that these estimates are equal, except for small numerical errors, also for finite data. This result has been validated both in simulations and using real data from a quadcopter.

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