An Evaluation of Swedish Municipal Borrowing via Nikkei-linked Loans

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Abstract

In this master thesis, we compare three different types of funding alternatives from a Swedish municipality’s point of view, with the main focus on analysing a Nikkei-linked loan. We do this by analysing the resulting interest rate and the expected exposures, taking collateral into consideration.

We conclude, with certainty, that there are many alternatives for funding and that they each need to be analysed and compared on many levels to be able to make a correct decision as to which ones to choose. An important part of this is to consider the implications of the newest regulations and risk exposure, as it might greatly influence the final price for contracts.

Between the cases that we considered, the SEK bond was the one with the lowest resulting spread, and the one which is the simplest considering the collateral involved. While other alternatives might be better depending on how profitable it is for the municipality to receive collateral, the SEK bond is the most transparent one and with least risk involved.
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Chapter 1

Introduction

When a municipality is looking for funding there are many options. One option, and perhaps the simplest, is to issue bonds. An ordinary, often called vanilla, bond pays either a fixed interest rate or a floating one, has regular coupon dates, dates when the interest is paid, and at the maturity date the principal is paid back. A schematic of this trade can be seen in figure 1.1 from a Swedish municipality’s point of view. It receives Swedish kronor (SEK) while paying an agreed upon interest rate, the SEK principal is of course paid back at the end of the contract although this is not showed in the figure.

An option which can yield a lower interest rate is to issue bonds in another currency and then use a cross-currency swap (CCS) to get the money and pay the interest in SEK, as Fujii et al. (2010) describe. A schematic is shown in figure 1.2 where Swedish municipalities issue bonds in Japan and pay the Stockholm inter-bank offer rate and some spread, \( y \), as interest. Since the CCS has some inherent costs (basis spread, \( x \) and cost for issuing/dealing with collateral) it is not certain that a lower

\[ \text{Figure 1.1: Example of how a municipality can use a vanilla bond to raise funds.} \]

\[ \text{Figure 1.2: Schematic of a cross-currency swap (CCS) to lower interest rates.} \]
interest in the foreign currency loan implies a lower total loan cost (Ang and Green 2013).

![Diagram](image)

*Figure 1.2: Example of how a municipality can use a bond to raise funds. X and y represents the spreads.*

A third option which will be studied is a more complicated one. Due to low interest rates during the past two decades many Japanese investors have turned to structured products for a potentially higher return. For example, the 10-year fixed rate is around 0.3%, and this creates a demand for financial products which pay a higher interest rate (without full exposure to the risk associated with the stock market). One such structure is a Nikkei-linked structured bond, that pays cash flows linked to the level of the Nikkei225 index (Stowell 2010).

This method can be used by municipalities to achieve a low interest rate because of their high credit rating, and with some conditions. Since the interest rate is linked to the Nikkei index, entering this structured bond means exposure to market risk. To hedge against this risk the structured bond is generally paired with a structured swap which exchanges the Nikkei-linked interest rate for e.g. a fixed rate or 3-month London inter-bank offer rate (Libor). These contracts are provided by a counterparty which brokers the deal, e.g. JP Morgan. Their incentive to broker this deal is that, while they receive the principal from the municipality at the end of the contract, the expected value they pay to the Japanese investors is less because of the knock-in feature explained in section 2.3.1 and thus they earn this spread. This and the municipalities’ high credit rating makes the spread in figure 1.3 where this trade is exemplified, be attractive, often negative. Thus this kind of structured product is appealing to all the involved parties.
When entering an over-the-counter (OTC) contract one has to take into consideration the counterparty’s credit risk (CCR), i.e. the risk that the counterparty enters default and is unable to sustain their part of the agreement. To counteract this risk in the contract there is a credit support annex (CSA). This states for example that, in case of default, the remaining cash flows will be netted. CSA also indicates that derivative exposures against a counterparty should be covered by collateral (credit support) which covers the exposure (approximately) and has a cost related to it. A core component of CCR under Basel III is credit valuation adjustment (CVA). It is computed on OTC derivatives and securities financing transactions. What it does is reflecting the market value of the cost of the credit spread’s volatility. (O’Kane 2016)

CVA has always existed in the financial world, but due to the financial crisis of 2008 the regulations in Basel III regarding CVA were strengthened because many financial institutions ignored it since they considered them selves "too big to fail". Within Basel III, CVA was designed with the intent to capture and measure losses on securities financing transactions and OTC derivatives due to the volatility in credit spread. Hence the recent crisis showed that the risk CVA should capture did not work accurately, as it should, within Basel II. This led to the Basel committee introducing a new capital charge in order to mitigate CVA losses. (Rosen and Saunders 2012)

There are about four different types of CCSs, of which two are considered primary types, since they are the most common: Floating-for-floating, or cross currency basis swap, has floating interest rates in both the pay and receive leg and this type of CCS is often used with major currency pairs (e.g. USD/JPY). Fixed-for-floating
CCS has a leg with fixed interest rate and one with floating interest rate, most often used with minor currencies against USD. (Flavell [2010])

It is worth noting that the more common interest rate swaps are not OTC but cleared by a clearing house. When they are cleared, the parties must have an initial margin (IM) which is 10 day value-at-risk for the portfolio. The IM is kept at a central bank or credit institution (which must be a third party, and it is subject to certain regulations as is explained in/by (Final Draft Regulatory Technical Standards)). Then the CVA is considered to be zero (applies for all cleared derivatives). Since a few years back, IM is gradually being required for OTC contracts also, as O’Kane (2016) explains. If IM (or other kind of securities) are implemented, they decrease the CVA, but add to the funding cost or funding value adjustment (FVA) since one needs to deal with the securities (Green [2016]).

Thus we can see that finding the actual cost for a loan requires a closer look than just considering the interest rate.

1.1 Purpose

The purpose of this thesis is to compare the cases and find which one is the most advantageous:

1. Loan (bond) in SEK.
2. Loan (bond) in a different foreign currency with CCS to SEK.
3. Nikkei-linked loan in JPY with a structured swap to USD and then CCS to SEK.

1.2 Delimitations

We will make the following delimitations in our thesis:
• We will use a one-factor model in our simulation, making the assumption that the interest rate is deterministic.

• We will assume that we only have one counterparty, so we will mainly focus on the expected exposure and not the probability of default, in this study.
Chapter 2

Methodology

It is our purpose to compare different means of raising funds which are somewhat different in their structure. First we describe how to value loan according to case 1 and 2, then go into the details of valuation for the third, more complicated case.

2.1 Valuation of a loan in a foreign currency

The loan will be in the form of a fixed rate bond and the bond will have a maturity date and regular coupons. This will then be paired with a fixed-for-floating CCS so that we receive the fixed leg in USD and pay a floating 3-month STIBOR rate. Since the CCS is an OTC contract, it includes a CSA. (Fares and Genest [2013])

2.2 Valuation of a Cross-Currency Swap

A swap is bilateral exchange with two legs, a receive and a pay one. Then the value of the swap is simply:

\[ V_{\text{swap}} = B_{\text{receive}} - B_{\text{pay}} \]  

(2.1)

that is, the difference between the two legs. Since the swap we are interested in is a CCS and one of the legs is in a foreign currency, (2.1) becomes:

\[ V_{\text{swap}} = B_{\text{domestic}} - S_0 B_{\text{foreign}} \]  

(2.2)
where $S_0$ is the spot exchange rate. Here we can observe that the pay and receive
legs are inverted for the other party. Each leg consists of a number of future cash
flows. Therefore the value of a leg is the present value of all the cash flows, which
is obtained by discounting the nominal value of each future cash flow:

$$B = \sum_{i=1}^{M} c_i D_i$$

(2.3)

where $M$ is the number of cash flows $c_i$ and $D_i$ is the discount factor used for each
cash flow $c_i$. The cash flows are known or agreed upon the start of the contract,
either fixed or floating (depending on e.g. Libor). Next we will discuss how to
choose the discount factor. (Henrard 2014; Hull 2017; Madura 2015)

2.2.1 Choosing discount rate

A widely used discount rate is the Libor-swap curve. A problem with simply using
this rate is that there is an observable basis spread in the CCS market. Also, using
the Libor rate directly doesn’t take into account the collateral agreements (CSA)
which have become standard in the market. To take these effects into account, Fujii,
Shimada, Takahashi et al. (2010) propose a method for constructing a yield curve
that is consistent with the market. Given that this method requires a recursive cre-
ation of the discount curve for each date, it might not be computationally feasible
to implement.

Since a contract with a CSA can be considered risk free (assuming the collateral is
posted frequently), the rate on collateral should be used as a proxy for the risk free
rate, e.g. Eonia or fed funds rate, depending on the currency (Piterbarg 2010). Of-
ten the overnight indexed swap (OIS) rate, together with the observed basis spread
(Smith 2013) is used. Thus the discount factor for a USD/SEK swap is USDOIS +
SEKUSDBS for the receive leg and SEKOIS for the pay leg.

This change of discounting rate has further implications as it changes the whole
pricing of interest rate derivatives. It means that one can no longer rely on only the
Libor curve. This is further explained in section 3.1.

2.3 Nikkei-linked loan

The Nikkei-linked loan is a structured bond with regular interest rate payments. The interest rate which is paid is either a high one or a low one, depending on the value of the Nikkei225 index at a fixing date (which is a predetermined number of business days before the payment is due). Assuming the notation where $S_0$ is the initial index value, $q$ is the strike level that decides which rate is paid. So the interest rate $r$ used for the cash flow $i$ is determined by the index level at date $t_i$:

$$r_i = \begin{cases} r_h & \text{if } S_{t_i} \geq qS_0 \\ r_l & \text{if } S_{t_i} < qS_0 \end{cases}$$ (2.4)

where $r_h$ and $r_l$ denote the high and the low interest rate. Some typical values might be $q = 0.85$, $r_h = 0.04$ and $r_l = 0.001$.

Another feature of the structured bond is that it has a knock-out and knock-in level, typically 105 % and 65 % respectively, see figure 2.1 - 2.3. The knock-out is triggered by the index value reaching a predetermined high level on a fixing date. The knock-out forces an early redemption of the bond where the principal is paid and the contract is terminated. The knock-in, is a low level of the index, and if it is reached at any time (not just on a fixing date), instead of the last payment being the principal $P$, it is now $P \cdot \min(S_T/S_0, 1)$. This means that if the index level would drop significantly, the buyer of the bond will receive less than the principal at the end of the contract.

To evaluate this contract we will use a Monte Carlo simulation of the factors which determine the price. This method is explained in more detail in section 2.4.2.
2.3.1 The different cases of the bond

To better illustrate how the structured bond works we look at the three different cases which can occur. In figure 2.1 - 2.3 there are three different graphs of the index movement used in the examples below. The first three cash flow dates are marked, while assuming the contract has a maturity longer than $t_3$.

1. **Neither knock-out nor knock-in:** This case is illustrated in figure 2.1. At $t_1$ and $t_3$ we have a payment of $r_h P$ while at $t_2$ we have $r_l P$. Because the index level keeps within the bounds of the knock-in/out then the end payment is simply $r_h P + P$.

2. **Knock-in:** This case is illustrated in figure 2.2. The first payments are the same as in the first case, but because the index value reached the knock-in level of 65% the end payment will not be as in the other cases but $r_i P + P \cdot \min \left( \frac{S_T}{S_0}, 1 \right)$. Say the index level at maturity will be at 80%, then the end payment will be $r_i P + 0.8P$.

3. **Knock-out:** This case is illustrated in figure 2.3. At the first cash flow date $t_1$ the index level is below the strike level so the payment is $r_l P$. At $t_2$ the index level is higher so the payment is $r_h P$. Then at $t_3$ the index level is above 105% so the contract is closed with the payment of $r_h P + P$. 
Figure 2.1: Neither knock-out nor knock-in. Described in case 1 above.

Figure 2.2: Knock-in. Described in case 2 above.
2.4 Structured Swap

As mentioned in the introduction, the purpose of the structured swap is to hedge against market risks such as the evolution of the Nikkei225 index and the FX rates of USD/JPY. The swap structure that hedges against the Nikkei-dependent rate of the structured bond is a swap that has a leg that replicates those cash flows while the other pays 3M USD Libor + spread. The spread may depend on our own probability of default (credit rating) and the characteristics of the structured leg. If we calculate the cash flows of the different legs we can price the structured bond using (2.3) and then (2.2).

A feature of the structured bond was that it can be "knocked out" which means that the contract is terminated before the maturity date, as explained in section 2.3. This means that the structured swap has the same property. If the structured bond is knocked out, so is the structured swap. Next we look at the cash flows of
the different legs.

### 2.4.1 Structured leg

The structured leg will be valued with Monte Carlo simulation.

### 2.4.2 Monte Carlo simulation

The general and the basic Monte Carlo simulation can be summarised in the following steps (Glasserman 2003):

1. Simulate market variables at time \( t \).
2. Evaluate contracts at time \( t \) with the simulated market variables from step 1.
3. Repeat step 1 and 2 a large number of times and take the average value of the contract prices as the expected value.

The contracts are evaluated using the methods described in this section. Since at least the Nikkei-linked contract is dependent on the index value at given time during the contract, the Monte Carlo simulation will be repeated at \( t_k \in (0, T] \) time points, where \( T \) is the maturity date.

To make sure we capture the correlation between the index and the FX rates we use a Cholesky decomposition of the correlation matrix (which is obtained by historical data). So, if we have the correlation matrix \( C \) then the Cholesky decomposition gives \( C = LL^T \) where \( L \) is lower triangular (Heath 2002). Given that we have a matrix \( u \) which has 3 rows of random normally distributed numbers, we can make the columns correlated by simply: \( u_{corr} = Lu \).

To reduce the variance in our samples (which may enable us to use fewer simulations) we will use Latin Hypercube sampling. With this method the sample space is divided in subsets which all have the same probability. Latin Hypercube sampling spreads the sample points to ensure that the set of outcomes is evenly covered.
2.4.3 Libor leg

As mentioned above, the Libor leg pays 3M USD Libor + spread on a quarterly basis. Assuming the structured bond lives until maturity, the Libor leg can be viewed as a floating rate note (FRN) with the same maturity date, which is basically a bond with variable coupon rate. Otherwise, if the bond is knocked out on date \( d \), it can be viewed as an FRN with maturity date \( d \). Assuming we know the probability that the contract is knocked out on a certain date, the Libor leg can be priced as the expected value of an FRN with stochastic maturity date:

\[
B_{Libor} = \sum_{k=1}^{n} (1 - p(k)) \cdot FRN(k),
\]

\( p(k) \) being the probability of a knock-out on the \( k \):th cash flow date, and \( FRN(k) \) is an FRN with that maturity date. (Smith 2014)

Now to price an FRN using the principles of (2.3) we must consider the cash flows. Every three months the 3M Libor (that was set at the start of the three month period) + spread is payed, and on the last payment the principal value \( P \) is payed back. The spread \( \delta \) is fixed over the length of the contract, while the Libor is set at the start of each period. Because the Libor will be set in the future (and thus impossible to know exactly beforehand) we use the forward rate over each period (R. White 2012). This gives us the following expression:

\[
FRN = P \cdot \sum_{j=1}^{N} \left( F(d_{k-1}, d_k) + \delta \right) \cdot \tau(d_{k-1}, d_k) \cdot D(0, d_k) + P \cdot D(0, d_N).
\]

By \( F(\cdot) \), \( \tau(\cdot) \) and \( D(\cdot) \) we mean the forward rate, length of the time period in years (using the appropriate day-count convention, typically Act/360 in USD and SEK) and discount factor between two dates, respectively. (Kenyon and Stamm 2012)

\( F(d_{k-1}, d_k) \) is described by the Libor3M curve. While the discount factors \( D(d_{k-1}, d_k) \) and \( D(0, d_k) \) is calculated from the OIS-curve. (Kenyon and Stamm 2012)
2.5 CVA

CVA is the difference between the true portfolio value, $\tilde{V}$, and the risk free portfolio value, $V$, where the true portfolio value takes the counterparty’s default in consideration. There are two key variables for valuation of CVA: probability of default (PD) and expected exposure (EE).

When calculating CVA the expectation should be taken under $\mathcal{Q}$, the risk-neutral measure. It is often divided into positive and negative exposure which are given by $\max(E(t), 0)$ and $\min(E(t), 0)$ respectively. In this study we will focus mainly on the expected positive exposure (EPE).

To be able to calculate the CVA we need to know the future default probabilities and exposure profiles. It is possible to find analytical expressions for these in some cases, as Pykhtin (2009) shows, but the most common way to do it is to use Monte Carlo methods to obtain the exposure profiles (Rosen and Saunders 2012).

2.5.1 Market variables

The market variables we need to simulate in order to be able to price the contracts are the Nikkei225 index ($S$) and the foreign exchange (FX) rates of the currencies of the swaps, i.e. USD/JPY ($FX_1$) and USD/SEK ($FX_2$). We model them using geometric Brownian motion, a widely used model (Hull 2017):

\[
\begin{align*}
\text{d}S(t) &= r_{\text{JPY}}(t)S(t)\text{d}t + \sigma_S(t)S(t)\text{d}W_S \\
\text{d}FX_1(t) &= (r_{\text{USD}}(t) - r_{\text{JPY}}(t))FX_1(t)\text{d}t + \sigma_{FX_1}(t)FX_1(t)\text{d}W_{FX_1} \\
\text{d}FX_2(t) &= (r_{\text{USD}}(t) - r_{\text{SEK}}(t))FX_2(t)\text{d}t + \sigma_{FX_2}(t)FX_2(t)\text{d}W_{FX_2}
\end{align*}
\]
where $r$ is the risk free rate, $\sigma$ is the volatility and $W$ is standard Brownian motion. Simulation is done by drawing random numbers which are normally distributed. The discrete-time variant of the model (using log return) is:

\[
\log \frac{S(t)}{S(0)} = (r_{JPY}(t) - \frac{1}{2} \sigma_S^2(t))t + \sigma_S(t) \sqrt{t} \epsilon_S
\]

(2.10)

\[
\log \frac{FX_1(t)}{FX_1(0)} = [(r_{USD}(t) - r_{JPY}(t)) - \frac{1}{2} \sigma_{FX_1}^2(t)]t + \sigma_{FX_1}(t) \sqrt{t} \epsilon_{FX_1}
\]

(2.11)

\[
\log \frac{FX_2(t)}{FX_2(0)} = [(r_{USD}(t) - r_{SEK}(t)) - \frac{1}{2} \sigma_{FX_2}^2(t)]t + \sigma_{FX_2}(t) \sqrt{t} \epsilon_{FX_2}
\]

(2.12)

where $\epsilon \sim \mathcal{N}(0, 1)$ (the standard normal distribution).
Chapter 3

Theoretical Framework

In this chapter we present the theory that will be used in our report. First we begin by examining the impact of the multi-curve framework and how to value the different instruments. Then we will consider how to compute the CVA for the different instruments. Lastly we will develop the theory necessary for the Monte Carlo simulation.

3.1 Multi-curve framework

Up until the crisis in 2007 interest rate derivatives were priced mainly using one curve, which was considered to be the risk-free curve and the curve relevant for Libor (more generally for the Ibor used in the specific currency). It was used to discount the fixed future cash flows and to price the theoretical deposits underlying the Libor index. This was the standard textbook approach, as can be seen in earlier editions of Hull (2017). This worked well in the current market, where banks were thought to have a negligible default risk and the spread between Libor and OIS was negligible. When the crisis started and the spread increased, it became clear that Libor could not be used as a proxy for the risk-free rate. Thus one could not use one single curve in the pricing of interest rate derivatives any more. (Henrard 2014)

Henrard (2007) was (to our knowledge) the first to propose a coherent valuation
framework where the index forward estimation was explicitly differentiated from
the discounting. This article was published shortly before the crisis, and focuses
on interest rate derivatives discounting, starting with the observation that different
instruments are valued using different curves which creates portfolio level arbitrage.
As such, it provides a more simplistic approach compared to later papers, reflecting
the then market practice, though it can easily be extended.

Soon after the crisis numerous literature relating to different aspects of what we now
call the multi-curve framework started to appear. Ametrano and Bianchetti (2009)
are the first to describe how the multi-curve framework impacts curve construction,
while Kijima et al. (2009) are the first to describe the impact of collateral. Bianchetti
(2010) proposes a description of a multi-curve approach, Moreni and Pallavicini
(2010) propose a parsimonious simultaneous modelling of both discounting and for-
ward curves while Mercurio (2009, 2010a, 2010b) proposes a comprehensive Libor
Market Model approach for discounting and forward curves. These are just some of
the first articles in this field. In order to have a comprehensive overview we chose
to base our approach on Henrard (2014), which is based mainly on Henrard (2010;
2013), but draws from all the literature up to date.

3.1.1 Discounting curves

The first, fundamental curves in the multi-curve framework are the discounting
curves, used to discount known cash flows. We define discount factors $P^D_X(t, u)$ as
the value in $t$ of an instrument paying one unit of currency $X$ at time $u$ (and the
superscript $D$ for ‘discounting’). With these discount factors the discount curve is
then built. The only restriction on the discount factors are that they should be
strictly positive, so there is no arbitrage.

To select the discounting curve, one needs to choose to impose a relationship between
some market instruments and the discounting curve. One such popular choice is to
use OIS-like instruments:

$$P^D_X(t, u) = P^O_X(t, u)$$  (3.1)
where the superscript $O$ denotes the OIS-like instrument value. This is often done since, after the crisis, Libor was no longer perceived to be risk free. Since an increasing number of instruments were collateralised, the rate for discounting should be a risk free rate, and the closest proxy is the OIS-rate, as explained by Hull and White (2012).

Another option is to impose a given spread $S$ between the discounting curve and an Ibor curve:

$$ P^D_X(t, u) = \exp \left( S(u - t) \right) P^I_X(t, u) $$  \hspace{1cm} (3.2)

where $I$ stands for IBOR. (Henrard 2014)

We stress here the need for different discounting curves when dealing with different currencies. Piterbarg (2012) shows that when there are instruments being collateralised in different currencies, different discounting curves must be used. In his framework the discounting rate is closely related with the collateral rate, and there exists a discounting curve for each collateral rate.

### 3.1.2 Discount Rate

Kenyon and Stamm (2012) demonstrate that for a cash deposit instrument, the market cash/deposit rate $r$ over a time period starting on, say date $d_s$ and ending on date $d_e$ is expressed as a simple rate so that the interest rate on an investment of 1 unit of currency paid over a time period $\tau(d_s, d_e)$ is:

$$ \text{interest} = r \cdot \tau(d_s, d_e) $$  \hspace{1cm} (3.3)

where $\tau(d_s, d_e)$ is the time period in the given day-count convention (i.e. Act/365, Act/360, 30/360 etc).
Thus the (forward) discount factor from a start date $d_s$, at $t_1$, to an end date $d_e$, at $t_2$, becomes:

$$D(d_s,d_e) = \frac{1}{1 + r\tau(d_s,d_e)}.$$  

(3.4)

The relationship between rates and discount factors is shown in figure 3.1. If we have the discount factor $D(0,d_s)$ from today to the start date available (usually from an overnight deposit) we can obtain the discount factor $D(0,d_e)$ from today to the the date $d_1$ as:

$$D(0,d_e) = D(0,d_s)D(d_s,d_e) \implies D(d_s,d_e) = \frac{D(0,d_e)}{D(0,d_s)}$$  

(3.5)

Given a yield curve, from which we can interpolate zero coupon rates and approximate the necessary discount factors, allows us to express the cash/deposit rate in terms of zero coupon rates. Assuming that the zero coupon rates are expressed as
continuous rates using the day-count convention Act/365, gives us that:

$$D(0, d) = e^{-r(t)t}$$  \hspace{1cm} (3.6)$$

where $t$ is the time period from 0 to $n$ in Act/365 and $r(t)$, is the zero coupon interest rate, interpolated from the yield curve. Having gone from zero coupon rates to discount factors we now solve:

$$\frac{1}{1 + \tau(d_s, d_e)} = \frac{D(0, d_e)}{D(0, d_s)}$$  \hspace{1cm} (3.7)$$

for $r$, giving:

$$r = \frac{D(0, d_s) - D(0, d_e)}{\tau(d_s, d_e)D(0, d_e)}$$  \hspace{1cm} (3.8)$$

as the cash/deposit rate calculated from a given yield curve. (Kenyon and Stamm 2012)

### 3.1.3 Forward curves

Our goal is to price derivatives of an IBOR or over-night index. Thus, another fundamental assumption of the multi-curve framework (which does not come as a consequence of the existence of a discounting curve) is the existence of instruments, $I^j_X$ where $j$ is the period e.g. 3M, which pay floating coupons based on IBOR or over night index. These are then priced indirectly through the forward curves $F^j_X$ which are defined by Henrard (2014) such that

$$I^j_X = P^D_X(t, v)\tau F^j_X(t, u, v)$$  \hspace{1cm} (3.9)$$

where $t$ is the time of pricing, $u$ is the starting date, $v$ the maturity date and $\tau$ is the accrual factor. The link between this definition of forward curves and market rates is that the IBOR rate fixing in $t_0$ is

$$I^j_X(t_0) = F^j_X(t_0, u, v).$$  \hspace{1cm} (3.10)$$
Now having defined the forward rates, there are multiple ways to implement them. We will present two of them here.

**Forward curves using pseudo-discount factors**

The most common way to model forward curves, as it is an evolution from the one-curve framework, is through pseudo-discount factors. The definitions is as follows: Let \( P^j_X(t, s) \) be a continuous function for \( t \leq s \) such that \( P^j_X(t, t) = 1 \) and an arbitrary strictly positive function for \( t \leq s < t + j, \ v = u + j \), then

\[
F^j_X = \frac{1}{\tau} \left( \frac{P^j_X(t, u)}{P^j_X(t, v)} - 1 \right). \tag{3.11}
\]

This definition resembles the forward rate in the one-curve world, where it was a consequence the no arbitrage condition. Here it is a definition, and has been called 'the wrong number in the wrong formula to give the correct result' (Henrard 2014).

**Direct forward curves**

The direct forward curve approach is more intuitive from the multi-curve framework point of view. We model the curve \( F^j_X(0, u, v) \) as a function of \( u \) directly from market data.

### 3.1.4 Interpolation

An important part in constructing the curves is to calibrate them to market data. We do this using market instruments, but there does not exist instruments which correspond to every possible combination of start date and maturity. Thus we have to employ some sort of interpolation scheme, and there are many to choose from. We will present a few examples below.

**Linear interpolation**

Linear interpolation is perhaps the simplest form of interpolation. Starting with the two point one desires to interpolate between, one simply draws a straight line
between them. Mathematically, this is formulated by Haug (2006) as follows:

\[ y_i = (y_2 - y_1) \frac{x_i - x_1}{x_2 - x_1} + y_1 \]  

(3.12)

using cartesian coordinates. This gives the coordinate \( y_i \) corresponding to chosen \( x_i \). Linear interpolation has the advantage of being very easy to implement, however when it comes to forward or discounting curves, it makes them have a saw-tooth pattern which is visually unappealing. (Henrard 2014)

**Natural cubic splines**

The cubic spline method is often seen as a more sophisticated technique, but does not necessarily mean that it is a better method, according to Haug (2006). To be able to get a smooth function that fits all the input points, this method uses all available points.

Let us assume that we have a table of points \([x_i, y_i]\) where \(i\) goes from 0 to \(n\) for the function:

\[ y = f(x) \]  

(3.13)

The cubic spline method is an incremental continuous curve that passes through each of the values in the given input. For each interval a separate cubic polynomial, with its own coefficients, is modelled.

\[ S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \quad x \in [x_i, x_{i+1}] \]  

(3.14)

where these polynomial segments \( S(x) \) are referred to as splines.

Now to define the spline we need to fix the coefficients. Since there are 4 for each spline, we have a total of \(4n\) coefficients and we need as many conditions. The first
conditions we get by requiring the function to be piece-wise continuous:

\[ S_i(x_i) = y_i, \quad S_i(x_{i+1}) = y_{i+1} \quad (3.15) \]

This means that each spline must start and end in the points it interpolates. This gives us \(2n\) conditions. Next we require that the function to be smooth, so we set constrains on the first and second order derivatives:

\[ S'_i(x_i) = S'_i(x_i), \quad S''_i(x_i) = S''_i(x_i) \quad (3.16) \]

but these conditions only apply to \(i = 1, 2, \ldots, n - 1\), which means we have \(2(n - 1)\) conditions. The last two conditions are the ones responsible for the ‘natural’ part of this method’s name:

\[ S''_0(x_0) = 0, \quad S''_{n-1}(x_n) = 0 \quad (3.17) \]

but this is not the only possible choice of boundary conditions.

The advantage of the natural cubic splines is that they create a smooth curve which is visually appealing. However, when applied to calibrating forward or discounting curves this method creates ‘ripples’ in the nodes farthest away from the origin. (Henrard 2014)

**Functional curves**

An option to using interpolation is to use functional (parameterised) functions as curve. The perhaps most known and used is the one developed by Nelson and Siegel (1987). It is easy to use and to interpret, but it has some aspects which can be problematic with respect to the stability of the curves, explained by Annaert et al. (2013) and Henrard (2014).
3.1.5 Calibration

Curve calibration refers to the building of the curves, specifically to ensure that the market instruments used in the calibration are consistent with the market when priced with the curves. What one uses as to measure 'consistency' with the market can differ. One way is to make sure that the present value of all instruments is zero when priced with the calibrated curve. Another is that the quotes (e.g. rate, price, clean price, yield) for the instruments computed from the curves are equal to actual market quotes.

To calibrate the curves, except in the simplest cases where a bootstrapping method suffices, a multidimensional root-finding algorithm is required. Often the implementation uses some Newton-Raphson style approach, with the Jacobian matrix used as a basis for some iterative process.

3.1.6 The Multi-Dimensional Newton-Raphson Algorithm

Atkinson (1989) explains that if we assume that we are faced with solving a 1-dimensional equation of the form $f(x) = 0$, where $f(x)$ is some function, we can solve it approximately by first making an initial guess of the solution as $x_0$. Then we can obtain a linear approximation of the function around $x_0$ as:

$$y = f(x_0) + f'(x_0)(x - x_0)$$  \hspace{1cm} (3.18)

As $y$ is an approximation to $f(x)$, solving the equation $f(x) = 0$ corresponds to solving $y = 0$ which means that:

$$0 = f(x_0) + f'(x_0)(x - x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$ \hspace{1cm} (3.19)
The function $f(x)$ is in the general case non-linear, the solution $x$ above will not solve $f(x) = 0$ exactly. So we repeat this iterative process:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (3.20)$$

until the value of $f(x)$ is below some threshold value. This converges to a solution, given a suitable initial point $x_0$.

If we generalise this to higher dimensions, we have an iterative method for solving non-linear systems. The assumption is that we have a vector valued function of several variables $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$

$$F(x) = [F_1(x_1, \ldots, x_m), \ldots, F_m(x_1, \ldots, x_m)] \quad (3.21)$$

To solve the system we can resort to a similar iterative strategy as for the one dimensional case above, i.e. we guess a starting point $x_0$ and make a linear approximation of $F(x)$ around it using Taylor's theorem according to Atkinson (1989):

$$y = F(x_0) + J(x_0) \cdot (x - x_0) \quad (3.22)$$

where $J(x_0) = \left[ \frac{\partial F_i}{\partial x_j}(x_0) \right]_{i,j=1,\ldots,m}$ is the Jacobian matrix at $x_0$. The linearisation of the equation $F(x) = 0$ around $x_0$ then corresponds to the linear system of equations

$$J(x_0) \cdot (x - x_0) = -F(x_0) \quad (3.23)$$

The matrix $J$ can be calculated either analytically or numerically. We can then solve the above system of equations to obtain $z = x - x_0$ which gives the solution

$$x = z + x_0. \quad (3.24)$$
3.2 Valuation of bonds, swaps and CCS

Having developed the theory of the multi-curve framework for creating the discounting and forward curves, we now look at how to price the instruments we need. First we price a standard bond, and then we look at interest rate and cross-currency swaps.

3.2.1 Bonds

A standard bond is an instrument which pays a fixed coupon \( c \) (a certain percentage of the principal, i.e. a fixed rate) at \( M \) number of dates \( t_i \) and at the last payment it also pays the principal \( P \). Often the coupon is quoted on a year basis, meaning that if the coupons are paid semiannually, and quoted as \( C \), then \( c = C/2 \). The (present) value of the bond is found by taking the sum of all discounted future cash flows:

\[
B(t) = P \sum_{i=1}^{M} c P^D_X(t, t_i) + P^D_X(t, t_M) P \tag{3.25}
\]

Now, if the coupons were not fixed but floating, we would have:

\[
B_{\text{floating}}(t) = P \sum_{i=1}^{M} P^D_X(t, t_i) \tau_i F^j_X(t, t_{i-1}, t_i) + P^D_X(t, t_M) P \tag{3.26}
\]

where \( \tau_i \) is the accrual factor for the period \([t_{i-1}, t_i]\), and \( t_i = t_{i-1} + j \). (Henrard 2014)

According to Madura (2015) a common way to represent the financing costs for the issuer of the bond is yield to maturity (YTM). YTM can be though of as the annualised yield the issuer pays during the bond’s life. The YTM can be calculated by solving for \( y \) in the following equation:

\[
P V(Bond) = \sum_{i=1}^{M} c \left( \frac{1}{1 + y/m} \right)^{i+T_p-1} + \frac{P}{(1 + y/m)^{M+T_p-1}} \tag{3.27}
\]

where \( m \) is the number of coupon payments per year and \( T_p \) is the fraction of the
period until next coupon payment. Note that the equation is non-linear and must be solved numerically. YTM can be used to compare bonds with different coupon periods and sizes as it shows the annual cost for the bond.

3.2.2 Swaps

An Interest Rate Swap (IRS) is an instrument which swaps one rate for another, often one fixed and the other floating. If a fixed rate is received and a floating one is paid, then the value of the swap is the difference between the cash flows. Using the same notation as in (3.25) and (3.26) but denoting the fixed leg cash flows $c_i$ at time $\tilde{t}_i$, the value of the IRS is

$$V_{IRS}(t) = \sum_{i=1}^{M} c_i P^D_X(t, \tilde{t}_i) - \sum_{i=1}^{N} P^D_X(t, t_i) \tau_i F^j_X(t, t_{i-1}, t_i)$$  (3.28)

We also define the forward swap rate, i.e. the fixed rate for which the present value of the IRS is 0:

$$S^j(t) = \frac{\sum_{i=1}^{N} P^D_X(t, t_i) \tau_i F^j_X(t, t_{i-1}, t_i)}{\sum_{i=1}^{M} P^D_X(t, \tilde{t}_i) \tilde{\tau}_i}$$  (3.29)

which is useful as it is often quoted on the market. We used $\tilde{\tau}_i$ to denote the accrual factor for the fixed leg.

Another swap which is widely used is the tenor swap (TS). We mention it because it shows another characteristic which applies to the CCS which we are interested in. A tenor swap exchanges two floating legs with different tenors, e.g. one paying 3M-Libor and the other paying 6M-Libor. The two legs are modelled as in (3.26), but with one modification. Since the credit risk is considered to be greater the longer the tenor, a spread is added to the leg with the shorter tenor to ensure that the swap is fair, as Fujii et al. [2010] explain, giving the following pricing formula:

$$V_{TS}(t) = \sum_{i=1}^{M} P^D_X(t, t_i) \tau_i^{T_2} (F^D_X(t, t_{i-1}, t_i) + \delta) - \sum_{i=1}^{N} P^D_X(t, t_i) \tau_i^{T_1} F^D_X(t, t_{i-1}, t_i)$$  (3.30)

where we use the superscripts $T_1$ and $T_2$ do differentiate between the tenors. The
spread, $\delta$, is often quoted on the market.

To value a cross-currency swap we use the same method as above, that is we compute the present value of the cash flows in their own currency, and with the relevant discounting curve. The cash flows in the foreign currency are paid according to the FX-rate set at the start of the contract, $FX(0)$. The total present value is then converted to the other currency using the spot exchange rate $FX(t)$ for the evaluation date, $t$. This makes the present value of the foreign leg dependant on the change in the FX-rate. Assuming the floating rate is paid in the foreign currency $Y$, (3.28) becomes:

$$ V_{CCS}(t) = \sum_{i=1}^{M} c_i P^D_X(t, \tilde{t}_i) \frac{FX(t)}{FX(0)} \sum_{i=1}^{N} P^D_Y(t, t_i) \tau_i (F^Y_Y(t, t_{i-1}, t_i) + \delta) $$

and the added spread as, similarly to the TS case, the floating rate is not risk free. To keep the framework coherent, one choice of curve in a currency must follow from the choice of curve in the other currency. Through multi-currency instruments like FX swaps we have links between the curves in different currencies. FX swaps are mainly interest rate instruments, and the main information they convey about the market is the difference in interest rates between two currencies, during a given time period. (Henrard 2014)

### 3.3 CVA

As explained in section 2.5, CVA is the difference between the true portfolio value, $\hat{V}$, and the risk free portfolio value, $V$, where the true portfolio value takes the counter-party’s default in consideration. We know from previous chapters that there are two key variables for valuation of CVA: PD and EE (EPE in our case). (3.32) is an approximation of CVA that is stated by Gregory (2010), where the assumption that there is no correlation between the exposure and default probability is made.

$$ CVA(T) = (1 - R) \int_0^T EE^*(t) dPD(t). $$
Here \( R \) is the part of the exposure that is recovered in case of a default, \( PD(t) \) denotes the default probability of the counter-party at a given time \( t \) and \( EE^*(t) \) is the risk-neutral discounted EE. When calculating CVA the expectation should be taken under \( Q \), the risk-neutral probability measure. \( EE^*(t) \) is defined by Zhu and Pykhtin (2007) as:

\[
EE^*(t) = E^Q\left[\frac{B_0}{B_t}E(t)\right]
\]

where \( B_t \) is the future value of one unit of the start currency that was invested today at the prevailing interest rate for maturity \( t \), which is now independent of counter-party default event. \( E(t) \) is the exposure at a time \( t \) and is simply the value of the contract at that time. It is often divided into positive and negative exposure which are given by \( \max(E(t), 0) \), \( \min(E(t), 0) \) respectively.

Furthermore, to estimate the default probabilities and exposure profiles by using CDS prices. Since, our delimitation that we assume we only use one counterparty so we mainly focus on the EPE and ENE and not on the default probabilities and exposure profiles.

Assuming we have used Monte Carlo simulation as described in section 2.4.2 (and more detailed in section 4.3.3) to obtain values for the contract and exposure at future time points, Rosen and Saunders (2012) show that (3.32) can be approximated as

\[
CVA \approx (1 - R) \sum_{k=1}^{K} \overline{EE}(t_k) \cdot (PD(t_k) - PD(t_{k-1}))
\]

where \( k = 1, ..., K \) so that \( t_k \in (0, T] \) and \( \overline{EE}(t_k) \) is a representative of the expected exposure during the period \((t_{k-1}, t_k)\). In the Basel III formula, a trapezoidal discretisation is used:

\[
\overline{EE}(t_k) = \frac{E(t_{k-1}) + E(t_k)}{2}
\]
but others, such as right- or left-point rule, might be used. Due to non-linearity we use Monte Carlo simulation to calculate EPE. EPE can be calculated analytically in linear cases, but otherwise it must be calculated through simulations.

3.4 Libor Leg

The Libor leg will pay Libor based cash flows, as shown in figure 3.2 until the structured leg is either knocked out or the contract matures. As explained earlier in 2.4.3 the Libor leg can be viewed as a floating rate note (FRN) given that, in this case, the structured bond lives until maturity.

Each cash flow date has a knock-out probability, thus the expected value of the Libor leg is the weighted sum over the values of a number of FRNs, ending on the different cash flow dates or on maturity. The expected value of the Libor leg is given by:

\[
E[\text{LiborLeg}] = \sum_{k=1}^{n} PV(\text{FRN}_k) (1 - p_k) \\
= \sum_{k=1}^{n} P^D_X(t, t_k) \tau_k (F^X_X(t, t_{k-1}, t_k) + \delta)(1 - p_k)
\]

Where \( p_k \) is the knock-out probability on the \( k \):th cash flow date and \( PV(\text{FRN}_k) \) is the PV of an FRN maturing on the \( k \):th cash flow date, given by discounting with the OIS-curve. (Smith [2014])
3.5 Black and Scholes PDE

The Black and Scholes partial differential equation (PDE) is an example of a parabolic PDE. This type of PDEs can be solved either by numerical methods or by a variety of analytical methods. One of those analytical methods is given by the Feynman-Kac theorem. This method produces a solution in the form of an expected value. With other words, by solving the Black and Scholes PDE with the Feynman-Kac method results in that the solution to the stochastic differential equation is given by the expected value of a stochastic process. (Wiersema 2008)

The Black-Scholes model describes the evolution of the stock price through the stochastic differential equation (SDE). The following equation (3.37), can be interpreted as modelling the changes (by percentage) in the stock price as increments of a Brownian motion (BM).

\[
\frac{dS(t)}{S(t)} = rdt + \sigma dW(t) \tag{3.37}
\]

Where \(W\) is a BM, \(r\) is the mean rate of return, \(\sigma\) is the volatility of the stock price. Implicitly are we describing the risk-neutral dynamics of the stock price by assuming that the rate of return is the same as the interest rate \(r\) in this case. (Glasserman 2003)

3.5.1 Monte Carlo

The solution to the Black and Scholes PDE is the expected value of a stochastic process, which can therefore be approximated by e.g Monte Carlo simulation. The Monte Carlo methods are based on the analogy between volume and probability. The concept of probability is characterised by the target measure, associates an event with a set of results, and defines the likelihood that the event should be it’s volume or measure relative to that of a universe of possible results. Monte Carlo calculates the volume of a set by interpreting the volume as a probability, i.e. uses
this identity in reverse. (Glasserman 2003)

To break it down let us assume that we have a room of possible outcomes that we
stochastically sample. We use a part of the stochastic features that fall into a given
set as an estimate of the volume of the set. The law of large numbers makes sure that
this estimate converges to the correct value as the number of draws increases. The
information about the likely extent of the error in the estimate after a finite number
of draws we can get from the central limit theorem (CLT). A small step takes us
from volumes to integrals, for example the problem of calculating the integral of a
given function, say \( f \), over the unit interval. This can be illustrated by:

\[
\alpha = \int_0^1 f(x) dx
\]  

(3.38)

The expected value of \( f \) is \( E[f(U)] \), where \( U \) is uniformly distributed between 0
and 1. Let us say that we can draw points \( U_1, ..., U_n \) independently and uniformly
from \([0, 1]\). If we now evaluate the function \( f \) at a random point \( n \), then the average
of the results gives us the Monte Carlo estimate.

\[
\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} f(U_i)
\]  

(3.39)

Assuming that \( f \) is integrable over \([0, 1]\) gives us (by the law of large numbers) that,
\( \hat{\alpha} \rightarrow \alpha \) with probability 1 while \( n \rightarrow \infty \). (Glasserman 2003)

3.5.2 Cholesky decomposition

Cholesky decomposition of correlation matrices can be used to create correlated
random numbers. With other words, Cholesky decomposition ensures that given
random variables are correlated. This is be explained by Hirsa (2012):

Let \( Z = Lx \) and \( a = LL^T \) be the Cholesky decomposition of the correlation matrix
\( C (N \times N) \) then we got that the expected value of \( ZZ^T \) is:

\[
\]

(3.40)

Where \( E[xx^T] = L \) since the random numbers in the vector \( x \) are independent (not correlated) and \( E[ZZ^T] = C \) means that the different random numbers in the vector \( Z \) are correlated with the correlations in the correlation matrix \( C \). We know that we need a semi-definite matrix for the Cholesky decomposition to be valid, explained next.

Positive Definite Matrices

Let \( x \) be a symmetric matrix. Then \( x \) is:

1. Positive definite if its eigenvalues are real and positive.
2. Negative definite if all its eigenvalues are real and negative.
3. Positive semi-definite if its eigenvalues are non-negative and real.
4. Negative semi-definite if its eigenvalues are non-positive and real.
5. Indefinite if none of 1-4 holds.

The big difference between semi-definite matrices and the positive/negative definite matrices is that: semi-definite matrices can be singular while the positive/negative definite matrices cannot. This is true because a matrix is singular if and only if at least one of the eigenvalues are 0. (Heath 2002)

A simple numerical explanation

Any given positive semi-definite matrix has a factorisation of the form \( a = LL^T \) where \( L \) is a lower triangular matrix. Solving for the lower triangular matrix \( L \) is straightforward. An example, suppose we wish to factor the positive definite matrix (Holton 2014):
A Cholesky factorisation takes the form:

\[
M = \begin{bmatrix}
4 & -2 & -6 \\
-2 & 10 & 9 \\
-6 & 9 & 14 \\
\end{bmatrix}
\]

(3.41)

By inspection, \( L_{1,1}^2 = 4 \), so we set \( L_{1,1} = 2 \). Then, \( L_{1,1}L_{2,1} = -2 \). Since we already have \( L_{1,1} = 2 \), we conclude \( L_{2,1} = -1 \), ..., this gives us the Cholesky matrix:

\[
a = \begin{bmatrix}
2 & 0 & 0 \\
-1 & 3 & 0 \\
-3 & 1 & 1 \\
\end{bmatrix}
\]

(3.43)

3.5.3 Latin Hypercube Sampling

One popular method of reducing the variance in a sample is Latin Hypercube Sampling (LHS). When drawing random samples there is no guarantee that parts of the sample space corresponding to significant outcomes will be sampled. An example in a two-dimensional sample space would be that all random samples happen to be drawn from only one half of the sample plane. LHS ensures a more even distribution of the drawn samples in the sample space. Glasserman (2003) describes LHS using the case of sampling from the uniform distribution over the hypercube.

For a dimension \( d \) and a sample size \( K \) we generate a stratified sample \( V_1^1, ..., V^K_i \) from the unit interval for every coordinate \( 1, ..., d \) using \( K \) equally probable subdi-
visions. This implies that each $V_i^j$ is uniformly distributed over $[(j - 1)/K, j/K)$, and if they are arranged as

$$
\begin{array}{cccc}
V_1^1 & V_2^1 & \cdots & V_d^1 \\
V_1^2 & V_2^2 & \cdots & V_d^2 \\
\vdots & \vdots & \ddots & \vdots \\
V_1^K & V_2^K & \cdots & V_d^K \\
\end{array}
$$

(3.44)

each row gives the coordinates of a point in subcubes along the diagonal of the unit hypercube, the first subcube being $[0, 1/K]^d$, the next $[1/K, 2/K]^d$ and so on. Next we randomly permute the values in each column. Let $\pi_1, \ldots, \pi_d$ be permutations of $1, \ldots, K$ such that all $K!$ such permutations are equally probable. Let $\pi_j(i)$ be the value to which the value $i$ is mapped by the $j$:th permutation. Thus (3.44) becomes

$$
\begin{array}{cccc}
V_{1}^{\pi_1(1)} & V_{2}^{\pi_2(1)} & \cdots & V_{d}^{\pi_d(1)} \\
V_{1}^{\pi_1(2)} & V_{2}^{\pi_2(2)} & \cdots & V_{d}^{\pi_d(2)} \\
\vdots & \vdots & \ddots & \vdots \\
V_{1}^{\pi_1(K)} & V_{2}^{\pi_2(K)} & \cdots & V_{d}^{\pi_d(K)} \\
\end{array}
$$

(3.45)

where each row now contains the coordinates of a point in a subcube no longer contained to the diagonal, only the point is now uniformly distributed over the unit interval. An illustration of LHS, in two-dimensions, can be seen in Figure 3.3. Note that in the two-dimensional case, LHS means only one sample point is taken from each row and column.
Now, to generate a LHS of sample size $K$ and dimension $d$ from a standard normal distribution $\mathcal{N}(0,1)$ we let $U_{i}^{j} \sim U(0,1)$ be independent for $i = 1, \ldots d$ and $j = 1, \ldots, K$. Then we set

$$V_{i}^{j} = \pi_{i}(j) - 1 + \frac{U_{i}^{j}}{K} \quad (3.46)$$

For the random permutation, one method is to first sample uniformly from \{1, ..., $K$\}, then repeat for the remaining values until only one remains. Finally we transform the samples:

$$Z_{i}^{j} = \Phi^{-1}(V_{i}^{j}) \quad (3.47)$$

where $\Phi^{-1}(\cdot)$ is the inverse normal cumulative distribution function.
Chapter 4

Implementation

In this chapter we will present how to implement the methods and theory mentioned in the earlier chapters. First we will present the market data needed for the calculations and then how to build the discounting and forward curves. Next we present how to price each contract, and finally how to calculate the expected exposure and CVA.

4.1 Market data

The first step is to collect the necessary market data used in the valuation of the contracts. For this we introduce notations in table 4.1.

<table>
<thead>
<tr>
<th>Market data</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD Libor Swap</td>
<td>( L(t, t) )</td>
</tr>
<tr>
<td>OIS JPY</td>
<td>( O_{\text{JPY}}(t, t) )</td>
</tr>
<tr>
<td>OIS USD</td>
<td>( O_{\text{USD}}(t, t) )</td>
</tr>
<tr>
<td>OIS SEK</td>
<td>( O_{\text{SEK}}(t, t) )</td>
</tr>
<tr>
<td>FX USD/JPY</td>
<td>( F_{\text{USD/JPY}}(t, t) )</td>
</tr>
<tr>
<td>FX USD/SEK</td>
<td>( F_{\text{USD/SEK}}(t, t) )</td>
</tr>
<tr>
<td>Volatility USD/JPY</td>
<td>( \sigma_{\text{USD/JPY}}(t, t) )</td>
</tr>
<tr>
<td>Volatility USD/SEK</td>
<td>( \sigma_{\text{USD/SEK}}(t, t) )</td>
</tr>
<tr>
<td>Volatility Nikkei225</td>
<td>( \sigma_{\text{Nikkei225}}(t, t) )</td>
</tr>
<tr>
<td>Date period Act/360</td>
<td>( \tau_i = (t_i - t_{i-1})/360 )</td>
</tr>
</tbody>
</table>
The FX-rates are the spot exchange rates for the day of the evaluation, whereas all other variables are vectors containing forward rates or volatilities, as is available on each market. Since we need values for them for dates not quoted on the market, we interpolate.

4.2 Curves

Now, having obtained the necessary market data, we move on to building the forward and discounting curves we need to price the contracts. We also discuss which interpolation method to choose.

4.2.1 Discounting curves

To get the discounting curve we need the discount factors $P^D_X(t, u)$ which we can get using OIS with different maturities $u$, ranging from 1 day to the maturity of the contracts. Using (3.6) which says $D(0, t) = e^{-r(t) t}$ combined with the choice of using OIS as the discounting rate (from (3.1)), we get:

$$
P^D_{JPY}(0, t) = \exp (-O_{JPY}(0, t) t)
$$

$$
P^D_{USD}(0, t) = \exp (-O_{USD}(0, t) t)
$$

$$
P^D_{SEK}(0, t) = \exp (-O_{SEK}(0, t) t)
$$

(4.1)

The forward discounting factors we get by applying (3.5):

$$
P^D_X(t_1, t_2) = \frac{P^D_X(0, t_2)}{P^D_X(0, t_1)}
$$

(4.2)

Since we do not have daily data on the OIS, we interpolate the missing values.

---

1The used volatility has been extracted from FIS’ Adaptiv.
4.2.2 Forward curve

For the forward curve we use IRS of different maturities as in the OIS case. Given quoted swap rates \( L(t) \) we use (3.29) to calculate \( F_{USD}^{3M}(0, t_{i-1}, t_i) \) for each time quoted on the market:

\[
L(t) = \frac{\sum_{i=1}^{N} P_{USD}^D(t, t_i) \tau_i F(t, t_{i-1}, t_i)}{\sum_{i=1}^{M} P_{USD}^D(t, \tilde{t}_i) \tilde{\tau}_i} \tag{4.3}
\]

Since we only will be using one forward curve, for the sake of brevity we drop the superscript and the subscript on \( F_{USD}^{3M}(0, 0, t_i) \). Every quoted swap rate is a function of multiple \( F(t, t_{i-1}, t_i) \), thus we get a system of equations to solve to get each \( F(t, t_{i-1}, t_i) \). We describe how to solve this in section 4.2.3.

A problem that arises in (4.3) is that the swap rates are quoted only for yearly maturities while the forward rates we need are quarterly. To remedy this we use linear interpolation to get quarterly swap rates, and thus have a swap rate for every forward rate date.

After we have calculated all \( F(t, t_{i-1}, t_i) \), we can calculate the implied discount factors, using (3.11). Below is the first step, using \( P(0, 0) = 1 \):

\[
F(0, 0, t_1) = \frac{1}{\tau} \left( \frac{1}{P(0, t_1)} - 1 \right) \implies P(0, t_1) = \frac{1}{1 + F(0, 0, t_1) \tau} \tag{4.4}
\]

After calculating all discount factors (for each 3 month period), they are then interpolated and used in (3.11) to get daily values for \( F(t, t_{i-1}, t_i) \), and this constitutes the forward curve.

4.2.3 Equation solving

Now that we have the expressions for swap rates (and eventually other instruments, such as cash/deposits) we can use a numerical solver to solve the following non-linear
system of equations, with the equations coming from (4.3):

\[
\begin{cases}
F(t, 0, t_1) = g_1(P_{USD}^D, \tau, \tilde{\tau}, L) \\
F(t, t_1, t_2) = g_2(P_{USD}^D, \tau, \tilde{\tau}, L, F) \\
\vdots \\
F(t, t_{M-1}, t_M) = g_M(P_{USD}^D, \tau, \tilde{\tau}, L, F)
\end{cases}
\]  \hspace{1cm} (4.5)

This is done so that we can interpolate on the pseudo-discount factors. To solve this system of equation we use the Multi-Dimensional Newton-Raphson algorithm, described in section 3.1.6.

Solving the system of equations, equation (3.21) can be written as \( F(x) = 0 \), assuming we move the right-hand-side to the left-hand-side

\[
\begin{cases}
F(t, 0, t_1) - g_1(P_{USD}^D, \tau, \tilde{\tau}, L) = 0 \\
F(t, t_1, t_2) - g_2(P_{USD}^D, \tau, \tilde{\tau}, L, F) = 0 \\
\vdots \\
F(t, t_{M-1}, t_M) - g_M(P_{USD}^D, \tau, \tilde{\tau}, L, F) = 0
\end{cases}
\]  \hspace{1cm} (4.6)

Since we have just solved a linearised version of the real non-linear system of equations, we do not have the exact solution but a better approximation of it than \( x_0 \) if the latter was chosen appropriately. We thus call the updated approximation \( x_1 \) and thus in general we obtain the iterative method

\[
J(x_k) \cdot z = F(x_k)  \hspace{1cm} (4.7)
\]

\[
x_{k+1} = z + x_k \hspace{1cm} (4.8)
\]

Thus in each step we solve a linear system of equations and then update the approximation \( x_k \) using the obtained solution. A suitable starting point \( r_0 \) for the zero coupon rates is the set of corresponding market rates, noting that the market rates fall on the same dates as the yield curve dates of the calibrated zero coupon curve.
4.2.4 Interpolation Function

To choose which interpolation method we consider the analysis Henrard (2014) makes, where he considers the visual aspect and the hedging aspect, i.e. if the forward curve constructed with the different methods give a reasonable hedge strategy. There he finds that using linear interpolation gives a reasonable hedge, while natural cubic spline gives counterintuitive one. On the other hand, when considering the visual aspect and ease of interpretation, the situation is reversed.

Since there are advantages and disadvantages with both methods, we will use simple linear interpolation (and flat extrapolation where necessary) described in (3.12):

\[ y_i = (y_2 - y_1) \frac{x_i - x_1}{x_2 - x_1} + y_1 \]  \hspace{1cm} (4.9)

because the implementation leads to a faster execution time without giving in to significant disadvantages.

We define a function that takes, as input arguments, a time point, an array of time points and their corresponding values. The task we want to complete is to interpolate the value for the time point from the arrays and their corresponding values. As a first step we find the position of the point in the array of points and then we perform linear interpolation between the points adjacent to the point.

4.3 Valuation of the contracts

With the market data and forward and discounting curves we have so far, we can price bonds and CCS, which we will do in the first part of this section. The second part deals with Monte Carlo simulation and the pricing of the structured swap.
4.3.1 Fix rate bond

A fix rate bond is priced using (3.25):

\[ B(t) = P \sum_{i=1}^{M} cP^D_X(t, t_i) + P^D_X(t, t_M)P \]  \hspace{1cm} (4.10)

with \( P \) being the principal, \( c \) the fixed rate and \( P^D_X \) chosen for the currency of the bond.

To be able to compare the bonds we will calculate the YTM in (3.27) with \( PV(Bond) \) being the value calculated in (4.10), at today’s valuation date:

\[ B(0) = \sum_{i=1}^{M} \frac{c}{(1 + y/m)^{t_i + T_p - 1}} + \frac{P}{(1 + y/m)^{M + T_p - 1}} \]  \hspace{1cm} (4.11)

The equation is solved numerically, with e.g. Excel’s solver.

4.3.2 CCS

According to (3.31) a fixed-for-floating USD/SEK CCS is priced as:

\[ V_{CCS}(t) = \sum_{i=1}^{M} c_i P^D_{SEK}(t, t_i) - \frac{FX_{USD/SEK}(t)}{FX_{USD/SEK}(0)} \sum_{i=1}^{N} P^D_{USD}(t, t_i) \tau_i(F(t, t_{i-1}, t_i) + \delta) \]  \hspace{1cm} (4.12)

\( FX_{USD/SEK}(0) \) is given by the market data, and \( FX_{USD/SEK}(t) \) by the generated path. The fix rate \( c_i = c \) is given by first calculating the present value of the structured swap, and then setting it so that the sum of their present value is zero. Thus we can see what fix rate is implied by the different contract setups.
A floating-for-floating CCS is obtained by having two floating legs:

\[
V_{CCS}(t) = \sum_{i=1}^{M} P_{SEK}^{D}(t, \tilde{t}_i) \tau_i (F(t, \tilde{t}_{i-1}, \tilde{t}_i) + \delta) - \frac{FX_{USD/SEK}(t)}{FX_{USD/SEK}(0)} \sum_{i=1}^{N} P_{USD}^{D}(t, t_i) \tau_i F(t, t_{i-1}, t_i)
\]

(4.13)

4.3.3 Monte Carlo simulation

The first part in the simulation is to generate random numbers, \( x \), to build the random paths. For the structured swap, we need two sets, \( X_1 \) and \( X_2 \). Then LHS is used according to:

\[
V_i^j = \pi_i(j) - 1 + \frac{U_i^j}{K}
\]

(4.14)

where \( x(j) = U^j \) and then transformed into normally distributed random variables using:

\[
Z_i^j = \Phi^{-1}(V_i^j)
\]

(4.15)

For \( x_1 \) and \( x_2 \), we use Cholesky decomposition to make them correlated. The Cholesky step can be explained in the two following steps, assuming we created vectors of random numbers \( x_1, ..., x_n \):

1. Let \( LL^T \) be the Cholesky decomposition of the correlation matrix \( C \ (N \times N) \).

2. Compute \( Lx \) where \( x = [x_1, ..., x_n]^T \).

Now are the numbers \( z = [z_1, ..., z_n] = Lx \) correlated according to the correlations in the correlation matrix \( C \).

Now that we have the random numbers to our liking, we simulate paths by first
rewriting (2.10)-(2.12) to:

\[
S(t) = S(0) e^{(r_{JPY}(t) - \frac{1}{2} \sigma_{S}^2(t)) t + \sigma_S(t) \sqrt{t} \epsilon_S}
\]  

(4.16)

\[
FX_1(t) = FX_1(0) e^{(r_{USD}(t) - r_{SEK}(t) - \frac{1}{2} \sigma_{FX_1}^2(t)) t + \sigma_{FX_1}(t) \sqrt{t} \epsilon_{FX_2}}
\]  

(4.17)

\[
FX_2(t) = FX_2(0) e^{(r_{USD}(t) - r_{SEK}(t) - \frac{1}{2} \sigma_{FX_2}^2(t)) t + \sigma_{FX_2}(t) \sqrt{t} \epsilon_{FX_2}}
\]  

(4.18)

Thus we can generate paths by iteratively calculating the next value for the index by:

\[
S(t_{i+1}) = S(t_i) e^{(O_{JPY}(t_i,t_{i+1}) - \frac{1}{2} \sigma_{Nikkei225}^2(t_i,t_{i+1})) \Delta t_i + \sigma_{Nikkei225}(t_i,t_{i+1}) \sqrt{\Delta t_i} \epsilon(i)}
\]  

(4.19)

Here \( \Delta t_i = t_{i+1} - t_i \), and we have replaced the notation to those for the market variables used. (4.17) and (4.18) can be rewritten in a similar manner. To make this into a Monte Carlo simulation, we create a large number of paths and price the contracts using values from these paths. Then we sum the prices and divide by the number of simulations to get an average value.

### 4.3.4 Structured swap

As explained previously, the structured swap is priced by calculating the difference between the values of the two legs. The structured leg is priced using Monte Carlo simulation. We simulate \( S(t) \) and \( FX_2(t) \) using (4.16) and (4.18). Having a path for the Nikkei225 index we can determine each future cash flow according to the rules of the contract, detailed in section 2.3. The future cash flows are then discounted with \( P^D_{JPY}(0, t) \) and added together constituting one possible value for the structured leg, i.e. one Monte Carlo simulation. The structured leg is thus priced by taking the average of 10 000 simulations.
For the Libor leg, we have from (2.5) that the present value of an FRN is:

\[
FRN = P \cdot \sum_{j=1}^{N} \left( F(t, t_{i-1}, t_i) + \delta \right) \cdot \tau_i \cdot P_{USD}^D(t, t_i) + P \cdot P_{USD}^D(t, t_N)
\]  

(4.20)

using the market data notation. \( P \) is the value in USD of the JPY principal using the FX-rate \( FX_{USD/JPY} \). No FX-part is here since we price the contract in USD, transferring instead the value of the structured leg to USD.

We get the expected value of the Libor leg by using (2.5):

\[
E[LiborLeg] = \sum_{k=1}^{n} FRN_k (1 - p_k)
\]  

(4.21)

The knock-out probabilities \( p_k \) are the results from the Monte Carlo simulation of the structured leg.

### 4.4 CVA

Once we have a way to price each contract, the expected exposure can be calculated. Then, we can calculate the CVA, given the counterparty’s probability of default and our rate of recovery. However, if the counterparty is the same for all contracts, or with a similar PD then the impact on the CVA will come almost solely from the expected exposure. Since we are mainly interested on how the contracts influence the CVA, we will only look at the expected exposure.

#### 4.4.1 Expected exposure

By pricing the contracts monthly for a future 3-year period, we can calculate the expected exposure, both positive and negative, in each future month. Exposure is simply the total value of the contracts at a time point, as explained in section 3.3, and the expected part is when dealing with future, uncertain values. We use Monte Carlo simulations to calculate the expected exposure by creating a path for the
index and FX. For this path we calculate the monthly exposure, both positive and negative. Then we do this a large number of times and then take the average to get the expected exposure. The discretisation rule used will be the left-point rule, and for the sake of visualisation, the values between the monthly data will be linearly interpolated.
Chapter 5

Analysis & Results

In this chapter we present our results, first for the SEK bond case, then the USD bond and CCS, and then for the structured case. After each case is presented and analysed, we compare them to see if any case is more advantageous than the other. Before this, in table 5.1 and 5.2 we present the nominal values and the exchange rates used in our simulations. The nominal values are based on the JPY which was set, the others being derived with the FX-rates.

Table 5.1: Nominal values in the different currencies.

<table>
<thead>
<tr>
<th>Currency</th>
<th>JPY</th>
<th>USD</th>
<th>SEK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>1,000,000,000</td>
<td>9,430,000</td>
<td>76,960,493.2</td>
</tr>
</tbody>
</table>

Table 5.2: FX-rates for the different currencies.

<table>
<thead>
<tr>
<th>Currency</th>
<th>JPY/USD</th>
<th>USD/SEK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate</td>
<td>0.00943</td>
<td>8.16124</td>
</tr>
</tbody>
</table>

5.1 Case 1: SEK bond

The first simple case we look at is a SEK bond. In table 5.3 we see a couple of Swedish municipal bonds where the rate (coupons) varies from 0.25 % to 2.5 %. We did our testing for a 3-year period, so we chose the bond K2109, with 1 % rate, for our comparisons.


Table 5.3: Swedish municipal bonds.

<table>
<thead>
<tr>
<th>Name</th>
<th>ISIN</th>
<th>Coupon</th>
<th>Buy Rate</th>
<th>Sale Rate</th>
<th>Maturity Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>K 1806</td>
<td>SE0006424990</td>
<td>1.000</td>
<td>100.19</td>
<td>100.20</td>
<td>2018-06-20</td>
</tr>
<tr>
<td>K 1903</td>
<td>SE0005131299</td>
<td>2.250</td>
<td>102.40</td>
<td>102.40</td>
<td>2019-03-12</td>
</tr>
<tr>
<td>K 2002</td>
<td>SE0008040786</td>
<td>0.750</td>
<td>101.99</td>
<td>102.05</td>
<td>2020-02-16</td>
</tr>
<tr>
<td>K 2012</td>
<td>SE0005705621</td>
<td>2.500</td>
<td>106.84</td>
<td>106.92</td>
<td>2020-12-01</td>
</tr>
<tr>
<td>K 2109</td>
<td>SE0006995064</td>
<td>1.000</td>
<td>103.22</td>
<td>103.29</td>
<td>2021-09-15</td>
</tr>
<tr>
<td>K 2206</td>
<td>SE0009269418</td>
<td>0.250</td>
<td>99.96</td>
<td>100.06</td>
<td>2022-06-01</td>
</tr>
<tr>
<td>K 2302</td>
<td>SE0009662943</td>
<td>0.750</td>
<td>101.34</td>
<td>101.44</td>
<td>2023-02-22</td>
</tr>
<tr>
<td>KOIO K2311</td>
<td>SE0010948240</td>
<td>1.000</td>
<td>101.78</td>
<td>101.92</td>
<td>2023-11-13</td>
</tr>
<tr>
<td>KOIO K2410</td>
<td>SE0010469205</td>
<td>1.000</td>
<td>100.74</td>
<td>100.87</td>
<td>2024-10-02</td>
</tr>
</tbody>
</table>

In figure 5.1 we have the PV for this bond. We can see it decreases sharply after each coupon payment.

![Figure 5.1: PV for a unit SEK bond over 3 years, which pays a coupon of 1% every year.](image)

Since we do not model any stochastic rate but assume it to be deterministic, the price of the bond is also deterministic, based on our discounting rate. This means that the PV of the bond is also the expected exposure, although it is not relevant...
for the bond, which is an outstanding loan and requires no securities to be issued.

When we calculate the YTM for this bond we get that it is -0.329 %, which is the annualised rate which the issuer of the bond would have to pay, i.e. the issuer receives a small rate for taking this loan. Using this rate, assuming it constant over the whole period we model an IRS which swaps this fix rate for Stibor 3M plus spread. The spread was set such that the PV of the contract is zero at the start date, namely -0.433 %. The PV can be seen in figure 5.2.

Besides paying Stibor, which is very small, even negative for short maturities, we have a negative spread which makes it even smaller. Also the PV end EPE are positive, so we expect to receive securities from this swap.

Figure 5.2: PV, EPE and ENE for the IRS which exchanges an annual fix rate of -0.329 % for Stibor 3M with a spread of -0.433 %.
5.2 Case 2: USD bond and CCS to SEK

In this second case we considered a USD bond (ISIN: XS1756423825), maturing in 2021, which pays a semianual coupon of 2.375 %, i.e. each coupon payment is of 1.1875 %. In figure 5.3 we have the PV for this bond.

![Graph of PV USD Bond](image)

Figure 5.3: PV for a unit USD bond over 3 years, which pays a coupon of 1.1875 % every 6 months.

Next, more interestingly, let us see what spread this implies for the CCS.

5.2.1 CCS

To easily compare the interest rate between the cases, we look at what the interest rate in SEK would be. As we need the funds in SEK, we use a fixed-to-floating CCS, the floating leg paying Stibor 3M + spread. The spread for the floating leg is taken such that the PV of the CCS is zero at the start of the valuation, which is what we call the implied spread. In this case, with an USD rate of 2.375 %, the
implied spread was 0.426 %, which is a higher than in case 1. In figure we have the PV for the CCS.

As we can see the PV is 0 at the beginning and then increases as the contract comes closer to maturity. This increase in PV is driven by the drift of the USD/SEK FX-rate. Both the positive and negative expected exposure increase from the start of the contract and start decreasing a little towards the end. This happens because as we near maturity, the time period is shorter and the risk decreases.

We also see that the EPE is generally larger than the ENE, which implies that we are exposed to a counterparty risk are are thus receiving collateral securities from our counterparty. This may have a positive influence on our total cost for this contract, depending on how the collateral is used.

Figure 5.4: PV, EPE and ENE for a CCS which swaps USD 2.375 % (semiannual) for Stibor 3M with 0.426 spread %.
5.3 Case 3: Nikkei-linked loan

The last case is with the structured Nikkei-linked bond which is then used in a structured swap to USD and a fixed-for-floating USD/SEK CCS. We first analyse the structured contracts and the CCS, and then their net result.

To make sure that we had code that was correct we did some check-ups and controls. One of the main controls, if not the most significant, is illustrated in figure 5.5. We wanted to make sure that the simulated index and FX curves look correct and that a knock-out or knock-in only occurs on the cash flow dates and not before or after. We made sure that the cash flows are not on the wrong day, as it can easily become a problem since we simulate for each month and the time to next cash-flow date changes every month by one month.

Figure 5.5 displays that if and only if the index hits or passes the knock-out level at a cash-flow date, the contract/instrument gets knocked-out. The cash-flow marking (red dots) indicates that we have a cash-flow date according to ACT/360, approximately every 90th day.
5.3.1 Nikkei-linked structured bond and swap

We first look at the expected PV of each leg of the swap separately, so we can see clearly how the structured bond performs, and then we look at them together. This is done with the help of (4.20) respectively (4.21). Thereafter, we analyse the EPE and ENE of the swap with the help of (3.33) where EPE and ENE is \( \max(E(t), 0) \) and \( \min(E(t), 0) \) respectively. We present result for three different correlation levels between the Nikkei225 index and the USD/JPY FX-rate, the correlations being \{-0.5, 0, 0.5\}.

Expected PV for respective leg

In figure 5.6 we have the expected PV for the Libor leg and the Structure leg (bond) of the swap, for correlation -0.5. Then, in figure 5.7 and 5.8 is the expected PV for correlations 0 and 0.5 respectively.
Figure 5.6: Expected PV for each leg in the Nikkei-linked bond with -0.5 correlation.

Figure 5.7: Expected PV for each leg in the Nikkei-linked bond with 0 correlation.
Figure 5.8: Expected PV for each leg in the Nikkei-linked bond with 0.5 correlation.

We can easily, by analysing the figures 5.6-5.8, get to the conclusion that the correlation does not have a major impact on the outcome of the expected PV for the Libor leg and structured leg in this case. We will see more clearly what different correlations do when we look at the PV for the swap as a whole.

By studying the appearance of the PVs, we can draw some conclusions. According to the definition of the structured bond and swap, we know that the structured leg depends heavily on the index. Since a knock-in (or a knock-out) occurs with some probability, this affects the PV, which drops to about half its nominal value toward the end of the contract.

**Expected PV for the Structured Bond and Swap**

Unlike the expected PV for respective leg in section 5.3.1, the difference between the correlations in expected PV for the structured contract is easier to spot in these figures.
In figure 5.9 we see the PV of the structured swap when the index and the FX-rate are uncorrelated. There is a slight decreasing trend which may be influenced by the FX-rate.

In figure 5.10 the correlation is -0.5. We can see that the PV has a similar shape, only the negative trend being more pronounced. This increases our belief that the FX-rate is causing it, since a negative correlation means that when the index decreases the FX-rate increases.
The last correlation can be seen in figure 5.11. The PV has a much lower decrease, in this case helped by the positive correlation, which now has an effect opposite to the one where the correlation was negative.

**EPE and ENE**

In figure 5.12-5.14 we present the EPE and ENE for the structured swap and the different correlations.

*Figure 5.11: Expected PV for the Nikkei-linked bond with 0.5 correlation.*

*Figure 5.12: EPE and ENE for the Nikkei-linked bond with -0.5 correlation.*
As we could conclude in previous sections, by analysing the figures for respective correlation, the expected positive and negative exposure do not differ, significantly, in percentage terms between the correlations. The same effects can also be observed in the expected exposures.

We can see that the ENE is significantly larger than the EPE and they are not symmetric around 0, as was the case for the CCS in previous section. This is because
the structured swaps starts with a negative expected PV. Also the exposures keep increasing until the end of the contract, unlike the CCS.

5.3.2 CCS

The CCS paired with the structured swap is a floating-for-floating CCS paying a Stibor + spread in SEK and receiving USD Libor. In figure 5.15 we have the PV, EPE and ENE of the CCS when the spread is set to give the contract a starting PV of 0, namely 0.742%.

![PV for the CCS](image)

*Figure 5.15: Comparison between the present value of a CCS from exchanging USD for SEK, EPE and ENE.*

Here we see that the PV of this CCS is positive when starting at 0 growing during the life of the contract, and the EPE being almost twice the size of the ENE.
In figure 5.16 we have the PV for the CCS with its spread at 0.580 %, the spread which makes the sum of its and the structured swap’s PVs be zero at the start date, which is the implied spread. The PV starts at the value of the structured swap’s PV as to offset it, but otherwise looks very similar to the CCS in figure 5.15.

5.3.3 Structured swap and CCS

Now it is time to look at the combined values of the structured swap and the CCS, which together make this third loan alternative. In figure 5.17 we see the PV, EPE and ENE for these contracts combined.
This result is what we could expect from viewing the separate contracts’ graphs earlier. The PV starts at 0 and then increases significantly. The exposures behave similarly as for the CCS, since they are larger than the exposures of the structured contract. Since the EPE is overall larger than the ENE, we would expect to receive collateral from our counterparty, to cover for the increase in the contracts’ value.

5.4 Comparisons

Now that we have analysed the different loan alternatives, can we say which one is the best? We first look at the rate which we pay in each case which we can see in table 5.4.

<table>
<thead>
<tr>
<th>Case</th>
<th>Spread (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.433</td>
</tr>
<tr>
<td>2</td>
<td>0.426</td>
</tr>
<tr>
<td>3</td>
<td>0.580</td>
</tr>
</tbody>
</table>

Alternative 1 has the lowest spread, even negative, and then alternative 2 and 3 are
both larger in that order. However, when considering the exposure things might be different.

We concluded that since the first case is simply a bond, no exposure profile is needed since no collateral needs to be exchanged. This means that this loan alternative is more transparent when considering the total cost involved. Even when coupling it with an IRS the spread is the lowest of the three compared, and the exposures are positive, so we expect to receive collateral.

The second case has a higher spread, but a larger (approximately 5 times larger) positive expected exposure and PV. This means that we expect to receive collateral from our counterparty to cover up the increased value of the contract. Depending on the specifics of the contract, it might mean that we have a positive inflow of cash which might offset the higher interest rate. In our simulation, the PV had increased by around 4% of the nominal value. This is something the municipality must consider, as it influences the total cost for this loan.

For the third case, similar conclusions may be reached. The spread is the highest, and the PV is positive and so is the exposure, meaning we expect to receive collateral to cover for the increase of the contract’s value, which was around 4% in our simulation. Depending on the specifics of the contract, the resulting gain may be such that it is more advantageous to choose this alternative with higher spread but higher expected receival of collateral. Another thing to consider with this type of contract is what happens when the structured bond and swap is knocked out and thus terminated, because the CCS is still active. How do we handle the refinancing, and what are the costs? Generally with this type of contract the alternatives are closing the CCS together with the other contracts and then entering new one, or have rolling FX-swaps to cover the remaining cash flows.
Chapter 6

Conclusions & Discussion

In this chapter we will present our conclusions but also discuss development opportunities and ethical aspects.

6.1 Conclusions

Given the results and analysis made in previous chapter, we now try to compile and conclude the outcome. Our purpose was to determine which of the three loan alternatives was the "best" when having as wide a perspective as possible, looking not only at the interest rate paid, but also at the exposure and what its impact might be. As we saw in the earlier chapter, determining the "best" alternative is not an easy task, especially when comparing more generally, without knowing the exact specifics of each contract.

When comparing the different alternatives, we need to consider both the spread and the expected exposure to know whether we expect to receive or to post collateral and what amount and kind of collateral that will be posted. Depending on the regulations and if we receive or post the collateral, a lower spread might not be the decisive factor in determining which loan to take.

What we can conclude with certainty is that there are many alternatives for funding
and that they each need to be analysed and compared on many levels to be able to make a correct decision as to which ones to choose. To consider the implications of the newest regulations and risk exposure is a crucial part of this, as it might greatly influence the final price for contracts.

In the cases that we compared, the lowest spread was achieved with the 'simplest' case, that is the SEK bond coupled with an fix-for-floating IRS. Since this case probably has the lowest uncertainty when considering collateral, we believe it is the most appropriate choice. However, this might change in the future when the Stibor is not as extremely low as is is today.

### 6.2 Further work

One the topic of this thesis, we consider there is a lot of room for further work. On of the most obvious, perhaps, is to expand the list of loan alternatives compared. Even if one does not wish to expand the types of contracts, there are many more markets and currencies that might be analysed for issuing bonds. It would also be necessary to improve the way domestic fix-rate bonds, as in case 1, are priced, because using YTM to approximate the yearly fix rate for the IRS is a rather simple approximation.

As mentioned earlier in the delimitation section, in this thesis we make the assumption that the interest rate is deterministic. Hence, by easy investigation on CCS we conclude that in this case the rates do not have such big impact that we consider the "complexity" of simulating stochastic rates worth it. We also use a flat volatility surface that we get from Bloomberg. Obviously, there is room for improvement on this topic, for there are many other more complex models which can be used to price the contracts.
6.3 Ethical aspects

First of, we considered, more specifically, municipalities looking for funding. This is an important task, as this funding is most often used for development projects in a community, be it school buildings or other infrastructure projects. The municipality will ultimately have to pay for the funding with tax money, and thus choosing a stable funding with low interest rate is paramount. It is therefore essential that, when looking for funding, municipalities take into consideration both the interest rate and risk exposure, as a very low interest rate might mean that a large risk is taken.

Furthermore, a more general aspect in this topic is regarding the regulations and focus on the financial institutions from the financial supervisory authority’s for respective country since the financial crisis in 2008. The institutions are obliged by law to do a proper valuation and control of their risks for each financial instrument. One of this new restrictions are highlighted in this thesis. We know from more updated studies that the multiple curve framework has replaced the single curve framework. Not only are the financial institutions obliged to present the most correct and most proper valuation of their risks and financial instruments, they also have a ethical responsibility as well.

To prevent an unstable financial market and future crises caused by the financial institutions, every institution, whether it is a commercial bank or a noncommercial bank, has a responsibility to do their best given the conditions and regulations. Therefore, it is essential to choose a relevant and good curve construction technique and not use methods for manipulations of the outcome.
Bibliography


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