SIAM J. MATH. ANAL.
Vol. 50, No. 6, pp. 5761–5782

BOUNDEDNESS OF THE STATIONARY SOLUTION TO THE BOLTZMANN EQUATION WITH SPATIAL SMEARING, DIFFUSIVE BOUNDARY CONDITIONS, AND LIONS' COLLISION KERNEL*

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Abstract. We investigate the Boltzmann equation with spatial smearing, diffusive boundary conditions, and Lions' collision kernel. Both the physical as well as the velocity space, are assumed to be bounded. Existence and uniqueness of a stationary solution, which is a probability density, has been demonstrated in [S. Caprino, M. Pulvirenti, and W. Wagner, *SIAM J. Math. Anal.*, 29 (1998), pp. 913–934] under a certain smallness assumption on the collision term. We prove that whenever there is a stationary solution then it is a.e. positively bounded from below and above.

Key words. Boltzmann equation, stationarity, spatial smearing, diffusive boundary conditions, Lions' collision kernel

AMS subject classifications. 76P05, 35Qxx

DOI. 10.1137/17M1160446

1. Introduction.

Over the past two decades, there has been a recurring interest in bounds on solutions to various forms of the Boltzmann equation. The different assumptions on the Boltzmann equation have led to different approaches.

For example, in [6] stochastic calculus (Malliavin calculus) is used to establish boundedness from below of solutions to a certain form of the Boltzmann equation. Furthermore, in [14] and [13] bounds in several function spaces and bounds by particular functions are derived by means of a detailed analysis of the collision kernel. In particular, regularity properties of the gain term are investigated. In [1] and [7] sophisticated comparison principles are provided in order to establish Maxwellian bounds.

The present paper uses a mathematical description of a rarefied gas in a vessel with diffusive boundary conditions introduced in [3]. The conditions on this form of the Boltzmann equation are physically motivated and allow us to demonstrate the existence of a unique probability density which is a stationary solution to this equation. In particular, boundedness of the physical space Ω, i.e., $\text{diam}(\Omega) := \sup\{|r_1 - r_2| : r_1, r_2 \in \Omega\} < \infty$, and strict positivity of the modulus of the velocity v of a particle, i.e., $0 < v_{min} < |v|$ imply that for all particles the free crossing time through the vessel is bounded by $\text{diam}(\Omega)/v_{min}$. Our analysis uses this assumption in (3.9), (3.19), and (4.28). In addition our analysis relies on the physically relevant hypothesis $|v| < v_{max} < \infty$. This assumption is crucial for the proof of Lemma 4.1 below.

From [3] and other references cited and discussed in [3] we take over the presence of spatial smearing in the collision operator and the boundedness of the collision kernel. These features entail the existence of a unique stationary solution which is a probability density, as demonstrated in the proof of Theorem 2.2 of [3].

The objective of the present paper is to show that whenever a Boltzmann equation in a form adapted from [3] and [11] has a stationary solution then it is a.e. positively bounded from below and above. The proof consists of two basic technical steps.

*Received by the editors December 8, 2017; accepted for publication (in revised form) October 3, 2018; published electronically November 15, 2018.
http://www.siam.org/journals/sima/50-6/M116044.html

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The first one is to provide suitable bounds on the gain term. This is carried out in section 3 by an iteration based on Lion's regularity result in [11, Theorem IV.1 and the subsequent Remark (ii)]. For this reason we use the particular form of the collision kernel in [11].

The second basic technical step in order to establish bounds on the stationary solution $g \equiv g(y,v)$, $y \in \overline{\Omega}$, $v \in V$, refers to the flux $J(g)$ in physical boundary points $r \in \partial\Omega$ and the total mass $\|g(y,\cdot)\|_{L^1(V)}$ in physical inner points $y \in \Omega$; the dot refers to integration over all possible velocities. In the proof of Lemma 4.1 below, we introduce an iteration to demonstrate that the function $\Omega \ni y \mapsto \|g(y,\cdot)\|_{L^1(V)}$ belongs to $L^q(\Omega)$ for all $1 \le q < \infty$ and that $J(g) \in L^q(\partial\Omega)$ for all $1 \le q < \infty$.

The results of the two basic steps then yield the a.e. positive lower and upper bounds on the stationary solution g; see Lemma 4.3 and Theorem 4.4.

2. Boltzmann equation with spatial smearing and diffusive boundary conditions. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded open set with smooth boundary, called the physical space. Furthermore, let $V := \{v \in \mathbb{R}^d : 0 < v_{min} < |v| < v_{max} < \infty\}$ be the velocity space and let $\lambda > 0$. Denote by $n(r)$ the outer normal at $r \in \partial\Omega$ and indicate the inner product in $\mathbb{R}^d$ by "$\circ$." Following [3], for $(r,v,t) \in \Omega \times V \times [0,\infty)$, consider the *Boltzmann equation*

$$\frac{d}{dt}\, p(r,v,t) = -v \circ \nabla_r p(r,v,t) + \lambda Q(p,p)\,(r,v,t)$$

with *diffusive boundary conditions*

$$p(r,v,t) = J(r,t)(p) M(r,v)\,, \quad r \in \partial\Omega,\ v \circ n(r) \le 0, \tag{2.1}$$

and initial condition $p(0,r,v) := p_0(r,v)$ or its *integrated (mild) version*

$$p(r,v,t) = S(t)\, p_0(r,v) + \lambda \int_0^t S(t-s)\, Q(p,p)\,(r,v,s)\, ds\,, \tag{2.2}$$

where we specify the following.

(i) The *flux* J is given by

$$J(r,t)(p) = \int_{\{v \in V : v \circ n(r) \ge 0\}} v \circ n(r)\, p(r,v,t)\, dv\,, \quad r \in \partial\Omega,\ t \ge 0.$$

(ii) The function M defined on the set $\{(r,v) : r \in \partial\Omega,\ v \in V,\ v \circ n(r) \le 0\}$ is continuous, has positive lower and upper bounds, and satisfies

$$\int_{\{v \in V : v \circ n(r) \le 0\}} |v \circ n(r)|\, M(r,v)\, dv = 1\,.$$

(iii) The semigroup $S(t)$, $t \ge 0$, in $L^1(\Omega \times V)$, called the *Knudsen semigroup*, is formally the solution to the initial boundary value problem

$$\left(\frac{d}{dt} + v \circ \nabla_r\right)(S(t)p_0)(r,v) = 0\,,$$
$$(S(t)p_0)(r,v) = J(r,t)(S(\cdot)p_0) M(r,v)\,, \quad r \in \partial\Omega,\ v \circ n(r) \le 0.$$

(iv) Denoting by χ the indicator function and setting $p := 0$ on $\Omega \times (\mathbb{R}^d \backslash V) \times [0,\infty)$, the *collision operator* Q is given by

$$Q(p,p)(r,v,t) = \int_\Omega \int_V \int_{S_+^{d-1}} B(v,v_1,e) h_\gamma(r,y) \chi_{\{(v^*,v_1^*) \in V \times V\}}(v,v_1,e)$$
$$\times \left(p(r,v^*,t)p(y,v_1^*,t) - p(r,v,t)p(y,v_1,t)\right)\, de\, dv_1\, dy\,.$$

Here S^{d-1} is the unit sphere. In addition, $S_+^{d-1} \equiv S_+^{d-1}(v-v_1) := \{e \in S^{d-1} : e \circ (v-v_1) > 0\}$, $v^* := v - e\circ(v-v_1)\, e$, $v_1^* := v_1 + e\circ(v-v_1)\, e$ for $e \in S_+^{d-1}$ as well as $v, v_1 \in V$, and de refers to the normalized Riemann–Lebesgue measure on S_+^{d-1}.

(v) The *collision kernel* $B \equiv B(v, v_1, e)$ is nonnegative, bounded, continuous on $V \times V \times S^{d-1}$, and symmetric in v and v_1. It satisfies $B(v^*, v_1^*, e) = B(v, v_1, e)$ for all $v, v_1 \in V$ and $e \in S_+^{d-1}$ for which $(v^*, v_1^*) \in V \times V$.

(vi) The *smearing* function h_γ is continuous on $\overline{\Omega \times \Omega}$, is nonnegative and symmetric, and vanishes for $|r - y| \geq \gamma > 0$.

Remark 2.1. For $t \geq 0$, let us regard

$$\Omega \times V \times [0, t] \ni (r, v, s) \mapsto S(t-s)\, Q(p,p)(r, v, s)$$

as a measurable function. Recall also that for every $t \geq 0$, $S(t) : L^1(\Omega \times V) \mapsto L^1(\Omega \times V)$ has operator norm 1. Noting that $L^1(\Omega \times V)$ is separable, the integral in (2.2) is a well-defined *Bochner integral* whenever

$$\int_0^t \|Q(p,p)(\cdot,\cdot,s)\|_{L^1(\Omega\times V)}\, ds < \infty\,.$$

Remark 2.2. We mention that the map

$$\mathbb{R}^{2d} \ni (v, v_1) \mapsto (v^*, v_1^*) := \big(v - e\circ(v-v_1)\, e\,,\ v_1 + e\circ(v-v_1)\, e\big)$$

has, for fixed $e \in S_+^{d-1}$, an inverse which we denote by (v^{-*}, v_1^{-*}). It is given by the relation

$$\mathbb{R}^{2d} \ni (v, v_1) = (v^* - e\circ(v^* - v_1^*)\, e\,,\ v_1^* + e\circ(v^* - v_1^*)\, e)\ .$$

In addition we mention that the absolute value of the Jacobian determinant of the map $(v, v_1) \mapsto (v^*, v_1^*)$ is one.

3. Analysis under Lions' assumptions on the collision kernel. In this section we examine the stationary solution to (2.2) under conditions that allow us to use results of [11]. In particular, we are interested in a certain upper bound on the gain term; see Lemma 3.4 below. Introduce $\theta := \arccos\,(e\circ(v-v_1)/|v-v_1|)$. Since we always suppose $e \in S_+^{d-1}(v-v_1)$ we have $\theta \in [0, \pi/2)$. In order to be compatible with [11], throughout this section we shall suppose that for all $(v, v_1) \in V \times V$ and $e \in S_+^{d-1}(v-v_1)$

$$B(v, v_1, e) = \mathbf{B}(|v-v_1|, \theta) \quad \text{for some} \quad \mathbf{B} \in C_c^\infty((0,\infty)\times(0,\pi/2))\,, \tag{3.1}$$

the space of all infinitely differentiable real functions with compact support contained in $(0,\infty)\times(0,\pi/2)$. We mention that B defined in this way satisfies (v) of section 2.

According to [3, Theorem 2.2] there is a $\lambda_0 > 0$ such that for $\lambda \leq \lambda_0$ we have the following. There exists a nonnegative a.e. on $\overline{\Omega} \times V$ defined real function $g \equiv g(\lambda)$ with $\|g\|_{L^1(\Omega\times V)} = 1$ such that $g_s(\cdot,\cdot,t) := g$ is the unique nonnegative stationary solution to (2.2) with $\|g_s(\cdot,\cdot,t)\|_{L^1(\Omega\times V)} = 1$, $t \geq 0$. The bound λ_0 is determined by (2.38), (2.17), and (3.9) of [3]. See also the remark after the proof of Theorem 2.2 in [3].

By the stationarity of $g_s(\cdot,\cdot,t)$ it is customary to write $Q(g,g)(r,v)$ instead of $Q(g_s, g_s)\ (r, v, t)$, $t \geq 0$. Because of

$$\int_0^t S(s)\, Q(g,g)(r,v)\, ds = \int_0^t S(t-s)\, Q(g,g)(r,v)\, ds$$

we have

$$g(r,v)=S(t)\,g(r,v)+\lambda\int_0^t S(s)\,Q(g,g)(r,v)\,ds\,,\quad t\ge 0; \tag{3.2}$$

for the precise meaning see Remark 3.2 below.

Let us introduce

$$\hat{B}(v,v_1):=\int_{S_+^{d-1}} B(v,v_1,e)\cdot\chi_{\{(v^*,v_1^*)\in V\times V\}}(v,v_1,e)\,de\,,\quad (v,v_1)\in V\times V,$$

and

$$g_\gamma(r,v_1):=\int_{y\in\Omega} g(y,v_1)\,h_\gamma(r,y)\,dy\,,\quad (r,v_1)\in\overline{\Omega}\times V.$$

We define now

$$\hat{B}_g(r,v):=\lambda\int_V \hat{B}(v,v_1)\,g_\gamma(r,v_1)\,dv_1\,,\quad (r,v)\in\overline{\Omega}\times V.$$

Furthermore, for $r\in\overline{\Omega}$ let us consider $g(r,\cdot)$ and $g_\gamma(r,\cdot)$ as functions defined on $\mathbb{R}^d$ by extending them by zero outside of V. Recalling the notation of section 2, in this section we shall use the decomposition $Q(g,g)=Q^+(g,g)-Q^-(g,g)$ of the collision operator specified by

$$\begin{aligned} &Q^+(g,g)(r,v)\\ &\quad=\int_V\int_{S_+^{d-1}} B(v,v_1,e)g(r,v^*)g_\gamma(r,v_1^*)\chi_{\{(v^*,v_1^*)\in V\times V\}}(v,v_1,e)\,de\,dv_1\\ &\quad=\int_{\mathbb{R}^d}\int_{S_+^{d-1}} B(v,v_1,e)g(r,v^*)g_\gamma(r,v_1^*)\,de\,dv_1\,. \end{aligned} \tag{3.3}$$

In fact, we have $\lambda Q^-(g,g)(r,v)=g(r,v)\hat{B}_g(r,v)$, where

$$\hat{B}_g(r,v)\le\lambda\|h_\gamma\|\|B\|\|g(\cdot,\cdot,t)\|_{L^1(\Omega\times V)}=\lambda\|h_\gamma\|\|B\|\,. \tag{3.4}$$

Remark 3.1. In other words, $\hat{B}_g$ is bounded on $\overline{\Omega}\times V$. Moreover, by (vi), the map $\overline{\Omega}\ni r\mapsto g_\gamma(r,\cdot)$ is bounded and uniformly continuous in $L^1(V)$. Thus by (v), $\hat{B}_g$ is bounded and continuous on $\overline{\Omega}\times V$.

Remark 3.2. It follows from Remark 2.2 that $\int_\Omega\int_V Q(g,g)(r,v)\,dv\,dr=0$. Thus we have

$$\begin{aligned} \|S(u)Q(g,g)\|_{L^1(\Omega\times V)}&\le\|Q(g,g)\|_{L^1(\Omega\times V)}\le 2\|Q^-(g,g)\|_{L^1(\Omega\times V)}\\ &=\frac{2}{\lambda}\|g\hat{B}_g\|_{L^1(\Omega\times V)}\le 2\|h_\gamma\|\|B\|<\infty\,,\quad u\ge 0, \end{aligned} \tag{3.5}$$

where, for the last estimate, we have taken into consideration $\|g\|_{L^1(\Omega\times V)}=1$ and we have applied (3.4). Recalling Remark 2.1, for $t\ge 0$ we may regard $\Omega\times V\times[0,t]\ni(r,v,u)\mapsto S(u)\,Q(g,g)(r,v,u)$ as a measurable function. By (3.5) and the separability of $L^1(\Omega\times V)$ the integral in (3.2) is a Bochner integral. Furthermore, according to [9, Appendix C] we may evaluate the integral a.e. on $\Omega\times V$.

For $(r,v) \in \overline{\Omega}\times V$ we will use the notation $T_\Omega \equiv T_\Omega(r,v) := \inf\{s>0 : r-sv \notin \Omega\}$, the first exit time from Ω of $[0,\infty) \ni t \mapsto r-tv$. Observe that for $(r,v) \in \partial\Omega \times V$ with $v \circ n(r) > 0$ we have $T_\Omega(r,v) > 0$ and that for $(r,v) \in \partial\Omega \times V$ with $v \circ n(r) < 0$ it holds that $T_\Omega(r,v) = 0$.

For $(y,v) \in \overline{\Omega} \times V$ let $y^- \equiv y^-(y,v) := y - T_\Omega(y,v)\,v$. Note that $y^- \in \partial\Omega$. Likewise, for $(r,v) \in \overline{\Omega} \times V$ define r^-. Furthermore, introduce $r^+ \equiv r^+(r,v) := r^-(r,-v)$ and observe that $r^-, r^+ \in \partial\Omega$ Let us also recall the definition of J in (i). Because of the stationarity of g in the sense of (3.2), $J(y,t)(g)$ is constant in the second argument t. We shall therefore write $J(y,\cdot)(g)$.

LEMMA 3.3. *Let g satisfy* (3.2) *in the sense of Remark* 3.2 *and let $(r,v) \in \Omega \times V$ or $(r,v) \in \partial\Omega \times V$ such that $v \circ n(r) \geq 0$. Then*

$$1 \leq \psi_g(r,v,t) := \exp\left\{\int_0^t \hat{B}_g(r-sv,v)\,ds\right\} \leq \sup \psi_g < \infty\,, \quad t \in [0, T_\Omega(r,v)],$$

where the supremum is taken over $\{(r,v,t) : (r,v) \in \Omega \times V,\ t \in [0,T_\Omega(r,v)]\}$. Suppose $g(r,v) < \infty$. Then

$$\begin{aligned} &g(r-tv,v) \\ &\quad = \psi_g(r,v,t)\left(-\int_0^t \frac{\lambda Q^+(g,g)(r-sv,v)}{\psi_g(r,v,s)}\,ds + g(r,v)\right), \quad t \in [0,T_\Omega(r,v)]. \end{aligned} \tag{3.6}$$

Proof. As already mentioned in Remark 3.1, $\hat{B}_g$ is bounded and continuous on $\overline{\Omega} \times V$. In particular, $[0,T_\Omega] \ni t \mapsto \hat{B}_g(r-tv,v)$ is continuous for all $(r,v) \in \Omega \times V$ or $(r,v) \in \partial\Omega \times V$ with $v \circ n(r) \geq 0$. According to (3.2) and Remark 3.2 we have for a.e. $(r,v) \in \Omega \times V$ and $t \in [0,T_\Omega]$ and, hence, also for a.e. $(r,v) \in \partial\Omega \times V$ with $v \circ n(r) \geq 0$ and $t \in [0,T_\Omega]$

$$\begin{aligned} g(r-tv,v) - g(r,v) &= -\int_0^t \lambda Q(g,g)(r-sv,v)\,ds \\ &= -\int_0^t \lambda Q^+(g,g)(r-sv,v)\,ds + \int_0^t \hat{B}_g(r-sv,v)\,g(r-sv,v)\,ds \end{aligned} \tag{3.7}$$

whenever $g(r,v) < \infty$. Again by the properties of $\hat{B}_g$ collected in Remark 3.1, the related homogeneous equation $\frac{d}{dt}\varphi(t) = \hat{B}_g(r-tv,v)\varphi(t)$, $t \in [0,T_\Omega]$, with initial value $\varphi(0) = g(r,v)$ has the unique solution

$$\varphi(t) = g(r,v)\,\psi_g(r,v,t) \equiv g(r,v)\exp\left\{\int_0^t \hat{B}_g(r-sv,v)\,ds\right\}, \quad t \in [0,T_\Omega], \tag{3.8}$$

whenever $g(r,v) < \infty$. En passant we note that, by (3.4), we have $1 \leq \psi_g$ and

$$\begin{aligned} \sup \psi_g &\leq \exp\left\{\sup_{(y,v)\in\overline{\Omega}\times V} T_\Omega(y,v)\cdot\lambda\|h_\gamma\|\|B\|\right\} \\ &\leq \exp\left\{\frac{\operatorname{diam}(\Omega)}{v_{min}}\cdot\lambda\|h_\gamma\|\|B\|\right\} < \infty, \end{aligned} \tag{3.9}$$

where the supremum on the left-hand side is taken over $\{(r,v,t) : (r,v) \in \Omega \times V,\ t \in [0,T_\Omega(r,v)]\}$.

Now recall (3.7) and keep in mind uniqueness of the related homogeneous equation. For $(r,v)\in\Omega\times V$ or $(r,v)\in\partial\Omega\times V$ with $v\circ n(r)\ge 0$ there is a unique solution to

$$\begin{aligned}&f(r-tv,v)-f(r,v)\\&\quad=-\int_0^t\lambda Q^+(g,g)(r-sv,v)\,ds+\int_0^t\hat{B}_g(r-sv,v)\,f(r-sv,v)\,ds\,,\end{aligned}$$

$t\in[0,T_\Omega]$, under the initial condition $f(0)=g(r,v)$ whenever $g(r,v)<\infty$. This solution is representable as the left-hand as well as the right-hand side of (3.6). ◻

LEMMA 3.4. *Let g satisfy* (3.2) *in the sense of Remark* 3.2 *and suppose* (3.1). *We have $g_\gamma\in L^\infty(\Omega\times V)$; note also Remark* 3.1. *Furthermore, there is a constant $0<c_Q<\infty$ independent of $(y,v)\in\Omega\times V$ such that*

$$Q^+(g,g)(y,v)\le c_Q\cdot\|g(y,\cdot)\|_{L^1(V)}\tag{3.10}$$

for a.e. $v\in V$ whenever $\|g(y,\cdot)\|_{L^1(V)}<\infty$.

Proof. Let $S'(\mathbb{R}^d)$ be the space of all tempered distributions on $\mathbb{R}^d$ and let $\hat{f}$ denote the Fourier transform of $f\in S'(\mathbb{R}^d)$. If $f\in L^1(\mathbb{R}^d)$, it is given by $\hat{f}(\xi)=\int_{\mathbb{R}^d}e^{-ix\circ\xi}f(x)\,dx$. For $s\in\mathbb{R}$ introduce the Bessel potential spaces

$$H^s(\mathbb{R}^d):=\left\{f\in S'(\mathbb{R}^d):\hat{f}\in L^2_{\mathrm{loc}}(\mathbb{R}^d),\ \int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|\hat{f}(\xi)\right|^2d\xi<\infty\right\}.$$

First we aim to show that for any $k\in\mathbb{N}$ there is a function $g^k\in H^{-2+\frac{k(d-1)}{2}}(\mathbb{R}^d)$ independent of $r\in\overline{\Omega}$ such that $g_\gamma(r,v)\le g^k(v)$ for $(r,v)\in\overline{\Omega}\times V$.

For a.e. $(r,v)\in\overline{\Omega}\times V$ we have by (3.6) and $\psi_g\ge 1$

$$\begin{aligned}g_\gamma(r,v)&=\int_{y\in\Omega}g(y,v)\,h_\gamma(r,y)\,dy\\&\le\|h_\gamma\|\int_{y\in\Omega}g(y,v)\,dy\\&\le\|h_\gamma\|\int_{y\in\Omega}g(y^-(y,v),v)\,dy\\&\quad+\|h_\gamma\|\int_{y\in\Omega}\int_0^{T_\Omega(y,v)}\lambda Q^+(g,g)(y-sv,v)\,ds\,dy\,.\end{aligned}\tag{3.11}$$

We shall treat both items on the right-hand side of (3.11) individually. By the definition of $y^-\equiv y^-(y,v)\in\partial\Omega$ in preparation of Lemma 3.3, we have $v\circ n(y^-)\le 0$ for all $(y,v)\in\Omega\times V$. In fact, v points from $y^-\in\partial\Omega$ to the inner of Ω while $n(y^-)$ is the outer normal at $y^-\in\partial\Omega$. Here, the boundary conditions (2.1) say that

$$g(y^-,v)=J(y^-,\cdot)(g)\cdot M(y^-,v)\,.$$

Next we recall that according to (ii), there exist $M_{\min},M_{\max}\in(0,\infty)$ such that $M_{\min}\le M(z,v)\le M_{\max}$ for all $z\in\partial\Omega$ and all $v\in V$ with $v\circ n(z)\le 0$. We fix $v\in V$

for the next chain of equations and inequalities and obtain

$$\begin{aligned}
\int_{y\in\Omega} g(y^-(y,v),v)\,dy &= \int_{y\in\Omega} J(y^-(y,v),\cdot)(g)M(y^-(y,v),v)\,dy \\
&\le M_{\max}\int_{y\in\Omega} J(y^-(y,v),\cdot)(g)\,dy \\
&= M_{\max}\int_{\{r\in\partial\Omega: v\circ n(r)\le 0\}}\int_{t\in[0,T_\Omega(r,-v)]} J(r,\cdot)(g)|v|\,dt\,\left(\frac{-v}{|v|}\right)\circ n(r)\,dr \\
&= M_{\max}\int_{\{r\in\partial\Omega: v\circ n(r)\le 0\}}\int_{t\in[0,T_\Omega(r,-v/|v|)]} J(r,\cdot)(g)\,dt\,\frac{|v\circ n(r)|}{|v|}\,dr \\
&= M_{\max}\int_{\{r\in\partial\Omega: v\circ n(r)\le 0\}} J(r,\cdot)(g)\cdot|r^+(r,v)-r|\,\frac{|v\circ n(r)|}{|v|}\,dr \\
&\le \operatorname{diam}(\Omega)M_{\max}\int_{\{r\in\partial\Omega: v\circ n(r)\le 0\}} J(r,\cdot)(g)\,dr,
\end{aligned}$$

where in the third line we have put $y^-(y,v) =: r$ which implies that $y = r+tv$ for some $t\in[0,T_\Omega(r,-v)]$. In particular, this substitution yields $dy = |v|\,dt\cdot(-v/|v|)\circ n(r)\,dr$.

Moreover, set

$$C_M := \left(M_{\min}\cdot\int_{\{w\in V: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|^2}{|w|^2}\,dw\right)^{-1}$$

and note that $C_M\in(0,\infty)$ is independent of $r\in\partial\Omega$. Taking into consideration (2.1) we verify that

$$\begin{aligned}
J(r,\cdot)(g) &\le C_M J(r,\cdot)(g)\int_{\{w\in V: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|^2}{|w|^2}M(r,w)\,dw \\
&= C_M\int_{\{w\in V: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|^2}{|w|^2}g(r,w)\,dw
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{y\in\Omega} g(y^-(y,v),v)\,dy \\
&\le \operatorname{diam}(\Omega)\cdot C_M M_{\max}\int_{r\in\partial\Omega}\int_{\{w\in V: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|^2}{|w|^2}g(r,w)\,dw\,dr \qquad (3.12)\\
&= \operatorname{diam}(\Omega)\cdot C_M M_{\max}\int_{w\in V}\int_{\{r\in\partial\Omega: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|^2}{|w|^2}g(r,w)\,dr\,dw
\end{aligned}$$

if the right-hand side is finite. Keeping in mind that Ω is a bounded domain with smooth boundary it turns out that there is a constant $C_\Omega>0$ only depending on Ω such that

$$\left|\frac{r-y}{|r-y|}\circ n(r)\right| \le C_\Omega|r-y| \quad y,r\in\partial\Omega. \tag{3.13}$$

As a consequence, we have $|w\circ n(r)|/|w| \le C_\Omega\,|r^+(r,w)-r|$ for all $w\in V$ and all $r\in\partial\Omega$ with $w\circ n(r)\le 0$. Thus, for any $w\in V$ and $r\in\partial\Omega$ such that $w\circ n(r)\le 0$ it holds that

$$\frac{|w\circ n(r)|}{|w|}\,g(r,w) \le C_\Omega\int_{t=0}^{T_\Omega(r^+,w)} g(r,w)|w|\,dt\,.$$

Since $g(r,w) = g((r+tw)-tw,w) \le \sup\psi_g\, g(r+tw,w)$ for $t\in[0,T_\Omega(r^+,w)]$ by (3.6), we have

$$\frac{|w\circ n(r)|}{|w|}\, g(r,w) \le C_\Omega \sup\psi_g \int_{t=0}^{T_\Omega(r^+,w)} g(r+tw,w)|w|\,dt\,. \tag{3.14}$$

This and

$$\int_{w\in V}\int_{\{r\in\partial\Omega: w\circ n(r)\le 0\}} \frac{|w\circ n(r)|}{|w|}\int_{t=0}^{T_\Omega(r^+,w)} g(r+tw,w)|w|\,dt\,dr\,dw$$
$$= \|g\|_{L^1(\Omega\times V)} = 1$$

imply that the right-hand side of (3.12) is finite. Writing $\tilde C$ for $C_\Omega\cdot\mathrm{diam}(\Omega)\cdot\sup\psi_g\cdot C_M M_{\max}$ we deduce from (3.12), (3.14), and the last calculation that

$$\int_{y\in\Omega} g(y^-(y,v),v)\,dy \le \tilde C\,. \tag{3.15}$$

In order to find an upper bound for the second item of (3.11) we shall apply the main result of [11, namely, Theorem IV.1 and Remark (ii)]. Note also the reformulation in Theorem L of [12]. In this regard let us recall the particular form of B, $B(v,v_1,e) = \mathbf{B}(|v-v_1|,\theta)$ for some $\mathbf{B}\in C_c^\infty((0,\infty)\times(0,\pi/2))$, where $\theta = \arccos\left(e\circ(v-v_1)/|v-v_1|\right)\in[0,\pi/2)$.

By the first line of (3.11) we have

$$g^0 := \sup_{r\in\overline{\Omega}} g_\gamma(r,\cdot)\in L^1(\mathbb{R}^d)\subseteq\{\hat f : f\in L^\infty(\mathbb{R}^d)\}\subseteq H^{-2}(\mathbb{R}^d). \tag{3.16}$$

For the last inclusion consult [8, Theorem 7.9.3]. Moreover,

$$\int_\Omega g(y,\cdot)\,dy\in L^1(\mathbb{R}^d)\,. \tag{3.17}$$

Now extend B to $\mathbb{R}^d\times\mathbb{R}^d$ by setting zero outside of $V\times V$ and introduce

$$\tilde g^1(y,v) := \lambda\int_{\mathbb{R}^d}\int_{S_+^{d-1}} B(v,v_1,e)\, g(y,v^*)g^0(v_1^*)\,de\,dv_1\,, \tag{3.18}$$

$y\in\Omega$, $v\in\mathbb{R}^d$. By (3.3) we obtain

$$\begin{aligned}
&\int_{y\in\Omega}\int_0^{T_\Omega(y,v)} \lambda Q^+(g,g)(y-sv,v)\,ds\,dy\\
&\quad = \frac{1}{|v|}\int_{y\in\Omega}\int_0^{T_\Omega(y,v)} \lambda Q^+(g,g)(y-sv,v)|v|\,ds\,dy\\
&\quad \le \frac{1}{|v|}\int_{y\in\Omega}\int_0^{T_\Omega(y,v)} \tilde g^1(y-sv,v)|v|\,ds\,dy\\
&\quad \le \frac{\mathrm{diam}(\Omega)}{v_{min}}\int_{y\in\Omega}\tilde g^1(y,v)\,dy\\
&\quad = \frac{\lambda\,\mathrm{diam}(\Omega)}{v_{min}}\int_{\mathbb{R}^d}\int_{S_+^{d-1}} B(v,v_1,e)\int_\Omega g(y,v^*)\,dy\, g^0(v_1^*)\,de\,dv_1\,.
\end{aligned} \tag{3.19}$$

Keeping (3.16) and (3.17) in mind, by the just mentioned result of Lions [11, Theorem IV.1 and the subsequent Remark (ii)], we may claim that

$$\int_{y\in\Omega} \tilde{g}^1(y,\cdot)\,dy \in H^{-2+\frac{d-1}{2}}(\mathbb{R}^d)\,. \tag{3.20}$$

In addition, using the integration by substitution of Remark 2.1 and $B(v^*, v_1^*, e) = B(v, v_1, e)$ for all $v, v_1 \in V$ and $e \in S_+^{d-1}$ for which $(v^*, v_1^*) \in V \times V$, we find

$$\begin{aligned}\int_{v\in\mathbb{R}^d}\int_{y\in\Omega} \tilde{g}^1(y,v)\,dy\,dv \\ &= \lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{S_+^{d-1}} B(v,v_1,e)\,\int_{\Omega} g(y,v)\,dy\,g^0(v_1)\,de\,dv_1\,dv \\ &\le \lambda \|h_\gamma\|\|B\| < \infty\,.\end{aligned}$$

Thus (3.20) defines an element

$$\int_{y\in\Omega} \tilde{g}^1(y,\cdot)\,dy \in H^{-2+\frac{d-1}{2}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\,. \tag{3.21}$$

Let $C_c^\infty(\mathbb{R}^d)$ denote the set of all infinitely differentiable real functions on $\mathbb{R}^d$ which have compact support. Set $C_1 := \|h_\gamma\| \cdot \tilde{C}$ and choose a nonnegative $\Phi_1 \in C_c^\infty(\mathbb{R}^d)$ with $C_1 \le \Phi_1$ on V. With

$$c_1 := \|h_\gamma\|\,\mathrm{diam}(\Omega)\,v_{min}^{-1}$$

it follows from (3.11), (3.15), (3.19), and (3.21) that

$$\begin{aligned}g_\gamma(r,\cdot) &\le \sup_{r\in\overline{\Omega}} g_\gamma(r,\cdot) = g^0 \\ &\le \Phi_1 + c_1 \int_{y\in\Omega} \tilde{g}^1(y,\cdot)\,dy =: g^1 \quad \text{on } V \text{ and hence on } \mathbb{R}^d\end{aligned}$$

for a.e. $r \in \overline{\Omega}$ and

$$g^1 \in H^{-2+\frac{d-1}{2}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\,.$$

We reiterate from (3.16) to here replacing g^0 by g^{k-1} and $\tilde{g}^1(y,v)$ by

$$\lambda \int_{\mathbb{R}^d}\int_{S_+^{d-1}} B(v,v_1,e)\,g(y,v^*)g^{k-1}(v_1^*)\,de\,dv_1 =: \tilde{g}^k(y,v)\,, \tag{3.22}$$

$y \in \Omega$, $v \in \mathbb{R}^d$ in order to obtain

$$\begin{aligned}g_\gamma(r,\cdot) &\le g^0 \le g^1 \le \cdots \le g^k \\ &:= \Phi_1 + c_1 \int_{y\in\Omega} \tilde{g}^k(y,\cdot)\,dy \in H^{-2+\frac{k(d-1)}{2}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\end{aligned} \tag{3.23}$$

for a.e. $r \in \overline{\Omega}$ and for all $k \in \mathbb{N}$. By Lions' theorem it also holds that

$$\tilde{g}^1(y,\cdot) \le \cdots \le \tilde{g}^{k+1}(y,\cdot) \in H^{-2+\frac{(k+1)(d-1)}{2}}(\mathbb{R}^d) \tag{3.24}$$

provided that $\|g(y,\cdot)\|_{L^1(V)} < \infty$.

We mention furthermore that, for those $l = -2+k(d-1)/2$ which are nonnegative integers, the Bessel potential space $H^{-2+\frac{k(d-1)}{2}}(\mathbb{R}^d)$ coincides with the Sobolev space $W^{l,2}(\mathbb{R}^d)$. In particular the norms are equivalent. In this case

$$W^{l,2}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d) \quad \text{continuously if} \quad l > \frac{d}{2}\,;$$

cf. [2, Corollary 9.13]. Relation (3.23) implies the a.e. boundedness of g_γ on $\overline{\Omega} \times V$. We have proved the first statement of the lemma.

In order to show (3.10) we use the norm estimate in [11, Remark (ii) to Theorem IV.1]. Let $0 < C < \infty$ be the constant introduced there. Moreover, let $0 < C_{l'} < \infty$ denote the constant from the Sobolev inequality between the spaces $H^{l'}(\mathbb{R}^d) = W^{l',2}(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. If $\|g(y,\cdot)\|_{L^1(V)} < \infty$ then according to (3.18) and (3.22)–(3.24) the following holds. If $l' := -2 + (k+1)(d-1)/2$ is a natural number and $l' > d/2$ then we have

$$\begin{aligned}\lambda Q^+(g,g)(y,v) &\le \tilde{g}^1(y,v) \le \tilde{g}^{k+1}(y,v) \le \|\tilde{g}^{k+1}(y,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\le C_{l'}\|\tilde{g}^{k+1}(y,\cdot)\|_{H^{l'}(\mathbb{R}^d)} \\ &\le C_{l'}\, C \cdot \|g^k\|_{H^l(\mathbb{R}^d)}\, \|g(y,\cdot)\|_{L^1(V)} \equiv \lambda c_Q \|g(y,\cdot)\|_{L^1(V)}\end{aligned}$$

for a.e. $v \in V$. ∎

4. Boundedness properties. In the proof of the subsequent lemma we shall use the notion of *narrow convergence* of finite measures on $\mathbb{R}^d$ in the sense of [4]. We say that a sequence of finite measures μ_n on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ converges *narrowly* to some finite measures μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for all real bounded continuous functions f on $\mathbb{R}^d$.

LEMMA 4.1. *Let g satisfy* (3.2) *in the sense of Remark* 3.2 *and suppose* (3.1). *The function $\Omega \ni y \mapsto \|g(y,\cdot)\|_{L^1(V)}$ belongs to $L^q(\Omega)$ for all $1 \le q < \infty$ and $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$ belongs to $L^q(\partial\Omega)$ for all $1 \le q < \infty$.*

Proof. Step 1. We establish an iteration to prove the lemma. The present Step 1 is dedicated to the particular case $q = 1$, the *initialization step* of the iteration. Because $\|g\|_{L^1(\Omega\times V)} = 1$ we just have to focus on $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$.

According to (2.1) and (3.6) we have

$$\begin{aligned}1 &= \int_\Omega \int_V g(y,v)\, dv\, dy \\ &\ge \frac{1}{\sup \psi_g} \int_\Omega \int_V g(y^-(y,v),v)\, dv\, dy \\ &\ge \frac{M_{\min}}{\sup \psi_g} \int_\Omega \int_V J(y^-(y,v),\cdot)(g)\, dv\, dy\,.\end{aligned} \tag{4.1}$$

We let $v = \alpha e$, where $\alpha \in (v_{min}, v_{max})$ and $e \in S^{d-1}$. Furthermore we denote by l_S the Riemann–Lebesgue measure on $(S^{d-1}, \mathcal{B}(S^{d-1}))$. Set

$$I_V(m) := \int_{v_{min}}^{v_{max}} \alpha^{m-1}\, d\alpha\,, \quad m \in \mathbb{N}.$$

It follows that

$$\begin{aligned}\int_V J(y^-(y,v),\cdot)(g)\,dv &= \int_{v_{min}}^{v_{max}} \alpha^{d-1} \int_{S^{d-1}} J(y^-(y,\alpha e),\cdot)(g)\,dl_S(e)\,d\alpha \\ &\geq I_V(d) \int_{S^{d-1}} J(y^-(y,\cdot\, e),\cdot)(g)\,dl_S(e),\end{aligned}$$

where we note that $y^-(y,\alpha e) \in \partial\Omega$ is independent of $\alpha \in (v_{min}, v_{max})$ and therefore appears as $y^-(y,\cdot\, e)$ in the second line. Let us denote

$$(\partial\Omega)_y := \{r \in \partial\Omega : \{r + a(y-r) : a \in (0,1)\} \subset \Omega\}, \quad y \in \overline{\Omega},$$

and

$$(\Omega)_r := \{y \in \Omega : \{r + a(y-r) : a \in (0,1)\} \subset \Omega\}, \quad r \in \partial\Omega\,.$$

We mention that for $y \in \Omega$ and $r \in \partial\Omega$ we have $r \in (\partial\Omega)_y$ if and only if $y \in (\Omega)_r$. Observe also that for $e \in S^{d-1}$, $y \in \Omega$, and $r := y^-(y,\cdot\, e)$ we have $r \in (\partial\Omega)_y$, $e = (y-r)/|y-r|$, and

$$dl_S(e) = |y-r|^{1-d}\,dr \cdot n(r)\circ(-e) = |y-r|^{1-d} \cdot \frac{n(r)\circ(r-y)}{|y-r|}\,dr\,.$$

Summarizing the preparations from relation (4.1) to here, we obtain

$$\begin{aligned}1 &\geq \frac{M_{\min}\cdot I_V(d)}{\sup\psi_g} \int_\Omega \int_{S^{d-1}} J(y^-(y,\cdot\, e),\cdot)(g)\,dl_S(e)\,dy \\ &= \frac{M_{\min}\cdot I_V(d)}{\sup\psi_g} \int_{y\in\Omega} \int_{r\in(\partial\Omega)_y} |y-r|^{1-d}\cdot \frac{n(r)\circ(r-y)}{|y-r|} J(r,\cdot)(g)\,dr\,dy \\ &= \frac{M_{\min}\cdot I_V(d)}{\sup\psi_g} \int_{r\in\partial\Omega} \int_{y\in(\Omega)_r} |y-r|^{1-d}\cdot \frac{n(r)\circ(r-y)}{|y-r|} J(r,\cdot)(g)\,dy\,dr \\ &= C_d \int_{\partial\Omega} J(r,\cdot)(g)\rho_d(r)\,dr\,,\end{aligned} \tag{4.2}$$

where $C_d := M_{\min}\cdot I_V(d)/\sup\psi_g$ and

$$\rho_d(r) := \int_{(\Omega)_r} |y-r|^{1-d}\cdot \frac{n(r)\circ(r-y)}{|y-r|}\,dy\,.$$

There is some $c_d > 0$, which depends on Ω but not on $r \in \partial\Omega$, such that $\infty > \rho_d(r) \geq c_d$ for all $r \in \partial\Omega$. Thus from (4.2) we may conclude that

$$\int_{\partial\Omega} J(r,\cdot)(g)\,dr \leq \frac{1}{c_d\,C_d}\,. \tag{4.3}$$

Step 2. In this step we assume that $\Omega \ni y \mapsto \|g(y,\cdot)\|_{L^1(V)}$ belongs to $L^q(\Omega)$ and that $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$ belongs to $L^q(\partial\Omega)$ for some $1 \leq q < \infty$. It is our aim to show that $\Omega \ni y \mapsto \|g(y,\cdot)\|_{L^1(V)}$ belongs to $L^{pq}(\Omega)$ and that $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$ belongs to $L^{pq}(\partial\Omega)$ for all $1 \leq p < d/(d-1)$. In other words, the present Step 2 is the *execution step* of the iteration.

According to (3.6) it holds with an appropriate constant $0 < c_q < \infty$ that

$$\begin{aligned} \|g(y,\cdot)\|^q_{L^1(V)} &\le c_q \int_V (g(y^-(y,v),v))^q\,dv \\ &\quad + c_q \left(\int_V \int_0^{T_\Omega} \lambda Q^+(g,g)(y-sv,v)\,ds\,dv \right)^q \quad \text{for a.e. } y \in \Omega, \end{aligned} \tag{4.4}$$

where we have taken into consideration that $\psi_g \ge 1$. The two integrals on the right-hand side are now treated separately. For this we let $v = \alpha e$ where $\alpha \in (v_{min}, v_{max})$ and $e \in S^{d-1}$. We recall that l_S denotes the Riemann–Lebesgue measure on $(S^{d-1}, \mathcal{B}(S^{d-1}))$. We have

$$\begin{aligned} &\int_V (g(y^-(y,v),v))^q\,dv \\ &\qquad = \int_{v_{min}}^{v_{max}} \alpha^{d-1} \int_{S^{d-1}} (J(y^-(y,\alpha e),\cdot)(g))^q (M(y^-,\alpha e))^q\,dl_S(e)\,d\alpha \\ &\qquad \le M^q_{\max} I_V(d) \int_{S^{d-1}} (J(y^-(y,\cdot\, e),\cdot)(g))^q\,dl_S(e), \end{aligned}$$

where we note again that $y^-(y,\alpha e) \in \partial\Omega$ is independent of $\alpha \in (v_{min}, v_{max})$ and therefore appears as $y^-(y,\cdot\, e)$ in the last line. We obtain

$$\begin{aligned} &\int_V (g(y^-(y,v),v))^q\,dv \\ &\qquad \le M^q_{\max} I_V(d) \int_{r\in(\partial\Omega)_y} |y-r|^{1-d} \cdot \frac{n(r)\circ(r-y)}{|y-r|} (J(r,\cdot)(g))^q\,dr \\ &\qquad \le M^q_{\max} I_V(d) \int_{r\in\partial\Omega} |y-r|^{1-d}\,(J(r,\cdot)(g))^q\,dr\,. \end{aligned} \tag{4.5}$$

Introduce

$$f(s) := \begin{cases} \exp\left\{-1/\left(1-s^2\right)\right\}, & s \in (-1,1)\,, \\ 0\,, & s \in \mathbb{R}\setminus(-1,1)\,, \end{cases}$$

$m_d := \left(\int_{\mathbb{R}^d} f(|x|)\,dx\right)^{-1}$, and the mollifier function $\psi_d(x) := m_d \cdot f(|x|)$, $x \in \mathbb{R}^d$. Define

$$\gamma_n(x) := n^d \int_{r\in\partial\Omega} \psi_d(n(x-r))(J(r,\cdot)(g))^q\,dr\,, \quad x \in \mathbb{R}^d,\ n \in \mathbb{N}.$$

We observe that the sequence of measures Γ_n, $n \in \mathbb{N}$, given by $\Gamma_n(A) := \int_A \gamma_n(x)\,dx$, $A \in \mathcal{B}(\mathbb{R}^d)$, converges narrowly as $n \to \infty$ to the measure Γ defined by

$$\Gamma(A) := \int_{A\cap\partial\Omega} (J(r,\cdot)(g))^q\,dr\,, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

We remark that the measure Γ is finite because of the assumption of Step 2 that $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$ belongs to $L^q(\partial\Omega)$.

Introduce $\Omega_1 := \{z \in \mathbb{R}^d : \inf_{y\in\Omega} |z-y| < 1\}$. The inequality (4.5) now implies

$$\begin{aligned}
&\int_V (g(y^-(y,v),v))^q\,dv \\
&\quad\le M_{\max}^q I_V(d) \int_{r\in\partial\Omega} |y-r|^{1-d}\,(J(r,\cdot)(g))^q\,dr \\
&\quad= M_{\max}^q I_V(d) \lim_{n\to\infty} \int_{x\in\Omega_1} |y-x|^{1-d}\gamma_n(x)\,dx\,,
\end{aligned} \tag{4.6}$$

where the sequence of functions $\mathbb{R}^d \ni y \mapsto \int_{x\in\Omega_1} |y-x|^{1-d}\gamma_n(x)\,dx$, $n\in\mathbb{N}$, is bounded in $L^p(\mathbb{R}^d)$ for $1 \le p < d/(d-1)$ because of

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \int_\Omega \left(\int_{x\in\Omega_1} |y-x|^{1-d}\gamma_n(x)\,dx\right)^p dy \\
&\le \sup_{n\in\mathbb{N}} \int_\Omega \left(\int_{x\in\Omega_1} \gamma_n(x)\,dx\right)^{p-1} \int_{x\in\Omega_1} |y-x|^{(1-d)p}\gamma_n(x)\,dx\,dy \\
&\le \sup_{n\in\mathbb{N}} \left(\int_{x\in\Omega_1} \gamma_n(x)\,dx\right)^p \sup_{x\in\Omega_1} \int_\Omega |y-x|^{(1-d)p}\,dy\,,
\end{aligned} \tag{4.7}$$

the definition of γ_n, $n\in\mathbb{N}$, and the finiteness of the measure Γ.

Furthermore, the limit in (4.6) is weakly in $L^p(\Omega)$ for $1 \le p < d/(d-1)$, i.e., $\Omega \ni y \mapsto \int_{r\in\partial\Omega} |y-r|^{1-d}\,(J(r,\cdot)(g))^q\,dr$, belongs also to $L^p(\Omega)$ for $1\le p < d/(d-1)$ under the assumption of Step 2 that $\int_{r\in\partial\Omega}(J(r,\cdot)(g))^q\,dr < \infty$. The mode of convergence in (4.6), i.e., weak convergence in $L^p(\Omega)$, can be justified as follows. On the one hand, for any test function $\varphi \in C_b(\Omega)$ and

$$\Phi(x) := \int_{y\in\Omega} \varphi(y)\,|y-x|^{(1-d)}\,dy\,, \quad x\in\mathbb{R}^d,$$

we have $\Phi \in C_b(\mathbb{R}^d)$ and

$$\begin{aligned}
&\int_{y\in\Omega} \varphi(y) \int_{x\in\Omega_1} |y-x|^{1-d}\gamma_n(x)\,dx\,dy \\
&\quad= \int_{x\in\Omega_1} \Phi(x)\,\gamma_n(x)\,dx \quad \underset{n\to\infty}{\longrightarrow} \quad \int_{r\in\partial\Omega} \Phi(r)(J(r,\cdot)(g))^q\,dr \\
&\quad= \int_{y\in\Omega} \varphi(y) \int_{r\in\partial\Omega} |y-r|^{1-d}\,(J(r,\cdot)(g))^q\,dr\,dy
\end{aligned}$$

by the narrow convergence of Γ_n to Γ as $n\to\infty$; note that $\Gamma_n(\mathbb{R}^d\backslash\Omega_1) = \Gamma(\mathbb{R}^d\backslash\partial\Omega) = 0$. On the other hand, by (4.7), and the assumption $\int_{r\in\partial\Omega}(J(r,\cdot)(g))^q\,dr < \infty$,

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \int_{x\in\Omega_1} |\cdot - x|^{1-d}\gamma_n(x)\,dx \right\|_{L^p(\Omega)}^p \\
&\quad\le \lim_{n\to\infty} \left(\int_{x\in\Omega_1} \gamma_n(x)\,dx\right)^p \sup_{x\in\Omega_1} \int_\Omega |y-x|^{(1-d)p}\,dy \\
&\quad= \left(\int_{r\in\partial\Omega} (J(r,\cdot)(g))^q\,dr\right)^p \sup_{x\in\Omega_1} \int_\Omega |y-x|^{(1-d)p}\,dy < \infty\,.
\end{aligned}$$

The last two chains of equalities and inequalities verify the claimed weak convergence in $L^p(\Omega)$ in (4.6).

Setting

$$C_{Q,v} := c_q c_Q^q \lambda^q (I_V(d-1))^q (l_S(S^{d-1}))^{q-1} \operatorname{diam}(\Omega)^{q-1}$$

for the second integral in (4.4) we obtain from (3.10)

$$\begin{aligned}
&\left(\int_V \int_0^{T_\Omega(y,v)} \lambda Q^+(g,g)(y-sv,v)\,ds\,dv\right)^q \\
&\quad = \left(\int_{v_{min}}^{v_{max}} \alpha^{d-1} \int_{S^{d-1}} \int_0^{T_\Omega(y,\alpha e)} \lambda Q^+(g,g)(y-s\,\alpha e,\alpha e)\,ds\,dl_S(e)\,d\alpha\right)^q \\
&\quad \le c_Q^q \lambda^q \left(\int_{v_{min}}^{v_{max}} \alpha^{d-2} \int_{S^{d-1}} \int_0^{T_\Omega(y,\alpha e)} \|g(y-s\,\alpha e,\cdot)\|_{L^1(V)}\,\alpha\,ds\,dl_S(e)\,d\alpha\right)^q \\
&\quad \le c_Q^q \lambda^q (I_V(d-1))^q \left(\int_{S^{d-1}} \int_0^{T_\Omega(y,e)} \|g(y-se,\cdot)\|_{L^1(V)}\,ds\,dl_S(e)\right)^q \\
&\quad \le \frac{C_{Q,v}}{c_q} \int_{S^{d-1}} \int_0^{T_\Omega(y,e)} \|g(y-se,\cdot)\|^q_{L^1(V)}\,ds\,dl_S(e) \\
&\quad \le \frac{C_{Q,v}}{c_q} \int_\Omega |x-y|^{1-d} \cdot \|g(x,\cdot)\|^q_{L^1(V)}\,dx\,.
\end{aligned} \tag{4.8}$$

With

$$C_{M,v} := c_q M^q_{\max} I_V(d)$$

it follows now from (4.4), (4.6), and (4.8) that for a.e. $y \in \Omega$

$$\begin{aligned}
\|g(y,\cdot)\|^q_{L^1(V)} \le C_{M,v} \lim_{n\to\infty} \int_{x\in\Omega_1} |y-x|^{1-d} \gamma_n(x)\,dx \\
+ C_{Q,v} \int_{x\in\Omega} |y-x|^{1-d} \|g(x,\cdot)\|^q_{L^1(V)}\,dx\,.
\end{aligned} \tag{4.9}$$

According to the argument below (4.7),

$$\Omega \ni y \mapsto \int_{r\in\partial\Omega} |y-r|^{1-d}\,(J(r,\cdot)(g))^q\,dr \quad \text{belongs to } L^p(\Omega)$$

for $1 \le p < d/(d-1)$ by the assumption $\int_{r\in\partial\Omega} (J(r,\cdot)(g))^q\,dr < \infty$. Furthermore, according to

$$\begin{aligned}
&\int_\Omega \left(\int_{x\in\Omega} |y-x|^{1-d} \|g(x,\cdot)\|^q_{L^1(V)}\,dx\right)^p dy \\
&\qquad \le \left(\int_{x\in\Omega} \|g(x,\cdot)\|^q_{L^1(V)}\,dx\right)^p \sup_{x\in\Omega} \int_\Omega |y-x|^{(1-d)p}\,dy\,,
\end{aligned}$$

the function

$$\Omega \ni y \mapsto \int_{x\in\Omega} |y-x|^{1-d} \|g(x,\cdot)\|^q_{L^1(V)}\,dx \quad \text{belongs to } L^p(\Omega)$$

for $1 \le p < d/(d-1)$ under the assumption of Step 2 that $\int_\Omega \|g(x,\cdot)\|^q_{L^1(V)}\, dx < \infty$. With this discussion in mind, it follows from (4.9) that

$$\begin{aligned} &\int_\Omega \|g(y,\cdot)\|^{pq}_{L^1(V)}\, dy < \infty \quad \text{if} \\ &\int_{\partial\Omega} (J(r,\cdot))^q\, dr < \infty \quad \text{and} \quad \int_\Omega \|g(x,\cdot)\|^q_{L^1(V)}\, dx < \infty. \end{aligned} \tag{4.10}$$

According to (3.6) we have for some suitable constant $0 < c_{J,1} < \infty$ only depending on V, p, and q,

$$\begin{aligned} (J(r,\cdot)(g))^{pq} &= \left(\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, g(r,v)\, dv \right)^{pq} \\ &\le c_{J,1} M^{pq}_{\max} \left(\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, J(r^-,\cdot)(g)\, dv \right)^{pq} \\ &\quad + c_{J,1} \left(\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r) \int_0^{T_\Omega(r,v)} \lambda Q^+(g,g)(r-sv,v)\, ds\, dv \right)^{pq} \end{aligned} \tag{4.11}$$

for a.e. $r \in \partial\Omega$. As above, the two items on the right-hand side will be treated separately. Recalling the definition of C_Ω in (3.13) we get for the first item of the right-hand side of (4.11)

$$\begin{aligned} &\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, J(r^-,\cdot)(g)\, dv \\ &\quad = \int_{v_{min}}^{v_{max}} \alpha^{d-1} \int_{e\in S^{d-1}_+(n(r))} \alpha e\circ n(r)\, J(r^-(r,\cdot e),\cdot)(g)\, dl_S(e)\, d\alpha \\ &\quad = \int_{v_{min}}^{v_{max}} \alpha^{d} \int_{e\in S^{d-1}_+(n(r))} e\circ n(r)\, J(r^-(r,\cdot e),\cdot)(g)\, dl_S(e)\, d\alpha \\ &\quad = I_V(d+1) \int_{y\in(\partial\Omega)_r} \frac{(r-y)\circ n(r)}{|r-y|}\, J(y,\cdot)(g)|r-y|^{1-d} \cdot \frac{n(y)\circ(y-r)}{|r-y|}\, dy \\ &\quad \le C_\Omega^2 I_V(d+1) \int_{y\in(\partial\Omega)_r} J(y,\cdot)(g)|r-y|^{3-d}\, dy \\ &\quad \le C_\Omega^2 I_V(d+1)(\operatorname{diam}(\Omega))^{3-d} \int_{y\in\partial\Omega} J(y,\cdot)(g)\, dy\,, \end{aligned} \tag{4.12}$$

where we are reminded that $d = 2$ or $d = 3$. Consequently, with some suitable constant $0 < c_{J,2} < \infty$ only depending on Ω, V, p, and q it holds that

$$\begin{aligned} &\left(\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, J(r^-,\cdot)(g)\, dv \right)^{pq} \\ &\qquad \le c_{J,2} \left(\int_{\{v\in V: v\circ n(r)\ge 0\}} (J(y,\cdot)(g))^q\, dy \right)^p . \end{aligned}$$

Recalling the assumption $\int_{y\in\partial\Omega} (J(y,\cdot)(g))^q\, dy < \infty$ we obtain

$$\int_{r\in\partial\Omega} \left(\int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, J(r^-,\cdot)(g)\, dv \right)^{pq} dr < \infty\,. \tag{4.13}$$

Let us turn to the second item in (4.11). Here we follow and slightly modify the calculations performed in (4.8) to obtain

$$
\begin{aligned}
&\int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\int_0^{T_\Omega(r,v)} \lambda Q^+(g,g)(r-sv,v)\,ds\,dv \\
&\qquad \le c_Q\lambda\int_{\{v\in V:v\circ n(r)\ge 0\}}\int_0^{T_\Omega(r,v)} \|g(r-sv,\cdot)\|_{L^1(V)}|v|\,ds\,dv \\
&\qquad = c_Q\lambda\int_{v_{min}}^{v_{max}} \alpha^{d-1}\int_{S_+^{d-1}(n(r))}\int_0^{T_\Omega(r,e)} \|g(r-se,\cdot)\|_{L^1(V)}\,ds\,dl_S(e)\,d\alpha \\
&\qquad \le c_Q\lambda I_V(d)\int_{x\in(\Omega)_r} |r-x|^{1-d}\cdot\|g(x,\cdot)\|_{L^1(V)}\,dx\,.
\end{aligned}
\tag{4.14}
$$

Noting that $\int_{x\in\Omega}\|g(x,\cdot)\|_{L^1(V)}\,dx=1$ as well as

$$c_3 := \sup_{y\in\partial\Omega}\int_\Omega |y-x|^{(1-d)p}\,dx<\infty \quad \text{for } 1\le p<d/(d-1)$$

and setting

$$c_{J,3} := c_3^{q-1}(c_Q\lambda I_V(d))^{pq}$$

we continue by

$$
\begin{aligned}
&\left(\int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\int_0^{T_\Omega(r,v)} \lambda Q^+(g,g)(r-sv,v)\,ds\,dv\right)^{pq} \\
&\qquad \le (c_Q\lambda I_V(d))^{pq}\left(\int_\Omega |r-x|^{(1-d)p}\|g(x,\cdot)\|_{L^1(V)}\,dx\right)^q \\
&\qquad \le c_{J,3}\int_\Omega |r-x|^{(1-d)p}\|g(x,\cdot)\|^q_{L^1(V)}\,dx\,.
\end{aligned}
\tag{4.15}
$$

Let us recall the definition $\Omega_1=\{z\in\mathbb{R}^d:\inf_{y\in\Omega}|z-y|<1\}$. Moreover, let us take into consideration

$$\int_{x\in\mathbb{R}^d}|r-x|^{(1-d)p}\cdot|x-z|^{1-d}\,dx = c_{p,d}\cdot|r-z|^{1+(1-d)p}\,,\quad z\in\mathbb{R}^d,$$

for some constant $0<c_{p,d}<\infty$ only depending on p and d. Furthermore observe that $1+(1-d)p>1-d$ for $1\le p<d/(d-1)$ and thus

$$\sup_{z\in\Omega_1}\int_{\partial\Omega}|r-z|^{1+(1-d)p}\,dr<\infty\,,\quad 1\le p<d/(d-1). \tag{4.16}$$

According to (4.9) we have

$$
\begin{aligned}
&\int_\Omega |r-x|^{(1-d)p}\|g(x,\cdot)\|^q_{L^1(V)}\,dx \\
&\qquad \le C_{M,v}\int_\Omega |r-x|^{(1-d)p}\left(\lim_{n\to\infty}\int_{\Omega_1}|x-z|^{1-d}\cdot\gamma_n(z)\,dz\right)dx \\
&\qquad\quad + c_{p,d}C_{Q,v}\int_\Omega |r-z|^{1+(1-d)p}\cdot\|g(z,\cdot)\|^q_{L^1(V)}\,dz\,.
\end{aligned}
$$

We remark that the integration $\int_\Omega |r-x|^{(1-d)p}\dots\,dx$ is the actual reason for the weak limit in $L^p(\Omega)$ in (4.6). Now we take advantage of the facts that $\Omega \ni x \mapsto |r-x|^{(1-d)p} \in L^{p/(p-1)}(\Omega)$ and $\Omega \ni x \mapsto \int_{\Omega_1} |x-z|^{1-d}\cdot\gamma_n(z)\,dz$ converge weakly in $L^p(\Omega)$ to $\Omega \ni x \mapsto \int_{\partial\Omega} |x-y|^{1-d}(J(y,\cdot))^q\,dy$ for $1 \le p < d/(d-1)$. For the latter recall the discussion below (4.7). We get

$$
\begin{aligned}
&\int_\Omega |r-x|^{(1-d)p}\|g(x,\cdot)\|^q_{L^1(V)}\,dx \\
&\qquad \le C_{M,v}\int_\Omega |r-x|^{(1-d)p}\int_{\partial\Omega}|x-y|^{1-d}(J(y,\cdot))^q\,dy\,dx \\
&\qquad\quad + c_{p,d}C_{Q,v}\int_\Omega |r-z|^{1+(1-d)p}\cdot\|g(z,\cdot)\|^q_{L^1(V)}\,dz \\
&\qquad \le c_{p,d}C_{M,v}\int_{\partial\Omega}|r-y|^{1+(1-d)p}(J(y,\cdot))^q\,dy \\
&\qquad\quad + c_{p,d}C_{Q,v}\int_\Omega |r-z|^{1+(1-d)p}\|g(z,\cdot)\|^q_{L^1(V)}\,dz\,.
\end{aligned}
\tag{4.17}
$$

Because of (4.16) the right-hand side and, hence, the left-hand side, of (4.17) is integrable with respect to the variable r and the Lebesgue measure on $(\partial\Omega,\mathcal{B}(\partial\Omega))$ under the assumptions of Step 2, $\int_{\partial\Omega}(J(y,\cdot))^q\,dy < \infty$ and $\int_\Omega \|g(z,\cdot)\|^q_{L^1(V)}\,dz < \infty$. We obtain from (4.15), (4.16), and (4.17)

$$
\int_{r\in\partial\Omega}\left(\int_{v\circ n(r)\ge 0} v\circ n(r)\int_0^{T_\Omega(r,v)}\lambda Q^+(g,g)(r-sv,v)\,ds\,dv\right)^{pq} dr < \infty \tag{4.18}
$$

under the assumptions $\int_{\partial\Omega}(J(y,\cdot))^q\,dy < \infty$ and $\int_\Omega \|g(z,\cdot)\|^q_{L^1(V)}\,dz < \infty$. Reviewing (4.11), (4.13), and (4.18) we may now conclude

$$
\int_{\partial\Omega}(J(y,\cdot))^{pq}\,dy < \infty \quad \text{if}
$$
$$
\int_{\partial\Omega}(J(y,\cdot))^q\,dy < \infty \quad \text{and} \quad \int_\Omega \|g(z,\cdot)\|^q_{L^1(V)}\,dz < \infty.
$$

Together with (4.10) the last line says that we have accomplished the execution step of the iteration. Summing up Steps 1 and 2, we have proved that

$$
\int_{\partial\Omega}(J(y,\cdot))^q\,dy < \infty \quad \text{and} \quad \int_\Omega \|g(z,\cdot)\|^q_{L^1(V)}\,dz < \infty \tag{4.19}
$$

for all $1 \le q < \infty$. ∎

For the sake of completeness we provide a fact which may be known to experts, even in a more general form. It is a part of [10, Theorem 5.8]. We mention that the measure $\omega \equiv \omega^{x_0}$ on $(\partial\Omega,\mathcal{B}(\partial\Omega))$ in this reference is the harmonic measure relative to Ω and some $x_0 \in \Omega$. However since $\partial\Omega$ is smooth in our setting, here ω is equivalent to the Lebesgue surface measure on $\partial\Omega$. The Radon–Nikodym derivative is the Poisson kernel $\tilde{k}(x_0,\cdot)$ on $\partial\Omega$ which is bounded on $\partial\Omega$. For the Poisson kernel on $\Omega\times\partial\Omega$ we refer to [5, Theorem 1.4 of Chapter 6, in particular (1.17)]. However we remark that in [5] the Poisson kernel is defined as the Radon–Nikodym derivative of the harmonic measures with respect to the normalized Lebesgue surface measure.

For $\alpha > 0$ introduce $\Gamma_\alpha(y) := \{x \in \Omega : |x-y| < (1+\alpha)\inf_{z\in\partial\Omega}|x-z|\}$.

LEMMA 4.2 (see [10, Theorem 5.8]). *Let $f \in L^1(\partial\Omega)$. Then there is a harmonic function h_f on Ω such that $f(y) = \lim_{\Gamma_\alpha(y)\ni x\to y} h_f(x)$ for a.e. $y \in \partial\Omega$ and all $\alpha > 0$. We have $h_f = \int_{\partial\Omega} f\, d\omega^x$, $x \in \Omega$.*

Let us continue with our analysis.

LEMMA 4.3. *Let g satisfy (3.2) in the sense of Remark 3.2 and suppose (3.1). The function $\partial\Omega \ni r \mapsto J(r,\cdot)(g)$ belongs to $L^\infty(\partial\Omega)$ and $\Omega \ni y \mapsto \|g(y,\cdot)\|_{L^1(V)}$ belongs to $L^\infty(\Omega)$.*

Proof. For a.e. $r \in \partial\Omega$ we have by (4.11), (4.12), and (4.14)

$$\begin{aligned} J(r,\cdot)(g) &= \int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, g(r,v)\, dv \\ &\le M_{\max} \int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r)\, J(r^-(r,v),\cdot)(g)\, dv \\ &\quad + \int_{\{v\in V: v\circ n(r)\ge 0\}} v\circ n(r) \int_0^{T_\Omega(r,v)} \lambda Q^+(g,g)(r-sv,v)\, ds\, dv \\ &\le C_{M,w} \int_{y\in\partial\Omega} J(y,\cdot)(g)\, dy + C_{Q,w} \int_\Omega |r-y|^{1-d}\, \|g(y,\cdot)\|_{L^1(V)}\, dy\,, \end{aligned}$$

where

$$C_{M,w} := M_{\max} C_\Omega^2 I_V(d+1)(\mathrm{diam}(\Omega))^{3-d} \quad \text{and} \quad C_{Q,w} := c_Q \lambda I_V(d).$$

Together with (4.19) it is immediate from Hölder's inequality and the boundedness of Ω that

$$\partial\Omega \ni r \mapsto J(r,\cdot)(g) \quad \text{belongs to } L^\infty(\partial\Omega)\,. \tag{4.20}$$

Similarly to (4.9) it follows from (4.4), the first line of (4.5), and (4.8) for $q = 1$, i.e., $c_q = 1$ in (4.4) and (4.8), that for a.e. $y \in \Omega$

$$\begin{aligned} \|g(y,\cdot)\|_{L^1(V)} &\le C_{M,v} \int_{r\in(\partial\Omega)_y} |y-r|^{1-d} \cdot \frac{n(r)\circ(r-y)}{|y-r|} J(r,\cdot)(g)\, dr \\ &\quad + C_{Q,v} \int_\Omega |y-x|^{1-d} \|g(x,\cdot)\|_{L^1(V)}\, dx\,. \end{aligned} \tag{4.21}$$

Again by Hölder's inequality and the boundedness of Ω, (4.19) implies that the second item on the right-hand side of (4.21) is bounded for $y \in \Omega$.

Regarding the first item on the right-hand side of (4.21), we observe that for all $r \in \partial\Omega$ the function $\mathbb{R}^d \setminus \{r\} \ni y \mapsto h_r(y) := |y-r|^{1-d} \cdot n(r)\circ(r-y)/|y-r|$ is a harmonic function on $\mathbb{R}^d \setminus \{r\}$. Furthermore, by (3.13) we have

$$\begin{aligned} &\int_{r\in\partial\Omega} |h_r(y)|\, dr \\ &\quad = \int_{r\in\partial\Omega} |y-r|^{1-d} \cdot \frac{|n(r)\circ(r-y)|}{|y-r|}\, dr \\ &\quad \le C_\Omega \sup_{r'\in\partial\Omega} \int_{r\in\partial\Omega} |r'-r|^{(2-d)}\, dr =: c_{\Omega,1} < \infty\,, \quad y \in \partial\Omega. \end{aligned} \tag{4.22}$$

With (4.20) we obtain

$$\int_{r\in\partial\Omega}|h_r(y)|J(r,\cdot)(g)\,dr\le c_{\Omega,1}\cdot\operatorname*{ess\,sup}_{r\in\partial\Omega}J(r,\cdot)(g)<\infty\,,\quad y\in\partial\Omega. \tag{4.23}$$

According to (4.22) we have $h_r|_{\partial\Omega}\in L^1(\partial\Omega)$, $r\in\partial\Omega$. It follows now from Lemma 4.2 that there is a harmonic function $h_{r,\Omega}$ on Ω for which $\lim_{\Gamma_\alpha(y)\ni x\to y}h_{r,\Omega}(x)=|h_r(y)|$ for a.e. $y\in\partial\Omega$ for all $\alpha>0$. Since both functions h_r as well as $h_{r,\Omega}$ are harmonic on Ω we can apply the representation via harmonic measures provided in Lemma 4.2 to both. The boundary value functions of h_r and $h_{r,\Omega}$ are, respectively, $h_r|_{\partial\Omega}$ and $|h_r(y)||_{\partial\Omega}$. Thus $h_r\le h_{r,\Omega}$ on Ω for all $r\in\partial\Omega$. The function

$$\Omega\ni y\mapsto\int_{r\in\partial\Omega}h_{r,\Omega}(y)J(r,\cdot)(g)\,dr=:H_\Omega(y)$$

is nonnegative and harmonic on Ω. Since H_Ω cannot have a local maximum on Ω, it is bounded by $c_{\Omega,1}\cdot\operatorname{ess\,sup}_{r\in\partial\Omega}J(r,\cdot)(g)$ according to (4.23). This allows us to conclude from $h_r\le h_{r,\Omega}$ on Ω that

$$\begin{aligned}\int_{r\in(\partial\Omega)_y}&|y-r|^{1-d}\cdot\frac{n(r)\circ(r-y)}{|y-r|}J(r,\cdot)(g)\,dr\\&=\int_{r\in(\partial\Omega)_y}h_r(y)J(r,\cdot)(g)\,dr\\&\le\int_{r\in(\partial\Omega)_y}h_{r,\Omega}(y)J(r,\cdot)(g)\,dr\\&\le H_\Omega(y)\le c_{\Omega,1}\cdot\operatorname*{ess\,sup}_{r\in\partial\Omega}J(r,\cdot)(g)\,,\quad y\in\Omega.\end{aligned}$$

In conclusion, both items on the right-hand side of (4.21) have turned out to be bounded for a.e. $y\in\Omega$. In other words,

$$\Omega\ni y\mapsto\|g(y,\cdot)\|_{L^1(V)}\quad\text{belongs to }L^\infty(\Omega).$$

We have completed the proof. ∎

THEOREM 4.4. *Let g satisfy* (3.2) *in the sense of Remark* 3.2 *and suppose* (3.1). *We have*

$$g\in L^\infty(\Omega\times V)\quad\text{and}\quad\frac{1}{g}\in L^\infty(\Omega\times V)\,.$$

Proof. Step 1. In this step we verify $1/g\in L^\infty(\Omega\times V)$. For this let us mention that, by Lemma 3.3, we have $1\le\psi_g$ and $\sup\psi_g<\infty$ where the supremum is taken over $\{(r,v,t):(r,v)\in\Omega\times V,\ t\in[0,T_\Omega(r,v)]\}$. Recall also the definition of ψ_g and (3.6).

We emphasize the following. Since for $(r,v)\in\partial\Omega\times V$ with $v\circ n(r)\ge0$ we have $r^-\equiv r^-(r,v)\in\partial\Omega$ and $v\circ n(r^-)\le0$, the boundary conditions (2.1) say that

$$g(r^-,v)=J(r^-,\cdot)(g)\cdot M(r^-,v)\,. \tag{4.24}$$

For the next chain of equations and inequalities fix an $r\in\partial\Omega$ such that $J(r,\cdot)(g)<\infty$ and recall that $J(y,\cdot)(g)<\infty$ holds for a.e. $y\in\partial\Omega$ by Lemma 4.3. We obtain from

(4.24)

$$\int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\cdot J(r^-(r,v),\cdot)(g)\cdot M(r^-,v)\,dv$$
$$= \int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\cdot g(r^-,v)\,dv$$
$$= \int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\cdot g(r-T_\Omega(r,v)v,v)\,dv \tag{4.25}$$
$$\le \sup\psi_g\cdot \int_{\{v\in V:v\circ n(r)\ge 0\}} v\circ n(r)\cdot g(r,v)\,dv$$
$$= \sup\psi_g\cdot J(r,\cdot)(g)\,,$$

where the second last line is a consequence of (3.6) and the last equality sign holds according to the definition of J in (i). Let us keep in mind that according to (ii), there exist constants $M_{\min}, M_{\max}\in(0,\infty)$ such that $M_{\min}\le M(r,v)\le M_{\max}$. Instead of one fixed $r\in\partial\Omega$, let us now consider (4.25) for a sequence $r_k\in\partial\Omega$, $k\in\mathbb{N}$, with $r_k \underset{k\to\infty}{\longrightarrow} r_\infty$ for some $r_\infty\in\partial\Omega$.

Assuming $J(r_k,\cdot)(g) \underset{k\to\infty}{\longrightarrow} 0$ on the right-hand side of (4.25), the left-hand side of (4.25) implies that $J(y,\cdot)(g)=0$ for a.e. $y\in(\partial\Omega)_{r_\infty}$. This can be seen as follows. We take into consideration that for a.e. $y\in(\partial\Omega)_{r_\infty}$ there is a $k_0\equiv k_0(y)$ such that $y\in(\partial\Omega)_{r_k}$ for $k>k_0$ which means that $y\in\bigcap_{l\in\mathbb{N}}\bigcup_{k>l}(\partial\Omega)_{r_k}$. In addition, for the left-hand side of (4.25) we get as in (4.12)

$$M_{\min}\cdot I_V(d+1)\int_{y\in(\partial\Omega)_{r_k}} J(y,\cdot)(g)\cdot\frac{(r_k-y)\circ n(r_k)\cdot n(y)\circ(y-r_k)}{|r_k-y|^{1+d}}\,dy$$
$$\le \int_{\{v\in V:v\circ n(r_k)\ge 0\}} v\circ n(r_k)\cdot J(r^-(r_k,v),\cdot)(g)\cdot M(r^-(r_k,v),v)\,dv$$
$$\underset{k\to\infty}{\longrightarrow}\ 0\,,$$

where the weight functions

$$y\in\partial\Omega\mapsto\sigma_d(r_k,y):=\frac{(r_k-y)\circ n(r_k)\cdot n(y)\circ(y-r_k)}{|r_k-y|^{1+d}}\cdot\chi_{(\partial\Omega)_{r_k}}(y)\,,$$

$k\in\mathbb{N}\cup\{\infty\}$, are uniformly bounded on $\partial\Omega$ for $d=2,3$ by (3.13) and therefore satisfy $\sigma_d(r_k,\cdot) \underset{k\to\infty}{\longrightarrow} \sigma_d(r_\infty,\cdot)$ in $L^1(\partial\Omega)$. We may now conclude that $\int_{y\in\partial\Omega} J(y,\cdot)(g)\cdot\sigma_d(r_\infty,y)\,dy=0$, i.e., $J(y,\cdot)(g)=0$ for a.e. $y\in(\partial\Omega)_{r_\infty}$.

Considering now the right-hand side of (4.25) for $y\in(\partial\Omega)_{r_\infty}$ instead of r, it follows from the left-hand side of (4.25) that even $J(y',\cdot)(g)=0$ a.e. on $y'\in(\partial\Omega)_y$. By iteration of the last conclusion we obtain $J(y,\cdot)(g)=0$ a.e. on $y\in\partial\Omega$.

The latter would imply $g=0$ a.e. on $\partial\Omega\times V$; recall (2.1) and (i). From (3.6) we would a.e. on $\{(r,v)\in\partial\Omega\times V: v\circ n(r)\ge 0\}$ obtain

$$0=\psi_g(r,v,T_\Omega(r,v))\cdot\int_0^{T_\Omega}\frac{\lambda Q^+(g,g)(r-sv,v)}{\psi_g(y,v,s)}\,ds\,.$$

Since $\psi_g\ge 1$ this would say $Q^+(g,g)=0$ a.e. on $\Omega\times V$. Together with $g=0$ a.e. on $\partial\Omega\times V$, by (3.6) this would yield $g=0$ a.e. on $\Omega\times V$.

Consequently, the above formulated assumption cannot hold, which means that $\inf_{r\in\partial\Omega} J(r,\cdot)(g) > 0$. From here and (2.1) as well as $0 < M_{\min} \le M$ we may now conclude

$$\inf\{g(y,v) : (y,v) \in \partial\Omega \times V : v \circ n(y) \le 0\} > 0. \tag{4.26}$$

On the other hand, relation (3.6) implies for all $(r,v) \in \partial\Omega \times V$ with $v \circ n(r) \ge 0$ such that $g(r,v) < \infty$ and $y := r^-(r,v)$

$$\begin{aligned} g(y,v) &= g(r - T_\Omega v, v) \\ &= \psi_g(r,v,T_\Omega)\left(-\int_0^{T_\Omega} \frac{\lambda Q^+(g,g)(r-sv,v)}{\psi_g(r,v,s)}\,ds + g(r,v)\right). \end{aligned} \tag{4.27}$$

Since $(y,v) \in \partial\Omega \times V$ in (4.27) satisfies $v \circ n(y) \le 0$ and $\sup \psi_g < \infty$ we obtain from (4.26) and (4.27)

$$\operatorname*{ess\,inf}_{(r,v)\in\partial\Omega\times V: v\circ n(r)\ge 0} \left\{-\int_0^{T_\Omega(r,v)} \frac{\lambda Q^+(g,g)(r-sv,v)}{\psi_g(r,v,s)}\,ds + g(r,v)\right\} > 0.$$

Now (3.6) implies positivity of $\operatorname{ess\,inf} g$, i.e., $1/g \in L^\infty(\Omega) \times V$.

Step 2. In this step we verify $g \in L^\infty(\Omega \times V)$. Recalling (3.6) and $1 \le \psi_g$ we may state that for a.e. $(y,v) \in \Omega \times V$ and $r = y^-(y,v)$ it holds that

$$g(y,v) \le g(r,v) + \int_0^{T_\Omega(y,v)} \lambda Q^+(g,g)(y-sv,v)\,ds\,.$$

Applying now (2.1) together with (ii) and (3.10), we verify

$$\begin{aligned} g(y,v) &\le M_{\max} J(r,\cdot)(g) + \frac{c_Q\lambda}{v_{min}} \int_0^{T_\Omega(y,v)} \|g(y-sv,\cdot)\|_{L^1(V)} |v|\,ds \\ &= M_{\max} J(r,\cdot)(g) + \frac{c_Q\lambda}{v_{min}} \int_0^{T_\Omega(y,v/|v|)} \|g(y-sv/|v|,\cdot)\|_{L^1(V)}\,ds \\ &\le M_{\max} \cdot \operatorname*{ess\,sup}_{r\in\partial\Omega} J(r,\cdot)(g) \\ &\qquad + \frac{c_Q\lambda}{v_{min}} \operatorname{diam}(\Omega) \cdot \operatorname*{ess\,sup}_{y\in\Omega} \|g(y,\cdot)\|_{L^1(V)} < \infty \quad \text{for a.e. } (y,v) \in \Omega \times V, \end{aligned} \tag{4.28}$$

the last inequality because of Lemma 4.3. We have completed the proof of the theorem. ∎

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