Local Conditions for Cycles in Graphs

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Abstract

A Hamilton cycle in a graph is a cycle that passes through every vertex of the graph. A graph is called Hamiltonian if it contains such a cycle. The problem of determining if a graph is Hamiltonian has been studied extensively, and there are many known sufficient conditions for Hamiltonicity.

A large portion of these conditions relate the degrees of vertices of the graph to the number of vertices in the entire graph, and thus they can only apply to a limited set of graphs with high edge density. In a series of papers, Asratian and Khachatryan developed local analogues of some of these criteria. These results do not suffer from the same drawbacks as their global counterparts, and apply to wider classes of graphs.

In this thesis we study this approach of creating local conditions for Hamiltonicity, and use it to develop local analogues of some classic results. We also study how local criteria can influence other global properties of graphs. Finally, we will see how these local conditions can allow us to extend theorems on Hamiltonicity to infinite graphs.

Sammanfattning

En Hamiltoncykel i en graf är en cykel som passerar genom varje hörn i grafen, och en graph är Hamiltonsk om den innehåller en sådan cykel. Problemet att avgöra om en graf är Hamiltonsk har studerats mycket, och det finns många kända villkor som garanterar Hamiltonicitet.

En stor del av dessa villkor sätter gradtalen för hörn i grafen i relation till antalet hörn i hela grafen, och de kan därför endast tillämpas på en begränsad mängd grafer med hög kanttäthet. I ett antal artiklar utvecklade Asratian och Khachatryan lokala motsvarigheter till några av dessa villkor. Dessa resultat har inte samma nackdelar som deras globala motsvarigheter, och kan appliceras på en större mängd grafer.

I denna avhandling undersöker vi detta tillvägagångssätt att skapa lokala villkor för Hamiltonicitet, och använder det för att ta fram lokala motsvarigheter till några klassiska resultat. Vi kommer också undersöka hur lokala villkor kan påverka andra globala egenskaper hos grafer. Slutligen kommer vi se hur dessa lokala villkor kan tillåta oss att utvidga Hamiltonicitetssatser till oändliga grafer.

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List of papers

The thesis is based on the following papers:

- I Armen S. Asratian, Jonas B. Granholm, Nikolay K. Khachatryan. *A localization method in Hamiltonian graph theory*. arXiv:1810.10430 [math.CO], 2018. Submitted.
- II Armen S. Asratian, Jonas B. Granholm, Nikolay K. Khachatryan. *Some local-global phenomena in locally finite graphs*. arXiv:1810.07023 [math.CO], 2018. Accepted for publication in Discrete Applied Mathematics.
- III Jonas B. Granholm. *Some cyclic properties of L_1 -graphs*. arXiv:1904.07183 [math.CO], 2019. Submitted.

In Paper I, I derived the main results of sections 4, 5 and 7, as well as Propositions 6.2 and 6.6. I also prepared all figures and was active in the writing of the entire paper. In Paper II, I proved Theorem 3.3 and Proposition 3.5. Furthermore, Theorem 5.4 was a collaboration by all authors. I also prepared all figures, and was active in the writing of the entire paper.

Parts of Paper III have been presented by me at the 26th British Combinatorial Conference in Glasgow, Great Britain, July 3 to 7, 2017, and parts of Paper I have been presented by me at the the conference Combinatorics 2018 in Arco, Italy, June 3 to 9, 2018. I also presented parts of these papers while visiting Laboratoire Bordelais de Recherche en Informatique, University of Bordeaux, March 2019.

Contents

1	Introduction	1
2	Fundamentals	3
3	Hamiltonicity and related concepts	7
4	Infinite graphs	13
5	Summary of papers	15
	Bibliography	17

Papers

I	A localization method in Hamiltonian graph theory	21
II	Some local–global phenomena in locally finite graphs	57
III	Some cyclic properties of L_1 -graphs	83

Chapter 1

Introduction

A *Hamilton cycle* in a graph is a cycle that passes through every vertex of the graph, and a graph containing a Hamilton cycle is called *Hamiltonian*. Hamiltonicity is one of the most fundamental notions in graph theory and has been studied extensively (see e.g. [23–25]). Problems related to Hamiltonicity arise in different areas of mathematics, as well as other branches of science.

The problem of determining if a graph is Hamiltonian is NP-complete [30], which roughly speaking means that no efficient method of finding Hamilton cycles is likely to exist. There are, however, many conditions that have been proven to imply Hamiltonicity. A classical result giving such a condition is Dirac’s Theorem [20], which states that a graph G on at least three vertices is Hamiltonian if each vertex $v \in V(G)$ has degree $d(v) \geq |V(G)|/2$. This result was later generalized by Ore [36] to give the following: A graph G on at least three vertices is Hamiltonian if every pair of non-adjacent vertices $u, v \in V(G)$ has degree sum $d(u) + d(v) \geq |V(G)|$.

A disadvantage with the above results, and others like them, is that they only apply to graphs with diameter 1 or 2, that is, graphs where the distance between any two vertices does not exceed 2. Furthermore, every graph satisfying one of these conditions is dense, that is, the number of edges in the graph is proportional to the square of the number of vertices.

One method, suggested by Asratian and Khachatryan (see e.g. [6–8]), that has received attention lately [18] is to replace global conditions with local analogues. This means that instead of using global properties, it suffices to consider balls of small radius, where a *ball of radius r* around a vertex v is the subgraph induced by the set of vertices at distance at most r from v . An advantage of such an

approach is that it allows for parallel processing of the graph. Another advantage of these localized theorems is that they, unlike their global equivalents, apply to families of sparse graphs with large diameter.

In this thesis we study this approach of creating local conditions for Hamiltonicity, and use it to develop local analogues of some classic results. We also study how local criteria can influence other global properties of graphs. Finally, we will see how these local conditions can allow us to extend theorems on Hamiltonicity to infinite graphs.

Chapter 2

Fundamentals

This chapter will define notions used in the thesis. For a more thorough introduction to graph theory, see [15].

A *graph* G is a pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where each edge connects two distinct vertices and no two vertices are connected by more than one edge. Two vertices are *adjacent* if they are connected by an edge. The *degree* of a vertex $v \in V(G)$, denoted $d_G(v)$ or simply $d(v)$, is the number of edges that are incident with v . A *k-regular graph* is a graph where every vertex has degree k . A *regular graph* is a graph that is k -regular for some k . A *complete graph* is a graph where every pair of vertices is connected by an edge. The complete graph with n vertices is denoted K_n . If X and Y are two disjoint subsets of $V(G)$, then the number of edges connecting one vertex in X and one vertex in Y is denoted $e(X, Y)$. An graph G is *finite* if $V(G)$ is finite, otherwise it is *infinite*. A graph is *locally finite* if all of its vertices have finite degree.

Two vertices are *independent* if they are not connected by an edge. An *independent set* S of vertices is a subset $S \subseteq V(G)$ such that all pairs of vertices in S are independent. The size of the largest independent set in a graph G is called the *independence number* of G , denoted $\alpha(G)$.

The *complement* of a graph G , denoted $\bar{G}$, is a graph such that $V(\bar{G}) = V(G)$, and two vertices in $\bar{G}$ are joined by an edge if and only if those vertices are not joined by an edge in G . Thus the complement of a complete graph K_n is an empty graph $\bar{K}_n$ without any edges. For any graph G , if we let $V' = V(G) = V(\bar{G})$ and $E' = E(G) \cup E(\bar{G})$, the graph $G' = (V', E')$ is complete.

A graph G' is a *subgraph* of the graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

If $S \subseteq V(G)$, then the graph G' where $V(G') = S$ and $E(G')$ contains every edge in $E(G)$ that joins two vertices of S is called the subgraph of G *induced by* S . The subgraph induced by the set $V(G) \setminus S$ is denoted $G - S$.

A *path* is a nonempty graph or subgraph of the form $P = (V, E)$ with $V = \{v_0, v_1, \dots, v_n\}$ and $E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n\}$ where all v_i are distinct. We say that P joins the vertices v_0 and v_n , or that P is a v_0 - v_n -*path*. The number of edges in a path is called the *length* of the path. A path containing n vertices is called an n -*path*, and is denoted P_n . Note that an n -path P_n has length $n - 1$. A *cycle* is a nonempty graph or subgraph of the form $C = (V, E)$ with $V = \{v_0, v_1, \dots, v_n\}$ and $E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n, v_nv_0\}$ where all v_i are distinct. The number of edges in a cycle is called the *length* of the cycle. A cycle containing n vertices is called an n -*cycle*, and is denoted C_n . A cycle of length three is called a *triangle*, and a cycle of length four is called a *square*.

We will usually represent a path as a sequence $v_0v_1v_2 \cdots v_n$ of its vertices. Similarly we represent a cycle as a sequence $v_0v_1v_2 \cdots v_nv_0$ of its vertices. To indicate that a path is traversed in a particular direction we use the notation $\vec{P}$, and we denote the same path in the reverse direction by $\bar{P}$. When we have specified a direction of a path, we use the notation v_i^+ to denote the successor of v_i on $\vec{P}$ and v_i^- to denote the predecessor. If $\vec{P}$ is the path $v_0v_1v_2 \cdots v_n$ directed from v_0 to v_n , then we denote a subpath $v_iv_{i+1} \cdots v_j$ of $\vec{P}$ from v_i to v_j by $v_i\vec{P}v_j$, and in the other direction it is denoted by $v_j\bar{P}v_i$. Analogous notation is used for cycles.

Two vertices are *connected* if there is a path joining them. A graph is connected if all pairs of vertices are connected, otherwise it is *disconnected*. A maximal connected subgraph of G is called a *component* of G . If S is a set of vertices such that removing the vertices of S and all edges incident with those vertices from G makes G disconnected, S is called a *vertex cut set*. If $S = \{v\}$, then v is called a *cut vertex*. A graph is k -*connected* if it contains at least $k + 1$ vertices but no vertex cut set with less than k vertices. The *connectivity* of G , denoted $\kappa(G)$, is the largest k such that G is k -connected. A graph G is *1-tough* if the induced subgraph $G - S$ contains at most $|S|$ components for any proper subset $S \subset V(G)$, that is, if removing k vertices cannot split G into more than k disconnected parts.

The *distance* between two vertices u and v , denoted $d_G(u, v)$ or simply $d(u, v)$, is the length of the shortest path joining u and v . If u and v are not connected by any path, then $d(u, v)$ is infinite. The *diameter* of a graph is the longest distance between any two vertices in the graph. If a graph is disconnected, then its diameter is infinite.

Figure 2.1: A graph, and the 1- and 2-ball around a vertex v .

The *square* of a graph G , denoted G^2 , is a graph such that $V(G^2) = V(G)$, and two vertices in G^2 are joined by an edge if and only if their distance in G is at most 2.

The *neighborhood* of a vertex v , denoted $N(v)$, is the set of vertices adjacent to v . The notation $N_r(v)$ is used to denote the set of vertices at distance r from v . The set of vertices at distance at most r from a vertex v (including v itself) is denoted $M_r(v)$. The *ball* of radius r around v , or just the r -ball around v , denoted $G_r(v)$, is the subgraph induced by $M_r(v)$. We say that the vertex u is an *interior vertex* of the ball $G_r(v)$ if the ball $G_1(u)$ is a subgraph of $G_r(v)$, that is, if u and all its neighbors are in $G_r(v)$. The set of interior vertices of the ball $G_r(v)$ is denoted $M_r^\circ(v)$. In Figure 2.1 we see a graph and two balls of differing radii in that graph.

A graph G is *bipartite* if the vertices of G can be partitioned into two sets such that every edge of G has one end in each of the sets. If the two sets have the same cardinality, the bipartite graph is *balanced*. A bipartite graph in which every two vertices from different parts of the bipartition are adjacent is called *complete bipartite*. The complete bipartite graph with m vertices in one of its parts and n vertices in the other is denoted $K_{m,n}$. As is well-known, a graph is bipartite if and only if it does not contain any cycle of odd length.

A graph is *claw-free* if it contains no induced subgraph isomorphic to the graph $K_{1,3}$, and *locally connected* if the subgraph induced by $N(v)$ is connected for every vertex $v \in V(G)$.

A *matching* in a graph G is a subset $M \subseteq E(G)$ such that no two edges in M have a common endpoint. A matching is *perfect* if every vertex of G is the endpoint of an edge in the matching.

The *union* of two graphs G_1 and G_2 , denoted $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *intersection* of two graphs G_1 and G_2 , denoted $G_1 \cap G_2$, is the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. If $G_1 \cap G_2 = \emptyset$, then G_1 and G_2 are said to be *disjoint*.

The *join* of two disjoint graphs G_1 and G_2 , denoted $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge. The *sequential join* of a sequence of disjoint graphs $G_1, G_2, \dots, G_n$, denoted $G_1 \vee G_2 \vee \dots \vee G_n$, is the graph obtained from $G_1 \cup G_2 \cup \dots \cup G_n$ by adding edges joining each vertex of G_1 to each vertex of G_2 , each vertex of G_2 to each vertex of G_3 , and so on.

Chapter 3

Hamiltonicity and related concepts

3.1 Hamiltonian graphs

In 1857 Sir William Rowan Hamilton introduced the following game: He drew a graph representing the edges of a dodecahedron (see Figure 3.1) and labeled each vertex with the name of a European city, and had the edges of the graph represent roads between the cities. Then he asked if it was possible to start in one city, visit each city exactly once, and finally return to the first city. A solution is not hard to find (see Figure 3.2), but the more general form of the puzzle – finding a cycle that passes through all vertices of an arbitrary graph – has proven to be trickier.

Definition 3.1. A cycle that passes through every vertex of a graph is called a *Hamilton cycle*. A graph containing a Hamilton cycle is *Hamiltonian*. A path that passes through every vertex of a graph is called a *Hamilton path*.

Figure 3.1: The graph in Hamilton's game.

Figure 3.2: A solution to Hamilton's game.

Hamilton cycles are important in many applications, and have been studied extensively (see e.g. [23–25]). It is easy to see that every Hamiltonian graph must be 2-connected, so a graph that is not 2-connected cannot contain a Hamilton cycle. This condition was strengthened by Chvátal:

Theorem 3.2 (Chvátal [16]). *If G is Hamiltonian, then G is 1-tough.*

Perhaps the most classic results that guarantees Hamiltonicity is the following:

Theorem 3.3 (Dirac [20]). *Let G be a graph on at least three vertices such that $d(v) \geq |V(G)|/2$ for every vertex $v \in V(G)$. Then G is Hamiltonian.*

This was later generalized by Ore:

Theorem 3.4 (Ore [36]). *Let G be a graph on at least three vertices such that $d(u) + d(v) \geq |V(G)|$ for every pair of non-adjacent vertices $u, v \in V(G)$. Then G is Hamiltonian.*

The condition in Ore's Theorem can be relaxed, if we allow a set of exceptions:

Theorem 3.5 (see e.g. Nara [33]). *Let G be a 2-connected graph such that $d(u) + d(v) \geq |V(G)| - 1$ for every pair of non-adjacent vertices $u, v \in V(G)$. Then G is Hamiltonian unless $G \in \mathcal{K}$, where*

$$\mathcal{K} = \{ G : K_{p,p+1} \subseteq G \subseteq K_p \vee \overline{K_{p+1}} \text{ for some } p \geq 2 \}.$$

A similar result was discovered earlier for the case of regular graphs:

Theorem 3.6 (Nash-Williams [34]). *Let G be a 2-connected regular graph such that $d(v) \geq (|V(G)| - 1)/2$ for each vertex $v \in V(G)$. Then G is Hamiltonian.*

For bipartite graphs, the bound can be relaxed significantly:

Theorem 3.7 (Moon–Moser [32]). *Let G be a balanced bipartite graph with $2n$ vertices, $n \geq 2$, such that $d(u) + d(v) > n$ for every pair of non-adjacent vertices u and v at odd distance in G . Then G is Hamiltonian.*

Figure 3.3: A non-Hamiltonian graph in which every ball that is not the whole graph is Hamiltonian.

This theorem will be expanded upon in Paper I, along with the following which takes three vertices into account:

Theorem 3.8 (Bauer–Broersma–Veldman–Rao [10]). *Let G be a 2-connected graph on at least three vertices such that*

$$d(x) + d(y) + d(z) \geq |V(G)| + \kappa(G)$$

for every triple of independent vertices $x, y, z \in V(G)$. Then G is Hamiltonian.

Asratian and his colleagues [4–7] have obtained local analogues of Theorems 3.3–3.6. The main idea of their method is to use the structure of balls of small radii. It is not as simple as checking if the balls are Hamiltonian, however; every ball in the graph in Figure 3.3 that is not the whole graph is Hamiltonian, but the whole graph is not (we will explore this further in Paper II). The following is a localization (and generalization) of Ore’s Theorem:

Theorem 3.9 (Asratian–Khachatryan [7]). *Let G be a connected graph on at least three vertices such that for every triple u, w, v with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ the following property holds:*

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|.$$

Then G is Hamiltonian.

Theorem 3.9 was later generalized to the following, using the set of exceptions $\mathcal{K}$ from Theorem 3.5:

Theorem 3.10 (Asratian–Broersma–van den Heuvel–Veldman [5]). *Let G be a connected graph on at least three vertices such that for every triple u, w, v with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ the following two properties hold:*

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$$

and $|N(u) \cap N(v)| \geq 2$. Then G is Hamiltonian unless $G \in \mathcal{K}$.

Theorem 3.10 is not a generalization of Theorem 3.5, as the condition $|N(u) \cap N(v)| \geq 2$ is too restrictive; for example, the graph C_5 is excluded. A localization that generalizes Theorem 3.5 was obtained by Asratian:

Theorem 3.11 (Asratian [4]). *Let G be a connected graph on at least three vertices such that for every triple u, w, v with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ the following property holds:*

$$d(u) + d(v) \geq |M_2(w)| - 1,$$

and furthermore every 2-ball in G is 2-connected. Then G is Hamiltonian unless $G \in \mathcal{K}$.

We will extend Theorem 3.10 in Paper III and Theorem 3.11 in Paper II.

3.2 Related concepts

A common technique for finding Hamilton cycles is to start with a cycle that does not cover all vertices and show that a contradiction arises if this cycle cannot be extended. In this process, it is sometimes useful to first establish the existence of a cycle covering all but an isolated set of vertices.

Definition 3.12. A cycle is called a *dominating cycle* if no two vertices outside the cycle are adjacent.

The following theorem, for example, is used in the proof of Theorem 3.8. We will develop a localization of it in Paper I.

Theorem 3.13 (Bondy [11]). *Let G be a 2-connected graph on at least three vertices such that*

$$d(x) + d(y) + d(z) \geq |V(G)| + 2$$

for every triple of independent vertices $x, y, z \in V(G)$. Then every longest cycle of G is dominating.

Many of the sufficient conditions for Hamiltonicity, such as Ore's Theorem, do not just imply Hamiltonicity, but also stronger properties like pancyclicity.

Definition 3.14. A graph G is *pancyclic* if it contains a cycle of every length from 3 up to $|V(G)|$.

In 1971 Bondy proved the following:

Theorem 3.15 (Bondy [12]). *Let G be a graph satisfying the conditions of Ore's Theorem. Then G is pancyclic unless G is a complete bipartite graph $K_{n,n}$ for some $n \geq 2$.*

This prompted him to make a famous metaconjecture [13]: “Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic. (There may be a simple family of exceptional graphs.)” Since then, many conditions that imply Hamiltonicity have been proven to imply pancyclicity; Asratian and Sarkisian [9], for example, showed that the condition in Theorem 3.9 implies pancyclicity, with the exception of $K_{n,n}$, and Aldred, Holton, and Min [1] showed that graphs satisfying the conditions of Theorem 3.5 are pancyclic, except for the cycle C_5 , the bipartite graphs $K_{n,n}$, and the graphs in the set $\mathcal{K}$.

Closely related to pancyclicity is the concept of extending cycles.

Definition 3.16. A graph G is *cycle extendable* if G contains at least one cycle, and for every non-Hamiltonian cycle C_n of length n in G there is a cycle C_{n+1} of length $n + 1$ containing every vertex of C_n .

Not every graph that satisfies the condition of Ore's Theorem is cycle extendable. However, Bondy noted the following property, which we will generalize in Paper III:

Theorem 3.17 (Bondy [14]). *Let G be a graph satisfying the conditions of Ore's Theorem. Then for every non-Hamiltonian cycle C_n of length n in G there is a cycle $C_{n+\ell}$ of length $n + \ell$, where $1 \leq \ell \leq 2$, such that $V(C_n) \subset V(C_{n+\ell})$.*

Hendry [27] found some classes of cycle extendable graphs. In particular he found a criterion for graphs satisfying the conditions of Ore's Theorem to be cycle extendable. Another class of cycle extendable graphs was found by Asratian:

Theorem 3.18 (Asratian [3]). *Let G be a connected graph such that for every vertex $v \in V(G)$, the ball $G_1(v)$ satisfies the conditions of Ore's Theorem. Then G is cycle extendable.*

Hamiltonian graph theory is not only focused on cycles. A related property that has also received attention is based on Hamilton paths.

Definition 3.19. A graph G is *Hamilton-connected* if for every pair of vertices $x, y \in V(G)$ there is a Hamiltonian x – y -path.

It is easy to see that every Hamilton-connected graph on at least three vertices is Hamiltonian. Furthermore, every Hamilton-connected graph on at least four vertices must be 3-connected, since any two vertices are the endpoints of a Hamilton path. If we add this condition to Theorem 3.9 we get the following:

Theorem 3.20 (Asratian [2]). *Let G be a 3-connected graph such that for every triple u, w, v with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ the following property holds:*

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|.$$

Then G is Hamilton-connected unless $G \in \mathcal{M}$, where

$$\mathcal{M} = \{ G : K_{p,p} \subseteq G \subseteq K_p \vee \overline{K_p} \text{ for some } p \geq 3 \}.$$

The relaxed conditions of Theorem 3.10 do not imply Hamilton-connectedness, as demonstrated by the graphs obtained by deleting a perfect matching from $K_{p,p}$ and $K_p \vee \overline{K_p}$ for $p \geq 4$. They do however imply a weaker property, which we will strengthen in Paper III:

Theorem 3.21 (Asratian–Broersma–van den Heuvel–Veldman [5]). *Let G be a connected graph on at least three vertices such that for every triple u, w, v with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ the following two properties hold:*

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$$

and $|N(u) \cap N(v)| \geq 2$. Then every pair of vertices x, y with $d(x, y) \geq 3$ is connected by a Hamilton path of G .

Chapter 4

Infinite graphs

Let G be a locally finite infinite graph. A *ray* in G is a one-way infinite path. We shall construct the *Freudenthal compactification* $|G|$ of G by defining “points at infinity” to which the rays of G converge. We say that two rays are equivalent if for each finite vertex set S there is a connected component of $G - S$ that contains infinitely many vertices of both rays. This is an equivalence relation, and we call the equivalence classes the *ends* of G . If an end ω is the equivalence class of a ray we say that the ray *converges* to ω .

The *Freudenthal compactification* $|G|$ is a topological space constructed by viewing the vertices of G as points and the edges of G as internally-disjoint line segments connecting the points corresponding to its endpoints. Finally the ends are added as points at the limit points of the rays converging to them.¹ An example can be seen in Figure 4.1. For a more thorough exposition, see [17].

Figure 4.1: A graph with one end and its Freudenthal compactification.

¹For the topologists out there: We view G as a cell complex with the usual topology. The topology of $|G|$ is generated by the open sets of G together with the basic open sets formed as follows: For each finite vertex set S and each end ω there is a unique component of $G - S$ in which every ray converging to ω has a subray. We say that ω *lives in* this component of $G - S$. Now for each finite vertex set S and each component C of $G - S$ we take as an additional basic open set the union of C (again viewed as a cell complex), the interior of all edges connecting vertices of S to vertices of C , and all ends of G that live in C .

In 2004, Diestel and Kühn [19] defined *Hamilton circles* as a generalization of Hamilton cycles to infinite locally finite graphs. A *circle* in $|G|$ is a homeomorphic image of the unit circle, that is, a curve that starts and ends at the same point and passes through every point of $|G|$ at most once. A *Hamilton circle* is a circle that passes through every vertex and every end exactly once. Note that for finite graphs these definitions coincide with ordinary cycles and Hamilton cycles.

Several theorems on Hamiltonicity of finite graphs have been extended to Hamilton circles in locally finite graph; for instance, Georgakopoulos [22] extended Fleischner's theorem [21] on the Hamiltonicity of the square of a 2-connected graph, and Heuer [28] and Hamann et al. [26] showed that locally finite, claw-free, locally connected graphs have Hamilton circles, extending a result by Oberly and Sumner [35]. Diestel [18] conjectured that Theorem 3.9 could be extended to locally finite graphs, and Heuer [29] proved the conjecture under the additional assumption that the graph is claw-free.

Kündgen, Li, and Thomassen [31] introduced another concept of Hamiltonicity for infinite locally finite graphs: A *closed curve* in the Freudenthal compactification $|G|$ is the image of a continuous embedding of the unit circle into $|G|$, and a *Hamilton curve* is a closed curve that meets every vertex exactly once. This notion is similar to that of Hamilton circles, with the difference that a Hamilton curve is allowed to meet the ends of $|G|$ multiple times. Kündgen, Li, and Thomassen proved the following theorem:

Theorem 4.1 (Kündgen–Li–Thomassen [31]). *The following are equivalent for any locally finite graph G .*

1. *For every finite vertex set S , G has a cycle containing S .*
2. *$|G|$ has a Hamilton curve.*

This theorem was then used in [31] to show that graphs satisfying the condition of Theorem 3.9 have Hamilton curves.

Chapter 5

Summary of papers

Paper I: A localization method in Hamiltonian graph theory

In this paper we formulate a general approach for finding localization theorems and use this approach to formulate local analogues of Theorems 3.7, 3.8, and 3.13. We also extend the localization of Theorem 3.8 to infinite locally finite graphs and show that it guarantees the existence of Hamiltonian curves.

Paper II: Some local–global phenomena in locally finite graphs

In this paper, we investigate connections between local and global properties in graphs. We show that if all balls of any fixed radius are k -connected, then so are all balls of larger radius. We also show that the analogous statement for Hamiltonicity is not true (not even if all balls that do not cover the entire graph are Hamiltonian); however, if all balls of radius 1 satisfy the condition of Ore’s Theorem then every ball of any radius is Hamiltonian. Finally, we extend Theorem 3.11 to infinite locally finite graphs and show that it guarantees the existence of Hamiltonian curves.

Paper III: Some cyclic properties of L_1 -graphs

In this paper, we show that not all graphs satisfying the conditions of Theorem 3.10 are pancyclic, unlike graphs satisfying the conditions of Theorems 3.5 and 3.9, but that any non-Hamiltonian cycle in such a graph can be extended to a larger cycle containing all vertices of the original cycle and at most two other vertices. We also prove a similar result for paths whose endpoints do not have any common neighbors, extending Theorem 3.21. Finally, we extend Theorem 3.10 to infinite locally finite graphs and show that it guarantees the existence of Hamiltonian curves.

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