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CONVERGENCE AND STABILITY PROPERTIES OF SUMMATION-BY-PARTS IN TIME

VIKTOR LINDERS *, JAN NORDSTRÖM †, AND STEVEN H. FRANKEL ‡

Abstract. We extend the list of stability properties satisfied by Summation-By-Parts (SBP) in time to include strong S-stability, dissipative stability and stiff accuracy. Further, it is shown that SBP in time is B-convergent for strictly contractive non-linear problems and weakly convergent for non-linear problems that are both contractive and dissipative.

Key words. SBP in time; Runge-Kutta methods; S-stability; Stiff accuracy; Dissipative stability; B-convergence

AMS subject classifications. 65L04, 65L05, 65L06, 65L20

1. Introduction.

Summation-By-Parts (SBP) in time was introduced in [30] as a new family of implicit time marching methods. SBP operators have traditionally been used for spatial discretisation, and originates within the finite difference community [21]. The method is designed to semi-discretely mimic the energy estimates obtained for well-posed initial boundary value problems (IBVPs). This useful property has found its way into other common methods of (spatial) discretisation, including finite volume [25, 26], spectral [4], discontinuous Galerkin [13] and flux reconstruction [33] methods.

In [1] it was shown that certain SBP operators can be associated with Runge-Kutta (RK) methods. It is natural to ask which aspects of the extensive classical RK theory that apply to the SBP in time framework. Efforts to answer this question was made in [30, 23, 1], where it was established that all SBP operators are A-, AN-, L-, B-, and BN-stable. These stability properties may be roughly described as follows:

A-stability: The numerical solution decays for the linear constant coefficient problem

$$\begin{aligned} u_t &= \lambda u, \quad 0 < t \leq T, \\ u &= g, \quad t = 0, \end{aligned} \tag{1}$$

with $\mathrm{Re}\lambda < 0$.

AN-stability: The numerical solution decays for the linear variable coefficient problem

$$\begin{aligned} u_t &= \lambda(t) u, \quad 0 < t \leq T, \\ u &= g, \quad t = 0, \end{aligned}$$

with $\mathrm{Re}\lambda(t) < 0$.

L-stability: The decay of the numerical solution is sufficiently fast when $\lambda \to -\infty$ in (1).

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B-stability: Two numerical solutions $u = v$ and $u = w$, corresponding to different initial data $g = g_u$ and $g = g_v$ for the non-linear problem

$$u_t = f, \quad 0 < t \leq T, \qquad u = g, \quad t = 0, \tag{2}$$

where $f = f(u)$ is autonomous, approach each other if the contractivity condition $(f(v) - f(w), v - w) \leq 0$ is satisfied.

BN-stability: B-stability for the non-autonomous problem (2), where $f = f(t, u)$.

Additionally, monotonicity (in [23] called energy stability) and algebraic stability were established in these works, where the latter is a particular algebraic condition that can be shown to be equivalent to AN-, BN- and B-stability for SBP in time. These results were obtained largely using energy arguments.

Considerably less attention has been payed to the convergence of SBP in time for different problems. In [23], the convergence rates for a stiff and a non-stiff linear autonumous problem were established. However, for non-linear problems, similar results are lacking.

In this paper we aim to extend the list of problems for which convergence of SBP in time is guaranteed. We will consider the generic (system of) ordinary differential equation(s) (2) and investigate the convergence and stability properties of SBP in time under several different assumptions on the regularity of $f(t, u)$. We will restrict our attention to real functions f, however the complex case can be treated in an analogous way. We focus on the following regularity assumptions:

- Contractive problems, $\langle f(t, u) - f(t, v), u - v\rangle \leq 0$,
- Semi-bounded / monotonic problems, $\langle f(t, u), u\rangle \leq 0$,
- Dissipative problems, $\langle f(t, u), u\rangle \leq \gamma - \omega\|u\|^2$,

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product, $\|\cdot\|$ the corresponding norm, and $\gamma \geq 0$ and $\omega > 0$ are constants.

Contractive problems have been the focus of a large portion of the classical literature on convergence and stability of Runge-Kutta methods. Semi-bounded / monotonic problems are relevant in the context of semi-discretisations of IBVPs. Dissipative problems are also of importance in this context as they arise in the discretisation of certain non-linear IBVPs for which semi-boundedness is not achieved. Note that the meaning of 'dissipative' here is different from that used in the study of numerical methods for IBVPs, due to the non-negative constant γ. We will elaborate on this later.

We will show that SBP in time enjoys several previously unreported stability properties. In particular, this includes showing that SBP operators that can be associated with Runge-Kutta methods are stiffly accurate, strongly S-stable as well as dissipatively stable; concepts that will be defined in section 4. We will also show that SBP in time converges for strictly contractive and semi-bounded problems (i.e. under strict inequality). In case of equality, e.g. if $\langle f, u\rangle = 0$, we establish weak convergence for problems that are both contractive and dissipative.

The approach taken in this paper is purely algebraic. Much of the literature on Runge-Kutta methods is concerned with algebraic conditions that imply different stability and convergence concepts. We will cover several such conditions and relate them to the symmetry properties of SBP operators. As such, this paper has the flavour of a short review of convergence and stability properties of SBP in time.

The paper is organised as follows: In section 2 we introduce necessary definitions and terminology. A discussion of precisely which SBP operators can be used to define RK methods is presented in section 3. Subsequently, section 4 contains an outline of stability and convergence properties satisfied by SBP in time. Some remarks on the unique solvability of the discrete problems are included in section 5. Finally, we summarise our findings in section 6.

2. Preliminaries. In this section we briefly review the defining features of SBP in time and Runge-Kutta methods.

2.1. Summation-By-Parts in time. We will use the following, somewhat extended definition of an SBP operator:

Definition 1. *A matrix D is said to be a Summation-By-Parts operator of order p on $\Omega = [T_a, T_b]$ if the following conditions are met:*

1. $D = P^{-1}Q$,
2. $P = P^\top > 0$,
3. $Q + Q^\top = -\mathbf{e}_0\mathbf{e}_0^\top + \mathbf{e}_n\mathbf{e}_n^\top = \mathrm{diag}(-1, 0, \dots, 0, 1) = B$,
4. $D\mathbf{t}^j = j\mathbf{t}^{j-1}, \quad j = 0, \dots p$,
5. $\mathbf{t} = (t_0, \dots, t_n)^\top$, *where* $T_A = t_0 < t_2 < \dots < t_{n-1} < t_n = T_b$.

Throughout the paper we will further demand that the matrix P is diagonal. This is essential in order to prove algebraic stability and stiff accuracy of SBP in time, which forms the basis of most results in this paper.

In Definition 1, $\mathbf{e}_0$ and $\mathbf{e}_n$ denote the first and last standard unit vectors respectively. Note that we have included the grid $\mathbf{t}$ in the definition of the operator, and that the grid points are distinct, ordered, and coincides with the boundaries of the domain Ω. The notation $\mathbf{t}^j$ should be interpreted as elementwise exponentiation. In particular, we take $\mathbf{t}^0 = \mathbf{1} = (1, \dots, 1)^\top$ as a definition.

Consider the initial value problem (1). The energy method (multiplying by the complex conjugated solution $\bar{u}$ and integrating in time) leads to the bound

$$|u(T)|^2 = 2\,\mathrm{Re}(\lambda)\|u\|^2 + g^2, \tag{3}$$

where $\|u\|^2 = \int_0^T |u|^2 \mathrm{d}t$.

Applying SBP in time to (1) involves replacing the continuous time derivative with the operator $D = P^{-1}Q$ as well as imposing the initial conditions in a weak sense. The resulting scheme reads

$$P^{-1}Q\mathbf{u} = \lambda\mathbf{u} + \sigma P^{-1}\mathbf{e}_0(\mathbf{e}_0^\top\mathbf{u} - g), \tag{4}$$

where σ is a scalar penalty parameter. Here, $\mathbf{u} = (u_0, \dots, u_n)^\top$ approximates the continuous solution u at each grid point (or *stage*) in time. The right-hand side is a Simultaneous Approximation Term (SAT) [5] that weakly imposes the initial condition.

Choosing $\sigma = -1$ and applying the discrete energy method to (4) (multiplying from the left by $\mathbf{u}^*P$, adding the conjugate transpose and using Definition 1) leads to

$$|\mathbf{e}_n^\top\mathbf{u}|^2 = 2\,\mathrm{Re}\,\lambda\|\mathbf{u}\|_h^2 + |g|^2 - |\mathbf{e}_0^\top\mathbf{u} - g|^2. \tag{5}$$

In (5), $\|\mathbf{u}\|_h^2 = \mathbf{u}^*P\mathbf{u}$, where $\mathbf{u}^*$ denotes the conjugate transpose of the vector $\mathbf{u}$. Note that (5) mimics (3) up to the small dissipative term $|\mathbf{e}_0^\top\mathbf{u} - g|^2$, which vanishes with grid refinements.

2.2. Runge-Kutta methods. A Runge-Kutta (RK) method is characterised by the triplet $(A, \mathbf{b}, \mathbf{c})$, where $A = (a_{i,j})_{i,j=0}^{n}$, $\mathbf{b} = (b_0, \ldots, b_n)^\top$, and $\mathbf{c} = (c_0, \ldots, c_n)^\top$. Consider again the ordinary differential equation (2). The $(n+1)$-stage RK scheme used to obtain the approximation $u_k \approx u(t_k)$ from $u_{k-1} \approx u(t_{k-1})$ reads

$$
\begin{aligned}
y_k^i &= u_{k-1} + h_k \sum_{j=0}^{n} a_{i,j} f(t_{k-1} + c_j h_k, y_k^j), \quad i = 0, \ldots, n, \\
u_k &= u_{k-1} + h_k \sum_{j=0}^{n} b_j f(t_{k-1} + c_j h_k, y_k^j).
\end{aligned} \tag{6}
$$

Here, $h_k = t_k - t_{k-1}$ denotes the local time step.

An RK method is said to be confluent if $c_i = c_j$ for some $i, j = 1, \ldots, n$ and $i \neq j$. Otherwise it is non-confluent. It is consistent if $\mathbf{b}^\top \mathbf{1} = 1$. This follows from letting $f = 1$ in (2) and inspecting (6).

3. Runge-Kutta characterisation of SBP in time. Let us divide the domain $[0, T]$ into N intervals $[t_k, t_{k+1}]$ of length h_k, $k = 0, \ldots, N-1$, where $t_0 = 0$ and $t_N = T$. This is implicitly done for the RK method in (6). In each interval we apply SBP in time in an analogous way to (4). The intervals are coupled together by SAT terms, hence this multi-block formulation poses no additional difficulty [30, 23, 1, 29].

In [1] it was shown that the SBP operator on the kth interval defines an RK method through the relations

$$
\tilde{D} = A^{-1}/h_k, \quad P = h_k \mathrm{diag}(\mathbf{b}), \quad \mathbf{t} = t_k \mathbf{1} + h_k \mathbf{c}. \tag{7}
$$

In other words, the multi-block formulation of SBP in time is equivalent to (6) with A, $\mathbf{b}$ and $\mathbf{c}$ given by (7). The normalisations by h_k are introduced so that the RK method is defined on the unit interval $[0, 1]$. In (7) we have introduced the notation $\tilde{D} = P^{-1}\tilde{Q} = P^{-1}(Q + \mathbf{e}_0 \mathbf{e}_0^\top)$. This matrix naturally arises from the discretisation (4) by choosing $\sigma = -1$ and collecting terms multiplying the solution $\mathbf{u}$.

REMARK 1. *A wider class of operators than those in Definition 1 can be used to define RK methods through the relations (7), namely the so called generalised SBP operators (gSBP) [11, 1]. We will not consider gSBP operators further in this paper since several of the results and proofs that follow do not immediately generalize. In particular, this is the case for those results that depend on stiff accuracy; see section section 4.*

LEMMA 2. *SBP in time is non-confluent and consistent.*

Proof. From (7) and Definition 1 it follows that $\mathbf{c} = (c_1, \ldots, c_n)$ satisfy $c_{j-1} < c_j$, $j = 1, \ldots, n$, hence the method is non-confluent. Further, we note that

$$
D\mathbf{t} = \mathbf{1} \Rightarrow \mathbf{1}^\top Q \mathbf{t} = \mathbf{1}^\top P \mathbf{1} = h_k \mathbf{b}^\top \mathbf{1}.
$$

Rearranging and using Definition 1 gives

$$
\mathbf{b}^\top \mathbf{1} = \frac{1}{h_k} \mathbf{1}^\top Q \mathbf{t} = \frac{1}{h_k} \mathbf{1}^\top (B - Q^\top) \mathbf{t} = \frac{t_{k+1} - t_k}{h_k} = 1,
$$

where we have used the fact that $\mathbf{1}^\top Q^\top = \mathbf{0}^\top$ and that the end points of $\mathbf{t}$ coincides with the boundaries of Ω. SBP in time is therefore consistent. ∎

In (7) it has been assumed that $\tilde{D}$ is invertible. This assumption should be addressed before we continue.

The elements of an SBP operator, as listed in Definition 1, all have clear interpretations in terms of the continuous problem: $\mathbf{t}$ is a discrete version of the domain $\Omega = [T_a, T_b]$; D approximates the time derivative $\mathrm{d}/\mathrm{d}t$ on Ω; P defines a quadrature rule on Ω (see [17, 22]); $\mathbf{e}_0$ and $\mathbf{e}_n$ project a given vector to the boundaries of Ω. It is less clear what invertibility of $\tilde{D}$ corresponds to in terms of the continuous problem. A possible interpretation can be made using the null-space of D:

Recall that the null-space (or *kernel*), $\ker(D)$ of the matrix D is the set of vectors $\{\mathbf{v} : D\mathbf{v} = \mathbf{0}\}$. This definition extends to continuous operators such as the derivative $\mathrm{d}/\mathrm{d}t$. Obviously the kernel of $\mathrm{d}/\mathrm{d}t$ is the span of the set $\{1\}$, i.e. all constants. It is reasonable to demand of the approximation D, that $\ker(D) = \mathrm{span}(\{\mathbf{1}\})$, i.e. all constant vectors. This motivates the following definition; see [38]:

DEFINITION 3. *An SBP operator D is said to be null-space consistent if*

$$\ker(D) = \mathrm{span}(\{\mathbf{1}\}).$$

Recall the well known fact that for any $(n+1)\times(n+1)$ matrix A,

$$\mathrm{rank}(A) + \dim\ker(A) = n+1. \tag{8}$$

We may now characterise those SBP operators with invertible $\tilde{D}$ as follows.

PROPOSITION 4. *The matrix $\tilde{D} = D + P^{-1}\mathbf{e}_0\mathbf{e}_0^\top$ is invertible if and only if D is null-space consistent.*

Proof. By definition, P is positive definite so with $Q = PD$ and $\tilde{Q} = P\tilde{D}$ we have

$$\ker(D) = \ker(Q), \qquad \ker(\tilde{D}) = \ker(\tilde{Q}). \tag{9}$$

In [30] it was shown that any vector $\mathbf{w} = (w_0, \dots, w_n)^\top \in \ker(\tilde{Q})$ satisfies $w_0 = w_n = 0$. Therefore, $\tilde{Q}\mathbf{w}$ is independent of the first (and last) column of $\tilde{Q}$, and consequently it is also independent of σ. Thus, $\mathbf{w} \in \ker(\tilde{Q})$ implies $\mathbf{w} \in \ker(Q)$. Hence, $\dim\ker(\tilde{Q}) \le \dim\ker(Q)$, and from (8) we have

$$\mathrm{rank}(\tilde{Q}) \ge \mathrm{rank}(Q). \tag{10}$$

Next, consider the matrix $J = I - \mathbf{1}\mathbf{e}_0^\top$, where I is the identity matrix. Note that

$$\tilde{Q}J = (Q + \mathbf{e}_0\mathbf{e}_0^\top)(I - \mathbf{1}\mathbf{e}_0^\top) = Q - (Q\mathbf{1})\mathbf{e}_0^\top + \mathbf{e}_0\mathbf{e}_0^\top - \mathbf{e}_0(\mathbf{e}_0^\top\mathbf{1})\mathbf{e}_0^\top = Q. \tag{11}$$

In (11) we have used the fact that $\mathbf{1} \in \ker(Q)$, which is an immediate consequence of Definition 1.

Let $\mathbf{v} \in \ker(Q)$ be a non-zero vector. It follows from (11) that

$$\mathbf{0} = Q\mathbf{v} = \tilde{Q}J\mathbf{v} = \tilde{Q}(\mathbf{v} - \mathbf{1}(\mathbf{e}_0^\top\mathbf{v})) = \tilde{Q}(\mathbf{v} - v_0\mathbf{1}), \tag{12}$$

where v_0 is the first element of $\mathbf{v}$. Note that in the special case when $\mathbf{v} \in \mathrm{span}\{\mathbf{1}\}$ we have $\mathbf{v} = v_0\mathbf{1}$ and, by (12), $J\mathbf{v} = \mathbf{v} - v_0\mathbf{1} = \mathbf{0}$. Hence, (12) implies that for each eigenvector $\mathbf{v} \in \ker(Q)$ there is a corresponding eigenvector $\mathbf{w} = J\mathbf{v} \in \ker(\tilde{Q})$ *except* when $\mathbf{v} \in \mathrm{span}\{\mathbf{1}\}$. It therefore holds that $\dim\ker(\tilde{Q}) \ge \dim\ker(Q) - 1$, and from (8) we have

$$\mathrm{rank}(\tilde{Q}) \le \mathrm{rank}(Q) + 1. \tag{13}$$

Taking (10) and (13) together it is clear that $\mathrm{rank}(\tilde{Q}) = \mathrm{rank}(Q) + 1$. But $\tilde{Q}$ is invertible if and only if $\mathrm{rank}(\tilde{Q}) = n+1$, i.e. if and only if $\mathrm{rank}(Q) = n$, and thus, by (8), if and only if $\ker(Q)$ is one-dimensional. However, recall from Definition 1 that $D\mathbf{1} = \mathbf{0}$ and consequently $Q\mathbf{1} = \mathbf{0}$. Thus, if $\ker(Q)$ is one-dimensional it is necessarily given by $\ker(Q) = \mathrm{span}\{\mathbf{1}\}$, which by Definition 3 and (9) makes D null-space consistent. ∎

Proposition 4 provides a characterisation of those SBP operators that can be associated with RK methods:

COROLLARY 5. *An SBP operator D defines an RK method through the relations (7) if and only if it is null-space consistent.*

Proof. The relations (7) are well defined if and only if $\tilde{D}$ is invertible. By Proposition 4, this holds if and only if D is null-space consistent. ∎

Not every SBP operator is null-space consistent. As counterexample can be constructed by letting $\mathbf{t} = (-3,-1,1,3)^\top/2$ and considering the following first order accurate SBP operator:

$$\underbrace{\begin{pmatrix} -1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 \end{pmatrix}}_{D} = \underbrace{\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}}_{Q}.$$

Evidently $\mathbf{v} = (0,1,-1,0)^\top \in \ker(D)$. By Proposition 4 and Definition 3, $\tilde{D}$ is therefore not invertible and does not define a Runge-Kutta method through the relations (7). This is of course obvious by inspection since the second and third rows of D, and hence of $\tilde{D}$, are identical and thus $\det(\tilde{D}) = 0$. A similar example was presented in [32].

REMARK 2. *To the best of our knowledge, the standard finite difference SBP operators with a repeated central interior stencil are null-space consistent.*

4. Stability and convergence of SBP in time. In this section we will outline several stability and convergence properties that an RK method might possess and show that they hold for SBP in time. Some results derived here are known but included with a novel proof that we believe provides new insight. Other results are reported for the first time.

We begin by establishing two properties of $A = \tilde{D}^{-1}/h_k$ that will be useful in the remainder.

LEMMA 6. *Let D be a null-space consistent SBP operator. Then the following relations are satisfied:*

$$A\mathbf{e}_0 = b_0\mathbf{1}, \tag{14a}$$

$$\mathbf{e}_n^\top A = \mathbf{b}^\top. \tag{14b}$$

Proof. By Proposition 4, invertibility of $\tilde{D}$ follows from null-space consistency. Consider the identity

$$h_k\tilde{D}\mathbf{1} = h_kP^{-1}(Q\mathbf{1} + \mathbf{e}_0\mathbf{e}_0^\top\mathbf{1}) = \mathbf{0} + h_kP^{-1}\mathbf{e}_0 = \frac{1}{b_0}\mathbf{e}_0, \tag{15}$$

where we have used the facts that $\mathbf{1} \in \ker(Q)$ and that P is diagonal and related to $\mathbf{b}$ through (7). Multiplying from the left by b_0A and using (7) gives (14a).

Next, consider the identity

$$\mathbf{1}^\top\tilde{Q} = \mathbf{1}(Q + \mathbf{e}_0\mathbf{e}_0^\top) = \mathbf{1}^\top(B - Q^\top + \mathbf{e}_0\mathbf{e}_0^\top) = \mathbf{1}^\top\mathbf{e}_n\mathbf{e}_n^\top = \mathbf{e}_n^\top, \tag{16}$$

where we have once again used the fact that $\mathbf{1} \in \ker(Q)$. Multiplying (16) from the right by $A = \tilde{Q}^{-1}P/h_k$ gives

$$\mathbf{1}^\top P/h_k = \mathbf{e}_n^\top A.$$

Since P is diagonal, (14b) follows from (7). ∎

4.1. Linear problems: Stiff accuracy, stiff order and S-stability. We begin our discussion in a linear context. Consider again the problem (1). Setting $f(t,u) = \lambda u$ in (6) leads to the relation

$$u_{k+1} = R(h_k\lambda)u_k, \tag{17}$$

where, for any complex number z, the *stability function* $R(z)$ is given by

$$R(z) = 1 + z\mathbf{b}^\top (I - zA)^{-1}\mathbf{1}; \tag{18}$$

see e.g. [16] for a derivation. The method is said to be A-stable if $|R(z)| < 1$ for every z with $\mathrm{Re}z < 0$. This reflects the property that the solution of (1) decays if $\mathrm{Re}\lambda < 0$. A-stability of SBP in time was established in [23].

To guarantee that stiff components of (1) are damped sufficiently fast, the concept of L-stability is used. An RK method is L-stable if it is A-stable and $\lim_{z\to\infty} R(z) = 0$. This ensures that $u_{k+1}/u_k \to 0$ as $\lambda \to -\infty$. L-stability of SBP in time was also established in [23] using energy arguments. Assuming null-space consistency, it may alternatively be shown using property (14a) in Lemma 6 by noting that

$$R(\infty) = 1 - \mathbf{b}^\top A^{-1}\mathbf{1} = 1 - \mathbf{b}^\top \mathbf{e}_0/b_0 = 1 - 1 = 0.$$

A similar argument can be put forward using property (14b); see [16].

An extension of A- and L-stability to inhomogeneous linear problems is introduced in [31]. There, it is suggested that one should consider the model problem

$$u_t = s_t(t) + \lambda(u - s(t)), \quad u(0) = s(0), \tag{19}$$

with solution $u(t) = s(t)$, where s is any function such that s_t is bounded. Clearly (1) is recovered from (19) by choosing $s(t) = 0$. This model problem is motivated by the observation that the accuracy of the solutions obtained for stiff problems might be lower than that expected by the method, which cannot be explained by analysing (1) alone. This effect is known as *order reduction*. Further, it turns out that stability for (19) is a much better indicator of the behaviour of the RK method when applied to non-linear problems than A- and L-stability [40], even though (19) itself is linear.

In [31] the following generalisation of A-stability is introduced:

Definition 7. *An RK method is said to be S-stable if, for the equation (19) and for any real positive constant λ_0, there exists a real positive constant h_0 such that*

$$\left\| \frac{u_{k+1} - s(t_{k+1})}{u_k - s(t_k)} \right\| < 1,$$

provided $u_k \neq s(t_k)$, for all $0 < h < h_0$ and all complex λ with $\mathrm{Re}(-\lambda) \leq \lambda_0$.

A-stability follows from S-stability by considering the special case $s = 0$. To see this, take the norm of the left- and right-hand sides of (17) and note that $\|u_{k+1}\|/\|u_k\| < 1$ is satisfied if and only if $|R(h_k\lambda)| < 1$. Conversely, A-stability is necessary but not sufficient for S-stability [31].

A generalisation of L-stability is also introduced in [31]:

Definition 8. *An RK method is strongly S-stable if it is S-stable and*

$$\frac{u_{k+1} - s(t_{k+1})}{u_k - s(t_k)} \to 0$$

as $\mathrm{Re}\lambda \to -\infty$.

L-stability is implied by strong S-stability by considering the case $s = 0$. However, the converse is not true since strong S-stability requires the RK method to be *stiffly accurate*, which is defined as follows:

DEFINITION 9. *Let an $n+1$-stage Runge-Kutta method be defined by the matrix A and the vectors $\mathbf{b}$ and $\mathbf{c}$. The method is said to be stiffly accurate if $c_n = 1$ and $\mathbf{e}_n^\top A = \mathbf{b}^\top$.*

We can immediately establish that SBP in time is stiffly accurate:

LEMMA 10. *A null-space consistent SBP operator defines a stiffly accurate RK method.*

Proof. By Definition 1 and (7), $c_n = 1$ and $\mathbf{e}_n^\top A = \mathbf{b}^\top$ is established in (14b) in Lemma 6. ∎

REMARK 3. *Here and elsewhere, the formulation "a null-space consistent SBP operator defines an RK method" should be interpreted such that D is null-space consistent while the associated RK method is obtained from P and $\tilde{D} = D + P^{-1}\mathbf{e}_0\mathbf{e}_0^\top$ through the relations (7).*

Stiff accuracy is a desirable property for several reasons. It is essential when solving singular perturbation problems and differential algebraic equations; see [16] for an overview. More interestingly, note that for an SBP operator, stiff accuracy implies that the final row of A is given by $\mathbf{b}^\top = \mathbf{1}^\top P/h_k$. Thus, the final stage of the method is as accurate as the quadrature rule P. It is known that for any SBP operator of order p, the matrix P defines a quadrature rule that is accurate at least of order $2p$ [17, 22]. This observation is intimately connected with the fact that the final stage of the SBP method is *super-convergent*, which has been discussed extensively in [23, 1].

With Lemma 10 established we are in position to show the following:

PROPOSITION 11. *A null-space consistent SBP operator defines an RK method that is strongly S-stable.*

Proof. It is shown in [31] that an RK method with $0 = c_0 < c_1 < \cdots < c_{n-1} < c_n = 1$ is S-stable if $|R(\infty)| < 1$ and

$$\lim_{z\to 0} \mathbf{b}^\top (A - zI)^{-1}(\mathrm{diag}(\mathbf{c}) - z\mathbf{e}_0\mathbf{e}_0^\top)$$

is finite. Further, it is shown that an S-stable RK method is strongly S-stable if and only if it is L-stable and stiffly accurate.

From Definition 1 and (7) it is clear that $\mathbf{c}$ fufills the necessary requirement, and $|R(\infty)| = 0$ by the definition of L-stability. Further, since A is invertible by Proposition 4, we have

$$\lim_{z\to 0} \mathbf{b}^\top (A - zI)^{-1}(\mathrm{diag}(\mathbf{c}) - z\mathbf{e}_0\mathbf{e}_0^\top) = \mathbf{b}^\top A^{-1}\mathrm{diag}(\mathbf{c}).$$

From (7) and Definition 1 we note that

$$\mathbf{b}^\top A^{-1}\mathrm{diag}(\mathbf{c}) = \mathbf{1}^\top P\tilde{D}\,\mathrm{diag}(\mathbf{c})/h_k = \mathbf{1}^\top \tilde{Q}\,\mathrm{diag}(\mathbf{c}) = \mathbf{e}_n^\top.$$

The final equality follows from (16) and the fact that $c_n = 1$. Obviously $\mathbf{e}_n^\top$ is finite, hence S-stability is ensured. Strong S-stability follows from L-stability and stiff accuracy as established in Lemma 10. ∎

For stiff problems one might be forced to take time steps that are much larger than the characteristic times of the problem. Examples arise in the simulation of high speed and turbulent flows. A consequence of this is that the accuracy of the RK method deteriorates. It is natural to ask what order of accuracy the RK method is reduced to for stiff problems. Such questions were a motivation for the introduction of the problem (19) in [31].

The order reduction phenomenon is reflected in the problem (19) by letting $h_k \gg |\lambda|^{-1}$. The *stiff limit* for the problem (19) is defined as the limits where $h \to 0$ while $\mathrm{Re}z \to -\infty$, where $z = \lambda h$ and $h = \max_k h_k$. Let

$$l_k = u_{k+1} - s(t_{k+1})$$

denote the local truncation error associated with the discretisation of problem (19). Then l_k depends both on h and λ, as well as on $s(t)$. The *stiff order* of the RK method is defined as the pair (r, q) for which

$$l_k = \mathcal{O}(h^{q+1} z^{-r})$$

holds in the stiff limit. Obviously it is desirable that q and r are as large as possible. If $r < 0$ the convergence rate would suffer drastically for very stiff problems, even if $q > 0$.

For a strongly S-stable RK method, q is given by the stage order of the method (i.e. p in Definition 1) and $r = 1$ [31]. We summarise this for SBP in time in the following proposition, which was shown in [23] using energy arguments:

Proposition 12. *A null-space consistent SBP operator of order p defines an RK method with stiff order $(1, p)$.*

Remark 4. *S-stable methods that are not stiffly accurate have $r = 0$. Methods that are not S-stable and not stiffly accurate may have $r < 0$; see [31].*

We close this discussion of linear problems by briefly mentioning another possible generalisation of the model problem (1), namely the variable coefficient problem

$$u_t = \lambda(t)u, \quad \mathrm{Re}\lambda(t) \leq 0.$$

Letting $f(t, u) = \lambda(t)u$ in (6) leads to the relation

$$u_{k+1} = K(Z)u_k, \quad K(Z) = 1 + \mathbf{b}^\top Z(I - AZ)^{-1}\mathbf{1};$$

see e.g. [16]. A non-confluent RK method is said to be AN-stable if $|K(Z)| < 1$ for every matrix $Z = \mathrm{diag}(z_0, \dots, z_n)$ satisfying $\mathrm{Re}z_k < 0$, $k = 0, \dots, n$. That SBP in time defines an AN-stable RK method was pointed out in [1]. We will see it again in the next section.

4.2. Non-linear problems: B-convergence, algebraic and dissipative stability.

We now leave the linear setting and consider instead the generic non-linear problem (2). Much of the classical theory of RK methods is concerned with this problem under the assumption that $f(t, u)$ satisfies some regularity condition. Here, we will focus on three different conditions that have particular importance either for the theory of RK methods, or for the study of partial differential equations. These are respectively contractive, semi-bounded / monotonic, and dissipative problems.

In the remainder we restrict our attention to real valued problems. The results presented in subsection 4.2.1 and subsection 4.2.2 can readily be extended to the complex valued setting. However, for the results of subsection 4.2.3 this is not the case. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the L^2 inner product and norm

$$\langle u, v \rangle = \int_0^T v^\top(t) H u(t) \mathrm{d}t, \quad \|u\| = \sqrt{\langle u, u \rangle},$$

where H is a symmetric positive definite matrix. In the following, we assume that $f(t, u)$ is a continuous function of u for almost all t and a measurable function of t for all u.

4.2.1. Contractive problems. We start by considering the case when (2) satisfies the one-sided Lipschitz condition

$$\langle f(t,u) - f(t,v), u - v\rangle \leq \beta \|u - v\|^2, \tag{20}$$

where u and v are two solutions to (2) with different initial data. Here, β is the one-sided Lipschitz constant. It can be shown that the solutions u and v in (20) satisfy the relation

$$\|u(t) - v(t)\| \leq e^{\beta t}\|u(0) - v(0)\|; \tag{21}$$

see e.g. [16]. The problem (2) is said to be *contractive* if (20) is satisfied with $\beta \leq 0$. Contractivity ensures that u and v do not diverge from each other as time progresses.

It is of course desirable that this property is preserved by the RK scheme. The RK method (6) is said to be BN-stable if (20) with $\beta \leq 0$ implies

$$\|u_k - v_k\| \leq \|u_0 - v_0\| \tag{22}$$

for the numerical solution obtained by the RK scheme. It is said to be B-stable if this implication holds and $f(t,u) = f(u)$. Obviously BN-stability implies B-stability. BN-stability of SBP in time was established in [23] using energy arguments.

An important result of classic RK theory is the existence of an algebraic condition, termed *algebraic stability*, that implies A-stability and is equivalent to AN-, B- and BN-stability for non-confluent methods [2, 6, 16]. This is a remarkable result given that A-, AN-, B- and BN-stability are defined for different test problems.

Definition 13. *An RK method defined by the triplet* $(A, \mathbf{b}, \mathbf{c})$ *is algebraically stable if the matrix*

$$M = \mathrm{diag}(\mathbf{b})A + A^\top \mathrm{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^\top$$

is positive semi-definite and if the elements of $\mathbf{b}$ *are non-negative.*

For SBP in time, only BN-stability has been proven directly in [23], however B-stability can be shown in exactly the same way. In [1] it is pointed out that AN- and algebraic stability follows due to the equivalence with BN-stability. Here, with the aid of Lemma 6, we will provide a direct proof of algebraic stability, which gives new insights that we will use in due course.

Proposition 14. *A null-space consistent SBP operator defines an RK method that is algebraically stable.*

Proof. From Definition 1 and (7) it follows that the elements of $\mathbf{b}$ are positive for any SBP operator. Next, from Definition 1, and with $\tilde{D} = D + P^{-1}\mathbf{e}_0\mathbf{e}_0^\top$, it is clear that

$$P\tilde{D} + \tilde{D}^\top P = \mathbf{e}_0\mathbf{e}_0^\top + \mathbf{e}_n\mathbf{e}_n^\top.$$

By Proposition 4, $\tilde{D}$ is invertible. Thus, we multiply from the left and right by $\tilde{D}^{-\top}/h_k = A^\top$ and $\tilde{D}^{-1}/h_k = A$ respectively and use (7) to obtain

$$\frac{1}{h_k}(A^\top P + PA) = A^\top \mathbf{e}_0\mathbf{e}_0^\top A + A^\top \mathbf{e}_n\mathbf{e}_n^\top A. \tag{23}$$

By Lemma 6, $\mathbf{e}_n^\top A = \mathbf{b}^\top$ (stiff accuracy), whence $A^\top \mathbf{e}_n\mathbf{e}_n^\top A = \mathbf{b}\mathbf{b}^\top$. By (7), $P = h_k\mathrm{diag}(\mathbf{b})$ and thus (23) may be rewritten as

$$\mathrm{diag}(\mathbf{b})A + A^\top\mathrm{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^\top = A^\top\mathbf{e}_0\mathbf{e}_0^\top A. \tag{24}$$

Algebraic stability follows by observing that the right-hand side of (24) is positive semi-definite. ∎

Apart from leading to the aforementioned A-, AN-, B- and BN-stability, algebraic stability plays an important role in the stability of gradient, conservative and Hamiltonian systems [37], as well as dissipative problems, which will be considered in subsection 4.2.3.

To address the order reduction phenomenon observed in [31] for the non-linear problem (2), the concept of B-convergence was introduced in [12].

DEFINITION 15. *For the problem (2) satisfying the one-sided Lipschitz condition (20), the RK method (6) is said to be B-convergent of order q if*

$$\|u_k - u(t_k)\| \le Ch^q, \tag{25}$$

where $h = \max_k h_k$. The function C may depend on β, t_k, h (and/or $h/\min_k h_k$) and bounds on the time derivatives $u^{(j)}(t)$, but not on $f(t, u)$.

REMARK 5. *In some texts C is allowed to depend on certain derivatives of $f(t, u)$, though not on $\partial f/\partial u$. In these cases, the concept defined in Definition 15 is referred to as* optimal *B-convergence.*

We may now show that the stiff convergence result in Proposition 12 translates to the non-linear contractive problem (2), (20) under the restriction $\beta < 0$:

PROPOSITION 16. *A null-space consistent SBP operator of order p defines an RK method that is B-convergent of order p for the problem (2) under the restriction (20) with $\beta < 0$.*

Proof. The bulk of the proof comes from [9] where consistent RK methods satisfying $0 = c_0 < c_1 < \cdots < c_{n-1} < c_n = 1$, and for which A is invertible, were considered. There it was shown that if the RK method is algebraically stable, has $|R(\infty)| < 1$ and if the elements of $\mathbf{b}$ are positive, then it is B-convergent with the stage order of the method for contractive problems with $\beta < 0$. For SBP in time, these properties are immediate consequences of Proposition 4 and Proposition 14 together with Definition 1, (7) and L-stability. ∎

4.2.2. Semi-bounded / monotonic problems. In the analysis of spatial discretisations of IBVPs, the goal is typically to design the numerical scheme such that the resulting system of ordinary differential equations satisfies the regularity condition

$$\langle f(t, u), u \rangle \le 0. \tag{26}$$

If this is achieved, time-stability (or *strict stability*) can be achieved [14, 20, 15]. In this context, property (26) is refered to as *semi-boundedness*. In the classical theory of RK methods, the same property is termed *monotonicity*.

If the problem (2) satisfies (26), then the solution u is bounded by the initial data:

$$\|u(t)\| \le \|u(0)\|, \quad t \ge 0. \tag{27}$$

It is of course a desirable property of a numerical scheme that (27) holds in the discrete setting, i.e. that

$$\|u_k\| \le \|u_0\|, \quad \forall k. \tag{28}$$

An RK method that fulfills (28) is said to be *monotone*. In [23] it was shown using energy arguments that SBP in time is a monotone method. Alternatively, we may resort to a result in [7] stating that every algebraically stable RK method is mototone, which extends to SBP in time by Proposition 14. We summarise in the following:

PROPOSITION 17. *A null-space consistent SBP operator defines a monotone RK method.*

Convergence for semi-bounded problems is a more delicate issue. For a semi-bounded problem that is also strictly contractive (i.e. that satisfies (20) with $\beta < 0$), we may resort to Proposition 16 to ensure convergence for the problem (2). This case typically arises for stable discretisations of parabolic PDEs such as the heat equation, and is thus of fundamental importance.

On the other hand, semi-bounded discretisations of hyperbolic PDEs typically have the contractivity bound $\beta \leq 0$ at best, unless additional artificial dissipation is added. This may happen e.g. in the discretisation of wave equations. In this case, Proposition 16 does not apply. We cannot in general expect to extend the results of Proposition 16 to $\beta = 0$ or higher values. A counter example was given in [35] for the Lobatto IIIC method of order $p \geq 2$, which belongs to the class of SBP operators [32]. This result does not imply that SBP in time will not converge for problems with $\beta \leq 0$ but merely demonstrates that such problems might exist.

It is worth pointing out that the particular first order accurate SBP operator

$$\underbrace{\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}}_{D} = \underbrace{\begin{pmatrix} 2 & \\ & 2 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{Q}$$

defines an RK method through the relations (7) that is B-convergent of order 3/2 even when $\beta = 0$; see [3, 8, 36]. The critical property that allows the improved B-convergence for this operator is the fact that $P\tilde{D} + \tilde{D}^\top P$ is positive definite;

$$P\tilde{D} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \Rightarrow \quad P\tilde{D} + \tilde{D}^\top P = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} > 0.$$

The purpose of this example is to show that a *special case* of an SBP operator can have better convergence properties than what the general theory predicts. For SBP operators of dimension $n + 1 \geq 3$, positive definiteness of $P\tilde{D} + \tilde{D}^\top P$ does not hold.

If problem (2) is contractive with $\beta \leq 0$ and strictly semi-bounded such that (26) is satisfied with strict inequality, convergence can still be ensured in certain circumstances. This is the topic of the next subsection. For examples of such discretisations, see e.g. [24, 27, 18].

4.2.3. Dissipative problems. There are important problems for which semi-boundedness is not easily achieved. Examples include semi-discretisations of the Euler and Navier-Stokes equations of fluid dynamics, where estimates of the form (26) are generally not obtainable. A notable exception is given in [28] where semi-boundedness was achieved with a spatial discretisation of the 2D incompressible Navier-Stokes equations.

If we relax the rather strong monotonicity condition (26), we obtain a larger family of problems: Let there be two constants $\gamma \geq 0$ and $\omega > 0$ such that for all u,

$$\langle f(t, u), u \rangle \leq \gamma - \omega \|u\|^2. \tag{29}$$

If (2) satisfies (29), the resulting system is said to be *dissipative* and is characterised by the property that it is monotonic outside the set $\{u : \|u\|^2 \leq \gamma/\omega\}$. In other words, the energy contained in the solution is guaranteed to decay if the amplitude of u is large enough. Condition (29) implies that there exists a time $t_c > 0$ and a constant $R_c > 0$ such that the solution satisfies the bound $\|u(t)\| \leq R_c$ for all $t > t_c$. To see this, note that

$$\|u\|_t^2 = 2(u_t, u) = 2(f, u) \leq \gamma - \omega \|u\|^2.$$

Integration in time gives

$$\|u\|^2 \leq \frac{\gamma}{\omega} + \left(\|g\|^2 - \frac{\gamma}{\omega}\right) e^{-\omega t}.$$

Thus, if $\|g\|^2 \leq \gamma/\omega$ we may choose $R_c = \gamma/\omega$. If, on the other hand, $\|g\|^2 > \gamma/\omega$, then $\|u\|^2$ approaches γ/ω monotonically from above. Hence, for every $t_c > 0$ there is a $\delta_c > 0$ such that $\|u\|^2 \leq R_c = \gamma/\omega + \delta_c$ for all $t > t_c$. Note that if $\gamma = 0$ we recover strict semi-boundedness for any $t > 0$.

It is sometimes possible to obtain a dissipative semi-discretisation of an IBVP when a semi-bounded one is not available. Examples include the forced incompressible Navier-Stokes equations with no-slip boundary conditions in two dimensions [39] and the Lorenz equations, describing atmospheric convection [37].

In order to analyse the stability of dissipative systems, the following definition was introduced in [37]:

DEFINITION 18. *An RK method is said to be dissipatively stable if, when applied to (2) under the condition (29), there are constants $\Delta t_d, R_d > 0$, both independent of the initial data g, such that for all $\Delta t \in (0, \Delta t_d)$ there exists an integer n_d for which*

$$\|u_k\| \leq R_d, \quad \forall k > n_d.$$

In [37] it was found that algebraic stability implies dissipative stability for any $\Delta t_d > 0$. By Proposition 14 we therefore immediately have

PROPOSITION 19. *A null-space consistent SBP operator defines a dissipatively stable RK method.*

Proposition 19 is an extension of the known BN-stability of SBP in time. However, it does not guarantee convergence for SBP discretisations of problem (2) under the dissipativity constraint (29). Steps in this direction have been taken in [10], where it is shown that under certain additional constraints, an RK method converges in a weak sense as follows.

Let the grid size $h_k = h$ be constant, set $h = T/N$ for some integer N and let u_k, $k = 0, \dots, N$ denote the numerical solution at time $t_k = kh$. We form the piecewise constant interpolant $\tilde{u}_N$ that takes the value u_{k+1} on the interval $(t_k, t_{k+1}]$. Then $\tilde{u}_N$ is a piecewise constant approximation of the exact solution $u(t)$ of (2). The sequence of approximations $(\tilde{u}_N)$ is said to be *weakly convergent* to u if $\langle \tilde{u}_N, v\rangle \to \langle u, v\rangle$ as $N \to \infty$ for every $v \in L^2([0,T])$.

Now, assume that there is a positive constant c such that $|f(t,u)| \leq c(1 + \|u\|)$. In [10] it was shown that the sequence (u_N) converges weakly to u if $f(t,u)$ is both contractive and dissipative and the RK method satisfies

$$\begin{aligned} \mathbf{e}_n^\top A &= \mathbf{b}^\top, \\ \operatorname{diag}(\mathbf{b})A + A^\top \operatorname{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^\top - \mathbf{d}\mathbf{d}^\top &\geq 0, \end{aligned} \tag{30}$$

where $\mathbf{d} = A^\top \operatorname{diag}(\mathbf{b}) A^{-1}\mathbf{1}$. Obviously, the conditions (30) are stronger than algebraic stability. Nevertheless, under the above assumptions on u and f we have the following result for SBP operators:

PROPOSITION 20. *A null-space consistent SBP operator defines an RK method that is weakly convergent for problem (2) under the dissipativity condition (29) and contractivity condition (20) with $\beta \leq 0$.*

Proof. The first property in (30) is precisely stiff accuracy, which is established by Lemma 10. For the second property, we have from (24) in Proposition 14 that

$$\operatorname{diag}(\mathbf{b})A + A^\top \operatorname{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^\top = A^\top \mathbf{e}_0 \mathbf{e}_0^\top A.$$

Now, using (7) and the identity $\tilde{Q}\mathbf{1} = Q\mathbf{1} + \mathbf{e}_0\mathbf{e}_0^\top\mathbf{1} = \mathbf{e}_0$ (or equivalently, property (14a) in Lemma 6) we find that

$$A^\top\mathbf{e}_0 = A^\top\tilde{Q}\mathbf{1} = A^\top P\tilde{D}\mathbf{1} = A^\top\mathrm{diag}(\mathbf{b})A^{-1}\mathbf{1} = \mathbf{d},$$

and thus $A^\top\mathbf{e}_0\mathbf{e}_0^\top A = \mathbf{d}\mathbf{d}^\top$. Consequently, for any null-space consistent SBP operator,

$$\mathrm{diag}(\mathbf{b})A + A^\top\mathrm{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^\top - \mathbf{d}\mathbf{d}^\top = 0, \tag{31}$$

and hence (30) is satisfied. ∎

Note that Proposition 20 allows contractivity with $\beta \leq 0$, i.e. it complements Proposition 16, which only applies for $\beta < 0$. While Proposition 20 ensures weak convergence of SBP in time for contractive and dissipative problems, it does not establish a convergence rate. This can be done under additional regularity assumptions on $f(u,t)$ and the initial data; see [10]. We shall not consider these assumptions here.

It is interesting to note that the proof of Proposition 20 is completely reversible and allows us to characterise those RK methods that can be associated with SBP operators through the relations (7). We summarise this in the following:

PROPOSITION 21. *An RK method $(A, \mathbf{b}, \mathbf{c})$ defines a null-space consistent SBP operator through the relations (7) if and only if $\mathbf{c}$ is ordered and includes the domain boundaries, the elements of $\mathbf{b}$ are positive, A is invertible, stiffly accurate and satisfies (31) as well as (14a).*

5. A few remarks on solvability.

An issue that we so far have not touched upon is the question whether the equations defined by the Runge-Kutta method (6) have a unique solution. The convergence results for all problems considered up to now implicitly rely upon this assumption. The assumption holds for the linear problem (1) and the non-linear contractive and/or dissipative problem (2) if the eigenvalues λ_j of A satisfy

$$\mathrm{Re}\lambda_j > 0, \quad j = 0, \dots, n; \tag{32}$$

see [19, 10]. In (32) we may of course replace A with $\tilde{D}$ if we so desire. The SBP operators satisfying (32) have not yet been adequately classified. It is known that the classical second order accurate finite difference method on SBP form [30] as well as the Lobatto IIIC methods of any order [34] satisfies (32). Obviously null-space consistency is a necessary condition. It is an open problem whether it is also sufficient. To the best of the authors' knowledge, no counter-example has been found.

However, it is worth pointing out that null-space consistent SBP operators exist for which $\tilde{Q}$ has imaginary eigenvalues. An example is given by

$$\underbrace{\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -1 & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{2} & 1 & -1 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}}_{D} = \underbrace{\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & \frac{1}{2} & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}}_{Q},$$

which is a first order accurate SBP operator on the grid $\mathbf{t} = (-2, -1, 0, 1, 2)^\top$. The eigenvalues of $\tilde{Q}$ and $\tilde{D}$ are listed in Table 1. In this example, $\tilde{Q}$ has the imaginary eigenvalues $\pm i\sqrt{3}/2$. However, $\tilde{D}$, and therefore also $A = \tilde{D}^{-1}/4$, still satisfies (32).

	$\tilde{Q}$	$\tilde{D}$
$\lambda_{1,2}$	$\pm\frac{\sqrt{3}}{2}i$	$0.153308 \pm 1.323715i$
$\lambda_{3,4}$	$0.288593 \pm 0.600143i$	$0.486329 \pm 0.801573i$
λ_5	0.422814	0.720726

Table 1: Eigenvalues of $\tilde{Q}$ and $\tilde{D}$ to six decimal places.

6. Summary. In this paper we have explored the stability and convergence properties of SBP in time. We have used the classical theory of Runge-Kutta methods to obtain all results. This has been achieved by noting that SBP operators define RK methods if and only if they are null-space consistent.

Previous work on the topic has shown that SBP in time is A-, L-, AN-, B-, BN- and algebraically stable. Here, we have provided a direct proof of algebraic stability. Further, we have shown that an RK method associated with an SBP operator is stiffly accurate and used these properties to establish strong S-stability and dissipative stability.

Convergence for linear and non-linear problems have been considered. We have shown that SBP in time has stiff order $(1, p)$. Further, the method is B-convergent of order p for contractive problems if the one-sided Lipschitz constant $\beta < 0$. In the less restrictive case $\beta \leq 0$ weak convergence has been established if the problem is also dissipative.

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