Linköping Studies in Science and Technology
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Coherent functors and asymptotic properties

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*To Maria, Chisomo, Mphaso,
Madaliso and Chimwemwe.*

Abstract

In this thesis we study properties of the so called coherent functors. Coherent functors were first introduced by Auslander in 1966 in a general setting. Coherent functors have been used since then as powerful tools for different purposes: to describe infinitesimal deformation theory, to describe algebraicity of a stack or to study properties of Rees algebras.

In 1998, Hartshorne proved that half exact coherent functors over a discrete valuation ring A are direct sums of copies of the identity functor, Hom-functors of quotient modules of A and tensor products of quotient modules of A . In our first article (Paper A), we obtain a similar characterization for half exact coherent functors over a much wider class of rings: Dedekind domains. This fact allows us to classify half exact coherent functors over Dedekind domains.

In our second article (Paper B), coherent functors over noetherian rings are considered. We study asymptotic properties of sets of prime ideals connected with coherent functors applied to artinian modules or finitely generated modules. Also considering quotient modules $M/\mathfrak{a}^n M$, where $\mathfrak{a}^n$ is the n^{th} power of an ideal $\mathfrak{a}$, one obtains that the Betti and Bass numbers of the images under a coherent functor of the quotient modules above are polynomials in n for large n . Furthermore, the lengths of these image modules are polynomial in n , for large n , under the condition that the image modules have finite length.

Populärvetenskaplig sammanfattning

Denna avhandling behandlar koherenta funktorer på kategorin av moduler över noetherska kommutativa ringar. Koherenta funktorer introducerades allmänt av Auslander 1966. Sedan dess har koherenta funktorer framgångsrikt använts som verktyg i kommutativ algebra och algebraisk geometri. 1998 fann Hartshorne en karakterisering, av så kallade halvexakta koherenta funktorer över vissa ringar: diskreta valuationsringar.

Denna avhandling består av två delar (var del i en artikel): I första delen hittar vi en liknande karakterisering av koherenta funktorer, som den givna av Hartshorne, över en bredare klass av ringar: Dedekindområden.

I andra delen studeras asymptotiska egenskaper hos koherenta funktorer på moduler över noetherska ringar. Vi studerar asymptotiska egenskaper när en koherent funktor appliceras på kvotmoduler av en modul genom höga potenser av ideal.

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Contents

I	Preliminaries and summary of papers	1
1	Category of modules	3
1.1	Categories and Functors	3
1.2	Abelian Categories	5
1.3	The category of modules	7
1.4	Homology functors	10
1.5	Limits and Completions	12
2	Coherent Functors	17
2.1	Coherent Functors	17
2.2	Left-, right- and half-exact coherent functors	19
3	Asymptotic stability	23
3.1	Associated primes ideals	23
3.2	Secondary representations and attached prime ideals	24
3.3	Asymptotic stability of Ass and Att	25
3.4	Depth and grade of a module	27
4	Graded modules and Hilbert polynomials	29
4.1	Length of a module	29
4.2	Graded and associated graded modules	30
4.3	Hilbert functions	31
4.4	Betti and Bass numbers	32
4.5	Betti numbers related to a half exact coherent functor over a DVR	33

5	An overview of the papers	37
5.1	Summary of Paper A	37
5.2	Summary of Paper B	38
	Bibliography	41
II	Included Papers	45
	Paper A	47
	Paper B	63

Part I

Preliminaries and summary of papers

1

Category of modules

This chapter introduces the relevant categorical notions that are used throughout this thesis. We refer the reader to [13] and [23] for the general perspective, to [10] and [12] for details on abelian categories and to [35] for details on the category of modules.

1.1 Categories and Functors

A *category* $\mathcal{A}$ consists of the following:

- (a) objects $A, B, C, D, \dots$
- (b) a set $\text{Hom}_{\mathcal{A}}(A, B)$ of morphisms $f : A \rightarrow B$ between each pair of objects A, B ; and
- (c) the law of composition, i.e, given two morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\mathcal{A}$, the composite $g \circ f$ is defined and satisfies

- (i) the *associative law*

$$(h \circ g) \circ f = h \circ (g \circ f)$$

where $h : C \rightarrow D$, and

- (ii) the *unit law*

$$f \circ 1_A = f \text{ and } 1_B \circ f = f$$

where $1_A : A \rightarrow A$ is the *identity* morphism.

Note, we say that $f : A \rightarrow B$ is a morphism in $\mathcal{A}$ to mean that f is a morphism in the set $\text{Hom}_{\mathcal{A}}(A, B)$ for some pair of objects A, B in $\mathcal{A}$.

Definition 1.1.1. Given two categories $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{B}$ is

- a) a *subcategory* of $\mathcal{A}$ if
 - (i) all the objects of $\mathcal{B}$ are also objects of $\mathcal{A}$; and
 - (ii) every morphism f in $\text{Hom}_{\mathcal{B}}(A, B)$ is also a morphism in $\text{Hom}_{\mathcal{A}}(A, B)$.
- b) a *full subcategory* of $\mathcal{A}$ if, for every pair of objects B, B' in $\mathcal{B}$, we have $\text{Hom}_{\mathcal{B}}(B, B') = \text{Hom}_{\mathcal{A}}(B, B')$.

Definition 1.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be categories; a *covariant functor* $F : \mathcal{A} \rightarrow \mathcal{B}$ with domain $\mathcal{A}$ and co-domain $\mathcal{B}$ sends

- (a) each object A in $\mathcal{A}$, to an object $F(A)$ in $\mathcal{B}$; and
- (b) each morphism $f : A \rightarrow B$ in $\mathcal{A}$, to a morphism $F(f) : F(A) \rightarrow F(B)$ in $\mathcal{B}$ such that the following conditions are satisfied
 - (i) $F(1_A) = 1_{F(A)}$ for every object A in $\mathcal{A}$; and
 - (ii) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ in $\mathcal{A}$.

The functor that sends every object in $\mathcal{A}$ to itself and every morphism in $\mathcal{A}$ to itself is called the *identity functor*, and is denoted by $1_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$. The composite $G \circ F$ of two functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ is also a functor, and is defined by

- (i) $(G \circ F)(A) = G(F(A))$ for every A in $\mathcal{A}$, and
- (ii) $(G \circ F)(f) = G(F(f))$ for every morphism f in $\mathcal{A}$.

Note that, given a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ and any pair of objects A, A' of $\mathcal{A}$, there is an induced morphism

$$\Phi_{A,A'} : \text{Hom}_{\mathcal{A}}(A, A') \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(A'))$$

defined by $\Phi_{A,A'}(f) = F(f)$.

Definition 1.1.3. Let $F, G : \mathcal{A} \rightarrow \mathcal{B}$ be functors. A *natural transformation* $\alpha : F \rightarrow G$ sends each object A in $\mathcal{A}$, to a morphism $\alpha_A : F(A) \rightarrow G(A)$ in $\mathcal{B}$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \downarrow F(f) & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

is commutative for every morphism $f : A \rightarrow B$ in $\mathcal{A}$.

Now, let $F, G, H : \mathcal{A} \rightarrow \mathcal{B}$ be functors and $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$ be natural transformations. For each object A in $\mathcal{A}$, the composite $\beta \circ \alpha : F \rightarrow H$ is defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. That $\beta \circ \alpha$ is a natural transformation follows from the commutative diagram below.

$$\begin{array}{ccccc}
 F(A) & \xrightarrow{\alpha_A} & G(A) & \xrightarrow{\beta_A} & H(A) \\
 \downarrow F(f) & & \downarrow G(f) & & \downarrow H(f) \\
 F(B) & \xrightarrow{\alpha_B} & G(B) & \xrightarrow{\beta_B} & H(B)
 \end{array}$$

Therefore, the collection of all functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and their natural transformations form the so called *functor category*.

1.2 Abelian Categories

The behavior of abelian categories is similar to that of abelian groups, hence the name.

Definition 1.2.1. Let A and B be objects in $\mathcal{A}$. A *product* of A and B is an object $A \times B$ in $\mathcal{A}$ together with two morphisms $p_A : A \times B \rightarrow A$ and $p_B : A \times B \rightarrow B$ such that for any other object X in $\mathcal{A}$ with morphisms $f_A : X \rightarrow A$ and $f_B : X \rightarrow B$, there exists a unique morphism $f : X \rightarrow A \times B$ making the diagram

$$\begin{array}{ccccc}
 & X & & & \\
 & \swarrow f_B & \downarrow f & \searrow f_A & \\
 B & \xleftarrow{p_B} & A \times B & \xrightarrow{p_A} & A
 \end{array}$$

commutative; i.e., $p_A \circ f = f_A$ and $p_B \circ f = f_B$.

Definition 1.2.2. Let A and B be objects in $\mathcal{A}$. A *coproduct* of A and B is an object $A \sqcup B$ in $\mathcal{A}$ together with two morphisms $\iota_A : A \rightarrow A \sqcup B$ and $\iota_B : B \rightarrow A \sqcup B$ such that for any other object C in $\mathcal{A}$ with morphisms $f_A : A \rightarrow C$ and $f_B : B \rightarrow C$, there exists a unique morphism $f : A \sqcup B \rightarrow C$ making the diagram

$$\begin{array}{ccccc}
 & C & & & \\
 & \swarrow f_B & \uparrow f & \searrow f_A & \\
 B & \xrightarrow{\iota_B} & A \sqcup B & \xleftarrow{\iota_A} & A
 \end{array}$$

commutative; i.e, $f \circ \iota_A = f_A$ and $f \circ \iota_B = f_B$.

In a category $\mathcal{A}$, an object Z such that there are unique morphisms $Z \rightarrow A$ and $A \rightarrow Z$ for each A in $\mathcal{A}$ is called a *zero object*. A *zero morphism* is any morphism that factors through a zero object, i.e, if $f : A \rightarrow B$ is a zero morphism, then there exist morphisms $g : A \rightarrow Z$ and $h : Z \rightarrow B$ such that $f = h \circ g$.

Both the zero object and the zero morphism are usually denoted by 0. The zero object, when it exists, is unique up to isomorphism.

Definition 1.2.3. A category $\mathcal{A}$ is said to be *additive* if

- (i) it has a zero object;
- (ii) it has a product $A \times B$ and a coproduct $A \sqcup B$ for each pair of objects A, B in $\mathcal{A}$; and
- (iii) its sets of morphisms are abelian groups and composition of morphisms is bilinear, i.e, if $f, f' \in \text{Hom}_{\mathcal{A}}(A, B)$ and $g, g' \in \text{Hom}_{\mathcal{A}}(B, C)$, then

$$(g + g') \circ f = g \circ f + g' \circ f \text{ and}$$

$$g \circ (f + f') = g \circ f + g \circ f'.$$

Definition 1.2.4. Let $\mathcal{A}$ be a category with the zero object 0 and let $f : A \rightarrow B$ be a morphism in $\mathcal{A}$.

- (i) The *kernel* of f is a morphism $k : K \rightarrow A$ such that $f \circ k = 0$, and such that for every morphism $g : K' \rightarrow A$ with $f \circ g = 0$ we have $g = k \circ g'$ for a unique morphism $g' : K' \rightarrow K$.
- (ii) The *cokernel* of f is a morphism $c : B \rightarrow C$ such that $c \circ f = 0$, and such that for every morphism $h : B \rightarrow C'$ with $h \circ f = 0$, we have $h = c \circ h'$ for a unique morphism $h' : C \rightarrow C'$.

We denote the kernel of f by $\text{Ker } f$ and the cokernel of f by $\text{Coker } f$. It is common practice to just say K is the kernel of f and C is the cokernel of f and we will adopt that.

Definition 1.2.5. Let $f : A \rightarrow B$ be a morphism in any category $\mathcal{A}$. Then, f is said to be

- (i) a *monomorphism*, or a *mono*, if for every pair of morphisms $g, h : K \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$.
- (ii) an *epimorphism*, or an *epi*, if for every pair of morphisms $g, h : B \rightarrow C$, $g \circ f = h \circ f$ implies $g = h$.
- (iii) an *isomorphism* if there is another morphism $f^{-1} : B \rightarrow A$ in $\mathcal{A}$ such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Definition 1.2.6. An *abelian* category is an additive category $\mathcal{A}$ in which

- (i) every morphism in $\mathcal{A}$ has a kernel and a cokernel; and
- (ii) a morphism $f : A \rightarrow B$ is a mono if and only if $\text{Ker } f = 0$, and it is an epi if and only if $\text{Coker } f = 0$.

Note that, in an abelian category, a morphism which is both a mono and an epi is an isomorphism.

Definition 1.2.7. Let $\mathcal{A}$ be an abelian category.

- (i) An object P in $\mathcal{A}$ is called a *projective* object if for every morphism $f : P \rightarrow B$ and every epimorphism $g : A \rightarrow B$, there is a morphism $h : P \rightarrow A$ such that $g \circ h = f$.
- (ii) We will say that $\mathcal{A}$ has *enough projectives* if for every object A in $\mathcal{A}$, there is an epimorphism $P \rightarrow A$ where P is a projective object.
- (iii) An object $Q \in \mathcal{A}$ is called an *injective* object if for every morphism $f : A \rightarrow Q$ and every monomorphism $g : A \rightarrow B$, there is a morphism $h : B \rightarrow Q$ such that $f = h \circ g$.
- (iv) We will say $\mathcal{A}$ has *enough injectives* if for every object A in $\mathcal{A}$, there is a monomorphism $A \rightarrow Q$ with Q an injective object.

1.3 The category of modules

Of interest to us is the abelian category $\mathcal{M}_A$ of A -modules and A -module homomorphisms and its full subcategory of finitely generated A -modules, where A is a noetherian commutative ring with unity. Note that $\mathcal{M}_A$ has enough projectives and enough injectives. The interested reader is referred to [31] and [35] for more details.

In this section and also the rest of the thesis, we will write $\text{Hom}_A(M, N)$ in place of $\text{Hom}_{\mathcal{M}_A}(M, N)$. This section introduces important covariant functors to this thesis, the so called *Hom-functors* $\text{Hom}_A(M, -) : \mathcal{M}_A \rightarrow \mathcal{M}_A$ where M is a fixed A -module.

Definition 1.3.1. A functor $F : \mathcal{M}_A \rightarrow \mathcal{M}_A$ is said to be *A-linear* if, for all pairs M, N of A -modules, the induced morphisms

$$\Phi_{M,N} : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(F(M), F(N))$$

are A -module homomorphisms.

We will show that the functor $\text{Hom}_A(M, -)$ is a projective object in the category of A -linear functors. We first give a description of this functor. Given two homomorphisms $f : M \rightarrow M'$ and $g : N \rightarrow N'$ in $\mathcal{M}_A$, there are homomorphisms

$$\text{Hom}_A(f, N) = \text{Hom}_A(f, 1_N) : \text{Hom}_A(M', N) \rightarrow \text{Hom}_A(M, N)$$

defined by $h \mapsto h \circ f$ for each homomorphism $h : M' \rightarrow N$, and

$$\text{Hom}_A(M, g) = \text{Hom}_A(1_M, g) : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N')$$

defined by $h \mapsto g \circ h$ for each homomorphism $h : M \rightarrow N$. Furthermore, the diagram

$$\begin{array}{ccc} \text{Hom}_A(M', N) & \xrightarrow{\text{Hom}_A(f, N)} & \text{Hom}_A(M, N) \\ \text{Hom}_A(M', g) \downarrow & & \downarrow \text{Hom}_A(M, g) \\ \text{Hom}_A(M', N') & \xrightarrow{\text{Hom}_A(f, N')} & \text{Hom}_A(M, N') \end{array} \quad (1.1)$$

is commutative.

We define the functor $\text{Hom}_A(M, -)$ as follows:

- (i) $\text{Hom}_A(M, -)(N) = \text{Hom}_A(M, N)$ for each A -module N ; and
- (ii) $\text{Hom}_A(M, -)(g) = \text{Hom}_A(M, g)$ for each homomorphism $g : N \rightarrow N'$. One can show that $\text{Hom}_A(M, -)(1_N) = 1_{\text{Hom}_A(M, N)}$ and that

$$\begin{aligned} \text{Hom}_A(M, -)(h \circ g) &= \text{Hom}_A(M, h) \circ \text{Hom}_A(M, g) \\ &= \text{Hom}_A(M, -)(h) \circ \text{Hom}_A(M, -)(g). \end{aligned}$$

Note that, the commutativity of (1.1) defines a natural transformation

$$\text{Hom}_A(f, -) : \text{Hom}_A(M', -) \rightarrow \text{Hom}_A(M, -)$$

for each $f : M \rightarrow M'$. Also, the functor $\text{Hom}_A(A, -)$ is isomorphic to the identity functor on $\mathcal{M}_A$.

Definition 1.3.2. Let M be any A -module. The *dual* M^* of M is defined by $M^* = \text{Hom}_A(M, A)$.

Note that, for every finitely generated projective A -module P , the dual P^* is also a finitely generated projective A -module. Furthermore, we have $P^{**} \cong P$ for every finitely generated projective A -module P .

Now, given any A -modules M, L and N , there is a natural morphism

$$\Psi : \text{Hom}_A(M, L) \otimes_A N \rightarrow \text{Hom}_A(M, L \otimes_A N) \quad (1.2)$$

given by $(f \otimes_A y)(x) = f(x) \otimes_A y$ for $f : M \rightarrow L$, $y \in N$ and $x \in M$. If M is a finitely generated projective A -module, Ψ is a natural isomorphism (see Proposition 2 in section 4.2 of Chapter II in [3]). Setting $L = A$ and assuming that M is a finitely generated projective A -module in (1.2), we get a natural isomorphism

$$M^* \otimes_A N \cong \text{Hom}_A(M, N),$$

and hence a natural isomorphism

$$M^{**} \otimes_A N \cong \text{Hom}_A(M^*, N).$$

Thus, if P is a finitely generated projective A -module, we have a natural isomorphism

$$P \otimes_A N \cong \text{Hom}_A(P^*, N)$$

for any A -module N .

To show that $\text{Hom}_A(M, -)$ is a projective object, we need to use the Yoneda lemma which we now state. We refer the reader to Proposition 1.4.3 in [13], pages 61–62 in [23] and Theorem 1.17 in [31] for details and the proof of the lemma.

Theorem 1.3.3 (Yoneda Lemma). *Let $F : \mathcal{M}_A \rightarrow \mathcal{M}_A$ be a covariant functor and M an A -module. There is a bijection:*

$$\text{Nat}(\text{Hom}_A(M, -), F) \cong F(M)$$

given by $\alpha \mapsto \alpha_M(1_M)$ and natural in both M and F . Here, $\text{Nat}(G, H)$ denotes the set of natural transformations between the functors G and H .

That the functor $\text{Hom}_A(M, -)$ is a projective object in the category of all A -linear functors on $\mathcal{M}_A$ follows from the Yoneda lemma and the commutative diagram

$$\begin{array}{ccc} \text{Nat}(\text{Hom}_A(M, -), F) & \xrightarrow{\cong} & F(M) \\ \alpha_{*M} \downarrow & & \downarrow \alpha_M \\ \text{Nat}(\text{Hom}_A(M, -), G) & \xrightarrow[\cong]{} & G(M) \end{array}$$

where $\alpha : F \rightarrow G$ is surjective.

We end this section by stating two useful and well known results, namely, the Snake lemma and the Five lemma. Details of these results can be found for instance in [1], [12], [21] and [31].

Theorem 1.3.4 (Snake Lemma). *Given a commutative diagram in $\mathcal{M}_A$ with exact rows,*

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ f \downarrow & & g \downarrow & & h \downarrow & & \\ 0 \longrightarrow & L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \end{array}$$

there is a well defined homomorphism $\delta : \text{Ker } h \rightarrow \text{Coker } f$ called the connecting homomorphism, such that the sequence

$$\text{Ker } f \xrightarrow{\sigma} \text{Ker } g \xrightarrow{\tau} \text{Ker } h \xrightarrow{\delta} \text{Coker } f \xrightarrow{\sigma'} \text{Coker } g \xrightarrow{\tau'} \text{Coker } h$$

is exact. Furthermore, σ is injective if α is injective and τ' is surjective if β' is surjective.

Theorem 1.3.5 (Five Lemma). *Consider the commutative diagram in $\mathcal{M}_A$ with exact rows.*

$$\begin{array}{ccccccccc}
 M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
 f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\
 N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5
 \end{array}$$

- (i) *If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.*
- (ii) *If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.*
- (iii) *If f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.*

1.4 Homology functors

In this section, we introduce the homology functor $H_n(-)$ with values in $\mathcal{M}_A$. We refer the reader to [8], [9], [12], [31], [35] and [37] for more details. Throughout this section, A is a commutative noetherian ring.

Definition 1.4.1. A *chain complex* in $\mathcal{M}_A$, denoted by $\mathbf{C}$, is a sequence

$$\mathbf{C} : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

of modules and homomorphisms in $\mathcal{M}_A$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 1.4.2. Given two chain complexes $\mathbf{C}$ and $\mathbf{C}'$ in $\mathcal{M}_A$, a *chain map* $\mathbf{f} : \mathbf{C} \rightarrow \mathbf{C}'$ is a sequence of homomorphisms $\{f_n : M_n \rightarrow M'_n\}_{n \in \mathbb{Z}}$ such that the diagram

$$\begin{array}{ccccccc}
 \mathbf{C} : & \cdots & \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \longrightarrow & \cdots \\
 \mathbf{f} \downarrow & & & f_{n+1} \downarrow & & f_n \downarrow & & f_{n-1} \downarrow & & \\
 \mathbf{C}' : & \cdots & \longrightarrow & M'_{n+1} & \xrightarrow{d'_{n+1}} & M'_n & \xrightarrow{d'_n} & M'_{n-1} & \longrightarrow & \cdots
 \end{array}$$

is commutative.

We will denote the identity chain map by $\mathbf{1}_{\mathbf{C}}$. Now, given two chain maps $\mathbf{f} : \mathbf{C} \rightarrow \mathbf{C}'$ and $\mathbf{g} : \mathbf{C}' \rightarrow \mathbf{C}''$, the composite $\mathbf{g} \circ \mathbf{f} : \mathbf{C} \rightarrow \mathbf{C}''$ is defined by $(g \circ f)_n = g_n \circ f_n$ for all $n \in \mathbb{Z}$. The chain complexes in $\mathcal{M}_A$ form an abelian category; see Proposition 5.100 in [31] for details.

Definition 1.4.3. A chain complex

$$\mathbf{C} : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

is *exact* if $\text{Im } d_{n+1} = \text{Ker } d_n$ for all $n \in \mathbb{Z}$.

Example 1.1

1. Every morphism $f : M_1 \rightarrow M_0$ of A -modules gives rise to a complex

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow M_1 \xrightarrow{f} M_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

where $M_n = 0$ for $n > 1$ and $M_n = 0$ for $n < 0$.

2. Similarly every short exact sequence

$$0 \longrightarrow M_2 \xrightarrow{f} M_1 \xrightarrow{g} M_0 \longrightarrow 0$$

gives rise to a complex

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow M_2 \xrightarrow{f} M_1 \xrightarrow{g} M_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

An exact sequence

$$\mathbf{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{f} M \longrightarrow 0 \quad (1.3)$$

in $\mathcal{M}_A$ is called a *projective resolution* of an A -module M if each P_n is a projective A -module for all $n = 0, 1, 2, \dots$. If (1.3) is a projective resolution for M , a *deleted projective resolution* of M is the chain complex

$$\mathbf{P}_M : \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

in $\mathcal{M}_A$.

Definition 1.4.4. Let

$$\mathbf{C} : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

be a chain complex in $\mathcal{M}_A$, $Z_n(\mathbf{C}) = \text{Ker } d_n$ and $B_n(\mathbf{C}) = \text{Im } d_{n+1}$. Then, for $n \in \mathbb{Z}$, the n^{th} homology $H_n(-)$ is defined on the complex $\mathbf{C}$ by

$$H_n(\mathbf{C}) = Z_n(\mathbf{C})/B_n(\mathbf{C}).$$

Since $\text{Im } d_{n+1} \subseteq \text{Ker } d_n$, we have an exact sequence

$$0 \rightarrow B_n(\mathbf{C}) \rightarrow Z_n(\mathbf{C}) \rightarrow H_n(\mathbf{C}) \rightarrow 0$$

for each n . Furthermore, a chain complex $\mathbf{C}$ is exact if and only if $H_n(\mathbf{C}) = 0$ for all n . The n^{th} homology $H_n(-)$ is a functor from the category of complexes to $\mathcal{M}_A$ for all $n \in \mathbb{Z}$, see for instance Proposition 6.8 in [31] for details.

Now, let $F : \mathcal{M}_A \rightarrow \mathcal{M}_A$ be a covariant functor and let

$$\mathbf{C} : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

be a chain complex in $\mathcal{M}_A$. Then $F(\mathbf{C})$:

$$\cdots \longrightarrow F(M_{n+1}) \xrightarrow{F(d_{n+1})} F(M_n) \xrightarrow{F(d_n)} F(M_{n-1}) \longrightarrow \cdots \quad (1.4)$$

is also a chain complex since $F(d_n) \circ F(d_{n+1}) = F(d_n \circ d_{n+1}) = F(0) = 0$ for all $n \in \mathbb{Z}$. For example, consider a chain complex

$$\mathbf{C} : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

in $\mathcal{M}_A$ and the covariant functor $- \otimes_A N$ for any A -module N . We get the chain complex $\mathbf{C} \otimes_A N$:

$$\cdots \rightarrow M_{n+1} \otimes_A N \xrightarrow{d_{n+1} \otimes_A 1_N} M_n \otimes_A N \xrightarrow{d_n \otimes_A 1_N} M_{n-1} \otimes_A N \rightarrow \cdots$$

and hence we can define, for all $n \in \mathbb{Z}$, the homology functor $H_n(\mathbf{C} \otimes_A -)$ by

$$H_n(\mathbf{C} \otimes_A N) = \text{Ker}(d_n \otimes_A 1_N) / \text{Im}(d_{n+1} \otimes_A 1_N).$$

1.5 Limits and Completions

In this section, A is a commutative ring, $\mathcal{M}_A$ is the category of A -modules and I is a directed set; i.e, a nonempty partially ordered set such that, for each pair $i, j \in I$, there is $k \in I$ with $i \leq k$ and $j \leq k$.

Definition 1.5.1. A *direct system* in $\mathcal{M}_A$ is an ordered pair $\{M_i, \varphi_j^i\}_{i \leq j}$ where

- (i) $\{M_i\}_{i \in I}$ is a family of A -modules, and
- (ii) $\{\varphi_j^i : M_i \rightarrow M_j\}_{i \leq j}$ is a family of homomorphisms with $\varphi_i^i = 1_{M_i}$ and such that, for $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_k^i} & M_k \\ & \searrow \varphi_j^i & \nearrow \varphi_k^j \\ & M_j & \end{array}$$

is commutative.

Definition 1.5.2. A *direct limit* of the direct system $\{M_i, \varphi_j^i\}_{i \leq j}$ is an A -module $\varinjlim M_i$ and a family of homomorphisms $\{q_i : M_i \rightarrow \varinjlim M_i\}_{i \in I}$ such that

- (i) $q_j \circ \varphi_j^i = q_i$ for all $i \leq j$, and
- (ii) for all A -modules X and each family of homomorphisms $\{g_i : M_i \rightarrow X\}_{i \in I}$ such that $g_j \circ \varphi_j^i = g_i$ for all $i \leq j$, there is a unique module homomorphism

$\theta : \varinjlim M_i \rightarrow X$ making the diagram

$$\begin{array}{ccccc}
 \varinjlim M_i & \xleftarrow{\quad \theta \quad} & X \\
 \uparrow q_i & & \uparrow g_i \\
 & M_i & \\
 \uparrow q_j & \downarrow \varphi_j^i & \uparrow g_j \\
 & M_j &
 \end{array}$$

commutative.

Definition 1.5.3. An *inverse system* in $\mathcal{M}_A$ is an ordered pair $\{M_i, \psi_i^j\}_{i \leq j}$ where

- (i) $\{M_i\}_{i \in I}$ is a family of A -modules, and
- (ii) $\{\psi_i^j : M_j \rightarrow M_i\}_{i \leq j}$ is a family of homomorphisms with $\psi_i^i = 1_{M_i}$ and such that, for $i \leq j \leq k$, the diagram

$$\begin{array}{ccc}
 M_k & \xrightarrow{\psi_j^k} & M_j \\
 \searrow \psi_i^k & & \swarrow \psi_i^j \\
 & M_i &
 \end{array}$$

is commutative.

Definition 1.5.4. An *inverse limit* of an inverse system $\{M_i, \psi_i^j\}_{i \leq j}$ is an A -module $\varprojlim M_i$ and a family $\{p_i : \varprojlim M_i \rightarrow M_i\}_{i \in I}$ of homomorphisms such that

- (i) $\psi_i^j \circ p_j = p_i$ for all $i \leq j$, and
- (ii) for all A -modules X and each family of homomorphisms $\{f_i : X \rightarrow M_i\}_{i \in I}$ such that $\psi_i^j \circ f_j = f_i$ for all $i \leq j$, there is a unique module homomorphism $\phi : X \rightarrow \varprojlim M_i$ making the diagram

$$\begin{array}{ccccc}
 \varprojlim M_i & \xleftarrow{\quad \phi \quad} & X \\
 \uparrow p_i & & \uparrow f_i \\
 & M_i & \\
 \uparrow p_j & \uparrow \psi_i^j & \uparrow f_j \\
 & M_j &
 \end{array}$$

commutative.

Given an inverse system $\{M_i, \psi_i^j\}_{i \leq j}$ in $\mathcal{M}_A$, there is a natural isomorphism

$$\mathrm{Hom}_A(X, \varprojlim M_i) \cong \varprojlim \mathrm{Hom}_A(X, M_i) \quad (1.5)$$

for every A -module X , see Proposition 5.21 in [31].

Definition 1.5.5. A *surjective system* in $\mathcal{M}_A$ is an inverse system $\{M_i, \psi_i^j\}_{i \leq j}$ in which each homomorphism $\psi_i^j : M_j \rightarrow M_i$ is surjective.

It is common practice to denote the inverse system $\{M_i, \psi_i^j\}_{i \leq j}$ by $\{M_i\}$ and we will use this notation from now onward.

Definition 1.5.6. Let $\{M_i\}$ and $\{N_i\}$ be two inverse systems (both indexed by I) in $\mathcal{M}_A$. A homomorphism $f : \{M_i\} \rightarrow \{N_i\}$ is a family of homomorphisms $\{f_i : M_i \rightarrow N_i\}$ such that the diagram

$$\begin{array}{ccc} M_j & \xrightarrow{f_j} & N_j \\ \psi_i^j \downarrow & & \downarrow \vartheta_i^j \\ M_i & \xrightarrow{f_i} & N_i \end{array}$$

is commutative.

We say that a sequence of inverse systems

$$0 \rightarrow \{X_i\} \rightarrow \{Y_i\} \rightarrow \{Z_i\} \rightarrow 0$$

is exact if, for each i , the sequence

$$0 \rightarrow X_i \rightarrow Y_i \rightarrow Z_i \rightarrow 0$$

is exact. Now, given an exact sequence of inverse systems

$$0 \rightarrow \{X_i\} \rightarrow \{Y_i\} \rightarrow \{Z_i\} \rightarrow 0,$$

the sequence

$$0 \rightarrow \varprojlim X_i \rightarrow \varprojlim Y_i \rightarrow \varprojlim Z_i$$

is always exact, see Proposition 10.2 in [1] for details.

Definition 1.5.7. Let $I = \mathbb{N}$ be the set of positive integers. An inverse system $\{M_i\}$ in $\mathcal{M}_A$ is said to satisfy the *Mittag-Leffler* condition if, for each i , the decreasing sequence $\{\psi_i^j(M_j)\}_{i \leq j}$ becomes constant for sufficiently large j .

The Mittag-Leffler condition is obviously satisfied when $\{M_i\}$ is a surjective system. Furthermore, if

$$0 \rightarrow \{X_i\} \rightarrow \{Y_i\} \rightarrow \{Z_i\} \rightarrow 0,$$

is an exact sequence of inverse systems and $\{X_i\}$ satisfies the Mittag-Leffler condition, then the sequence

$$0 \rightarrow \varprojlim X_i \rightarrow \varprojlim Y_i \rightarrow \varprojlim Z_i \rightarrow 0$$

is also exact, see Proposition 10.3 in [21] for details.

Now, for any proper ideal $\mathfrak{a}$ in A and any A -module M , we have a descending sequence of submodules

$$M \supseteq \mathfrak{a}M \supseteq \mathfrak{a}^2M \supseteq \mathfrak{a}^3M \supseteq \dots \quad (1.6)$$

where $\mathfrak{a}^0 = A$. For $m \geq n$, the inclusions $\mathfrak{a}^mM \subseteq \mathfrak{a}^nM$ induce a family of homomorphisms

$$\{\psi_m^n : M/\mathfrak{a}^mM \rightarrow M/\mathfrak{a}^nM\}.$$

Thus, $\{M/\mathfrak{a}^nM\}$ is a surjective inverse system in $\mathcal{M}_A$. The inverse limit of this system is called the $\mathfrak{a}$ -adic completion and is denoted by $\widehat{M}$, i.e., $\widehat{M} = \varprojlim M/\mathfrak{a}^nM$. If $M = A$, $\widehat{A}$ is said to be the completion ring of A and for each A -module M , $\widehat{M}$ is an $\widehat{A}$ -module.

If A is noetherian and M is finitely generated, then $\widehat{A} \otimes_A M \cong \widehat{M}$. Furthermore, given an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of finitely generated A -modules, the sequence

$$0 \rightarrow \widehat{M}' \rightarrow \widehat{M} \rightarrow \widehat{M}'' \rightarrow 0$$

is exact.

2

Coherent Functors

Most of the material we present in this chapter is found in [2] and [11]. Throughout this chapter, A is a commutative noetherian ring, $\mathcal{M}_A$ is the category of A -modules and all functors are A -linear on M_A .

2.1 Coherent Functors

Definition 2.1.1. An A -linear functor F is said to be *coherent* if there is an exact sequence

$$h_N \rightarrow h_M \rightarrow F \rightarrow 0 \quad (2.1)$$

where M, N are finitely generated A -modules. Here, h_X denotes the covariant functor $\text{Hom}_A(X, -)$, where X is a finitely generated A -module.

Obviously, coherent functors send finitely generated A -modules to finitely generated A -modules. Note also that, the morphism $h_N \rightarrow h_M$ is induced by a unique module homomorphism $f : M \rightarrow N$. However, the choice of the exact sequence (2.1) is not unique.

Also note that the category of coherent functors has enough projectives since h_X is a projective object in the category of A -linear functors (see section 1.3). Setting $C = \text{Coker}(f : M \rightarrow N)$, we obtain an exact sequence

$$0 \rightarrow h_C \rightarrow h_N \rightarrow h_M \rightarrow F \rightarrow 0.$$

Coherent functors have, among others, the following properties:

- (i) Given an exact sequence $0 \rightarrow F \rightarrow H \rightarrow G \rightarrow 0$ of A -linear functors where F and G are coherent, then H is also coherent.

- (ii) If $\alpha : F \rightarrow G$ is a natural transformation between coherent functors, then $\text{Ker } \alpha$, $\text{Im } \alpha$ and $\text{Coker } \alpha$ are also coherent.
- (iii) Coherent functors preserve direct limits: Let $\{X_i\}$ be a direct system and consider the commutative diagram.

$$\begin{array}{ccccccc}
 h_N(\varinjlim X_i) & \longrightarrow & h_M(\varinjlim X_i) & \longrightarrow & F(\varinjlim X_i) & \longrightarrow & 0 \\
 \cong \downarrow & & \cong \downarrow & & \downarrow \Phi & & \\
 \varinjlim h_N(X_i) & \longrightarrow & \varinjlim h_M(X_i) & \longrightarrow & \varinjlim F(X_i) & \longrightarrow & 0
 \end{array}$$

By 1.3.5, the induced map Φ is an isomorphism .

We refer the reader to [2] or [11] for proofs of properties (i) and (ii). We now give some ‘well known’ examples of coherent functors.

Example 2.1

1. Let M be a finitely generated A -module. The functor $h_M = \text{Hom}_A(M, -)$ is coherent since the sequence

$$h_0 \rightarrow h_M \rightarrow h_M \rightarrow 0$$

is exact.

2. Let $P_1 \rightarrow P_2 \rightarrow M \rightarrow 0$ be a projective resolution of M in $\mathcal{M}_A$ with each P_i , $i = 1, 2$, finitely generated. Due to the natural isomorphism

$$P \otimes_A N \cong \text{Hom}_A(P^*, N)$$

(see section 1.3), the sequence

$$h_{P_1^*} \rightarrow h_{P_2^*} \rightarrow M \otimes_A - \rightarrow 0$$

is exact. Thus, $M \otimes -$ is a coherent functor.

3. Let $\mathbf{P} : \dots P_{i+1} \rightarrow P_i \rightarrow P_{i-1} \rightarrow \dots$ be a complex in $\mathcal{M}_A$, where for each i , P_i is finitely generated. Consider the chain complex

$$\mathbf{P} \otimes_A - : \dots \longrightarrow P_{i+1} \otimes_A - \xrightarrow{\alpha} P_i \otimes_A - \xrightarrow{\beta} P_{i-1} \otimes_A - \longrightarrow \dots$$

Let $B_i(-) = \text{Im } \alpha$ and $Z_i(-) = \text{Ker } \beta$. Then, $Z_i(-)$ and $B_i(-)$ are coherent functors by property (ii) above. Since

$$0 \rightarrow B_i(-) \rightarrow Z_i(-) \rightarrow H_i(\mathbf{P} \otimes -) \rightarrow 0$$

is exact, again by property (ii), the homology functor $H_i(\mathbf{P} \otimes_A -)$ is coherent for each i . In particular, for any finitely generated A -module M , the functor $\text{Tor}_i^A(M, -)$ defined on each A -module N by $\text{Tor}_i^A(M, N) = H_i(\mathbf{P}_M \otimes_A N)$ is coherent for each i . Here, $\mathbf{P}_M$ is a deleted projective resolution for M .

4. Similarly, the cohomology functor $H^i(\text{Hom}_A(\mathbf{P}, -))$ is coherent for each i . In particular, for any finitely generated A -module M , the functor $\text{Ext}_A^i(M, -)$ defined on each A -module N by $\text{Ext}_A^i(M, N) = H^i(\text{Hom}_A(\mathbf{P}_M, N))$ is coherent for each i .

2.2 Left-, right- and half-exact coherent functors

An A -linear functor F is said to be

- (i) *exact* if $0 \rightarrow F(M') \rightarrow F(M) \rightarrow F(M'') \rightarrow 0$ is exact,
- (ii) *half exact* if $F(M') \rightarrow F(M) \rightarrow F(M'')$ is exact,
- (iii) *left exact* if $0 \rightarrow F(M') \rightarrow F(M) \rightarrow F(M'')$ is exact,
- (iv) *right exact* if $F(M') \rightarrow F(M) \rightarrow F(M'') \rightarrow 0$ is exact,

whenever $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact in $\mathcal{M}_A$.

Let F be any A -linear functor and define a natural transformation

$$\alpha : F(A) \otimes_A - \rightarrow F$$

as follows: From $\mu_m : A \rightarrow M$, given by $\mu_m(a) = am$, we have the morphism $F(\mu_m) : F(A) \rightarrow F(M)$. For any A -module M , define $\alpha_M : F(A) \otimes M \rightarrow F(M)$ by $\alpha_M(\xi \otimes m) = F(\mu_m)(\xi)$.

We therefore have an exact sequence

$$F(A) \otimes - \xrightarrow{\alpha} F \longrightarrow F_0 \longrightarrow 0 \quad (2.2)$$

where $F_0 = \text{Coker } \alpha$. Since α_A is an isomorphism by definition, we have $F_0(A) = 0$. Also, F_0 will be coherent whenever F is coherent.

It is clear from (2.2) that F is right exact if α is an isomorphism. Conversely, suppose that F is right exact. Since α_A is an isomorphism, for any free A -module L , α_L is also an isomorphism. Let L_0 and L_1 be free A -modules such that

$$L_1 \rightarrow L_0 \rightarrow M \rightarrow 0$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccc} F(A) \otimes L_1 & \longrightarrow & F(A) \otimes L_0 & \longrightarrow & F(A) \otimes M & \longrightarrow & 0 \\ \cong \downarrow \alpha_{L_1} & & \cong \downarrow \alpha_{L_0} & & \downarrow \alpha_M & & \\ F(L_1) & \longrightarrow & F(L_0) & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

It follows by 1.3.5 that α_M is an isomorphism.

Suppose now that F is half exact, and let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence in $\mathcal{M}_A$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 F(A) \otimes M' & \longrightarrow & F(A) \otimes M & \longrightarrow & F(A) \otimes M'' & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 F(M') & \longrightarrow & F(M) & \longrightarrow & F(M'') & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 F_0(M') & \longrightarrow & F_0(M) & \longrightarrow & F_0(M'') & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

A diagram chase shows that F_0 is half exact.

We now state a result that has been useful in proving our main results in our first paper.

Theorem 2.2.1. [11, Proposition 3.12] *Given that F is a coherent functor, there exists a finitely generated A -module X , unique up to isomorphism, and a natural transformation $\beta : F \rightarrow h_X$ such that*

- (i) β_Q is an isomorphism for every injective A -module Q . Hence, if $F_1 = \text{Ker } \beta$, then $F_1(Q) = 0$ whenever Q is an injective A -module
- (ii) F is left exact if and only if β is an isomorphism. In, particular, F is left exact if and only if $F \cong h_X$ for some finitely generated A -module X .

Corollary 2.2.2. *Left exact coherent functors preserve inverse limits of directed sets.*

Proof. Since every left exact coherent functor is isomorphic to h_N for some finitely generated module N , it preserves inverse limits by (1.5). $\square$

Definition 2.2.3. An A -module C is said to be a *cogenerator* of $\mathcal{M}_A$ if, for every A -module M and every nonzero element m in M , there exists a homomorphism $h : M \rightarrow C$ with $h(m) \neq 0$.

Definition 2.2.4. An injective A -module Q is said to be an *injective hull* (or *injective envelope*) of an A -module M , denoted by $\text{Env}(M)$, if $M \subset Q$ and there is no injective A -module Q' such that $M \subset Q' \subsetneq Q$.

Proposition 2.2.5. *For any coherent functor F , the following are equivalent:*

- (i) $F(C) = 0$ when C is an injective cogenerator of $\mathcal{M}_A$.
- (ii) $F(\text{Env}(A/\mathfrak{m})) = 0$, where $\mathfrak{m}$ is a maximal ideal of A .

(iii) Let M and N be finitely generated A -modules. Every $f : M \rightarrow N$ such that $h_N \rightarrow h_M \rightarrow F \rightarrow 0$ is exact, must be an injective homomorphism.

(iv) Let M and N be finitely generated A -modules. There is an injective homomorphism $f : M \rightarrow N$ such that $h_N \rightarrow h_M \rightarrow F \rightarrow 0$ is exact.

Proof. (i) $\Rightarrow$ (ii): Since $A/\mathfrak{m}$ is simple for every maximal ideal $\mathfrak{m}$ of A , by Theorem 19.8 in [20], the cogenerator C of $\mathcal{M}_A$ contains $\text{Env}(A/\mathfrak{m})$ as a direct summand. Hence, $F(\text{Env}(A/\mathfrak{m})) = 0$ since F preserve direct summands.

(ii) $\Rightarrow$ (iii): Let M and N be finitely generated A -modules and let $f : M \rightarrow N$ be a homomorphism such that $h_N \rightarrow h_M \rightarrow F \rightarrow 0$ is exact. Since by hypothesis $F(\text{Env}(A/\mathfrak{m})) = 0$,

$$h_N(\text{Env}(A/\mathfrak{m})) \rightarrow h_M(\text{Env}(A/\mathfrak{m})) \rightarrow 0$$

is exact. Now, if $K = \text{Ker } f$, we also have an exact sequence

$$h_N(\text{Env}(A/\mathfrak{m})) \rightarrow h_M(\text{Env}(A/\mathfrak{m})) \rightarrow h_K(\text{Env}(A/\mathfrak{m})) \rightarrow 0.$$

This shows that $h_K(\text{Env}(A/\mathfrak{m})) = \text{Hom}_A(K, \text{Env}(A/\mathfrak{m})) = 0$. Thus for all maximal ideals $\mathfrak{m}$ of A , we have $\text{Hom}_{A/\mathfrak{m}}(K/\mathfrak{m}, \text{Env}(A/\mathfrak{m})) = 0$. But this, by Theorem 18.6 in [24], implies that $K/\mathfrak{m} = 0$ for all $\mathfrak{m}$. Hence $K = 0$, and this shows that $f : M \rightarrow N$ is injective.

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (i): Let C be an injective cogenerator of $\mathcal{M}_A$. By hypothesis, we have that $h_N(C) \rightarrow h_M(C) \rightarrow 0$ is exact and hence $F(C) = 0$. □

We end the chapter by stating two results which we extended in our first paper.

Theorem 2.2.6. [11, Proposition 3.5] *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k . Then, for any coherent functor F and any finitely generated A -module M , the natural map $\widehat{F(M)} \rightarrow \varprojlim F(M/\mathfrak{m}^n M)$ is an isomorphism.*

Theorem 2.2.7. [11, Proposition 6.1] *Let A be a discrete valuation ring (DVR) with parameter t . If F is a half exact coherent functor, then F is a direct sum of copies of the identity functor, functors of the form $A/t^n \otimes -$, and functors of the form $\text{Hom}_A(A/t^n, -)$, for various $n \geq 1$.*

3

Asymptotic stability

In this chapter, we discuss known results concerning asymptotic associated and attached prime ideals. More details on associated prime ideals and the support of an A -module can be found in [1], [4], [21], [24] and [35]. For attached prime ideals, see [22] and the appendix to chapter 2 section 6 in [24]. An extensive background to asymptotic stability of sequences of sets of prime ideals and some results can be found in [25] and [30].

We begin with the definitions of associated and attached prime ideals and describe their main properties. Throughout this chapter, A is a noetherian commutative ring and F is a coherent functor on $\mathcal{M}_A$.

3.1 Associated primes ideals

Definition 3.1.1. Let M be an A -module.

- (i) The *annihilator* of $x \in M$, denoted by $(0 :_A x)$ or $\text{Ann } x$, is the ideal in A defined by

$$(0 :_A x) = \{a \in A : ax = 0\}.$$

- (ii) The annihilator of M is the ideal in A defined by

$$(0 :_A M) = \{a \in A : ax = 0, \text{ for all } x \in M\}.$$

Definition 3.1.2. Let M be an A -module. A prime ideal $\mathfrak{p}$ in A is said to be *associated* to M if $\mathfrak{p} = (0 :_A x)$ for some $x \in M$. The set of prime ideals associated to M is denoted by $\text{Ass}_A M$.

Note that, $\mathfrak{p}$ is an associated prime ideal of M if and only if there is an injective homomorphism $A/\mathfrak{p} \rightarrow M$.

Definition 3.1.3. The *support* of an A -module M , denoted $\text{Supp}_A M$, is the set of primes ideals $\mathfrak{p}$ in A such that $M_{\mathfrak{p}} \neq 0$.

Let M be a finitely generated A -module and let $\mathfrak{a} = (0 :_A M)$. Then $\text{Supp}_A(M) = V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of all prime ideals containing $\mathfrak{a}$. For ideals $\mathfrak{a}$ and $\mathfrak{b}$, one has $V(\mathfrak{a}) = V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

The set of associated prime ideals has the following properties:

- (i) If M is a finitely generated A -module, then $\text{Ass}_A M$ is always a finite set. In fact, $\text{Ass}_A M$ consists of the prime ideals which belong to the primary submodules of M in a so called reduced primary decomposition of the zero submodule of M , see Theorem 6.8 in [24] or Lemma 7.1.8 in [35].
- (ii) For any A -module M , $\text{Ass}_A M \subset \text{Supp}_A M$ and for each $\mathfrak{p} \in \text{Supp}_A M$, there is $\mathfrak{q} \in \text{Ass}_A M$ such that $\mathfrak{q} \subset \mathfrak{p}$. In particular, the two sets $\text{Ass}_A M$ and $\text{Supp}_A M$ have the same minimal elements (under inclusion).
- (iii) If M is a finitely generated A -module and N is any A -module, then

$$\text{Ass}_A \text{Hom}_A(M, N) = \text{Supp}_A M \cap \text{Ass}_A N$$

(see Proposition 10 in section 1.4 of Chapter IV in [4]).

- (iv) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of finitely generated A -modules, then we have

$$\text{Ass}_A M' \subset \text{Ass}_A M \subset \text{Ass}_A M' \cup \text{Ass}_A M''$$

3.2 Secondary representations and attached prime ideals

Definition 3.2.1. A nontrivial A -module M is said to be *secondary* if, for each element a in A , either $aM = M$ or a is in the radical $\sqrt{(0 :_A M)}$ of the annihilator of M .

Note that $\mathfrak{p} = \sqrt{(0 :_A M)}$ is a prime ideal, and we say that M is $\mathfrak{p}$ -secondary.

Proposition 3.2.2. [22, 1.1] *All finite direct sums and nontrivial quotients of $\mathfrak{p}$ -secondary A -modules are $\mathfrak{p}$ -secondary.*

It follows from 3.2.2 and the natural map $\bigoplus_{i=1}^n M_i \rightarrow \sum_{i=1}^n M_i$ that a finite sum of $\mathfrak{p}$ -secondary submodules of an A -module is $\mathfrak{p}$ -secondary.

Definition 3.2.3. Let M be an A -module.

- (i) We say that M has a *secondary representation* if M can be expressed as a finite sum $M = M_1 + M_2 + \cdots + M_n$ where each M_i is a secondary A -submodule of M .

- (ii) A secondary representation $M = M_1 + M_2 + \cdots + M_n$ of M is said to be *minimal* if the prime ideals $\mathfrak{p}_i = \sqrt{(0 :_A M_i)}$, $1 \leq i \leq n$, are all distinct and M cannot be expressed as a sum of a proper sub-collection of the M_i .
- (iii) Given a minimal secondary representation $M = M_1 + M_2 + \cdots + M_n$ of M , the prime ideals $\mathfrak{p}_i = \sqrt{(0 :_A M_i)}$, $1 \leq i \leq n$, are called the *attached* prime ideals of M . The set of attached prime ideals of M is denoted by $\text{Att}_A(M)$.

If $M = M_1 + M_2 + \cdots + M_n$ is a minimal secondary representation of an A -module M , the set $\text{Att}_A M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ depends only on M . This means that, $\text{Att}_A M$ is independent of the chosen secondary representation of M .

Definition 3.2.4. A nontrivial A -module M is said to be *sum-irreducible* if any sum of two proper submodules of M is a proper submodule of M .

Proposition 3.2.5. [22, 5.1] *Any sum-irreducible artinian A -module is a secondary A -module*

This Proposition is used to prove

Proposition 3.2.6. [22, 5.2] *Every artinian A -module has a secondary representation.*

Proposition 3.2.7. [22, 2.4 and 4.1] *If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of artinian A -modules, then we have*

$$\text{Att}_A M'' \subset \text{Att}_A M \subset \text{Att}_A M' \cup \text{Att}_A M''.$$

3.3 Asymptotic stability of Ass and Att

A sequence of elements $\{x_n\}_{n=1}^\infty$ is said to be *asymptotically stable* if there is an integer m such that $x_n = x_m$ for all $n \geq m$, i.e, it is eventually constant.

The topic of asymptotic stability of sequences of sets of prime ideals is classical in commutative algebra. It arose from the following question by Ratliff which appeared in 1976 in his paper [29]:

Question. *Let $\mathfrak{p}$ be a prime ideal in a domain A . If $\mathfrak{p}$ is in $\text{Ass}_A(A/\mathfrak{a}^k)$ for some $k \geq 1$, does it imply that $\mathfrak{p}$ is in $\text{Ass}_A(A/\mathfrak{a}^n)$ for all large n ?*

In the paper that appeared in 1979, Brodmann gave a counterexample to this question [5, Example 9]. He then posed and answered a related question which then triggered a lot of research in this new direction. He proved

Theorem 3.3.1. *Let $\mathfrak{a}$ be an ideal of A and let M be a finitely generated A -module. Then, the two sequences of sets*

$$\{\text{Ass}_A(\mathfrak{a}^n M / \mathfrak{a}^{n+1} M)\}_{n=1}^\infty \text{ and } \{\text{Ass}_A(M / \mathfrak{a}^n M)\}_{n=1}^\infty$$

are asymptotically stable.

Many generalizations of 3.3.1 exists in the literature. Rush [32] extended 3.3.1 by showing that the sequence $\{\text{Ass}_A(M/\mathfrak{a}^n N)\}_{n=1}^\infty$ is asymptotically stable, where M is a finitely generated A -module and N is a submodule of M . In [14], Katz, McAdam and Ratliff Jr. proved that, if $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ are ideals of A satisfying some conditions, the sequence $\{\text{Ass}_A(A/\mathfrak{a}_1^{g_n(1)} \dots \mathfrak{a}_s^{g_n(s)})\}_{n=1}^\infty$ is asymptotically stable. Here $\{g_n(1), \dots, g_n(s)\}_{n=1}^\infty$ is an increasing sequence (in the sense $g_i(j) \leq g_{i+1}(j)$) of s -tuples of positive integers.

Also, Kingsbury and Sharp [16] proved that, if $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ are ideals of A and $N \subset M$ is a submodule of a finitely generated A -module M , the sequence $\{\text{Ass}_A(M/\mathfrak{a}_1^{g_n(1)} \dots \mathfrak{a}_s^{g_n(s)} N)\}_{n=1}^\infty$ is asymptotically stable. They also showed that the sequence $\{\text{Att}_A(N' :_N \mathfrak{a}_1^{g_n(1)} \dots \mathfrak{a}_s^{g_n(s)})\}_{n=1}^\infty$ is asymptotically stable whenever $N' \subset N$ is a submodule of an artinian A -module N . Again $\{g_n(1), \dots, g_n(s)\}_{n=1}^\infty$ is an increasing sequence of s -tuples of positive integers. These results are of interest for future research.

Of interest to this thesis are the following theorems:

Theorem 3.3.2. [26, Theorem 1] *Let $\mathfrak{a}$ be an ideal of A and M be a finitely generated A -module. Then the sequences of sets*

$$\{\text{Ass}_A \text{Tor}_i^A(M, \mathfrak{a}^n/\mathfrak{a}^{n+1})\}_{n=1}^\infty \text{ and } \{\text{Ass}_A \text{Tor}_i^A(M, A/\mathfrak{a}^n)\}_{n=1}^\infty$$

are asymptotically stable.

Theorem 3.3.3. [15, Corollary 3.5] *Let M and N be finitely generated A -modules and M' be a submodule of M . Then the two sequences of sets*

$$\{\text{Ass}_A(\text{Tor}_i^A(N, M/\mathfrak{a}^n M'))\}_{n=1}^\infty \text{ and } \{\text{Ass}_A(\text{Ext}_A^i(N, M/\mathfrak{a}^n M'))\}_{n=1}^\infty$$

are asymptotically stable.

Recently, Se generalized 3.3.1, 3.3.2 and 3.3.3 to coherent functors. He proved

Theorem 3.3.4. [33, Theorem 1.11] *Let $\mathfrak{a}$ be an ideal of A , M a finitely generated A -module and let F be a coherent functor. Then the two sequences of sets*

$$\{\text{Ass}_A F(\mathfrak{a}^n M/\mathfrak{a}^{n+1} M)\}_{n=1}^\infty \text{ and } \{\text{Ass}_A F(M/\mathfrak{a}^n M)\}_{n=1}^\infty$$

are asymptotically stable.

In paper B (in part II of our thesis), we have proved Se's result using a different approach.

On the other hand, the theory of attached prime ideals was developed by I.G. Macdonald [22] and others, and is seen to be the dual to the theory of associated prime ideals. The first results concerning asymptotic stability of attached prime ideals appeared in 1986 in the paper by Sharp [34]. He proved

Theorem 3.3.5. *Let $\mathfrak{a}$ be an ideal of A and let N be an artinian A -module. Then the two sequences of sets*

$$\{\text{Att}_A(0 :_N \mathfrak{a}^n/0 :_N \mathfrak{a}^{n-1})\}_{n=1}^\infty \text{ and } \{\text{Att}_A(0 :_N \mathfrak{a}^n)\}_{n=1}^\infty$$

are asymptotically stable.

In the paper that appeared in 1990, Melkersson [27] proved 3.3.5 using a different approach. Rush [32] extended 3.3.5 by showing that, if N is an artinian A -module and N' is a submodule of N , the sequence of sets of prime ideals $\{\text{Att}_A(N' :_N \mathfrak{a}^n)\}_{n=1}^\infty$ is asymptotically stable. In [26], Melkersson and Schenzel proved that, if $\mathfrak{a}$ is an ideal of A and N an artinian A -module, the two sequences of sets

$$\{\text{Att}_A(\text{Ext}_A^i(\mathfrak{a}^n/\mathfrak{a}^{n-1}, N))\}_{n=1}^\infty \text{ and } \{\text{Att}_A(\text{Ext}_A^i(A/\mathfrak{a}^n, N))\}_{n=1}^\infty$$

are asymptotically stable. Notice that 3.3.5 is recovered when $i = 0$ in the second assertion.

We have generalized 3.3.5 to coherent functors. See summary of Paper B in chapter 5.

3.4 Depth and grade of a module

Let M be a finitely generated A -module and let $\mathfrak{b}$ be an ideal in A such that $M \neq \mathfrak{b}M$. The *depth* of M is defined by

$$\text{depth}_{\mathfrak{b}}(M) = \min\{i : \text{Ext}_A^i(A/\mathfrak{b}, M) \neq 0\}.$$

If $M \neq 0$ and $\mathfrak{b} = (0 :_A M)$, the *grade* of M is defined by

$$\text{grade}(M) = \text{depth}_{\mathfrak{b}} M$$

i.e., the grade of M is the least $i \geq 0$ such that $\text{Ext}_A^i(A/\mathfrak{b}, M) \neq 0$.

One can show that

$$\text{depth}_{\mathfrak{b}}(M) = \min\{i : \text{Ext}_A^i(N, M) \neq 0\}$$

for any finitely generated A -module N with $\text{Supp}_A(N) = V(\mathfrak{b})$.

Brodmann proved

Theorem 3.4.1. [6, Theorem 2(i)] *Let M be a finitely generated A -module, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $M \neq (\mathfrak{a} + \mathfrak{b})M$. Then the values $\text{depth}_{\mathfrak{b}} M/\mathfrak{a}^n M$ are independent of n for large n .*

Se extended this result and proved

Theorem 3.4.2. [33, Theorem 1.11] *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A , M a finitely generated A -module and F a coherent functor. Then $\text{depth}_{\mathfrak{b}} F(M/\mathfrak{a}^n M)$ is independent of n for large n .*

We have given a different proof to 3.4.2.

4

Graded modules and Hilbert polynomials

Most of the material in this chapter can be found in [1], [21], [24], and [35]. Throughout this chapter, A is a commutative ring.

4.1 Length of a module

Definition 4.1.1. An A -module M

- (i) is *noetherian* if, under inclusion, every set of submodules of M has a maximal element.
- (ii) is *artinian* if, under inclusion, every set of submodules of M has a minimal element.

Note that if M is noetherian (respectively artinian), then submodules and quotient modules of M are also noetherian (respectively artinian). Also, M being noetherian is equivalent to every submodule of M being finitely generated. It follows that every finitely generated module over a noetherian ring is noetherian.

Now, let M be a finitely generated A -module. Consider a chain of submodules of M of length n

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = 0. \quad (4.1)$$

The chain (4.1) is a *composition series* of M if, for $1 \leq i \leq n$, the quotient M_{i-1}/M_i is isomorphic to $A/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ in A . If M has a composition series of length n , then any other composition series of M has length n . The *length* of M , denoted by $l_A(M)$, is the length of any composition series

of M . Note that, an A -module has a composition series if and only if it is both noetherian and artinian, and such a module is called a *module of finite length*.

Now, given an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of modules of finite length, we have $l_A(M') - l_A(M) + l_A(M'') = 0$.

We refer the reader to Propositions 6.7, 6.8 and 6.9 in [1] for more details on these facts.

Proposition 4.1.2. *Let A be a commutative noetherian ring. Every coherent functor on $\mathcal{M}_A$ sends artinian A -modules (respectively A -modules of finite length) to artinian A -modules (respectively A -modules of finite length).*

Proof. Let M be a finitely generated A -module. There is a surjective homomorphism $A^k \rightarrow M \rightarrow 0$ for some positive integer k . Hence, the sequence

$$0 \rightarrow \operatorname{Hom}_A(M, X) \rightarrow \operatorname{Hom}_A(A^k, X) \cong X^k$$

is exact for any A -module X . Since X^k is artinian whenever X is artinian, $\operatorname{Hom}_A(M, X)$ is artinian. Now, if M and N are finitely generated A -modules such that $h_N \rightarrow h_M \rightarrow F \rightarrow 0$ is exact, then $F(X)$ is an artinian A -module whenever X is artinian. Similarly, $F(X)$ has finite length if X has finite length. $\square$

4.2 Graded and associated graded modules

A *graded ring* R is a family $(R_n)_{n \geq 0}$ of subgroups of R such that $R = \bigoplus_{n \geq 0} R_n$ and $R_n R_m \subset R_{n+m}$ for all $n, m \geq 0$. Note that R_0 is a subring of R , and for each n , R_n is an R_0 -module. Further, R is noetherian if and only if R_0 is noetherian and R is finitely generated as an R_0 -algebra. See Proposition 10.7 in [1] or Theorem 13.1 in [24] for details.

Definition 4.2.1. A graded ring R is said to be *standard graded* or *homogeneous* if it is generated as an R_0 -algebra by elements of R_1 .

Let R be a graded ring. An R -module M is a *graded module* if $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $R_n M_m \subset M_{n+m}$ for all n, m . Note that M_n is an R_0 -module for each n . If $R = \bigoplus_{n \geq 0} R_n$ is a noetherian graded ring and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded R -module, then, for each n , M_n is finitely generated as an R_0 -module. In particular, R_n is finitely generated as an R_0 -module.

Let A be a ring and $\mathfrak{a}$ an ideal of A . The *Rees ring* of A with respect to $\mathfrak{a}$ is the subring $A[\mathfrak{a}t] = \bigoplus_{n \geq 0} \mathfrak{a}^n t^n$ of the polynomial ring $A[t]$. If $\mathfrak{a}$ is generated by a finite number of elements in A , say $a_1, \dots, a_r$, the Rees ring is written as $A[\mathfrak{a}t] = A[a_1 t, \dots, a_r t]$. Thus $A[\mathfrak{a}t]$ is noetherian if A is noetherian. The graded module over the Rees ring $M[\mathfrak{a}t] = \bigoplus_{n \geq 0} \mathfrak{a}^n M t^n$ is called the *Rees module* of M with respect to $\mathfrak{a}$.

Now, let $\mathfrak{a}$ be an ideal in A . The *associated graded ring* of A with respect to $\mathfrak{a}$ is the graded ring $G_{\mathfrak{a}}(A)$ given by $G_{\mathfrak{a}}(A) = \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$. It is a quotient of the Rees ring $A[\mathfrak{a}t]$, namely,

$$G_{\mathfrak{a}}(A) \cong A[\mathfrak{a}t] / \mathfrak{a} A[\mathfrak{a}t].$$

If A is noetherian, then $G_{\mathfrak{a}}(A)$ is noetherian.

Let M be an A -module. The graded $G_{\mathfrak{a}}(A)$ -module

$$G_{\mathfrak{a}}(M) = \bigoplus_{n \geq 0} \mathfrak{a}^n M / \mathfrak{a}^{n+1} M$$

is called *the associated graded module* of M with respect to $\mathfrak{a}$.

Now, let A be a noetherian ring, $\mathfrak{a}$ an ideal of A and $G_{\mathfrak{a}}(A) = \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ the associated graded ring of A . For an artinian A -module N , Kirby [17] introduced the graded module

$$G^{\mathfrak{a}}(N) = \bigoplus_{n \leq 0} (0 :_N \mathfrak{a}^{-n+1}) / (0 :_N \mathfrak{a}^{-n})$$

dual to the associated graded ring. It is a graded artinian $G_{\mathfrak{a}}(A)$ -module and $(0 :_N \mathfrak{a}^{-n+1}) / (0 :_N \mathfrak{a}^{-n})$ is an artinian A -module for each n . See Theorem 1 in [17] and the proof of Theorem 1.2 in [27] for details.

4.3 Hilbert functions

An *integer-valued* polynomial is a polynomial $f \in \mathbb{Q}[X]$ such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. For example, for each integer $r \geq 0$,

$$f_r(X) = \binom{X+r}{r} = \frac{(X+r)(X+r-1)\dots(X+1)}{r!}$$

is an integer-valued polynomial. Every integer-valued polynomial f can be expressed uniquely as

$$f = \sum_{i=0}^r a_i f_i(X)$$

where $a_i \in \mathbb{Z}$, $a_r \neq 0$ and r is the degree of f .

Definition 4.3.1. A function $H : \mathbb{Z} \rightarrow \mathbb{Z}$ is said to be of *polynomial type* if there is an integer-valued polynomial $P(X)$ (necessarily unique) such that $H(n) = P(n)$ for $n \gg 0$.

Let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated R_0 -algebra such that R_0 is an artinian ring and let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated graded R -module. Then R_0 is noetherian and M_n is a finitely generated R_0 -module. Hence, $l_{R_0}(M_n)$ is finite.

Definition 4.3.2. Let R be a finitely generated R_0 -algebra with R_0 artinian and let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated graded R -module. The *Hilbert function* $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$ of M is given by $H_M(n) = l_{R_0}(M_n)$.

If $R = R_0[r_1, \dots, r_n]$, $r_i \in R_1$, the Hilbert function H_M is of polynomial type, see Theorem 4.1.3 in [7] or Theorem 9.2.1 in [35]. And the polynomial $P_M(X)$ such that $H_M(n) = P_M(n)$ is called the *Hilbert polynomial* of M .

Now, let $\mathfrak{a}$ be an ideal of a noetherian ring A such that $A/\mathfrak{a}$ is an artinian (local) ring and let M be a finitely generated A -module. Then $\mathfrak{a}^n M / \mathfrak{a}^{n+1} M$ is a finitely generated A -module annihilated by $\mathfrak{a}$, hence a finitely generated $A/\mathfrak{a}$ -module. Thus, $l_A(\mathfrak{a}^n M / \mathfrak{a}^{n+1} M)$ is finite. Also $l_A(M / \mathfrak{a}^n M)$ is finite, see for instance section 9.3 on page 164 in [35]. The Hilbert function $H_{\mathfrak{a}, M}$ with respect to $\mathfrak{a}$ given by $H_{\mathfrak{a}, M}(n) = l_A(M / \mathfrak{a}^n M)$ is of polynomial type, see Proposition 11.4 in [1], Theorem 4.1.3 in [7] or Theorem 9.3.1 in [35] for the proof.

Dually, if N is an artinian A -module and $0 :_N \mathfrak{a}$ has finite length, then $0 :_N \mathfrak{a}^n$ has finite length for all n . Kirby proved

Theorem 4.3.3. [17, Proposition 2] *Let N be an artinian A -module and let $l_A(0 :_N \mathfrak{a})$ be finite, then the function $H_{\mathfrak{a}, N}(n) = l_A(0 :_N \mathfrak{a}^n)$ is of polynomial type.*

Later, Kodiyalam proved

Theorem 4.3.4. [19, Theorem 2] *Let $\mathfrak{a}$ be an ideal of a noetherian ring A and M, N be finitely generated A -modules. Then the functions $l_A(\text{Ext}_A^i(M, N / \mathfrak{a}^n N))$ and $l_A(\text{Tor}_i^A(M, N / \mathfrak{a}^n N))$ are of polynomial type provided $l_A(M \otimes_A N)$ is finite.*

That $l_A(M \otimes_A N)$ is finite means that $\text{Supp}_A(M) \cap \text{Supp}_A(N)$ consists of finitely many maximal ideals in A . To include cases where $l_A(M \otimes_A N)$ is not finite, Theoredoscu [36] imposed the condition that

$$\text{Supp}_A M \cap \text{Supp}_A N \cap V(\mathfrak{a})$$

should be a finite set of maximal ideals in A . Here, $V(\mathfrak{a})$ is the set of all prime ideals in A containing $\mathfrak{a}$. Note that, this condition is equivalent to saying that

$$(M \otimes_A N) / \mathfrak{a}(M \otimes_A N) \cong M \otimes_A (N / \mathfrak{a} N)$$

has finite length since $\text{Supp}_A M \cap \text{Supp}_A N \cap V(\mathfrak{a}) = \text{Supp}_A M \cap \text{Supp}_A N / \mathfrak{a} N$.

In this thesis, we have generalized 4.3.3 and 4.3.4 to coherent functors, see chapter 5.

4.4 Betti and Bass numbers

In this section, A is a noetherian commutative local ring with maximal ideal $\mathfrak{m}$ and residue field k .

Let M be a finitely generated A -module. For a non-negative integer i , the i^{th} *Betti number* of M , denoted by $\beta_i^A(M)$, is given by

$$\beta_i^A(M) = l_A(\text{Tor}_i^A(k, M))$$

and the i^{th} Bass number of M , denoted by $\mu_A^i(M)$, is given by

$$\mu_A^i(M) = l_A(\text{Ext}_A^i(k, M)).$$

If $\mathfrak{a}$ is an ideal in A , it is known that $l_A(\mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n)$ is of polynomial type (see for instance section 3 Theorem 1 in [28]). This is equivalent to $\beta_0^A(\mathfrak{a}^n) = l_A(\text{Tor}_0^A(k, \mathfrak{a}^n))$ being of polynomial type via the isomorphism $\mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n \cong k \otimes_A \mathfrak{a}^n$.

In extending this result, Kodiyalam proved

Theorem 4.4.1. [19, Theorem 1] *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous graded ring with $R_0 = A$. Let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated R -module. Then, for each $i \geq 0$, the Betti numbers $\beta_i^A(M_n)$ and the Bass numbers $\mu_A^i(M_n)$ are of polynomial type.*

Now, let $\mathfrak{a}$ be an ideal in A , M a finitely generated A -module and R the Rees ring of A . It follows from 4.4.1 that, for each $i \geq 0$, the Betti numbers $\beta_i^A(\mathfrak{a}^n M)$ and $\beta_i^A(\mathfrak{a}^n M/\mathfrak{a}^{n+1} M)$ are of polynomial type. And also, for each $i \geq 0$, the Bass numbers $\mu_A^i(\mathfrak{a}^n M)$ and $\mu_A^i(\mathfrak{a}^n M/\mathfrak{a}^{n+1} M)$ are of polynomial type. To cater for modules of the form $M/\mathfrak{a}^n M$, Kodiyalam proved

Theorem 4.4.2. [19, Corollary 7] *Let $\mathfrak{a}$ be an ideal in A . For any fixed integer $i \geq 0$, both $\beta_i^A(M/\mathfrak{a}^n M)$ and $\mu_A^i(M/\mathfrak{a}^n M)$ are of polynomial type.*

Note that 4.4.2 follows from 4.3.4 with $M = k$. We have made a generalization of 4.4.2 to coherent functors, see chapter 5.

4.5 Betti numbers related to a half exact coherent functor over a DVR

In this section, we compute the Betti numbers of $F(M/\mathfrak{a}^n M)$ where F is a half exact coherent functor, A is a discrete valuation ring (DVR) with parameter t and $k = A/\mathfrak{m}$ with $\mathfrak{m} = (t)$. We will show that, for a proper ideal $\mathfrak{a}$ in A , the i^{th} Betti number $\beta_i^A(F(M/\mathfrak{a}^n M))$ is a (constant) polynomial for all n .

Since, every non-zero ideal $\mathfrak{a}$ of A is generated by a positive power of t , we will consider $\mathfrak{a} = (t)$. Throughout this section, t^r will denote the ideal (t^r) .

Now, every finitely generated module over a DVR is a direct sum of copies of A and modules of the form A/t^m for various $m \geq 1$. We will consider these cases. Let $M = A/t^m$, then

$$\begin{aligned} M/t^n M &= (A/t^m)/(t^n(A/t^m)) \\ &\cong (A/t^m)/((t^n, t^m)/t^m) \\ &\cong A/(t^n, t^m) \\ &= \begin{cases} A/t^n, & 1 \leq n \leq m \\ A/t^m, & n \geq m \end{cases} \end{aligned} \tag{4.2}$$

Now, consider the exact sequence

$$0 \rightarrow A \rightarrow A \rightarrow k \rightarrow 0$$

where $A \rightarrow A$ is multiplication by t , $k = A/t$ and $A \rightarrow k$ is the natural map. Then, for any A -module M , we have an exact sequence

$$0 \rightarrow 0 \rightarrow \operatorname{Tor}_1^A(k, M) \rightarrow M \rightarrow M \rightarrow k \otimes_A M \rightarrow 0$$

where $M \rightarrow M$ is multiplication by t . Thus

$$\operatorname{Tor}_i^A(k, M) = \begin{cases} k \otimes_A M, & i = 0 \\ 0 :_M t, & i = 1 \\ 0, & i > 1, \end{cases} \quad (4.3)$$

If $M = A/t^n$ for $n \geq 1$, we get

$$\begin{aligned} \operatorname{Tor}_0^A(k, A/t^n) &\cong A/t \otimes_A A/t^n \\ &\cong (A/t^n)/(t(A/t^n)) \\ &\cong A/(t, t^n) \\ &= A/t \\ &= k \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \operatorname{Tor}_1^A(k, A/t^n) &= 0 :_{A/t^n} t \\ &\cong (t^n :_A t)/t^n \\ &\cong t^{n-1}/t^n \\ &\cong k. \end{aligned} \quad (4.5)$$

From (4.3), (4.4) and (4.5) we conclude that, for $n \geq 1$,

$$\operatorname{Tor}_i^A(k, A/t^n) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases} \quad (4.6)$$

Now, let F be a half exact coherent functor over a DVR A with parameter t . Since every half exact coherent functor over a DVR is a direct sum of copies of the identity functor, functors of the form $A/t^s \otimes_A -$ and of the form $\operatorname{Hom}_A(A/t^s, -)$ for various $s \geq 1$ (see Theorem 2.2.7), we consider each of these cases separately.

Case 1 Suppose that F is the identity functor.

i) If $M = A$, we have by (4.6) that

$$\operatorname{Tor}_i^A(k, F(M/t^n M)) = \operatorname{Tor}_i^A(k, A/t^n) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

ii) If $M = A/t^m$, then

$$\begin{aligned} \operatorname{Tor}_i^A(k, F(M/t^n M)) &= \operatorname{Tor}_i^A(k, (A/t^m)/(t^n(A/t^m))) \\ &\cong \begin{cases} \operatorname{Tor}_i^A(k, A/t^n), & 1 \leq n \leq m \\ \operatorname{Tor}_i^A(k, A/t^m), & n \geq m \end{cases} \end{aligned} \quad (4.7)$$

where the isomorphism is due to (4.2). Thus, in any case, we have by (4.6) that

$$\mathrm{Tor}_i^A(k, (A/t^m)/(t^n(A/t^m))) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

Case 2 Suppose that $F = A/t^s \otimes_A -$, for $s \geq 1$.

i) If $M = A$, we have

$$\begin{aligned} F(M/t^n M) &= A/t^s \otimes_A A/t^n \\ &\cong A/(t^n, t^s) \\ &= \begin{cases} A/t^s, & 1 \leq s \leq n \\ A/t^n, & s \geq n \end{cases} \end{aligned} \quad (4.8)$$

Hence, (4.6) shows that

$$\mathrm{Tor}_i^A(k, A/t^s \otimes_A A/t^n) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

ii) If $M = A/t^m$, by using (4.2) followed by (4.8), we get

$$\begin{aligned} F(M/t^n M) &= A/t^s \otimes_A ((A/t^m)/t^n(A/t^m)) \\ &\cong \begin{cases} A/t^s \otimes_A A/t^n, & 1 \leq n \leq m \\ A/t^s \otimes_A A/t^m, & n \geq m \end{cases} \\ &\cong \begin{cases} A/t^s, & 1 \leq s \leq n \\ A/t^n, & n \leq s \leq m \\ A/t^m, & s \geq m \end{cases} \end{aligned}$$

Again by (4.6) we have

$$\mathrm{Tor}_i^A(k, A/t^s \otimes_A ((A/t^m)/t^n(A/t^m))) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

Case 3 Suppose that $F = \mathrm{Hom}_A(A/t^s, -)$, for $s \geq 1$.

i) If $M = A$, we have

$$\begin{aligned} F(M/t^n M) &= \mathrm{Hom}_A(A/t^s, A/t^n) \\ &\cong 0 :_{A/t^n} t^s \\ &= (t^n :_A t^s)/t^n \\ &\cong \begin{cases} t^{n-s}/t^n, & 1 \leq s \leq n \\ A/t^n, & s \geq n \end{cases} \\ &\cong \begin{cases} A/t^s, & 1 \leq s \leq n \\ A/t^n, & s \geq n \end{cases} \end{aligned} \quad (4.9)$$

and again by (4.6) we get

$$\mathrm{Tor}_i^A(k, \mathrm{Hom}_A(A/t^s, A/t^n)) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

ii) If $M = A/t^m$, by using (4.2) followed by (4.9), we have

$$\begin{aligned} F(M/t^n M) &= \mathrm{Hom}_A(A/t^s, (A/t^m)/t^n(A/t^m)) \\ &\cong \begin{cases} \mathrm{Hom}_A(A/t^s, A/t^n), & 1 \leq n \leq m \\ \mathrm{Hom}_A(A/t^s, A/t^m), & n \geq m \end{cases} \\ &\cong \begin{cases} A/t^s, & 1 \leq s \leq n \\ A/t^m, & n \leq s \leq m \\ A/t^n, & s \geq m \end{cases} \end{aligned}$$

Thus, in this case also, (4.6) shows that

$$\mathrm{Tor}_i^A(k, \mathrm{Hom}_A(A/t^s, A/t^n)) = \begin{cases} k, & i = 0, 1 \\ 0, & i > 1 \end{cases}$$

Therefore, the i^{th} Betti number $\beta_i^A(F(M/\mathfrak{a}^n M))$ is a (constant) polynomial for all large n .

5

An overview of the papers

This chapter gives an overview of the papers that form part of this thesis.

5.1 Summary of Paper A

The main aim of this paper is to extend Hartshorne's result 2.2.7 to Dedekind domains. That is, to characterize half-exact coherent functors over principal ideal domains (PIDs) and more generally over Dedekind domains. Our proof has similar steps as Hartshorne's proof. Throughout, we assume that A is a noetherian commutative ring and F is a coherent functor over the category $\mathcal{M}_A$.

Now, for any ideal $\mathfrak{a}$ in A and any finitely generated A -module M , there is a natural map

$$\widehat{F(M)} \rightarrow \varprojlim F(M/\mathfrak{a}^n M)$$

which arises by taking the inverse limit of $F(M)/\mathfrak{a}^n F(M) \rightarrow F(M/\mathfrak{a}^n M)$. Here $\widehat{F(M)}$ is the completion of $F(M)$ in the $\mathfrak{a}$ -adic topology. In section 2, we show that this natural map is an isomorphism whenever $\text{Supp}(M/\mathfrak{a}M)$ consists of only maximal ideals of A . This generalizes 2.2.6. The condition on the support is equivalent to saying that $M/\mathfrak{a}M$ has finite length. To prove the result, we make use of the Mittag-Leffler condition (see section 1.5) to obtain an exact sequence

$$\varprojlim h_Y(M/\mathfrak{a}^n M) \rightarrow \varprojlim h_X(M/\mathfrak{a}^n M) \rightarrow \varprojlim F(M/\mathfrak{a}^n M) \rightarrow 0$$

where X, Y are finitely generated A -modules such that $h_Y \rightarrow h_X \rightarrow F \rightarrow 0$ is exact. We then show that

$$\varprojlim h_N(M/\mathfrak{a}^n M) \cong \widehat{h_N(M)}$$

for any finitely generated A -module N , and use an appropriate commutative diagram and 1.3.5 to complete the proof.

In section 3, we devote ourselves to the construction of a functor $\tilde{F}$ which also appears in [11] in a more general setting. For a multiplicative subset $S \subset A$ and a coherent functor F , we use the fact that every finitely generated $S^{-1}A$ -module M is isomorphic to $S^{-1}N$ for some finitely generated A -module N to define $\tilde{F}$ on finitely generated $S^{-1}A$ -modules. We then show that $\tilde{F}$ is left-, right- or half-exact whenever F is left-, right- or half-exact. It also turns out that $\tilde{F}$ is coherent whenever F is. When $S = A \setminus \mathfrak{p}$, we denote $\tilde{F}$ by $F_{\mathfrak{p}}$ and we use this in the proof of our main theorem.

In section 4, we prove our main results. For any A -linear functor F , we consider the exact sequence

$$F(A) \otimes - \xrightarrow{\alpha} F \longrightarrow F_0 \longrightarrow 0 \quad (5.1)$$

(see section 2.2). We first show that $\alpha_M : F(A) \otimes M \rightarrow F(M)$ is injective for any finitely generated A -module M with projective dimension at most 1. We then show that if F is a half exact coherent functor over a PID, then F_0 is left exact and the sequence (5.1) splits, i.e. $F \cong F(A) \otimes - \oplus F_0$. From this, we deduce that F is a direct sum of copies of the identity functor, functors of the form $\text{Hom}_A(A/p^s, -)$ and functors of the form $A/p^s \otimes -$ for various $s \geq 1$, where p is an irreducible element in A .

When A is a Dedekind domain, by localizing to prime ideals, we show that F_0 is left exact and hence the sequence (5.1) splits. We again deduce from this that F is a direct sum of functors of the form $\text{Hom}_A(P, -)$, $\text{Hom}_A(A/\mathfrak{p}^s, -)$ and $A/\mathfrak{p}^s \otimes -$ for various $s \geq 1$, where P is a finitely generated projective A -module and $\mathfrak{p}$ is a prime ideal in A .

5.2 Summary of Paper B

In this paper we study asymptotic stability of sets of prime ideals and Hilbert polynomials of graded modules related to coherent functors. Throughout the paper, we assume that A is a commutative noetherian ring and $R = A[a_1, \dots, a_r]$ is a graded A -algebra where a_i is an element of degree 1 in R .

In section 2, we first show that if $\mathcal{S}$ is a thick subcategory (i.e. is closed under kernels, cokernels and extensions) of the category of graded R -modules, then for each coherent functor F and every U in $\mathcal{S}$, the graded R -module $F(U)$ is also in $\mathcal{S}$. We then show that if $\mathcal{S}$ is a Serre subcategory (i.e. is closed under subobjects, quotients and extensions) of the category of graded R -modules and $f : U \rightarrow V$ is a graded R -homomorphism, then $\text{Ker } F(f)$ and $\text{Coker } F(f)$ are in $\mathcal{S}$ whenever $\text{Ker } f$ and $\text{Coker } f$ are in $\mathcal{S}$. This is true in particular for the category of finitely generated modules and the category of artinian modules.

Next, we give a different proof to 3.3.4. We then end the section by extending Sharp's result 3.3.5. We prove that, if N is an artinian A -module, $\mathfrak{a}$ an ideal of A and F a coherent functor, the two sequences $\{\text{Att}_A F((0 :_N \mathfrak{a}^n)/(0 :_N \mathfrak{a}^{n-1}))\}_{n=1}^{\infty}$ and $\{\text{Att}_A F(0 :_N \mathfrak{a}^n)\}_{n=1}^{\infty}$ are asymptotically stable. We obtain the first assertion

by using the fact that $G^{\mathfrak{a}}(N) = \bigoplus_{n \leq 0} (0 :_N \mathfrak{a}^{-n+1}) / (0 :_N \mathfrak{a}^{-n})$ is an artinian $G(A)$ -module (see section 4.2). Hence, we have that $F(G^{\mathfrak{a}}(N))$ is an artinian $G(A)$ -module and Theorem 3.1 in [34] gives the result.

For the second assertion, we applying F to the exact sequence

$$0 \longrightarrow U \xrightarrow{f} V \longrightarrow W \longrightarrow 0$$

where U, V and W are graded modules over the Rees ring $R = A[\mathfrak{a}t]$ which in degree $-n$, $n = 1, 2, 3, \dots$, are defined by $U_{-n} = 0 :_N \mathfrak{a}^{n-1}$, $V_{-n} = 0 :_N \mathfrak{a}^n$ and $W_{-n} = (0 :_N \mathfrak{a}^n) / (0 :_N \mathfrak{a}^{n-1})$. Taking $C = \text{Coker } F(f)$, we consider the exact sequence in degree $-n$

$$F(0 :_N \mathfrak{a}^{n-1}) \longrightarrow F(0 :_N \mathfrak{a}^n) \longrightarrow C_{-n} \longrightarrow 0$$

Since C will be an artinian graded R -module, $\text{Att}_A(C_{-n})$ is asymptotically stable. The result now follows from the inclusions

$$\text{Att}_A C_{-n} \subset \text{Att}_A F(0 :_N \mathfrak{a}^n) \subset \text{Att}_A F(0 :_N \mathfrak{a}^{n-1}) \cup \text{Att}_A(C_{-n})$$

and the fact that the sets in the above inclusions are finite.

In section 3, we study Hilbert polynomials related to coherent functors. We obtain two results that extends Kirby's result 4.3.3. In the first, we show that if N is an artinian A -module and $0 :_N \mathfrak{a}^n$ has finite length, then the length of $F(0 :_N \mathfrak{a}^n)$ is of polynomial type (see section 4.3 for the definition). We achieve this by first showing that the length of $\text{Hom}_A(X, 0 :_N \mathfrak{a}^n)$ is of polynomial type when X is a finitely generated A -module. Letting Z to be the cokernel of the map $X \rightarrow Y$ such that $\text{h}_Y \rightarrow \text{h}_X \rightarrow F \rightarrow 0$ is exact, we obtain the exact sequence

$$0 \rightarrow \text{h}_Z(0 :_N \mathfrak{a}^n) \rightarrow \text{h}_Y(0 :_N \mathfrak{a}^n) \rightarrow \text{h}_X(0 :_N \mathfrak{a}^n) \rightarrow F(0 :_N \mathfrak{a}^n) \rightarrow 0.$$

The result now follows by the additive property of the length function.

In the second result, we assume that $F(0 :_N \mathfrak{a}^n)$ has finite length for all n , and prove that the length of $F(0 :_N \mathfrak{a}^n)$ is of polynomial type. We also prove its dual that, if M is a finitely generated A -module and $F(M/\mathfrak{a}^n M)$ has finite length for all n , then $F(M/\mathfrak{a}^n M)$ is of polynomial type.

Next, we show that the composite of two coherent functors is coherent. We then use this to prove two results. The first is Se's result 3.4.2. We also show that $\text{grade } F(M/\mathfrak{a}^n M)$ is independent of n for large n . The second is an extension of Kodiyalam's result 4.4.1. We show that, if A is local, the i^{th} Betti number and the i^{th} Bass number of $F(M/\mathfrak{a}^n M)$ are of polynomial type.

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Part II

Included Papers

Papers

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