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A COMPARISON OF TWO ENTROPY STABLE DISCONTINUOUS GALERKIN SPECTRAL ELEMENT APPROXIMATIONS FOR THE SHALLOW WATER EQUATIONS WITH NON-CONSTANT TOPOGRAPHY

ANDREW R. WINTERS* AND GREGOR J. GASSNER

ABSTRACT. In this work, we compare and contrast two provably entropy stable and high-order accurate nodal discontinuous Galerkin spectral element methods applied to the one dimensional shallow water equations for problems with non-constant bottom topography. Of particular importance for numerical approximations of the shallow water equations is the well-balanced property. The well-balanced property is an attribute that a numerical approximation can preserve a steady-state solution of constant water height in the presence of a bottom topography. Numerical tests are performed to explore similarities and differences in the two high-order schemes.

Keywords: entropy stable, discontinuous Galerkin, skew-symmetric, shallow water equations, well-balanced, method comparison

1. INTRODUCTION

Fluid flows in lakes, rivers, and near oceanic shores are of great interest in hydrology, oceanography and climate modeling. Common to all of these flows is the fact that vertical scales of motion are much smaller than the horizontal scales. By this and the assumption of hydrostatic balance [30], the incompressible Navier-Stokes equations can be simplified to the so-called shallow water equations, which in one dimension are

$$(1.1) \quad \partial_t \underline{u} + \partial_x \underline{f} = \begin{cases} \frac{\partial h}{\partial t} + \frac{\partial(vh)}{\partial x} = 0, \\ \frac{\partial(vh)}{\partial t} + \frac{\partial(v^2h + gh^2/2)}{\partial x} = -gh \frac{\partial b}{\partial x}. \end{cases}$$

Here, h is the height of the water and v is the velocity. The constant g is the acceleration due to gravity and the function $b = b(x)$ represents the bottom topography of the surface over which the fluid flows. The shallow water system with topography (1.1) amounts to a system of balance laws.

If the bottom topography is flat, i.e. $b \equiv \text{constant}$, then Eq. (1.1) reduces to the standard shallow water equations without topography, which is a strictly hyperbolic system of conservation laws,

$$(1.2) \quad \partial_t \underline{u} + \partial_x \underline{f} = 0.$$

It is well-known that solutions of the conservation law (1.2) as well as solutions of the balance law (1.1), may develop shock discontinuities in finite time, independent of the smoothness of the

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initial data. Hence, solutions of balance laws (1.1) are considered in the weak sense and are well-defined provided the source $s = -ghb_x$ remains uniformly bounded, i.e., weak solutions of (1.1) are well-defined under the assumption that the topography function $b \in W^{1,\infty}(\mathbb{R})$, see e.g. [9].

The preservation of a steady-state solution known colloquially as the “lake at rest” condition:

$$(1.3) \quad h + b = \text{constant}, \quad v = 0,$$

is especially important for the shallow water equations (1.1). This is because relevant waves in a flow may be viewed as small perturbations of (1.3), see [2]. A good numerical method for the shallow water equations should accurately capture both the steady states and their small perturbations (quasi-steady flows). Such a property diminishes the appearance of unphysical waves with magnitude proportional to the grid size (a so-called “numerical storm” [7]), which are normally present for numerical schemes that cannot preserve (1.3). A numerical method that exactly preserves the “lake at rest” steady state (1.3) is said to be well-balanced, see e.g. [2, 12, 15, 22].

Of particular interest is the issue of robustness and the ability of the method to remain stable and accurate, even if discontinuities develop in the solution of a PDE. Recent work has focused on the use of high-order discontinuous Galerkin (DG) approximations to create robust numerical methods for the solution of systems of conservation laws, e.g., [4, 15, 20]. These robust high-order DG methods may be derived from the perspective of entropy conservation, e.g. [4, 23, 29], or reformulate the PDE in a skew-symmetric formulation to maintain conservation, e.g. [15, 20]. The motivation behind the two approaches are similar [27].

The aim of this paper is to compare two entropy stable schemes and their use to solve the shallow water equations. The first is a skew-symmetric formulation of (1.1) which, additionally, is well-balanced by construction [15]. The second scheme, originally developed by Carpenter et al. for the solution of the Navier-Stokes equations [4], is entropy stable by construction. However, we will show that the method of Carpenter et al. is not well-balanced. Although it is possible to introduce a correction to recover the well-balanced property, entropy stability is no longer ensured.

The remainder of this paper is organized as follows: Sec. 2 provides a brief overview of the skew-symmetric, entropy conserving scheme for the shallow water equations developed in [15]. In Sec. 3 we describe and highlight the entropy conserving scheme of Carpenter et al. fully described in [4]. We describe provably, entropy stable versions of both schemes in Sec. 4. Numerical results are presented in Sec. 5. We make direct comparisons between the two entropy stable schemes as well as explore the accuracy of the skew-symmetric scheme for challenging quasi-steady state problems. Sec. 6 presents concluding remarks.

2. SKEW-SYMMETRIC, ENTROPY CONSERVATIVE DISCONTINUOUS GALERKIN SPECTRAL ELEMENT METHOD

The authors in [15] design an arbitrarily high-order accurate nodal discontinuous Galerkin spectral element type method. The discretization uses a skew-symmetric formulation of the one dimensional shallow water equations and was shown to exactly preserve the local mass, momentum, and entropy. We collect the major results below with full details given in [15].

Remark 1. (Nonlinear Correction Terms) For numerical schemes constructed from a skew-symmetric form it is not obvious that the approximation conserves momentum discretely. However, from the Lax-Wendroff theorem, conservation is a fundamental requirement to correctly predict shock speeds.

Fisher et al. [10] proved important properties for diagonal norm summation-by-parts (SBP) operators, such as the discontinuous Galerkin spectral element method with Gauss-Lobatto quadrature [13, 14]. First, Fisher et al. demonstrated that $\underline{D}f$ can be recast to a subcell flux-difference

formulation such that the Law-Wendroff theorem holds. Furthermore, they proved that for point wise nodal representations of functions α and β then

$$(2.1) \quad \underline{\underline{\alpha}} \underline{\underline{D}} \underline{\underline{\beta}} + \underline{\underline{\beta}} \underline{\underline{D}} \underline{\underline{\alpha}},$$

is a consistent and conservative approximation of the derivative of the corresponding discrete flux

$$(2.2) \quad \underline{\underline{D}} \underline{\underline{\alpha}} \underline{\underline{\beta}}.$$

If we subtract the discrete flux (2.2) from (2.1), we obtain an error in the associated product rule

$$(2.3) \quad \underline{\underline{s}}_{\alpha\beta} := -\underline{\underline{D}} \underline{\underline{\alpha}} \underline{\underline{\beta}} + \underline{\underline{\alpha}} \underline{\underline{D}} \underline{\underline{\beta}} + \underline{\underline{\beta}} \underline{\underline{D}} \underline{\underline{\alpha}}.$$

In [14], it was shown that the discrete mean value of terms with the structure of (2.3) is exactly zero.

Definition 1. (ECDGSEM) To construct a nodal discontinuous Galerkin spectral element method [19] for the skew-symmetric form of the shallow water equations [15] on each element we do the following:

- (1) Multiply the governing equations by a square integrable test function φ and integrate over the domain.
- (2) Divide the domain into K non-overlapping grid cells.
- (3) Integrate by parts twice on each element, generating boundary terms, to obtain a strong form DGSEM approximation [18].
- (4) Approximate all quantities by piecewise polynomials of degree N on each element.
- (5) Select the test function φ from a subspace of L_2 , namely, piecewise polynomials.
- (6) Replace integrals by Gauss-Lobatto quadrature.
- (7) Choose Lagrange nodal polynomial basis functions, $\ell_j(\xi)$, $j = 0, \dots, N$, with the interpolation nodes collocated with the quadrature nodes.
- (8) Replace the boundary flux by a numerical flux, f^* .

This yields the approximation on each element

$$(2.4) \quad \begin{aligned} \mathcal{J} \partial_t \underline{\underline{u}}_1 + \underline{\underline{D}} \underline{\underline{f}}_1 &= \underline{\underline{S}} \left[\underline{\underline{f}}_1^* - \underline{\underline{f}}_1 \right], \\ \mathcal{J} \partial_t \underline{\underline{u}}_2 + \underline{\underline{D}} \underline{\underline{f}}_2 + g \underline{\underline{h}} \underline{\underline{D}} \underline{\underline{b}} + \underline{\underline{s}}_{hv^2} + \underline{\underline{s}}_{h^2} &= \underline{\underline{S}} \left[\underline{\underline{f}}_2^* - \underline{\underline{f}}_2 \right], \end{aligned}$$

where $\mathcal{J} = \frac{\Delta x}{2}$ is the element Jacobian, we have the physical fluxes

$$(2.5) \quad \begin{aligned} \underline{\underline{f}}_1 &= \underline{\underline{h}} \underline{\underline{v}}, \\ \underline{\underline{f}}_2 &= \underline{\underline{h}} \underline{\underline{v}}^2 + \frac{g}{2} \underline{\underline{h}}^2, \end{aligned}$$

and skew-symmetric nonlinear correction terms of the form (2.3)

$$(2.6) \quad \begin{aligned} \underline{\underline{s}}_{hv^2} &= \frac{1}{2} \left[-\underline{\underline{D}} \underline{\underline{h}} \underline{\underline{v}}^2 + \underline{\underline{h}} \underline{\underline{v}} \underline{\underline{D}} \underline{\underline{v}} + \underline{\underline{v}} \underline{\underline{D}} \underline{\underline{h}} \underline{\underline{v}} \right], \\ \underline{\underline{s}}_{h^2} &= \frac{g}{2} \left[-\underline{\underline{D}} \underline{\underline{h}}^2 + 2 \underline{\underline{h}} \underline{\underline{D}} \underline{\underline{h}} \right]. \end{aligned}$$

In combination with the entropy conserving numerical flux

$$(2.7) \quad f^* = f^{*,ec} = \left(\begin{array}{c} \{v\} \{h\} \\ \{v\}^2 \{h\} + \frac{1}{2} g \{h^2\} \end{array} \right),$$

we get discrete exact conservation of the mass, momentum and total energy. The total energy is an entropy function for the shallow water equations [12], hence we exactly preserve the entropy of the system and therefore call this discretization the entropy conservative discontinuous Galerkin spectral element method (ECDGSEM). Furthermore, if we approximate the bottom topography b so that it is continuous across element interfaces and use an initial condition so that the jumps in the water height h are zero for all interfaces, the ECDGSEM is well balanced.

The matrix

$$(2.8) \quad D_{ij} = \ell'_j(\xi_i), \quad i, j = 0, \dots, N,$$

is the standard polynomial derivative matrix of the Lagrange interpolating polynomial evaluated at the quadrature nodes. By collocating the Gauss-Lobatto quadrature points and the interpolation nodes of the Lagrange basis functions ℓ , the derivative matrix $\underline{\underline{D}}$ possesses the following relationship for all polynomial degrees N , see e.g. [4, 13, 14]

$$(2.9) \quad \underline{\underline{Q}} := \underline{\underline{M}} \underline{\underline{D}},$$

with

$$(2.10) \quad \underline{\underline{Q}} + \underline{\underline{Q}}^T = \underline{\underline{B}}, \quad \text{or} \quad (\underline{\underline{M}} \underline{\underline{D}}) + (\underline{\underline{M}} \underline{\underline{D}})^T = \underline{\underline{B}},$$

where $\underline{\underline{B}}$ and $\underline{\underline{M}}$ are diagonal matrices

$$(2.11) \quad \underline{\underline{M}} = \text{diag}[\omega_0, \dots, \omega_N], \quad \underline{\underline{B}} = \text{diag}[-1, 0, \dots, 0, 1],$$

where $\{\omega_j\}_{j=0}^N$ denote the Gauss-Lobatto quadrature weights ordered from left to right within the cell. Usually, the matrix $\underline{\underline{M}}$ is called the mass matrix and $\underline{\underline{B}}$ the boundary matrix. The relationship of the derivative operator (2.10) is called the summation-by-parts (SBP) property and is common in the finite difference community, e.g. [6, 21, 25, 26]. We use the matrices (2.11) to define the surface matrix

$$(2.12) \quad \underline{\underline{S}} := -\underline{\underline{M}}^{-1} \underline{\underline{B}} = \text{diag}\left(\frac{1}{\omega_0}, 0, \dots, 0, -\frac{1}{\omega_N}\right).$$

The operator $\{\!\!\{\cdot\}\!\!\}$ in (2.7) is the average value at an interface. Consider an interface $m + \frac{1}{2}$ between the grid cells m and $m + 1$. We define the average from a nodal DGSEM quantity a at the interface as

$$(2.13) \quad \{\!\!\{a\}\!\!\} := \frac{1}{2}(a_0^{m+1} + a_N^m).$$

3. HIGH-ORDER, ENTROPY CONSERVATIVE DISCONTINUOUS GALERKIN SCHEME OF CARPENTER ET AL.

We briefly describe the entropy conservative, high-order DG approximation developed by Carpenter et al. that is described at length in [4]. Conveniently, their scheme is also derived from a summation-by-parts, strong form DGSEM approximation with a Gauss-Lobatto quadrature. A key to the Carpenter et al. scheme is the use of complementary grids, where the approximation is reminiscent of a finite volume method. We denote quantities on the complementary grid using a bar.

Definition 2. (SCSCE-DG) The discretization (3.1) presents the entropy conserving, spectral collocation element, semi-discrete DG approximation of Carpenter et al. in indicial form. For consistency with the work of Carpenter et al. [4] we denote this discretization by SCSCE-DG. From the entropy conserving flux (3.4) the high-order method looks almost identical to a finite volume type update, where the boundaries of each element incorporate the numerical flux (2.7)

$$(3.1) \quad (\mathcal{J}\underline{u})_i = \begin{cases} - \left[\frac{\bar{f}_1^{(S)} - \bar{f}_0^{(S)}}{\omega_0} - \frac{f^{*,ec}|_{-1} - f_0}{\omega_0} + RHS_0 \right], & i = 0, \\ - \left[\frac{\bar{f}_{i+1}^{(S)} - \bar{f}_i^{(S)}}{\omega_i} + RHS_i \right], & i = 1, \dots, N-1, \\ - \left[\frac{\bar{f}_{N+1}^{(S)} - \bar{f}_N^{(S)}}{\omega_N} + \frac{f^{*,ec}|^{+1} - f_N}{\omega_N} + RHS_N \right], & i = N. \end{cases}$$

where f_0 and f_N are the physical flux (2.5) evaluated at the endpoints. For compactness we introduced a right hand side term in (3.1) of the form

$$(3.2) \quad RHS_i = \begin{bmatrix} 0 \\ gh_i b'_i \end{bmatrix}, \quad i = 0, \dots, N$$

where the bottom topography is discretized in the traditional nodal DG way and the prime notation represents a slice of the matrix-vector product that computes the discrete DG derivative, e.g.,

$$(3.3) \quad b'_i = \sum_{j=0}^N D_{ij} b_j.$$

To compute a high-order, entropy conservative flux $\bar{f}^{(S)}$ we use the result of the following theorem, presented without proof. (The proof appears in [11], Thm. 3.1).

Theorem 1. *A two-point entropy conservative flux can be extended to high-order with formal boundary closures by using the form*

$$(3.4) \quad \bar{f}_0^{(S)} = f_0, \quad \bar{f}_i^{(S)} = \sum_{k=i}^N \sum_{l=0}^{i-1} 2 Q_{lk} f^{ec}(u_l, u_k), \quad i = 1, \dots, N, \quad \bar{f}_{N+1}^{(S)} = f_N,$$

where f is the physical flux (2.5), the entropy conserving flux f^{ec} is given by (2.7), and Q_{lk} is the $(l, k)^{th}$ entry of the SBP matrix $\underline{Q}$.

Remark 2. The high-order, entropy conservative flux (3.1) relies on the existence of a two-point, non-dissipative flux, denoted $f^{ec}(u_l, u_k)$, that satisfies a local version of Tadmor's entropy conservation condition [28]. Such a two-point flux, with low enough computational cost to be practical, has been constructed in a variety of contexts [12, 15, 17]. We demonstrate the construction of f^{ec} using the analysis tools of Ismail and Roe [17] in the next section.

4. ENTROPY STABLE APPROXIMATIONS

A problem with entropy conservative formulations is they may suffer breakdown if used without dissipation to capture shocks. Physically, entropy must be dissipated at a shock. However, an isentropic algorithm does not allow the capture of this physical process, which results in the generation of large amplitude oscillations around the shock [4]. Another dire issue is that entropy

conservative formulations can not converge to the weak solution as there is no mechanism to admit the dissipation physically required at the shock.

In their work Carpenter et al. outline procedures to guarantee the SCSCE-DG scheme is entropy stable [4]. In contrast, the ECDGSEM scheme is introduced as a baseline scheme with zero viscosity. So, some form of entropy consistent viscosity must be added to the ECDGSEM to yield an entropy stable DG scheme for the shallow water equations. We introduce dissipation into the ECDGSEM method with Riemann solver type numerical flux functions. We also introduce a procedure for interior stabilization analogous to that proposed by Carpenter et al. Both dissipative mechanisms for the ECDGSEM are motivated to preserve the entropy inequality discretely.

To derive an entropy stable numerical interface flux we use entropy analysis techniques, similar to those used by Ismail and Roe [17], in the context of DG approximations. This new interface flux will remedy stability issues of entropy conservative schemes. We will derive a dissipation term, motivated by the entropy inequality, to add at grid cell interfaces and stabilize the approximation. In turn, this creates a provably stable DG approximation for the shallow water equations (1.1). Detailed analysis of the entropy properties of the shallow water equations has also been performed by Fjordholm et al. [12] from the perspective of finite volume methods.

4.1. Entropy Stable Riemann Flux at Cell Interfaces. For the shallow water equations (1.1) we have the entropy function

$$(4.1) \quad E(\underline{u}) = \frac{1}{2} (gh^2 + u^2h) + ghb,$$

which consists of the kinetic energy $hv^2/2$ and the gravitational potential energy $gh^2/2 + ghb$. We compute the entropy variables directly from the entropy function (4.1)

$$(4.2) \quad \underline{q} = \partial_{\underline{u}} E = \begin{bmatrix} g(h+b) - \frac{v^2}{2} \\ v \end{bmatrix}.$$

We can use the entropy function (4.1) and the entropy variables (4.2) to rewrite the shallow water equations (1.1) as an entropy conservation law valid for *smooth* solutions [12]

$$(4.3) \quad \partial_t E(\underline{u}) + \partial_x F(\underline{u}) = 0,$$

with the associated entropy flux

$$(4.4) \quad F(\underline{u}) = \frac{1}{2} (hv^3) + ghv(h+b).$$

The balance between the entropy and entropy flux (4.3) must be modified to take into account the presence of possible discontinuous solutions of (1.1). This leads to the entropy inequality [12]

$$(4.5) \quad \partial_t E(\underline{u}) + \partial_x F(\underline{u}) \leq 0.$$

In the absence of a source term ($b \equiv \text{constant}$), (4.5) amounts to the usual entropy condition for conservation laws. Additionally, the entropy variables (4.2) are equipped with the Jacobian matrices $\underline{\underline{H}} = \partial_{\underline{q}} \underline{u}$ and $\underline{\underline{H}}^{-1} = \partial_{\underline{u}} \underline{q}$

$$(4.6) \quad \underline{\underline{H}} = \frac{1}{g} \begin{bmatrix} 1 & v \\ v & v^2 + c^2 \end{bmatrix}, \quad \underline{\underline{H}}^{-1} = \frac{1}{h} \begin{bmatrix} v^2 + c^2 & -v \\ -v & 1 \end{bmatrix},$$

where $c = \sqrt{gh}$ is the wave celerity. As it will be of use in later derivations we note that

$$(4.7) \quad \underline{q} \cdot \underline{f} = \frac{1}{2} ghv^2 + F,$$

where F is the entropy flux (4.4).

To create an optimal correction term (i.e. minimal dissipation required for stability [17]) to the numerical flux created by the entropy conserving flux (2.7) and create an entropy stable scheme, we first examine the flux Jacobian in conservative variables. Our goal is to relate the right eigenvectors of the flux Jacobian to the entropy Jacobian matrix $\underline{\underline{H}}$ (4.6). The conservative flux Jacobian is

$$(4.8) \quad \partial_{\underline{u}} \underline{f} = \begin{bmatrix} 0 & 1 \\ gh - v^2 & 2v \end{bmatrix}.$$

We find the eigendecomposition $\underline{\underline{R}} \underline{\underline{\Lambda}} \underline{\underline{L}}$ of the Jacobian matrix (4.8) to be

$$(4.9) \quad \underline{\underline{R}} = \alpha \begin{bmatrix} 1 & 1 \\ v+c & v-c \end{bmatrix}, \quad \underline{\underline{\Lambda}} = \begin{bmatrix} v+c & 0 \\ 0 & v-c \end{bmatrix},$$

and $\underline{\underline{L}} = \underline{\underline{R}}^{-1}$. We determine the scaling factor α on the right eigenvectors to satisfy the Merriam identity [23], which relates the matrices $\underline{\underline{H}}$ and $\underline{\underline{R}}$. The Merriam identity states that there exists a matrix such that we can write the entropy Jacobian matrix as

$$(4.10) \quad \underline{\underline{H}} = \underline{\underline{Y}} \underline{\underline{Y}}^T,$$

where, with the proper selection of the constant α in (4.9), we achieve that $\underline{\underline{Y}} = \underline{\underline{R}}$. A quick calculation yields that we use the scaled matrix of right eigenvectors

$$(4.11) \quad \underline{\underline{Y}} = \underline{\underline{R}} = \frac{1}{\sqrt{2g}} \begin{bmatrix} 1 & 1 \\ v+c & v-c \end{bmatrix}.$$

We next present a general method to derive an entropy stable numerical flux. In doing so we rederive the entropy conserving flux (2.7), but in a very different way than the authors in [15]. To derive the entropy conserving flux at interface jumps we have the following constraint to guarantee the entropy conservation law (4.3) is satisfied discretely [17]

$$(4.12) \quad \llbracket \underline{q} \cdot \underline{f} \rrbracket - \llbracket \underline{q} \rrbracket^T \underline{f} = \llbracket F \rrbracket,$$

where $\llbracket \cdot \rrbracket = (\cdot)_R - (\cdot)_L$ is the jump across an interface. Using the earlier calculation (4.7) we rewrite the constraint (4.12) as

$$(4.13) \quad \llbracket \underline{q} \rrbracket^T \underline{f} = \frac{1}{2} g \llbracket v h^2 \rrbracket.$$

Our goal is to derive the entropy conserving $\underline{f}^{*,ec}$ to be used with the shallow water equations. Again, quantities in braces indicate the arithmetic mean (2.13). We begin with the constraint, for the shallow water equations, to determine this entropy conserving flux $\underline{f}^{*,ec}$

$$(4.14) \quad \llbracket \underline{q} \rrbracket^T \underline{f}^{*,ec} = \frac{1}{2} g \llbracket v h^2 \rrbracket.$$

The most important steps in deriving an entropy conserving flux are:

- Selecting an ansatz for the flux $\underline{f}^{*,ec}$, usually based on the physical flux so that consistency is trivially shown.
- Choosing a parameterization that allows us to rewrite the jump terms in the condition (4.14). This is achieved using the following properties of the linear jump operator $\llbracket \cdot \rrbracket$
  - $\llbracket ab \rrbracket = \{ \{ a \} \} \llbracket b \rrbracket + \{ \{ b \} \} \llbracket a \rrbracket$,
  - $\llbracket a^2 \rrbracket = 2 \{ \{ a \} \} \llbracket a \rrbracket$.

- Separating the coefficients of each linear jump term (after applying the parameterization) and solving.

We expand the entropy conservation constraint (4.14) and find two equations for the two unknown flux components f_1^* and f_2^*

(4.15)

$$\begin{aligned} \left\{ g \llbracket h + b \rrbracket - \frac{1}{2} \llbracket v^2 \rrbracket \right\} f_1^* + \llbracket v \rrbracket f_2^* &= \frac{1}{2} g \llbracket v h^2 \rrbracket, \\ \{ g \llbracket h \rrbracket + g \llbracket b \rrbracket - \llbracket v \rrbracket \llbracket v \rrbracket \} f_1^* + \llbracket v \rrbracket f_2^* &= \frac{1}{2} g \llbracket h^2 \rrbracket \llbracket v \rrbracket + g \llbracket v \rrbracket \llbracket h \rrbracket \llbracket h \rrbracket, \\ \{ g \llbracket h \rrbracket + 0 - \llbracket v \rrbracket \llbracket v \rrbracket \} f_1^* + \llbracket v \rrbracket f_2^* &= \frac{1}{2} g \llbracket h^2 \rrbracket \llbracket v \rrbracket + g \llbracket v \rrbracket \llbracket h \rrbracket \llbracket h \rrbracket, \\ \llbracket h \rrbracket \{ g f_1^* - g \llbracket v \rrbracket \llbracket h \rrbracket \} + \llbracket v \rrbracket \left\{ - \llbracket v \rrbracket f_1^* + f_2^* - \frac{1}{2} g \llbracket h^2 \rrbracket \right\} &= 0, \end{aligned}$$

where we used the assumption that the approximation of the bottom topography b is continuous across cell interfaces, i.e. $\llbracket b \rrbracket = 0$. Now we solve for the coefficients and determine the entropy conserving flux to be

(4.16)

$$\underline{f}^{*,ec} = \left[\frac{\llbracket v \rrbracket \llbracket h \rrbracket}{\llbracket v \rrbracket^2 \llbracket h \rrbracket + \frac{1}{2} g \llbracket h^2 \rrbracket} \right],$$

where the constraint (4.14) is immediately satisfied and we can trivially check the numerical flux's consistency. The numerical flux (4.16) also matches the work of Fjordholm et al. [12].

The design of the entropy stable Riemann solver uses the entropy conserving flux (4.16) and incorporates, matching the work of Ismail & Roe [17], a dissipation term that guarantees entropy stability. This numerical flux is of the form

(4.17)

$$\underline{f}^{*,es} = \underline{f}^{*,ec} - \frac{1}{2} \underline{R} | \underline{\Lambda} | \underline{R}^T \llbracket \underline{q} \rrbracket.$$

We previously scaled the matrix $\underline{R}$ (4.11) to relate the right eigenvectors of the flux Jacobian to the entropy Jacobian. This relationship, sometimes referred to as the Merriam identity [23], is essential to guarantee that the entropy produced by the numerical flux (4.17) is negative. Explicitly, the entropy produced by the numerical flux (when applied to conservative variables $\underline{u}$) is [17]

(4.18)

$$- \frac{1}{2} \llbracket \underline{q} \rrbracket^T \underline{R} | \underline{\Lambda} | \underline{L} \llbracket \underline{u} \rrbracket,$$

which can become positive in the presence of very strong shocks [3]. To remedy this issue we use the reformulation for infinitesimal disturbances [3]

(4.19)

$$\begin{aligned} \frac{1}{2} \llbracket \underline{q} \rrbracket^T \underline{R} | \underline{\Lambda} | \underline{L} \llbracket \underline{u} \rrbracket &\simeq \frac{1}{2} \llbracket \underline{q} \rrbracket^T \underline{R} | \underline{\Lambda} | \underline{L} \left(\partial_{\underline{q}} \underline{u} \right) \llbracket \underline{q} \rrbracket, \\ &= \frac{1}{2} \llbracket \underline{q} \rrbracket^T \underline{R} | \underline{\Lambda} | \underline{L} \underline{R}^T \llbracket \underline{q} \rrbracket, \\ &= \frac{1}{2} \llbracket \underline{q} \rrbracket^T \underline{R} | \underline{\Lambda} | \underline{R}^T \llbracket \underline{q} \rrbracket, \end{aligned}$$

where $\underline{L} = \underline{R}^{-1}$. The result (4.19) is a positive definite quadratic form, thus the entropy produced by the numerical flux is guaranteed negative.

Summarising this section, we now have that the ECDGSEM and SCSCE-DG use the entropy stable numerical flux (4.17) at cell interfaces. We reiterate that the dissipation term in (4.17) is

the amount required to guarantee entropy stability, it *does not and is not designed* to eliminate overshoots near shocks. Both entropy conserving schemes are provably stable and high-order approximation using the proof techniques from Carpenter et al. [4]. Therefore, we will designate the schemes with the new abbreviated names ESDGSEM and SSSCE-DG.

4.2. Interior Cell Entropy Stability. An advantage of the SSSCE-DG is that because the method is derived directly from the general entropy inequality (4.5). Carpenter et al. provide a straightforward approach for entropy stabilization inside grid cells [4]. Later in this section we will motivate a similar interior stabilization procedure for the skew-symmetric ESDGSEM.

For the SSSCE-DG the idea of adding dissipation on the interior of a grid cell arises from a version of Tadmor's pointwise, local conditions for entropy stability [29]

$$(4.20) \quad (q_{i+1} - q_i)^T (\bar{f}_i - \bar{f}_i^{(S)}) \leq 0, \quad i = 1, \dots, N,$$

where $\bar{f}$ is the vector of physical fluxes interpolated onto the staggered grid and $\bar{f}^{(S)}$ is defined in (3.4). The conditions (4.20) are enforced discretely using the limiter function from Carpenter et al. [4]

$$(4.21) \quad \begin{aligned} \bar{f}_0^{(SS)} &= \bar{f}_0^{(S)}, \\ \bar{f}_i^{(SS)} &= \bar{f}_i + \delta(\bar{f}_i^{(S)} - \bar{f}_i), \quad \delta = \frac{\sqrt{\beta^2 + \varepsilon^2} - \beta}{\sqrt{\beta^2 + \varepsilon^2}}, \quad \beta = (q_{i+1} - q_i)^T (\bar{f}_i^{(S)} - \bar{f}_i), \quad \varepsilon = 10^{-12}, \\ \bar{f}_{N+1}^{(SS)} &= \bar{f}_{N+1}^{(S)}, \end{aligned}$$

where we designate $\bar{f}^{(SS)}$ as the entropy stable flux. The limiter function δ ensures that a linear combination of the physical and entropy conservative flux is used, with the correct sign, to satisfy the conditions (4.20) and maintain stability. The pointwise entropy stability conditions (4.20) and the entropy stable flux (4.21), are valid for any pair of fluxes $\bar{f}$ and $\bar{f}^{(S)}$, provided that both can be expressed in a telescopic form [4]. It is important to note that the local conditions (4.20) *do not* provide insight on the magnitude of dissipation required to achieve a non-oscillatory shock.

Now we want a similar mechanism to add dissipation within grid cells for the skew-symmetric ESDGSEM. Instead of the entropy conditions (4.20) indicating if stabilization within a cell should occur we will use at each Gauss-Lobatto node

$$(4.22) \quad \beta = -v \underline{s}_{hv^2},$$

where the multiplication is componentwise and $\underline{s}_{hv^2}$ is defined in (2.6). The choice of β in (4.22) is motivated by the entropy inequality (4.5). Multiplying the correction $\underline{s}_{hv^2}$ by the velocity v is analogous to scaling by the second entropy variable and the negation allows us to use the previously defined limiter function δ . Now, when the advective correction term is computed in the ESDGSEM we can add extra dissipation internally by

$$(4.23) \quad (\tilde{s}_{hv^2})_i = \delta_i (s_{hv^2})_i, \quad \delta_i = \frac{\sqrt{\beta^2 + \varepsilon^2} - \beta}{\sqrt{\beta^2 + \varepsilon^2}}, \quad \beta_i = -v_i (s_{hv^2})_i, \quad \varepsilon = 10^{-12}$$

Because the correction term $\underline{s}_{hv^2}$ is already computed, the computational overhead for the limiter procedure (4.23) is minimal.

4.3. A Correction Term to Render the SSSCE-DG Well-Balanced. An advantage of the skew-symmetric ESDGSEM formulation is that the approximation is, by design, well-balanced for an arbitrary, continuous bottom topography b [15]. The entropy stable numerical flux (4.17) does not effect the well-balanced property, as it uses the same assumption of continuous approximation of b . If the bottom topography for a problem is constant, i.e. $b_x = 0$, then the SSSCE-DG is well-balanced. Unfortunately, for non-constant bottom topography the SSSCE-DG loses the well-balanced property and spurious waves on the order of the grid size are generated. We demonstrate this phenomenon in Fig. 1.

FIGURE 1. (*left*): An initial constant, water height solution with non-constant bottom. (*right*): Closeup of the “numerical storm” generated from a non-constant topography by the SSSCE-DG. The dashed line is the true, constant water height solution.

Motivated by the skew-symmetric DGSEM we find that adding an appropriate correction term to the momentum equation the SSSCE-DG approximation can be made well-balanced.

Theorem 2. (*Well-Balanced SSSCE-DG*) *If the correction term $\underline{s}_{h^2}$ (2.6) is added to the momentum equation of the SSSCE-DG approximation, then the method is well-balanced.*

Proof. We consider the steady-state solution

$$(4.24) \quad h(x, t) = C - b(x), \quad v(x, t) = 0,$$

where C is a constant water height and $b(x)$ is the bottom topography. Where, again, we assume that the approximation of the bottom topography is continuous across grid cell interfaces.

Just as was the case in the skew-symmetric form, we immediately achieve well-balancedness in the continuity equation because $\underline{h} \underline{v} = 0$ discretely for the ansatz (4.24). Thus, it is our goal to show that, with the addition of the correction term $\underline{s}_{h^2}$ (2.6),

$$(4.25) \quad \mathcal{J} \dot{u}_2 = 0,$$

discretely.

For the SSSCE-DG approximation of the momentum equation we have from (3.1) in indicial form

$$(4.26) \quad (\mathcal{J}\dot{u}_2)_i = \begin{cases} - \left[\frac{\bar{f}_1^{SS} - \bar{f}_0^{SS}}{\omega_0} - \frac{f^{*,ec}|_{-1} - f_0}{\omega_0} + gh_0 b'_0 + \frac{g}{2} (2h_0 h'_0 - (h^2)'_0) \right], & i = 0, \\ - \left[\frac{\bar{f}_{i+1}^{SS} - \bar{f}_i^{SS}}{\omega_i} + gh_i b'_i + \frac{g}{2} (2h_i h'_i - (h^2)'_i) \right], & i = 1, \dots, N-1, \\ - \left[\frac{\bar{f}_{N+1}^{SS} - \bar{f}_N^{SS}}{\omega_N} + \frac{f^{*,ec}|^{+1} - f_N}{\omega_N} + gh_N b'_N + \frac{g}{2} (2h_N h'_N - (h^2)'_N) \right], & i = N, \end{cases}$$

where the prime notation represents a slice of the matrix-vector product that computes the discrete derivative, e.g.,

$$(4.27) \quad b'_i = \sum_{j=0}^N D_{ij} b_j.$$

By construction we know that at either endpoint the entropy stable flux is equal to the physical flux

$$(4.28) \quad \bar{f}_0^{SS} = f_0, \quad \bar{f}_{N+1}^{SS} = f_N.$$

For the well-balanced problem it is straightforward to show that the entropy stable flux (4.21) simplifies to become the entropy conserving flux (3.4), i.e.,

$$(4.29) \quad \bar{f}_i^{SS} = \bar{f}_i^{(S)} = \sum_{k=i}^N \sum_{l=0}^{i-1} 2 Q_{lk} f_2^{ec}(u_l, u_k), \quad i = 1, \dots, N,$$

where Q_{lk} is the $(l, k)^{\text{th}}$ component of the summation-by-parts matrix (2.9) and f_2^{ec} is the second component of the entropy conserving flux (2.7)

$$(4.30) \quad \begin{aligned} f_2^{ec}(u_l, u_k) &= \llbracket v_{lk} \rrbracket^2 \llbracket h_{lk} \rrbracket + \frac{g}{2} \llbracket h_{lk}^2 \rrbracket, \\ &= 0 + \frac{g}{2} \left(\frac{h_l^2 + h_k^2}{2} \right), \\ &= \frac{g}{4} (h_l^2 + h_k^2). \end{aligned}$$

We next directly show that $(\mathcal{J}\dot{u}_2)_0 = 0$ in (4.26). Showing that the terms $(\mathcal{J}\dot{u}_2)_i$ for $i = 1, \dots, N$ are also zero uses the same method and makes repeated use of the fact that, excluding the boundary components, the summation-by-parts matrix $\underline{\underline{Q}}$ is skew-symmetric. We begin with the $i = 0$ term

$$(4.31) \quad (\mathcal{J}\dot{u}_2)_0 = - \left[\frac{\bar{f}_1^{SS} - \bar{f}_0^{SS}}{\omega_0} - \frac{f^{*,ec}|_{-1} - f_0}{\omega_0} + gh_0 b'_0 + \frac{g}{2} (2h_0 h'_0 - (h^2)'_0) \right].$$

From the relationship (4.28) we reorganize (4.31) to obtain

$$(4.32) \quad (\mathcal{J}\dot{u}_2)_0 = - \left[\frac{\bar{f}_1^{SS} - f^{*,ec}|_{-1}}{\omega_0} + gh_0 (h_0 + b_0)' - \frac{g}{2} (h^2)'_0 \right].$$

We assumed that the bottom topography is approximated smoothly across grid cell interfaces. If we also take the initialization of the quantity h to be continuous across grid cell interfaces, then we know for the well-balanced problem (4.24) that

$$(4.33) \quad gh_0(h_0 + b_0)' = 0,$$

due to the consistency of the derivative matrix $\underline{\underline{D}}$ in the sense that the derivative of a constant is exactly zero. So, (4.32) becomes

$$(4.34) \quad (\mathcal{J}\dot{u}_2)_0 = - \left[\frac{\bar{f}_1^{SS} - f^{*,ec}|_{-1}}{\omega_0} - \frac{g}{2} (h^2)'_0 \right].$$

To show that $(\mathcal{J}\dot{u}_2)_0$ vanishes, we must expand each term in (4.34) to see the explicit cancellations.

To demonstrate these explicit cancellations we use several properties of the SBP matrix (2.9). From the definition of the matrix $\underline{\underline{Q}}$ and the consistency of the derivative matrix $\underline{\underline{D}}$ we know that the rows of the matrix $\underline{\underline{Q}}$ sum to zero, see, for example, [10]. Also, the SBP matrix is nearly skew-symmetric, a property we make use of below. So, for clarity, we illustrate the structure of the SBP matrix $\underline{\underline{Q}}$ for $N = 5$:

$$(4.35) \quad \underline{\underline{Q}} = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} & Q_{03} & Q_{04} \\ Q_{10} & Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{20} & Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{30} & Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{40} & Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & Q_{01} & Q_{02} & Q_{03} & Q_{04} \\ -Q_{01} & 0 & Q_{12} & Q_{13} & Q_{14} \\ -Q_{02} & -Q_{12} & 0 & Q_{23} & Q_{24} \\ -Q_{03} & -Q_{13} & -Q_{23} & 0 & Q_{34} \\ -Q_{04} & -Q_{14} & -Q_{24} & -Q_{34} & \frac{1}{2} \end{bmatrix}.$$

First, we expand the entropy stable flux component of (4.34) to find

$$(4.36) \quad \begin{aligned} \bar{f}_1^{SS} &= \sum_{k=1}^N 2Q_{0k} f_2^{ec}(u_0, u_k), \\ &= 2Q_{01} \left(\frac{g}{4}(h_0^2 + h_1^2) \right) + \cdots + 2Q_{0k} \left(\frac{g}{4}(h_0^2 + h_k^2) \right) + \cdots + 2Q_{0N} \left(\frac{g}{4}(h_0^2 + h_N^2) \right), \\ &= \frac{g}{2} \left\{ \frac{h_0^2}{2} + 2Q_{01} \frac{h_1^2}{2} + \cdots + 2Q_{0N} \frac{h_N^2}{2} \right\}, \end{aligned}$$

where we used the simplified formula (4.30) for the entropy conserving flux. Also, the first row of the SBP matrix sums to zero, so from (4.35) we manipulate to determine

$$(4.37) \quad 2Q_{01} + 2Q_{02} + \cdots + 2Q_{0N} = 1.$$

Next we expand the boundary term

$$(4.38) \quad \begin{aligned} f^{*,ec}|_{-1} &= f^{ec}|_{-1}, \\ &= \frac{g}{4}(h_0^2 + h_N^{2,L}), \end{aligned}$$

where the L indicates that we use the value of h_N from the left neighboring grid cell. Finally, we expand the derivative of h^2 to find

$$\begin{aligned}
 (h^2)'_0 &= \sum_{j=0}^N D_{0j} \underline{h}^2, \\
 &= \frac{1}{\omega_0} \sum_{j=0}^N Q_{0j} \underline{h}^2, \\
 &= \frac{1}{\omega_0} (Q_{00} h_0^2 + Q_{01} h_1^2 + \cdots + Q_{0N} h_N^2), \\
 &= \frac{1}{\omega_0} \left(-\frac{h_0^2}{2} + Q_{01} h_1^2 + \cdots + Q_{0N} h_N^2 \right),
 \end{aligned}
 \tag{4.39}$$

as we know that $Q_{00} = -1/2$. Now we combine the results (4.36), (4.38), and (4.39) to show that the time derivative of the momentum equation vanishes

$$\begin{aligned}
 \mathcal{J} \dot{u}_{2,0} &= - \left[\frac{\bar{f}_1^{SS} - f^{*,ec}|_{-1}}{\omega_0} - \frac{g}{2} (h^2)'_0 \right], \\
 &= -\frac{g}{4\omega_0} h_N^{2,L} + \frac{g}{2\omega_0} (Q_{01} h_1^2 + \cdots + Q_{0N} h_N^2) - \frac{g}{2\omega_0} \left(-\frac{h_0^2}{2} + Q_{01} h_1^2 + \cdots + Q_{0N} h_N^2 \right), \\
 &= -\frac{g}{4\omega_0} h_N^{2,L} + \frac{g}{4\omega_0} h_0^2, \\
 &= -\frac{g}{4\omega_0} (C - b_N^L)^2 + \frac{g}{4\omega_0} (C - b_0)^2, \\
 &= 0.
 \end{aligned}
 \tag{4.40}$$

■

Remark 3. (Well-Balanced, Standard DGSEM) We note that the same correction term s_{h^2} can be used to render a standard formulation of the strong-form GL-DGSEM

$$\begin{aligned}
 J \partial_t \underline{u}_1 + \underline{\underline{D}} \underline{f}_1 &= \underline{\underline{S}} \left[\underline{f}_1^* - \underline{f}_1 \right], \\
 J \partial_t \underline{u}_2 + \underline{\underline{D}} \underline{f}_2 + g \underline{\underline{h}} \underline{\underline{D}} \underline{b} &= \underline{\underline{S}} \left[\underline{f}_2^* - \underline{f}_2 \right],
 \end{aligned}
 \tag{4.41}$$

well-balanced.

Remark 4. (Stability Not Guaranteed) The s_{h^2} correction to recover well-balancedness causes the standard DGSEM and SSSCE-DG to lose provable entropy stability. We see this immediately by converting the problem to entropy variables, including the correction term. For the SSSCE-DG we find

$$\partial_t E(\underline{u}) + \partial_x F(\underline{u}) + v s_{h^2} = 0,
 \tag{4.42}$$

and for the standard DGSEM

$$\partial_t E(\underline{u}) + \partial_x F(\underline{u}) - v s_{hv^2} = 0.
 \tag{4.43}$$

In either case, we know that the sign of the remaining term scaled by the velocity in (4.42) or (4.43) is unknown. Therefore, it is no longer guaranteed that the update to the entropy is negative.

Thus, the entropy inequality near a shock (4.5) may be violated. It is unclear to the authors at this time if there exists a simple correction term that recovers well-balancedness and maintains provable entropy stability.

5. NUMERICAL RESULTS

Now, we compare two, provably stable approximation techniques: the ESDGSEM and the SSSCE-DG *without* the well-balanced correction term from Sec. 4.3. To draw meaningful conclusions about the two methods we consider shock problems as well as several perturbations from the “lake at rest” condition with non-constant bottom topographies. Each method uses its respective spatial discretization and the low storage five stage fourth order accurate Runge-Kutta time integrator of Carpenter and Kennedy [5].

We also provide verification that the ESDGSEM and the SSSCE-DG schemes are high-order using a smooth, exact solution for the shallow water equations. For each numerical example, unless otherwise noted, we set the gravitational constant to $g = 1$.

5.1. Convergence of the ESDGSEM and the SSSCE-DG. We consider a test problem with an exact solution to verify the high-order accuracy of both entropy stable schemes. Specifically, we take the smooth functions

$$(5.1) \quad h(x, t) = 2 + \cos(x) \cos(t), \quad v(x, t) = \frac{\sin(x) \sin(t)}{h},$$

to be the analytical solution of the shallow water equations with the bottom topography

$$(5.2) \quad b_1(x) = \begin{cases} \sin\left(\frac{\pi x}{4}\right) & \text{if } |x - 10| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the solution (5.1) has a vanishing source term for the continuity equation, and for the momentum equation we have an additional source term on the right hand side

$$(5.3) \quad s(x, t) = \sin(x) \cos(t) + \frac{v \sin(t)(2 \cos(t) \cos^2(x) + \cos(t) \sin^2(x) + 4 \cos(x))}{h^2} - h \cos(t) \sin(x).$$

The physical domain is taken to be $\Omega = [0, 20]$. The exact solution (5.1) is used to prescribe the initial condition and the boundary conditions for the convergence test.

In Tables 1 and 2 respectively present for the ESDGSEM and SSSCE-DG of a grid convergence study for an even and odd choice of the polynomial degree. For the convergence test we fix the time step at $\Delta t = 1/200$ to ensure the error is dominated by the spatial approximation. As observed in previous work [13, 14, 15], the stabilized entropy conserving scheme obtains the optimal convergence order for odd ($N = 3$) and even ($N = 4$) tests. We note that if we use the entropy conserving version of either scheme the convergence rate will remain optimal for even polynomial degrees but becomes suboptimal for odd polynomial degrees, see Fisher et al. [10] for details.

Fig. 2 shows exponential convergence of the spatial approximation with 10 elements at $T = 1.0$ until $N = 24$ for the ESDGSEM and $N = 30$ for the SSSCE-DG, where time integrator errors become dominant. For both schemes we find when the value of Δt is halved, the error in the approximation is reduced by a factor of 16, as expected for the fourth order time integration scheme used.

# elements	L_∞ error	EOC	# elements	L_∞ error	EOC
$N = 3$			$N = 4$		
20	9.18E-04	—	20	7.67E-05	—
40	3.72E-05	4.63	40	1.53E-06	5.65
80	2.32E-06	4.00	80	4.78E-08	5.00
160	1.52E-07	3.93	160	1.48E-09	5.01

TABLE 1. Experimental order of convergence (EOC) of the ESDGSEM for $N = 3$ and $N = 4$.

# elements	L_∞ error	EOC	# elements	L_∞ error	EOC
$N = 3$			$N = 4$		
20	2.57E-03	—	20	1.81E-04	—
40	1.63E-04	3.98	40	6.00E-06	4.91
80	9.57E-06	4.09	80	1.64E-07	5.19
160	5.77E-07	4.05	160	4.53E-09	5.18

TABLE 2. Experimental order of convergence (EOC) of the SSSCE-DG for $N = 3$ and $N = 4$.

FIGURE 2. (left): Semi-log plot shows the spectral convergence of the ESDGSEM scheme applied to a smooth solution. (right): Semi-log plot shows the spectral convergence of the SSSCE-DG scheme applied to a smooth solution.

5.2. Dam Break and Shock Capturing. Now, we compare the two entropy stable schemes for a dam break problem on a flat bottom topography $b = 0$. We will see that the skew-symmetric

ESDGSEM and the SSSCE-DG schemes produce similar results for a dam break problem

$$(5.4) \quad h(x, 0) = \begin{cases} 2.0, & \text{if } x < 10 \\ 1.5, & \text{if } x > 10 \end{cases}, \quad v(x, 0) = 0,$$

on the domain $\Omega = [0, 20]$ divided into a grid of 50 uniform grid cells. We take the polynomial order $N = 3$ in each grid cell and integrate the solution up to the final time $T = 1.5$. We use the dam break problem to make explicit comparisons between the entropy stable scheme of Carpenter et al. [4] and the skew-symmetric scheme of [15]. In all the plots to follow a solid line indicates a reference solution, a line marked with triangles indicates the ESGSEM and a line marked with diamonds indicates the SSSCE-DG.

First we consider the problem that is stabilized only by the boundary flux $f^{*,es}$ (4.17) that contains minimal dissipation. There is no interior stabilization. This numerical example highlights similarities of the two methods and demonstrate their near equivalence. We present the computations with detail near the shock and rarefaction regions in Fig. 3. The computed solution for both schemes are identical, in a visual sense, in the pre-shock and expansion regions, the right plot in Fig. 3. From the left of Fig. 3 we see that the ESGSEM and SSSCE-DG both resolve the shock well and generate similarly sized overshoots. We see in the shocked region that the ESGSEM is slightly more dissipative than the SSSCE-DG.

FIGURE 3. The solution of the dam break problem (5.4) at $T = 1.5$ for the ESGSEM (triangles) and the SSSCE-DG (diamonds). The solid line is a reference solution. (*left*): Zoomed comparison of the shocked region. (*right*): Zoomed comparison of the pre and post regions.

Next, the interior stabilization penalty terms for both schemes, discussed in Sec. 4.2, are also included. This numerical example demonstrates a small reduction in overshoot size, but we again see that the dissipation added is minimal. The detailed plots are presented in Fig. 4. We see that the SSSCE-DG has become more dissipative in the pre-shock region. But, as seen on the left in

Fig. 4, the ESDGSEM with interior stabilization (4.22) widens the shock and is more dissipative in the shocked region than the interior stabilized SSSCE-DG.

FIGURE 4. The solution of the dam break problem (5.4) at $T = 1.5$ for the ESDGSEM (triangles) and the SSSCE-DG (diamonds) both with interior entropy stabilization. The solid line is a reference solution. (*left*): Zoomed comparison of the shocked region. (*right*): Zoomed comparison of the pre and post regions.

The results for the entropy stable approximations with interior stabilization in Fig. 4 are quite similar to the computations with entropy stability provided only at element interfaces in Fig. 3. This is often the case for weak shock problems that the interior stabilization offers little. However, when testing stronger shock problems the additional stabilization helps to stabilize the computation and has a more direct impact on the results and shock front sharpness.

5.3. Well-Balancedness for Moving Water Equilibrium. In addition to the “lake at rest” conditions (1.3), the shallow water equations admit another class of steady state solutions often referred to as a general moving water equilibria [16, 24]:

$$(5.5) \quad m := hv = \text{constant}, \quad E := \frac{1}{2}v^2 + g(h + b) = \text{constant},$$

where m and E are the moving water equilibrium variables [16]. There are several possible configurations of the moving water steady state: subcritical, subcritical with a discontinuity, and supercritical depending on the value of the Froude number

$$(5.6) \quad Fr := \frac{|v|}{\sqrt{gh}},$$

complete details can be found in [8, 16]. The “lake at rest” conditions (1.3) (or still water steady state) is a special case of the moving water equilibrium. Preservation of the moving water equilibrium is more difficult than the still water equilibrium and typically requires a special discretization of the source term, see for example [12, 31].

We consider the subcritical moving water equilibrium test case for both entropy stable approximations. This steady state problem is a widely used classical test case for numerical schemes that approximate shallow water equations [12, 24, 32, 31]. For the subcritical test we consider the bottom topography

$$(5.7) \quad b_2(x) = \begin{cases} 0.2 - 0.05(x - 10)^2 & \text{if } |x - 10| \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

on the physical domain $\Omega = [0, 25]$ with $g = 9.812$. The initial condition for the subcritical steady state are

$$(5.8) \quad m = 4.42, \quad E = 22.06605,$$

together with the boundary condition of $m = 4.42$ at the upstream, and $h = 2$ at the downstream. To convert the initial condition from the equilibrium variables to the conserved variables we apply a Newton method to solve for the water height h in the formulation (5.5), for complete details see [24].

Previously, it was demonstrated that the ECDGSEM scheme maintains the well-balanced property for a moving water subcritical steady state problem with bottom topography $b_2(x)$ [15]. The same remains true for the ESDGSEM. We compute the solution of the subcritical steady state up to $T = 5$ using 100 elements with polynomial order $N = 6$ or $N = 7$ in each element. We show in Tbl. 3 the L_∞ error in the approximation of the water height h and the discharge hv to demonstrate that the steady state is maintained up to numerical round-off error. This verifies the desired well-balanced property for the subcritical moving water equilibrium configuration. The water height for the subcritical moving water steady state is shown in Fig. 5.

	h	hv
$N = 6$	6.41E-13	5.98E-13
$N = 7$	3.21E-14	2.75E-14

TABLE 3. The L_∞ error of the ESDGSEM for the water height h and discharge hv for the subcritical moving water equilibrium (5.8) at time $T = 5$ simulated with 100 elements.

Next, we approximate the subcritical moving water test (5.8) with the SSSEC-DG scheme. The parameters of the computation are identical to those used for the ESDGSEM run. Though the SSSEC-DG scheme is not well balanced for the still water equilibrium, it is well-balanced for this particular moving water configuration. The L_∞ errors of the computation are given in Tbl. 4. We see that the errors in the water height and discharge are, again, on the order of machine precision.

The well-balanced property for the “lake at rest” condition appears unimportant for the SSSEC-DG scheme to recover the subcritical moving water steady state. However, in Secs. 5.4 and 5.5 we will demonstrate that the lack of well-balancedness of the SSSEC-DG scheme will lead to significant spurious oscillations, generated by the bottom topography, in steady water regions.

5.4. Perturbation of “Lake at Rest”. We know analytically and have verified numerically that the ESDGSEM is a well-balanced approximation [15]. The recovery of the “lake at rest” solution (1.3) is of particular importance when the shallow water equations are used to model rivers and

FIGURE 5. The computed water height for the subcritical moving water equilibrium test problem at time $T = 5$.

	h	hv
$N = 6$	9.11E-13	8.00E-13
$N = 7$	4.25E-14	2.05E-14

TABLE 4. The L_∞ error of the SSSEC-DG for the water height h and discharge hv for the subcritical moving water equilibrium (5.8) at time $T = 5$ simulated with 100 elements.

coastal flows [2]. A good numerical method for the shallow water equations should accurately capture both the steady states and their small perturbations (quasi-steady flows).

With the importance of capturing waves in a quasi-steady flow in mind, we consider a small perturbation of the “lake at rest” problem given by

$$(5.9) \quad h(x, 0) = \begin{cases} 1.01 - b_1(x), & \text{if } |x - 6| < 0.25 \\ 1.00 - b_1(x), & \text{otherwise} \end{cases}, \quad v(x, 0) = 0,$$

with bottom topography (5.2). Because the perturbations are very small, they will not be clearly visible in a plot showing both the height and the bottom topography. Therefore we provide a close

up of the computed, dashed line, and exact, solid line, of the left and right traveling perturbations in Fig. 6.

We run the computation with periodic boundary conditions on the domain $\Omega = [0, 20]$ divided into 50 uniform grid cells with $N = 3$ in each. The final time is $T = 1.5$ and the bottom topography (5.2). Hence, the perturbation is a very small disturbance of the lake at rest and we seek to study how this disturbance propagates in time. A closeup of the resulting height is shown at left in Fig. 6. This closeup clearly shows that the ESDGSEM is able to approximate both waves. This is a consequence of its ability to preserve the steady state. There are oscillations trailing both left and right going waves; again, this is to be expected. The entropy stable numerical flux (4.17) adds the minimal dissipation needed for stability, but does not provide the dissipation required to achieve non-oscillatory shocks. This example and findings are analogous to one presented in [12]. On the right of Fig. 6 we see that the SSSCE-DG scheme can approximate the waves; however, the solution includes small spurious waves generated from the non-constant bottom topography.

FIGURE 6. (*left*): Closeup of the ESDGSEM computation of left and right traveling computed (circles) and exact (solid) waves generated by a perturbed “lake at rest” initial condition at $T = 1.5$. (*right*): Closeup of the SSSCE-DG computation of the “lake at rest” perturbation. Note the small spurious waves generated by the bottom topography.

5.5. Small Perturbations of Steady Water. The following quasi-stationary test case was proposed by LeVeque [22]. It was chosen to demonstrate the capability of a proposed scheme to compute the solution of a rapidly varying flow over a smooth bed, and the perturbation of a stationary state.

We consider the domain $\Omega = [0, 2]$ with the bottom topography

$$(5.10) \quad b_3(x) = \begin{cases} 0.25(\cos(10\pi(x - 1.5)) + 1), & \text{if } |x - 1.5| < 0.1, \\ 0, & \text{otherwise.} \end{cases}$$

and initial conditions

$$(5.11) \quad h(x, 0) = \begin{cases} 1 - b_3(x) + \varepsilon, & \text{if } |x - 1.15| < 0.05 \\ 1 - b_3(x), & \text{otherwise} \end{cases}, \quad v(x, 0) = 0,$$

where ε is a non-zero perturbation constant.

We run two cases $\varepsilon = 0.2$ (big pulse) and $\varepsilon = 10^{-3}$ (small pulse). Theoretically, for small ε , the disturbance should split into two waves, propagating left and right at the characteristic speeds $c = \pm\sqrt{gh}$. Many numerical methods have difficulty with the calculations involving such small perturbations of the water surface [22].

For each run we consider the periodic problem on the domain $\Omega = [0, 2]$ divided into 50 uniform grid cells with polynomial order $N = 4$ in each. We run the computation up to a final time of $T = 0.6$. At this point in time the right-traveling wave has been influenced by the bottom topography and leaves a slight wake. Capturing this wake accurately can be difficult and for schemes which are not well-balanced the solution may become unstable [22, 33]. We present the results of the ESDGSEM for $\varepsilon = 0.2$ on the left and $\varepsilon = 10^{-3}$ on the right in Fig. 7. An analogous use of this example can be found in [33]. The solid black line in each plot show the results of a high resolution computation.

FIGURE 7. (*left*): The surface level $h + b$ for LeVeque's quasi-steady state problem with $\varepsilon = 0.2$ at $T = 0.6$. (*right*): A closeup of the surface level $h + b$ for LeVeque's quasi-steady state problem with $\varepsilon = 10^{-3}$ at $T = 0.6$.

We perform the same tests on the SSSCE-DG. We see in Fig. 8 that the non-well-balanced scheme produces significant errors in the wake region of the bottom topography. This is particularly true for the small pulse problem.

5.6. Dam Break Over a Bump. Next, we simulate the dam break problem over the sine bump (5.2). The problem will exhibit a shallow, stagnant downstream layer and a propagating surge front

FIGURE 8. (*left*): The surface level computed by the SSSCE-DG for the big pulse problem. There are visible errors in the wake region. (*right*): Surface level for the small pulse problem exhibits significant spurious waves generated by the non-well-balanced approximation.

[1]. The bottom topography influences the shape solution significantly, in particular causing the stagnant region immediately in front of the topography. Similar results may be found in, e.g., [1].

This problem highlights a difficulty one may encounter in shallow water equation simulations where the water height h is near the bottom topography. If oscillations in the solution are too large they may produce a non-physical negative water height. For both entropy stable schemes the computation will crash if a negative water height is encountered. In fact, a negative value for h is problematic when computing the eigenvalues $\lambda = v \pm \sqrt{gh}$, which renders the system not hyperbolic and not well posed [33].

We consider the initial condition

$$(5.12) \quad h(x, 0) = \begin{cases} 1.60 - b_1(x), & \text{if } x \leq 10 \\ 1.05 - b_1(x), & \text{if } x > 10 \end{cases}, \quad v(x, 0) = 0.$$

We compute the solution to the dam break problem with outflow and inflow boundary conditions on the domain $\Omega = [0, 20]$ up to a final time of $T = 4.5$ on a uniform grid of 100 cells with polynomial order $N = 3$ in each cell. The left plot of Fig. 9 shows the surface level $h + b$ and the bottom topography. The change in the rarefaction region is clearly visible as well as the stagnant layer. The right plot of Fig. 9 shows the computed velocity where the stagnant layer can be easily identified. The solid black line in each plot show the results of a high resolution computation.

When we apply the SSSCE-DG to the dam break over a bump problem we encounter unstable behavior. Because the approximation is not well-balanced and the water height is very near the bottom topography, the calculation becomes unstable because the spurious oscillations cross the bottom topography, i.e., h becomes negative. The computation run with SSSCE-DG including the correction to restore well-balancedness (2.6) is also unstable. The lack of provable stability causes

FIGURE 9. (*left*): The surface level $h + b$ and bottom topography for the dam break problem at $T = 4.5$. (*right*): The computed velocity v .

the computation to crash. Again, a negative water height is invalid under the assumptions of the shallow water equations model [12, 33].

5.7. Water Flow Near a Shallow Region. As a final numerical example we consider a modification of a test case proposed by de Boer [8] designed to test the ability of a method to handle shallow water depths. The initial conditions are

$$(5.13) \quad h(x, 0) = 1.1 - b_1(x), \quad v(x, 0) = \begin{cases} -5.25, & \text{if } x \leq 5 \\ 5.25, & \text{if } x > 5 \end{cases},$$

on the domain $\Omega = [0, 20]$ with the sine bottom topography (5.2) placed at the center. The solution consists of two rarefaction waves traveling in opposite directions. The right rarefaction wave interacts with the bottom topography creating a large pulse on the upwind side of the bump and a steepening stagnation region on the downwind side of the bump along with a region of expansion to the original water height. As time progresses this stagnation region becomes very close to the bottom topography.

We compute the solution to the two rarefaction problem with periodic boundary conditions up to the final time $T = 2.3$ such that the waves do not interact with the boundary. The spatial discretization uses a uniform grid of 100 cells with polynomial order $N = 3$ in each cell. We present the initial setup and the computed solution found with the ESDGSEM scheme in Fig. 10. The solid black line in each plot shows a reference solution created with a high resolution computation.

When we apply the SSSCE-DG to the two rarefaction test problem we encounter unstable behavior near the end of the computation corresponding to when the stagnation region is close to the bottom topography. As the computed solution becomes close to the bottom spurious oscillations generated by the lack of well-balancedness cause negative water heights and the computation crashes.

FIGURE 10. (*left*): The initial configuration of $h + b$ for the two rarefaction propagation over near a bottom topography. (*right*): The computed water height at the time $T = 2.3$. The black line is a reference solution.

6. CONCLUSIONS

In this work we compare the provable stability and well-balancedness of two entropy conserving, high-order DG approximations for the shallow water equations. We found that the base entropy conservative algorithm of Carpenter et al. applied to the shallow water equations with entropy conserving/stable numerical fluxes is remarkably robust despite large oscillations at flow discontinuities. Also, when entropy stabilization was included at interior nodes the amount of dissipation added appeared optimal, in an entropy preserving sense. As we found in numerical experiments the entropy stabilized skew-symmetric scheme was slightly more dissipative and widened shocks more so than the algorithm of Carpenter et al.

Unfortunately, the procedure of Carpenter et al. is not well-balanced. It was found that for a certain moving water steady state equilibrium the lack of well-balancedness was insignificant. The ESDGSEM and SSSEC-DG schemes could recover the moving water steady state to machine precision. However, we found in numerical examples that the lack of well-balancedness of the SSSEC-DG scheme leads to significant errors in quasi-steady state computations. Also, for demanding test cases with low water height we found that the lack of well-balancedness lead to unstable behavior for the SSSEC-DG scheme. We note that the method was not designed for the solution of the shallow water equations, so a lack of well-balancedness is not surprising. Through numerical experimentation we found that a scheme that fails to maintain the “lake at rest” condition generates spurious waves near the bottom topography which propagate through the domain. The correction term found by the authors to maintain well-balancedness causes the loss of provable stability for the scheme. It is an open question of can a simple correction term be found which also maintains the provable entropy stability.

Inspired by the work of Carpenter et al., we explored stability for a skew-symmetric, entropy conserving scheme designed for the shallow water equations. We showed provable entropy stability,

but with the additional feature of well-balancedness for an arbitrary, continuous bottom topography. We applied this scheme to several known, challenging quasi-steady state and shock problems to demonstrate the scheme's robustness and ability to model complex phenomena in the shallow water equations.

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