Estimation of fatigue life by using a cyclic plasticity model and multiaxial notch correction

Master Thesis in Solid Mechanics

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Abstract

Mechanical components often possess notches. These notches give rise to stress concentrations, which in turn increases the likelihood that the material will undergo yielding. The finite element method (FEM) can be used to calculate transient stress and strain to be used in fatigue analyses. However, since yielding occurs, an elastic-plastic finite element analysis (FEA) must be performed. If the loading sequence to be analysed with respect to fatigue is long, the elastic-plastic FEA is often not a viable option because of its high computational requirements.

In this thesis, a method that estimates the elastic-plastic stress and strain response as a result of input elastic stress and strain using plasticity modelling with the incremental Neuber rule has been derived and implemented. A numerical methodology to increase the accuracy when using the Neuber rule with cyclic loading has been proposed and validated for proportional loading. The results show fair albeit not ideal accuracy when compared to elastic-plastic finite element analysis. Different types of loading have been tested, including proportional and non-proportional as well as complex loadings with several load reversions.

Based on the computed elastic-plastic stresses and strains, fatigue life is predicted by the critical plane method. Such a method has been reviewed, implemented and tested in this thesis. A comparison has been made between using a new damage parameter by Ince and an established damage parameter by Fatemi and Socie (FS). The implemented algorithm and damage parameters were evaluated by comparing the results of the program using either damage parameter to fatigue experiments of several different load cases, including non-proportional loading. The results are fairly accurate for both damage parameters, but the one by Ince tend to be slightly more accurate, if no fitted constant to use in the FS damage parameter can be obtained.

Keywords: Multiaxial load, fatigue, notch, elastic-plastic, stress-strain, non-proportional load, incremental Neuber, damage parameter
Preface

This master thesis has been carried out at the division of Solid Mechanics at the University of Linköping during the spring semester of 2019. The study has been performed in cooperation with Volvo Construction Equipment.

I would like to express my deepest gratitude to my two supervisors, Carl-Johan Thore at Linköping University and Magnus Andersson at Volvo Construction Equipment. The feedback and advice they gave me was vital in order to reach an adequate result.
Abbreviations

BS Bannantine-Socié
FS Fatemi-Socié
FEA Finite Element Analysis
FEM Finite Element Method
LRS Local Reference System
LC Load Criterion

Nomenclature

\( b \) Fatigue strength exponent
\( C \) Hardening constant
\( c \) Fatigue ductility exponent
\( D \) Material damage
\( E \) Elastic/Young’s modulus
\( e \) Deviatoric strain
\( \Delta e \) Deviatoric strain increment
\( \Delta e^a \) Deviatoric elastic-plastic/actual strain increment
\( \Delta e^{ae} \) Deviatoric elastic-plastic/actual strain increment
\( \Delta e^e \) Deviatoric elastic/pseudo-elastic strain increment
\( \Delta e^{e0} \) Deviatoric elastic/pseudo-elastic strain increment
\( F \) Yield function
\( F_1 \) Active hardening surface
\( F_2 \) Target hardening surface
\( G \) Shear modulus
\( H \) Plastic modulus
\( K' \) Cyclic strength coefficient
\( N_f \) Number of cycles to failure
\( n \) Number of cycles
\( n' \) Cyclic strain hardening exponent
\( R \) Size of hardening surface
\( S \) Deviatoric stress
\( \Delta S \) Deviatoric stress increment
\( \Delta S^a \) Deviatoric elastic-plastic/actual stress increment
\( \Delta S^{ea} \) Deviatoric elastic-plastic/actual stress increment
\( \Delta S^e \) Deviatoric elastic/pseudo-elastic stress increment
\( \Delta S^{e0} \) Deviatoric elastic/pseudo-elastic stress increment
\( \Delta S^{e0} \) Deviatoric elastic/pseudo-elastic stress reference system
\( \Delta S^{e0} \) Deviatoric elastic-pseudo-elastic stress reference system
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1. Introduction

This chapter covers the background of the problem at hand as well as a strategy to find a suitable solution.

1.1 Problem formulation

Components used in many different applications often has notches with small radii, e.g. if the component is manufactured by welding, a notch will appear at the weld toe as seen in fig. 1. These areas are likely to be subjected to loading that induce plastic flow during service, since stress concentrations are present. Strength and fatigue analyses are often focused on these areas since they will be critical to the structural integrity of the component. The loading of components is also often multiaxial, such as combined bending and torsion [1]. Currently, often when designing with respect to fatigue life, multiaxial load histories are used with Wöhler curves to calculate fatigue life. Wöhler, or S-N curves where S represents stress amplitude and N represents the number of load cycles to failure, relates an input load to a prediction of life expectancy of a component. This method does not take the elasto-plastic material behaviour, e.g. relaxation effects, into account, nor is the effect of mean stress correction considered, which are known to affect life prediction [2,3]. The inaccuracy of the current method may cause designers to oversize components and critical areas, i.e. making conservative design choices. It is common that welded component could be optimized with respect to the weld joints without affecting fatigue properties [4]. In turn this would lead to reductions of manufacturing costs. [5]

Figure 1: CAD-model of a component welded to a frame.
Using a method that does that is able to incorporate mean stress correction and relaxation effects would be more exact than using Wöhler curves. One alternative to achieve this is to perform a transient elastic-plastic, non-linear, Finite Element Analysis. Elastic-plastic FEA’s are able to accurately predict elastic-plastic response from time-dependent loadings. The resultant stress-strain histories could then be used in a rainflow cycle scheme and the damage of each cycle could be calculated by using a so-called damage parameter. However, for long time histories this is not a viable option since the computation would require large amounts of time and/or memory.

Less computationally intense methods to calculate stress and strain have been presented by Singh [6] as well as stress and strain and fatigue calculation by Ince [7] and Gates [8]. These methods consist of one part where approximate elasto-plastic stress-strain histories are calculated in a simplified way from pseudo-elastic stress and strain, and one part where the approximate stress-strain curves are used to calculate fatigue life. Pseudo-elastic stress and strain are results of performed FEA’s where the response is always linear-elastic.

By applying the described method to compute transient elastic-plastic stress and strain, critical points, or nodes, in the performed linear-elastic FEA would be identified and its computed stress and strain history corrected. This is achieved in a numerical context by finding all elements that are subjected to stress levels above the yield limit, i.e. at some point undergoes plastic flow. The actual, elastic-plastic stress and strain would then be computed by using the mathematical model for these nodes. In general, these nodes are located at notches of the structure, and subjected to stress concentrations. These locations are also those that should be used in fatigue analysis, since cracking and failure are most likely to originate here.

The method found in Singh’s dissertation was originally posted in 1998, as well as Ince’s in 2013. Since then, slight modifications have been made, by Ince and others [9-14], but the computational model is essentially the same, in that they compute elastic-plastic stress and strain from pseudo-elastic stress and strain and later compute multiaxial fatigue by employing a damage parameter.

The objective of this thesis is to create a mathematical model that acts like a function which predicts fatigue life as a function of the input pseudo-elastic stress histories. To this end, understanding the theory required to create such a model is necessary. The model is able to incorporate the effects of material plasticity which results in better fatigue life predictions than using the conventional method. Verification of the mathematical model is to be performed for different load cases which are likely to occur in a real application.

The aim is to be able to obtain higher accuracy when conducting strength and fatigue calculations of structures possessing notches. This in turn could lead to less material being used, e.g. shorter, fewer or smaller welds. Less required material would imply cost reductions when manufacturing the components prone to fatigue [5].
1.2 Approach

In this study, existing mathematical models of material mechanics are coupled to calculate stress and strain at notches. The chosen method is used to approximate elastic-plastic stress and strain response and in turn fatigue life. Finally, numerical implementation of the mathematical model is performed in the numerical programming tool Matlab.

To evaluate the resulting algorithm, approximate elastic-plastic stress-strain histories are compared to stress and strain histories from performing transient elastic-plastic stress analysis in the finite element program Ansys. The fatigue life predictions are compared and evaluated against experimental results where the stress and strain histories are known.

The response of unit loads of finite element analyses will not be computed in this thesis, neither will any load histories be measured. The input to this thesis is instead given pseudo-elastic stress histories. Only material zones at notches will be considered, the model presented will not be valid for any arbitrary zone of a component. Thermal loads are not considered.
2. Modelling of elastic-plastic stress and strain behaviour

This chapter describes the mathematical relations of mechanics that are required to establish a model that may be a solution to the problem which has been formulated.

2.1 The stress – strain state

This initial section will cover general theory of stress and strain that emerges as a component is loaded.

2.1.1 Linear elasticity

Forces can be classified as either surface loads or volume forces. Surface loads are also referred to as external loads and are applied to the boundary of a material body. Volume forces are external forces, continuously distributed throughout the body. External forces give rise to internal forces, resulting in material stress. A body is elastic if it is able return to its original shape after applied forces have been removed. An elastic body subjected to some loads is shown in fig. 2, and as a result of applied loads is deformed, resulting in elastic stress and strain. The current stresses and strains of the body due to loads at any position in time is called the stress-strain state. [15]

![Figure 2: Stress state of an arbitrarily loaded body.](image)

If an imagined cut is made and the body is divided into two parts. The force applied to one part of the body is generally not in equilibrium with the external forces of the other body part. To guarantee equilibrium of forces after applying external loads, section forces arise at the cut of the body. These are called internal forces.
The intensity at some point in the cut surface is defined as

$$\lim_{A \to 0} \frac{F}{A} = t$$

Here, \( F \) is force, \( A \) is surface area and \( t \) is the traction vector. The stress state is described by use of the Cauchy axiom, where a normal vector of the cut is used. The stress tensor \( \sigma \) is usually described using normal vectors in the directions of a Cartesian coordinate system, \( x_i \), where \( i = 1, 2, 3 \). The traction on a surface with normal related to \( n = (n_1, n_2, n_3) \) is related to the stress tensor, via

$$t = \sigma \cdot n$$

Where the stress state of a body is described using the Cauchy stress tensor

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

where each component \( \sigma_{ij} i,j = 1, 2, 3 \) is related to the three surfaces parallel to the coordinate planes, depicted in fig. 3. Components of the stress vector parallel to one of the coordinate planes spanned by the coordinate axes are called shear stresses, denoted by symbols of different indexes such as \( \sigma_{13} \). Equilibrium of moments of a volume element implies that the Cauchy tensor is symmetric, so that

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{13} = \sigma_{31}, \quad \sigma_{23} = \sigma_{32}.$$  \hspace{1cm} (2)

The stress-strain state of a point in the body can be described using an infinitesimally small volume element shaped as a cube, subjected to nine force vectors where six are unique, of which three are normal and three are shear forces. This state is shown in fig 3.

Figure 3: Stress state of material volume element.
The linear strains that appear as a result of deformation of a body can be expressed by the Cauchy strain tensor

$$
\varepsilon = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix},
$$

(3)

where the components are

$$
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

where \( u_i \) is the displacement with respect to the direction \( i \). From this expression, it follows that the strain tensor, (3), is symmetric. To relate stress and strain, some constitutive model is required. For small deformations, Hooke’s law is most often applied [15]. According to Hooke’s law, the stress and strain tensors are related via

$$
\sigma = C : \varepsilon
$$

where \( C \) is the fourth order elastic modulus tensor. The operator “;” implies matrix multiplication where the result will lead to a double contraction. It has 81 components, but because of the symmetry of \( \sigma \) and \( \varepsilon \), the number of independent components reduces to 36. For an isotropic material, Hooke’s law reads in Voigt notation as

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & \nu \\
\nu & 1 - \nu & \nu \\
\nu & \nu & 1 - \nu \\
0 & 1 - 2\nu & 0 \\
0 & 0 & 1 - 2\nu \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{23}
\end{bmatrix}
$$

(4)

where \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio [12].
2.1.2 Regions at material surfaces

For material which lies at a free surface, such as a notch tip, the stress component directed outward of the surface is zero. An example of this is shown in fig. 4. Equilibrium requires that the opposing forces at the opposing surface of an infinitesimal material element must also be zero. Situations such as this are cases of *plane stress*, since one normal stress component, say $\sigma_{11}$ is zero. That leaves three stress components of eq. (1) and four strain components of eq. (3), which means that the stress and strain tensors can be simplified to [11]

$$\mathbf{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

and

$$\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}.$$  

Figure 4: Stress state of surface element with outward normal direction.

2.1.3 Total strain theory

The total strain may be viewed as a superposition of elastic and plastic strains, i.e.

$$\mathbf{\varepsilon} = \mathbf{\varepsilon}^e + \mathbf{\varepsilon}^p,$$

where $\mathbf{\varepsilon}^e$ is elastic strains and $\mathbf{\varepsilon}^p$ is plastic strains. To find the total strain, one can use Hooke’s law to calculate elastic strains, but to calculate the plastic strains, another method is required. One alternative is to use a so-called *plasticity model*, which is implemented to predict the stress-strain response for a body being subjected to an imposed stress or strain path inducing plastic flow. The plasticity model is often formulated in an incremental manner, so that actual stress-strain increments are estimated for each cycle of imposed stress or strain.
The incremental version of eq. (7) is
\[ d\varepsilon = d\varepsilon^e + d\varepsilon^p. \] (8)

To calculate the elastic part \( d\varepsilon^e \), Hooke’s law is applied. Hooke’s law, eq. (4), in incremental, index form for an isotropic material is expressed as
\[ d\varepsilon^e_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{ij} \delta_{ij} \] (9)

Where \( \delta_{ij} \) is the Kronecker delta and \( d\sigma_{ij} \) is an increment of stress. However, calculating the plastic strains of eq. (8) requires a different approach. In the following chapters, a method to calculate elastic-plastic stresses and strains by using linear-elastic stresses as a function of time is presented [16]. For zones at a surface which is the focus of this study, finding the stress-strain state at a point in time during loading, requires seven unique time-dependent equations.

2.2 Pseudo-elastic stress histories by finite element unit loads

An elastic FEA can be used to calculate stresses in an arbitrary component, including notches or not. Assuming linear elasticity and neglecting inertia, the Finite Element Method (FEM), gives the following linear relation between external load and nodal displacements:
\[ Kd(t) = F(t). \]

Here, \( F \) is the external load vector, \( K \) is the stiffness matrix, \( d \) is the displacement vector and \( t \) is time. The displacement vector may be used to compute stress as
\[ \sigma(t) = E Bd(t) \]

where \( \sigma \) is the stress vector, \( E \) is the elasticity matrix and \( B \) is the strain displacement matrix [17].

Machine components are often subjected to complex loading with large variation, so that performing FEA for every unique load case may not be viable from the aspect of time and computational power. This problem can be solved by using a property from the linear-elastic FEA. Since the relations between them are linear, superposition by several load cases can be used to find the resultant displacements and stresses.
For example, in the case of two load cases, this is described by

\[ \sigma(t) = \sigma^1(t) + \sigma^2(t) \]

where \( \sigma(t) \) is the resultant transient stress matrix from two super positioned load cases \( \sigma^1(t) \) and \( \sigma^2(t) \). Because of the linearity of the governing equations, one may perform a linear-elastic FEA using only a so-called unit load. The resultant stresses are then found by using a time dependent scalar function. This can be used to find the transient stress state for one load case, albeit with changing load size. This is described by,

\[ \sigma(t) = \sum_{i=1}^{n_{\text{unit}}} p(t)^i \sigma^i_{\text{unit}}. \]

Where \( \sigma \) is the transient stress matrix of one load case, \( p(t) \) is a time dependent scalar function and \( \sigma_{\text{unit}} \) is the resultant stress state of a unit load case. In this case, \( n_{\text{unit}} \) is the number of different unit load cases and scalar functions required to create the desired load case and \( i \) corresponds to both the time-dependent scalar value of \( p \) and the resultant stress of one FEA, \( \sigma_{\text{unit}} \). Superposition and scaling of several unit load cases is depicted in fig. 5.

![Figure 5: Scaling and superposition of linear FEA's.](image)

The large drawback of this method is that results of linear-elastic FEA are only valid for stress levels up to the yield limit of the material. Therefore, the stress histories are called pseudo-elastic when the linear-elastic stress transcend the yield limit. To correct the result by using this method, the stresses must be calculated by using a plasticity model [11].
2.3 Plasticity modelling

As a metallic specimen is loaded such that the stress exceeds the yield limit, subsequent loading will induce permanent deformation. These deformations are known as plastic deformation. Many methods to model plasticity are available; however, they share a few properties. Namely that they consist of a yield function, a flow rule and a hardening rule. Incremental plasticity modelling can be used to analyse material behaviour that is load path dependent [11,18].

2.3.1 Yield function

When modelling material behaviour, it is necessary to be able to determine when plasticity occurs and when it does not, i.e. the loading is elastic. By introducing a yield criterion, this distinction can be made. A common criterion for isotropic materials is the von Mises yield criterion, which states that plastic loading occurs when

\[ F(\sigma, \sigma_y) = \sigma_{eq}(\sigma) - \sigma_y = 0 \]  \hspace{1cm} (10)

Here \( \sigma_{eq} \) is the von Mises equivalent stress and \( \sigma_y \) is the yield limit of the material in question. When the value of the yield function, \( F \), is below zero, there is elastic material behaviour. If the yield function is exactly zero, neutral loading occurs. Neutral loading implies that the applied stress coincides with the surface that is spanned in stress-space by the used yield criterion, so that the material is no longer elastic, nor will plastic flow occur. However, in plasticity modelling neutral is regarded as elastic loading. If, within the iterative plasticity model, the value of the yield function is positive, plastic loading occurs [16]. The plastic material behaviour which is described here is summarized by

\[ \begin{align*}
F < 0 \quad &\rightarrow \quad d\varepsilon^p = 0; \quad \text{Elastic loading} \\
F = 0 \quad &\rightarrow \quad d\varepsilon^p \neq 0; \quad \text{Plastic loading}
\end{align*} \]  \hspace{1cm} (11)

The conditions in eq. (11) are used to determine whether plastic flow occurs [16].
2.3.2 Flow rule

To find the plastic strains for each increment, one may use a so-called flow rule. The flow rule relates stresses and plastic strains when plastic deformation occurs. A widely used flow rule was once proposed by Drucker [19] based on the normality postulate, that increments of plastic strain occur in the normal direction to the yield surface during plastic flow. This phenomenon is called associative plasticity and reads,

\[ \text{where } d\lambda \text{ is a scalar valued function, which can be viewed as the size of the increment of plastic flow. The term, } \frac{\partial F}{\partial \sigma_{ij}} \text{ represents the direction of plastic flow.} \]

\[ de_{ij}^p = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \quad (12) \]

2.3.3 Hardening rule

One important rule to apply when modelling plastic behaviour of materials is the consistency condition. It states that the yield surface follows the stress in stress space during plastic loading, so that the stress remains on the yield surface. To keep this condition fulfilled, a hardening rule must be used. It determines how the yield surface moves and changes during plastic flow. From the consistency condition, the yield surface in some way follows the direction of stress in the stress space. The remaining necessary details to describe the motion and changes of the yield surface can be found by employing one of three categories of hardening models [16]:

1. **Isotropic Hardening** – the yield surface increases in size during plastic flow. Using isotropic hardening the yield criterion becomes

   \[ F(\sigma, \sigma_y) = \sigma_{eq}(\sigma) - \sigma_y(\sigma) = 0, \]

   i.e. the yield surface expands as a function of stress.

2. **Kinematic Hardening** – the yield surface translates during plastic flow.

   \[ F(\sigma, \alpha, \sigma_y) = \sigma_{eq}(\sigma - \alpha) - \sigma_y = 0 \quad (13) \]

The term \( \alpha \) gives the position of the yield surface in stress-space and is referred to as the back stress.
3. Mixed/combined hardening – the yield surface may both expand and translate during plastic flow, i.e.

\[ F(\sigma, \alpha, \sigma_y) = \sigma_{eq}(\sigma - \alpha) - \sigma_y(\sigma) = 0. \]

2.3.4 Yield function for kinematic hardening

In this study a kinematic hardening model is used based on the von-Mises equivalent stress as [20]

\[ \sigma_{eq} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \] (14)

where \( S_{ij} \) is the deviatoric stress tensor and the Einstein-summation is used, i.e. there is an implicit summation over the index \( i \) and \( j \). The deviatoric stress tensor is calculated by removing the hydrostatic part, also called the mean, of the Cauchy stress tensor as,

\[ S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}. \] (15)

The use of only the deviatoric part of the stress tensor is motivated by the fact that the hydrostatic stress does not induce plastic flow as much as the deviatoric part [45]. The zero components of the stress matrix, eq. (5), implies that the Cauchy stress tensor can be obtained from the deviatoric stress tensor. Based on eq. (13), the yield function may be written as

\[ F(\sigma_{ij}, \alpha_{ij}, \sigma_y) = \frac{1}{\sigma_y} \sqrt{\frac{3}{2} (S_{ij} - \xi_{ij})(S_{ij} - \xi_{ij}) - 1} \] (16)

where \( \xi_{ij} \) is the deviatoric part of the backstress tensor, calculated in the same manner as the deviatoric stress tensor as

\[ \xi_{ij} = \alpha_{ij} - \frac{1}{3} \alpha_{kk} \delta_{ij}. \] (17)
2.3.5 Size of plastic flow

For material modelled using the aforementioned von Mises yield function, the size of a plastic strain increment, \( d\lambda \), of eq. (12) is said to be equal to the equivalent plastic strain [21,22], i.e.

\[
d\lambda = d\varepsilon_{eq}^p.
\]  
(18)

Where \( d\varepsilon_{eq} \) is the increment of equivalent plastic strain calculated by

\[
d\varepsilon_{eq}^p = \sqrt{\frac{2}{3}} \varepsilon_{ij}^p \varepsilon_{ij}^p.
\]  
(19)

Substituting

\[
d\varepsilon_{eq}^p = \frac{d\sigma_{eq}}{C},
\]  
(20)

in eq. (18) gives

\[
d\lambda = \frac{d\sigma_{eq}}{C}
\]  
(21)

where \( d\sigma_{eq} \) is increment of equivalent stress, \( C \) is a constant related to the plastic modulus \( H \). The increment of equivalent stress is calculated by [23]

\[
d\sigma_{eq} = \frac{3 S_{ij} dS_{ij}}{2 \sigma_{eq}}
\]  
(22)

For a surface element, the sum in the nominator is computed as

\[
dS_{ij} = dS_{11}S_{11} + dS_{22}S_{22} + dS_{33}S_{33} + 2dS_{23}S_{23},
\]  
(23)

where \( dS_{ij} \) is the increment of deviatoric stress. To calculate the constant \( C \), one uses (20) specialized to a uniaxial case, i.e.

\[
d\varepsilon_x^p = \frac{1}{C} \frac{\sigma_x d\sigma_x}{3 \sigma_x^2} 2\sigma_x = \frac{2}{3C} d\sigma_x.
\]  
(24)
This gives

\[ C = \frac{2}{3} \frac{d\sigma_x}{d\varepsilon_x^p} = \frac{2}{3} H, \]  

(25)

in which

\[ H = \frac{d\sigma_x}{d\varepsilon_x^p} \]  

(26)

is the plastic modulus. It is obtained by use of the Ramberg-Osgood equation, which defines the relation between elastic-plastic strain and stress as [24]

\[ C = \frac{2}{3} H = \frac{2}{3} K' n' \left( \frac{\sigma}{K'} \right)^{\frac{n'-1}{n'}}. \]  

(27)

Here the constants \( K' \) and \( n' \) are the cyclic strength coefficient and cyclic strain hardening exponent and are determined by uniaxial tensile testing.

**2.3.6 Direction of plastic flow**

For a kinematic material, the term representing the direction of plastic flow in equation (12) is written as [22]

\[ \frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2} S_{ij} - \xi_{ij} \]  

(28)

This expression can be interpreted as a normalized vector spanning from the centre of the yield surface, \( \xi \) to a point where it coincides with the surface, which is shown in deviatoric form in fig. 6.
Combining the equations for size and direction of plastic flow, eq. (21) and (28) in (12) gives

\[ d\varepsilon_{ij}^p = d\lambda \frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2} \frac{d \sigma_{eq} \left( S_{ij} - \xi_{ij} \right)}{C \sigma_y}. \] (29)

By substituting this expression in eq. (7) with Hooke’s law to calculate elastic strains, the total strain increment becomes

\[ d\varepsilon = d\varepsilon^e + d\varepsilon^p = \frac{1 + \nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{ij} \delta_{ij} + \frac{3}{2} \frac{d \sigma_{eq} \left( S_{ij} - \xi_{ij} \right)}{C \sigma_y}. \] (30)

The elastic deviatoric strain components are given by, [7]

\[ d\varepsilon_{ij}^e = \frac{dS_{ij}}{2G} \] (31)

where the shear modulus is

\[ G = \frac{E}{2(1 + \nu)}. \] (32)
Since the plastic strains are already deviatoric in eq. (29), the equation for total deviatoric strain is written as,

\[
d e_{ij} = \frac{dS_{ij}}{2G} + \frac{3}{2} \frac{d\sigma_{eq}}{C} \left( S_{ij} - \xi_{ij} \right) \sigma_y. \tag{33}
\]

This gives one equation per strain component, i.e. maximum six unique equations. However, in the case where a surface material element is considered, only four deviatoric strain components are non-zero, c.f. eq. (6). This entails that there are four constitutive equations to be solved. The numbers of total unknowns in these are seven, since additional three stress components are necessary to calculate deviatoric stress components. Therefore, three more equations are required to get a unique solution. These are obtained by coupling the constitutive equations to Neuber’s notch correction relations in section 2.4.

2.3.7 Loading criteria

Before stresses and strains are computed, loading conditions must be checked, to determine whether the current stress-strain increment will lead to loading or unloading. The loading condition may be one of three modes: elastic unloading, neutral and elastic-plastic (active) loading. The current mode is determined by using a loading criterion. As the stress decreases, i.e. the stress increment moves inward from the yield surface in stress space, elastic unloading is occurring. Mathematically, this is characterized by [7]

\[
\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} < 0. \tag{34}
\]

If the stress increment moves tangentially to the yield surface, neutral loading take place, i.e.

\[
\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} = 0. \tag{35}
\]

This mode is regarded as elastic unloading in a numerical environment. Plastic loading occurs if the current stress state moves outward from the yield surface, i.e.

\[
\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} > 0. \tag{36}
\]

Which is only valid in a numerical context before updating the current backstress to translate the yield surface, since F > 0 is impossible according to the flow rule.
For numerical implementations using stress increments, the active mode, or loading criterion, is based on

\[ LC = (S_{ij} - \xi_{ij})d\sigma_{ij}. \]  

(37)

Here, \( d\sigma_{ij} \) is an increment of elastic stress, which is output from the linear-elastic FEA. [7]

The current loading mode is thus determined by:

- \( LC < 0 \rightarrow \) Elastic Unloading
- \( LC > 0 \rightarrow \) Loading
- \( LC = 0 \rightarrow \) Neutral Loading

\[ \text{(38)} \]

**2.4 Calculation of elastic-plastic stress and strain at notches**

This section covers the method that is used to calculate elastic-plastic stress and strain for zones at the surface of a notch. By coupling a certain set of equations of the mechanical behaviour of material at notches to the previously established equations, (32) of elastic-plastic behaviour, a mathematical model including a set of seven equations is required. That is, one for each of the components in the stress and strain matrices of eq. (5-6).

**2.4.1 Stress and strain calculation at notches**

Machine components usually possess geometrical discontinuities, such as notches. These entail that the component will have stress concentrations throughout its structure, which will often lead to crack initiation, propagation and failure. To be able to approximate the stress in the vicinity of notches, stress concentration factors are often used with nominal stress. The most common equation to calculate stress due to stress concentrations was developed by Neuber [7] and reads \( \sigma^a\varepsilon^a = K^t S e \) where \( \sigma^a \) and \( \varepsilon^a \) is actual, elastic-plastic stress and strain, respectively. \( K_t \) is the stress concentration factor, \( S \) is nominal stress and \( e \) is nominal strain. Neuber’s rule may also be given in the form of

\[ \sigma^a\varepsilon^a = \sigma^e\varepsilon^e, \]  

(39)

where \( \sigma^e = K_t S \) and \( \varepsilon^e = K_t e \) are the pseudo-elastic stress and strain, respectively. Neuber’s method is valid under the condition that the zone undergoing plastic flow is small and contained within an elastic region. In other words, this method should not be used when global plasticity occurs.
When applying Neuber’s method for calculating stress concentrations, the stress state at the root of a notch may be viewed as uniaxial. In addition, the method assumes that the product of stress and local strain be independent of plastic flow. This is the basis of eq. (39). The estimate solution that defines the elastic-plastic stress-strain state is found where eq. (39) intersects the hardening curve of the material, while also at one point, coinciding with the elastic line. This is described in fig. 7.

![Figure 7: Solutions of the Neuber equation.](image)

A multiaxial notch-root analysis for stress and strain by Buczynski and Glinka [25] has been introduced to approximate actual, elastic-plastic stresses and strains by using the result from linear elastic FEA of components with notches. This is done by relating the elastic and elastic-plastic strain energy densities at notches, using the stress-strain behaviour of the material. The equations in incremental form are given by

\[
S_{ij}^e \, d\varepsilon_{ij}^e + e_{ij}^e \, dS_{ij}^e = S_{ij}^a \, d\varepsilon_{ij}^a + e_{ij}^a \, dS_{ij}^a
\]

where \( S^e, S^a \) and \( \varepsilon^e, \varepsilon^a \) is the pseudo elastic and elastic-plastic deviatoric stress and strain tensors, respectively and no Einstein summation is used over the indices. This equation is known as the incremental Neuber rule, and graphically depicted as shown in fig. 8. The areas spanned by the two terms on either side in eq. (40) are required to be equal and thus the relation eq. (40) between pseudo-elastic and elastic-plastic stress and strain is found.
The incremental Neuber rule, eq. (40), gives an additional three equations necessary to solve for all unknown plastic stress and strain increments.

When using deviatoric stress, another component will appear that was zero in the Cauchy stress tensor, see eq. (5) and (15), which entails that there will be a total of eight unknowns. However, the trace of the stress deviatoric tensor is zero, and this supplies another equation. When modelling cyclic loading, using the total and incremental stresses and strains in absolute terms is not preferable, and will give incorrect results [26,27]. To remedy this, local reference systems are used and described in the next chapter.

2.4.2 Neuber’s rule for cyclic loading

When using Neuber’s rule for cyclic loading, it must be specialized for each loading branch. Each loading branch originates from the point of latest load reversal, i.e. the point where plastic loading stops, and elastic unloading commences. The use of local reference systems (LRS), was initially introduced for calculating stress-strain behaviour by Chu [28], stating that

$$\left(\sigma_{ij}^e - \sigma_{ij}^{e\theta}\right)d\varepsilon_{ij}^e = \left(\sigma_{ij}^a - \sigma_{ij}^{a\theta}\right)d\varepsilon_{ij}^a. \quad (41)$$

Here, the left- and right-hand side represents elastic-plastic and elastic work, or energy, respectively. The upper index, $\theta$ represents the location of the LRS. It is either elastic or elastic-plastic.
For multiaxial stress-strain cases, a generalization can be made based on the energy concept, and written as [27]

\[
(\sigma_{ij}^e - \sigma_{ij}^{e0})(\varepsilon_{ij}^e - \varepsilon_{ij}^{e0}) = (\sigma_{ij}^a - \sigma_{ij}^{a0})(\varepsilon_{ij}^a - \varepsilon_{ij}^{a0}).
\] (42)

By coupling equations (40–42) in terms of deviatoric stress, the remaining required three equations necessary to solve for the actual, elastic-plastic stress-strain state at each increment is constructed, where the equations are [29]

\[
(S_i^e - S_i^{e0})de_i^e + (e_i^e - e_i^{e0})dS_i^e = (S_i^a - S_i^{a0})de_i^a + (e_i^a - e_i^{a0})dS_i^a.
\] (43)

Here, \(S_i^{e0}, S_i^a, e_i^{e0}, e_i^a\) is the value of deviatoric elastic or elastic-plastic stress or strain of the current active local reference system. These systems are described in the following sections.

2.4.3 Full system of equations

By combining (33) and (43), a set of seven equations is found. However, in deviatoric terms the non-zero components are eight. The term of elastic-plastic equivalent stress increment is calculated by eq. (22). An additional equation is found using the fact that the trace of deviatoric stress must be zero consistently. The resultant system of eight equations to be used when solving for actual, elastic-plastic stress, \(dS_i^a\) and strain \(de_i^a\) increments is

\[
de_{11}^a = \frac{dS_{11}^a}{2G} + \frac{3}{2} \frac{d\sigma_{eq}^a}{C} \frac{(S_{11}^a - \xi_{11})}{\sigma_y},
\]

\[
de_{22}^a = \frac{dS_{22}^a}{2G} + \frac{3}{2} \frac{d\sigma_{eq}^a}{C} \frac{(S_{22}^a - \xi_{22})}{\sigma_y},
\]

\[
de_{33}^a = \frac{dS_{33}^a}{2G} + \frac{3}{2} \frac{d\sigma_{eq}^a}{C} \frac{(S_{33}^a - \xi_{33})}{\sigma_y},
\]

\[
de_{23}^a = \frac{dS_{23}^a}{2G} + \frac{3}{2} \frac{d\sigma_{eq}^a}{C} \frac{(S_{23}^a - \xi_{23})}{\sigma_y},
\] (44)

\[
(S_{22}^e - S_{22}^{e0})de_{22}^e + (e_{22}^e - e_{22}^{e0})dS_{22}^e = (S_{22}^a - S_{22}^{a0})de_{22}^a + (e_{22}^a - e_{22}^{a0})dS_{22}^a,
\]

\[
(S_{33}^e - S_{33}^{e0})de_{33}^e + (e_{33}^e - e_{33}^{e0})dS_{33}^e = (S_{33}^a - S_{33}^{a0})de_{33}^a + (e_{33}^a - e_{33}^{a0})dS_{33}^a,
\]

\[
(S_{23}^e - S_{23}^{e0})de_{23}^e + (e_{23}^e - e_{23}^{e0})dS_{23}^e = (S_{23}^a - S_{23}^{a0})de_{23}^a + (e_{23}^a - e_{23}^{a0})dS_{23}^a,
\]

\[
\text{trace}(S^e) = \text{trace}(dS^e) = dS_{11}^a + dS_{22}^a + dS_{33}^a = 0.
\]

This system is also given in matrix form in Appendix A.
2.4.4 Active local reference system

At the start of elastic unloading from a point of plastic flow, a local reference system $O$, is defined in terms of deviatoric stress and strain for both elastic the solution, $\sigma_{ij}^0$, $\epsilon_{ij}^0$ and elastic-plastic solution $\sigma_{ij}^{a0}$, $\epsilon_{ij}^{a0}$, see fig. 9 [26, 27]. The latest defined LRS is then used in the elastic-plastic stress-strain calculations of eq. (43), meaning that it is denoted as active.

![Local reference system](image)

Figure 9: Local reference systems for Neuber’s rule.

There are however some exceptions to employ when defining the local reference systems. For complex cyclic load histories, many LRS’s will be defined since the loads may oscillate. If a component is loaded such that it undergoes plastic flow and then unloaded, see fig. 10 and fig. 11, a new LRS, $O_2$ will be placed at the initial point of unloading. If it is then reloaded, a closed hysteresis loop will be created by this smaller loading branch, as in fig. 12. As the stress-strain increases and plastic flow begins to occur, the Neuber equations will give an incorrect result, shown in fig. 13 where an artefact is present. To remedy this, the LRS should be reverted to the initial one, $O_1$ which is described in fig. 14. Consequently, the stress-strain curve now follows the same curve as if the unloading branch had not occurred [30].

![Loading](image)

![Unloading](image)

Figure 10: Loading. Figure 11: Unloading.
Figure 12: Re-loading.

Figure 13: Re-loading with artefact.

Figure 14: Proper material behaviour without artefact.
2.4.5 Proposed method to avoid artefacts during loading

The previously described behaviour of correcting the active LRS may be achieved by a simple method. In order to implement this method, elastic and elastic-plastic stress and strain are stored when the load is reversed, at which point in time a so-called peak has been found. These points are determined by evaluating the load criterion, (38) to determine whether the current stress increment is loading or unloading. If the LC classification of the current increment is different to that of the previous increment, the current point is stored. In pseudo-code,

\[ \text{If } \text{sign}(LC(\sigma_{ij}^c, \varepsilon_{ij}^c, \alpha_{ij}^c)) \neq \text{sign}(LC(\sigma_{ij}^{c-1}, \varepsilon_{ij}^{c-1}, \alpha_{ij}^{c-1})) \]

\[ \text{Store } \sigma_{ij}^c, \varepsilon_{ij}^c, \sigma_{ij}^e, \varepsilon_{ij}^e \text{ in } P \]

where \( c \) is the current increment number and \( P \) is the matrix where the stresses and strains are stored.

A method that can determine whether the current active LRS should be moved is necessary. Such a method is presented in [30] and is based on the so-called J-function,

\[ J(\sigma_{ij}^e, \varepsilon_{ij}^e) = \sigma_{eq}^e \cdot \varepsilon_{eq}^e, \] (45)

where \( \sigma_{eq}^e \) is equivalent elastic stress and is calculated by eq. (14). \( \varepsilon_{eq}^e \) is equivalent elastic strain and is calculated as

\[ \varepsilon_{eq}^e = \sqrt{\frac{2}{3} \varepsilon_{ij}^e \varepsilon_{ij}^e}. \] (46)

The J-function is to be used to determine whether a full cycle of peaks has been created, subsequently leading to a reversion of the active LRS. For a full cycle to be present, one of two conditions must be fulfilled. If the level of the found peaks increases during a certain time interval, the so-called load direction (LD) is positive. For such a case, the level of the latest peak, peak number \( n \), must be greater than that of peak \( n-2 \). If, however, the level of the curve decreases, the LD is negative, then the level of peak \( n \) must be smaller than peak \( n-2 \). These examples are described in fig. 15-16. For a uniaxial case, the loading direction could easily be found by using the sign of the load increment. However, for multiaxial loadings, a different methodology must be used, in this case the J-function.
Unfortunately, using this function has one important drawback. Since neither equivalent stress nor strain can be negative, nor can the J-function. This entails that the curve of the J-function will show additional reversals as the J-function goes to zero. As an example, take a uniaxial tension load-case where the loading goes from tension to compression. The J-function will revert around zero even though no reversal of loading has been performed, as shown in fig. 17.

An effect of this phenomena is that the functions cannot be used to find full cycles of smaller loops, as a result of the altered shape of the curve. To remedy the drawback of using the method from [24], a new method is proposed in which the J-function is manipulated so that it will not reach zero.
This is achieved by finding the maximum level of the $J$-function for a certain load history and then subtracting elastic stress and strain terms when calculating equivalent stress and strain in the $J$-function. The new function then reads

$$J'(\sigma_{ij}, \varepsilon_{ij}, \sigma_{ij,\text{max}}, \varepsilon_{ij,\text{max}}) = \sigma_{eq}(\sigma_{ij} - \sigma_{ij,\text{max}}) \cdot \varepsilon_{eq}(\varepsilon_{ij} - \varepsilon_{ij,\text{max}})$$ \hspace{1cm} (47)$$

where $\sigma_{ij,\text{max}}$ and $\varepsilon_{ij,\text{max}}$ is elastic stress and strain at the point of maximum value of $J$, respectively, for the entire used load history.

By employing $J'$, the definition of LD reads

$$LD(\sigma_{ij}^c, \varepsilon_{ij}^c, \sigma_{kl}^n, \varepsilon_{kl}^n) = \text{sign} \left( J'(\sigma_{ij,c}, \varepsilon_{ij,c}, \sigma_{ij,\text{max}}, \varepsilon_{ij,\text{max}}) - J'(\sigma_{ij,n}, \varepsilon_{ij,n}, \sigma_{ij,\text{max}}, \varepsilon_{ij,\text{max}}) \right)$$ \hspace{1cm} (48)$$

where the index $n$ represents the latest peak defined and index $c$ represents the current increment. The algorithm that identifies whether a full cycle has been created and removes stored peaks associated to that cycle is shown in pseudo-code below. Consequently, two peaks are removed, and the active LRS is reverted back to the latest peak defined during plastic flow. The first line is used to determine whether a cycle could possibly have been created, to decrease the required computational time. It says that the algorithm should only be used if the current value of the $J'$-function is equal to that of the peak defined just prior to the latest.

```
if $J'_c \cong J'_{n-1}$

    if $LD > 0 \& J'_{n-2} < J'_n$
        $P_{n-1}, P_n = []$
        Remove peak $n$ and $n-1$.
    
    else if $LD < 0 \& J'_{n-2} > J'_n$
        $P_{n-1}, P_n = []$
        Remove peak $n$ and $n-1$.

end
```

end
2.5 Translation of the hardening surfaces

To keep the consistency condition fulfilled, see section 2.3.3, i.e. that the yield surface coincides with the stress in stress-space during plastic flow, a method that in some way moves the yield surface is required. Many different models with different translation rules are available, in this study the model according to Mróz and Garud [7] will be implemented. The Mróz-Garud is a piecewise linear model suitable to use when the applied loading includes reversions. It is often used for its computational efficiency when modelling non-linear kinematic hardening [22].

2.5.1 The Mróz-Garud cyclic plasticity model

In Mróz’ multi surface model, the mechanical behaviour is described using an approximative kinematic hardening model by defining a number of nested, i.e. hardening surfaces, where the innermost surface represents the yield limit and the outermost represent the ultimate limit of the material. Each of these surfaces correspond to a specific value of hardening, i.e. the slope of the hardening curve. Two important terms in Mróz-models are the active and target surfaces. The active surface is the largest surface with which the stress tensor is currently in contact with, i.e. the largest translating surface. The target surface is the surface that is just external to the active. Initially, the innermost surface is the active and as the stress and backstress increase, the current active surface will switch to one of the larger surfaces. If the stress reaches the target surface the current active surface is changed to that surface. Which surface that is at any point the active surface, determines what part of the hardening curve to use as the parameter C when calculating plastic strain in eq. (44). The value of C is described as constant for each nested surface, making the plastic modulus piecewise linear, as shown in fig. 18. The plastic modulus for each nested surface is calculated using eq. (46) with $\sigma = \sigma_{eq}$.

![Figure 18: Piecewise linear plastic modulus of hardening surfaces.](image-url)
To implement the Mróz-Garud multi-surface model, a number of hardening surfaces are defined with certain radii, representing the limit of each surface. Initially, the centre of each surface, given by the back stress, $\alpha$, is at the origin. This is only true if the material used is a virgin material. Every hardening surface translates as the material hardens, i.e. the back stress is moved in stress-space. However, only the current active, initially the innermost, surface is implicitly translated as a result of increased stress. If an inner hardening surface is translated so that it coincides with an outer surface, it “drags” the outer surface with it [31].

Garud [32] modified Mróz’ method such that the yield surfaces are prevented from intersecting one-another, by re-directing the direction of translation. According to this model, the movement of the stress surface depends not only on current stress, but also the direction of stress increment. The Garud method for moving the yield surfaces are summarized in [7,11,33].

In a numerical context, translating the nested surfaces according to the Mróz-Garud method should follow the following procedure, where it is assumed that the current stress state has settled at a point $\sigma$, on the active surface $f_1$ with radius $R_{f_1}$ located internally to the target surface $f_2$ with radius $R_{f_2}$. The centre of these two surfaces are located at positions $\alpha_{f_1}$ and $\alpha_{f_2}$. The stress state will then be updated by an increment of stress, $d\sigma$, aimed outwards of the active surface such that the position must be updated to keep the yield surface coincident with the stress.

The algorithm to translate the surfaces for this arbitrary case are described below and shown in figure 19. A detailed algorithm is available in appendix B.

1. Extend the stress increment by a scalar value, $x$, in its direction so that it intersects the target surface at point $A_1$.
2. Find the normal vector $n_{A_1}$ of the surface $f_2$ at the point $A_1$.
3. Find the point on the active surface where its normal vector is parallel to the one that has been found on the target surface. This point is denoted $A_G$.
4. Connect the points $A_1$ and $A_G$ to find the direction of translation. This vector is denoted $A_1A_G$.
5. Translate the active surface $f_1$ from its position $\alpha_{f_1}$ to an updated position $\alpha_{f_1}'$ along $A_1A_G$ such that the updated stress $\sigma + d\sigma$ lies on this surface.
The surface that represents the ultimate stress of the material, i.e. the outermost surface and denoted the failure surface, will not translate. If the next-to outermost surface reaches the failure surface, the material will fracture due to ductility exhaustion. During plastic flow, all inner surfaces to the active surface translate together with the active surface. Then, the centre of each surface does not move relative to each other. This implies that that the translation rules according to the Mróz-Garud model only need to be implemented to the active surface. It can be described by

\[
d\alpha_{ij}^k = \begin{cases} d\alpha_{ij} & \text{if } k = A \\ 0 & \text{otherwise} \end{cases}
\]

where \(d\alpha\) is the incremental translation of each surface and \(A\) represents the number of the active hardening surface [33]. Instead of translating all surfaces, using the fact that the inner hardening surfaces to the active surface are mutually tangent at the current stress state enables for a simple method. The current back stress of the inner surfaces can be calculated by

\[
\alpha_{ij}^k = \sigma_{ij} - \frac{R_k}{R_{k+1}} (\sigma_{ij} - \alpha_{ij}^{k+1}) \quad \text{for } k = 1, ..., A - 1
\]

where \(\sigma\) is the current stress state, \(\alpha^k\) and \(R_k\) and is the back stress and size of each surface, \(\alpha^A\) and \(R_A\) is the back stress and size of the current active hardening surface [33].
2.5.2 Bi-section of increments

If the surface $f_1$ intersect the outer surface, $f_2$, the outer surface should be translated together with $f_1$ so that the position of updated stress lays tangent to both $f_1$ and $f_2$. In other words, the inner surface pushes the outer surface with it. To achieve this, the yield function can be used to determine if intersection of surfaces will occur. Instead of the position and size of the inner surface, one uses the current values of the inner surface which gives the conditions

$$ F(\sigma + d\sigma, \alpha f_2 + d\alpha f_2, R_{f_2}) \geq 0 $$

(51)

i.e. if the yield function with respect to the outer surface $f_2$ is larger than or equal than zero, intersection occurs. In addition, the current stress and increment of stress must be updated, since the model is based on that the current stress-state during plastic loading lies on the yield surface and the increment of stress starts on the yield surface. If intersection occurs, it implies that the factor $x$ described in the previous section will be less than one, since not all of the stress increment is required to reach the outer surface [6]. The phenomena of one surface pushing the outer surface can be achieved by calculating $x$, then defining the stress increment as

$$ d\sigma_{ij} = (1 - x) \cdot d\sigma_{ij} $$

(52)

The current total stress will then be

$$ \sigma_{ij} = \sigma_{ij} + x \cdot d\sigma_{ij}. $$

(53)

This defines the stress increment as the remainder of stress increment, after intersection of the outer surface. The same procedure as previously described is then implemented, starting with calculating a new value of $x$ to find the position of $A1$. When two surfaces coincide, the outer surface is chosen as the new active surface and correspondingly, the current plastic modulus $H$ is chosen as the value associated with this surface.
During unloading from a point at the yield surface, the initial behaviour is elastic [7]. In a numerical context, at the event of unloading, the active surface is set to the smallest nested surface, i.e. the yield surface.
3. Fatigue estimation

The theory presented thus far has described a method to estimate elastic-plastic stress and strain using a linear-elastic input. In the section which follows from here on, the resultant transient stress and strain will be used to estimate fatigue and predict the fatigue life of a component.

Fatigue estimation is very complex if the loading is multiaxial, rather than uniaxial. Unfortunately, complex multiaxial loading is also very common in machine components. Examples include shafts subjected to combined tension and torsion or sheet metal subjected to bending in several axes. Another factor is that different sources of stress act with different frequencies and phase which adds to complexity [18].

If the load path is such that the components of the stress tensor change proportionally with respect to each other so that the direction of principal stresses are constant, the loading is said to be proportional. Otherwise, if the direction of principal stresses change, the loading is non-proportional [11,18].

To estimate the fatigue life of a component subjected to multiaxial loadings, four components are required. First, one needs a cyclic stress-strain model as has been described previously. Secondly, a cycle counting method to identify cycles during loading. Third, a damage model that is used to calculate the damage for each cycle. Lastly, a damage accumulation model that computes total damage after a number of loading cycles is also required.

3.1 The critical plane method

One often used method to approximate multiaxial damage is the so called-critical plane method [30]. It is based on finding the orientation of a material volume element such that it is subjected to maximum damage, and then calculate fatigue life with respect to that orientation. This orientation is found by an iterative approach where the material element is rotated around two axes and evaluated at each position. Fig. 21 shows a case where a material element has been rotated an angle $\varphi$ around the 3-axis, and as a result the forces subjected upon the element changes from $\mathbf{1}$ to $\mathbf{1}'$. To evaluate the damage, a damage parameter is used to calculate the damage for a specific load case as a consequence of the current spatial orientation. The plane of maximum damage is then chosen as the critical plane. For material at the surface, such as at notches, only one angle such as in fig. 21 is a variable. The stress and strain components of a plane can be found by Mohr's circle, as depicted in fig. 22 where the stress state is shown. This state can be represented by viewing fig. 21 where the surface is the plane spanned by the $\mathbf{1}$ and $\mathbf{2}$ axes.
The resulting normal and shear stresses and strains of an element in plane stress, found by Mohr’s circle, are [34,35]

\[
\sigma(\phi) = \sigma_x \cos^2 \phi + \sigma_y \sin^2 \phi + 2\tau_{xy} \sin \phi \cos \phi
\]
\[
\tau(\phi) = \frac{\sigma_y - \sigma_x}{2} \sin 2\phi + \tau_{xy} \cos 2\phi
\]
\[
\varepsilon(\phi) = \varepsilon_x \cos^2 \phi + \varepsilon_y \sin^2 \phi + \frac{\gamma_{xy}}{2} \sin \phi \cos \phi
\]
\[
\gamma(\phi) = (\varepsilon_y - \varepsilon_x) \sin 2\phi + \gamma_{xy} \cos 2\phi
\]

where \(\phi\) is the angle between the x-axis and the plane. The methodology for fatigue in this chapter is implemented numerically to compute an approximate life prediction using the corrected, elastic-plastic stress-strain history and is described in the next chapter.
In a numerical context, an assumption regarding the number of angles to search for the critical orientation must be made, since the number of orientations possible is infinite. One may for example choose to search all possible angles in the interval $0^\circ – 180^\circ$ using steps of $5^\circ$. Using smaller steps increases the computational time but decreases the risk of missing critical angles where the highest damage is found.

### 3.2 Rainflow cycle counting

When monitoring a component subjected to many different sources of loading, the resulting stress-strain state will often lack any apparent patterns. To be able to estimate the fatigue life this needs to be simplified in some manner and a method often used is the rainflow cycle counting method [24]. The rain flow cycle counting method is used to reduce a complex and irregular load history into a number of cycles which can be evaluated with respect to fatigue.

Using some chosen entity as a function of time, e.g. stress or strain, the rainflow cycle counting method defines cycles as closed stress-strain hysteresis loops by pairing two strain-states together, which is described in fig. 23. The chosen entity to be used when defining the endpoints of cycles is called the **main or counting channel**. The shear strain is often chosen as the counting channel, although the normal strain may be chosen if normal strain is generally larger than shear strain in a loading sequence or depending on which damage parameter is used. The other entities that is not the main/counting channel is called **auxiliary channels**. Different numerical algorithms are available to carry out the rainflow counting. In this study, rainflow counting according to the standard ASTM E 1049 – 2009 is applied.

![Figure 23: Identification of load cycles by rainflow cycle counting.](image-url)
When applying rainflow cycle counting for a multiaxial load case, transient effects on fatigue may be lost. Since rainflow cycle counting is performed using the main channel, the effects of the fluctuation of the auxiliary channels could be lost. Within a cycle defined by e.g. shear strain, the maximum level of normal stress and strain could occur at any time and will affect the life of the component. This behaviour is synonymous with non-proportional, or out-of-phase, loading. As an example, take a case where a component is subjected to normal and shear stress non-proportionally. Since the loading is out-of-phase and if shear strain is used as the counting channel, the points in time defining hysteresis curves of shear stress and strain will not register the normal stress and strain where they are maximum. This leads to non-conservative life predictions for the component. This phenomenon is somewhat remedied by using the so called Bannantine-Socie (BS) multiaxial cycle counting method [36]. According to the BS cycle counting method, the entities required for fatigue evaluation that are calculated by any of the auxiliary channels, are defined sometime during the cycle. Whereas the start and end points are found by using the main channel, as shown in fig. 24. For the aforementioned case, the maximum normal stress and strain could now be found when they are at their maximum level [36].

![Figure 24: Definition of entities according to Bannantine-Socie method.](image)

3.3 Damage parameters

A damage parameter quantifies the amount of fatigue damage depending on material state variables, such as stress and strain. Using a damage parameter, the relation between damage and fatigue life is approximated. Many demands are put on a damage parameter: it should be able to incorporate mean stress effects, be load path dependent and be applicable for many load cases, e.g. uniaxial and multiaxial including proportional and non-proportional loading.
Many different damage parameters have been proposed but in this study a damage parameter proposed in the mentioned Ph. D thesis by Ince [7] is used. This damage parameter where shear strain is used as the counting channel, is expressed as:

\[
\max \varphi \left( \frac{\tau_{\text{max}}}{\sigma_f' \sqrt{3}} \frac{\Delta \gamma^e}{2} + \frac{\Delta \gamma^p}{2} + \frac{\sigma_{n,\text{max}}}{\sigma_f'} \frac{\Delta \varepsilon_n^e}{2} + \frac{\Delta \varepsilon_n^p}{2} \right) = \left[ \frac{(1 + \nu_e)}{E} (2N_f)^{2b} + \frac{(1 + \nu_p)}{E} (2N_f)^c \right] + \left[ \frac{(1 - \nu_e)}{E} (2N_f)^{2b} + \frac{(1 - \nu_p)}{E} (2N_f)^c \right],
\]

where, for a given cycle, \( \tau_{\text{max}} \) is maximum shear stress, \( \sigma_f' \) is the fatigue strength coefficient, \( \Delta \gamma^e \) and \( \Delta \gamma^p \) is elastic and plastic shear strain range, respectively. Furthermore, \( \Delta \varepsilon_n^e \) and \( \Delta \varepsilon_n^p \) is elastic and plastic normal strain range; and \( \sigma_{n,\text{max}} \) is maximum normal stress on the plane of maximum shear strain. \( \nu_e \) and \( \nu_p \) is elastic and plastic Poisson’s ratio. Finally, \( b \) and \( c \) are the axial fatigue and ductility exponent, respectively. The elastic and plastic normal and shear ranges are calculated by

\[
\Delta \varepsilon = \epsilon_{\text{max}} - \epsilon_{\text{min}}
\]

where the max and min indices represent the maximum and minimum level of either elastic or plastic normal or shear strain of a cycle. \( N_f \) is number of load cycles to failure. To find the number of cycles to failure, a Newton-solver is used, here \textit{fsolve} in Matlab.

### 3.4 Cumulative damage

The idea of material damage from a fatigue perspective is not well defined, because of insufficient understanding of the phenomena. It is described as a deterioration of material that causes the mechanical properties, such as stiffness, to worsen. Another effect of damage is increasing stress, since the deterioration of material decreases the cross-section area of a given position.

Material damage is here denoted as \( D \), and quantified in the range \( 0 \rightarrow 1 \), where a value of 0 represents virgin material, and a value of 1 describes a material state where all material has deteriorated. This is a purely symbolic state that is not realistically obtained, since the material will fail at a lower value, called critical damage. Accumulated damage after \( n \) cycles can be calculated by Miner’s rule, as

\[
D = \frac{n}{N_f}
\]
The damage of several different load cycles is be summarized linearly, as [37]

\[ D = \sum_i \frac{n_{i}}{N_{f,i}} \]

Here, \( n_i \) is number of performed cycles of a specific type of cycle and \( N_{f,i} \) is number of cycles to failure for that cycle type. By summating over all types of cycles the accumulated damage, \( D \), is found.
4. Implementation

This chapter covers how the reviewed theories are used in a numerical context to; first correct pseudo-elastic to actual elastic-plastic stresses and strains and then perform the fatigue analysis by applying the damage parameter and estimating life by the critical plane method. The implementation is performed in Matlab.

4.1 Calculating elastic-plastic stress and strain

To approximate the actual elastic-plastic stresses, the cyclic plasticity model described in the previous chapter is used. A load history in incremental form is input to the program, together with necessary material data. The material data includes the yield and ultimate limit, Young’s modulus, Poisson’s ratio as well as the radius and plastic tangent modulus of each nested surface to be used in the multi surface model. The program then enters its iterative section, where for each iteration an elastic trial step is calculated by adding an elastic increment to the current stress. Initially, loading conditions are checked using eq. (37) and compared to the previous increment. If the loading differs from the last increment, the stress-strain state of both elastic and elastic-plastic solution is stored, since a peak, see section 2.4.5, has been found. If plastic flow occurred during the previous increment, the newly found peak is set as the current active local reference system to be used when calculating elastic-plastic stress and strain. If the current mode determined by the LC is unloading, the material behaviour is elastic, and elastic unloading is the current mode. However, if loading occurs, the yield criterion, eq. (16), is calculated to determine whether plastic flow occurs. If $F < 0$, elastic loading occurs. However, if $F > 0$, plastic flow occurs, the stress must be corrected. The flow-chart to this algorithm is shown in fig. 25. In short, this program classifies each iteration into three different material behaviours when loading, elastic unloading, elastic loading or elastic-plastic loading, detailed in fig. 26. These are further described next.
Figure 25: Flow-chart of cyclic plasticity model, main program.
Elastic loading entails that the stress-strain state is inside the yield surface and that the trial elastic increment taken is equal to actual stress and strain. Therefore, only to save the elastic trial step as actual is what is done during this type of loading. Then, a new iteration is performed, and the new elastic trial step is taken. Likewise, as elastic unloading occurs, the elastic trial step is saved as actual stress and strain. What differs during this mode, is that the active hardening surface is reset to the innermost one, i.e. the yield surface. During plastic flow, the hardening surface has likely progressed to a larger one.

As plastic flow occurs, the stress-strain state of the elastic trial step is not equal to the actual, elastic-plastic state but needs to be calculated. Firstly, the stored peaks are checked for possible full hysteresis loops so that the correct local reference system is used. The elastic-plastic stress-strain state is found by eq. (44) and further described in the next section. Lastly the hardening surfaces must be translated, which gives an updated back stress value. This is also further described below. A new iteration is then afterwards performed.

![Flow-chart of the three loading scenarios.](image-url)
4.2 Solving for elastic-plastic stress and strain increments

One crucial element in the cyclic plasticity model is the section where the actual, elastic-plastic stresses and strains are found by using eq. (44). Since it is linear, it can be solved by Gaussian elimination. The system of equations is solved by formulating the system in matrices, in the form \( \mathbf{Ax} = \mathbf{b} \) where the unknowns, \( dS_i \) and \( de_{ij} \), are assembled in the vector \( \mathbf{x} \). After the solution is found, the size of plastic strain \( d\lambda \) can be calculated using eq. (21), and its direction by eq. (28) and thereafter also the plastic strain increment, \( d\sigma^p \) by eq. (29). The elastic strain increment, \( de^e \), is found by eq. (9). The resultant increment of stress deviator, \( dS_i \) \( a \) is afterwards used in the Mróz-Garud surface translation algorithm.

![Flow chart of numerical implementation of stress-strain correction.](image)

Figure 27: Flow chart of numerical implementation of stress-strain correction.
4.3 Translating surfaces according to the Mróz-Garud model

The methodology presented in section 2.5 is implemented numerically. To be able to carry out the translation, several input arguments are required. The inputs are: the radii of all nested surfaces, $R_{ik}$, and their positions in stress-space, i.e. the back-stress tensor $\alpha_{ij}$. It is also required to know which surface the current active surface is. Lastly, one need the stress of the last increment, $S_{ij}$ and the current increment of stress $dS_{ij}$ which has been found by eq. (44).

The general algorithm for this step is shown in fig. 28. Since the output stress from the notch correction is in deviatoric form, it first needs to be converted to Cauchy stress, using eq. (15). To know when the active surface will translate such that it will coincide or intersect with its outer/target surface, a yield criterion with respect to that outer surface is calculated, eq. (51). If the result is larger than or equal to zero, the increment $dS_{ij}$ is bi-sectioned, meaning that the stress increment is set to that portion of the stress increment that is outside the current active surface, the current total stress is then set to be on the active surface. This is described in eq. (52) and (53). The current active surface is also updated to the larger surface. The surfaces are then translated according to the method described in the algorithm in section 2.5.1.
Figure 28: Flow chart for the numerical implementation of Mróz-Garud surface translation.
4.4 Predicting fatigue life

Since the critical plane method of section 3.1 is to be used, the algorithm to estimate fatigue life requires an iterative search throughout the interval of possible angles that orients the plane. The input to the program is the elastic-plastic stress-strain history resulting from a pseudo-elastic load history. Then, for the entire load history, the resultant stresses and strains are calculated by eq. (54-57) using the current angle. Rainflow cycle counting is used to identify cycles of the stresses and strains and for each of these cycles, the number of load cycles to failure is found by solving eq. (58) using a non-linear equation solver, where the amplitude entities are found by eq. (59). In this study the command *fsolve* in Matlab was used. The damage of each cycle is then found by using eq. (60) and the total accumulated damage is calculated by summation of the damage of each cycle, eq. (61). This procedure is repeated for angles, \( \varphi \), of 0° in increments of 5° up to 180°. The critical plane is then found by selecting it as the plane where the highest damage has occurred. Using the accumulated damage of that plane, the fatigue life is estimated by eq. (61) and output from the function. The flow-chart of this program is shown in fig. 29.
Figure 29: Flow-chart for the numerical implementation of fatigue life estimation.
5. Numerical results

In this chapter, selected results from the numerical implementation are presented. Firstly, calculated transient elastic-plastic stress and strain for different load cases are displayed. Secondly, results from the method for adapting Neuber’s rule for cyclic loading are presented. Lastly, results of fatigue life predictions for different load cases are shown and compared to experiments.

5.1 Elastic-plastic stress and strain

To test the model, a number of load cases is executed. Two different FEA’s in Ansys is performed, one transient linear-elastic analysis where the resultant stress is used as input to the developed mathematical model. The result of the established mathematical model is then compared to the result of a performed elastic-plastic transient FEA. The FEA model used is a solid cylinder, with outer diameter 10 mm, with a circumferential groove of diameter 1 mm which centre is located at a depth 2 mm. It is loaded such that a small region where the material is undergoing plastic flow appears. The full model is shown in fig. 30 and the groove in fig. 31. The mathematical model was carried out in Matlab 2017a. The FEA was performed in ANSYS R17.1 using 37248 brick elements. All nodes of one of the two planar sides of the structure in fig. 30 is locked in all directions, and the loading is applied to the nodes of the other side.

For the following load cases of sections 5.1 the numerical parameters used is shown in table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus, $E$</td>
<td>200 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Yield limit, $\sigma_y$</td>
<td>226 MPa</td>
</tr>
<tr>
<td>Ultimate limit, $\sigma_u$</td>
<td>800 MPa</td>
</tr>
<tr>
<td>Cyclic strength coefficient, $K'$</td>
<td>704.7</td>
</tr>
<tr>
<td>Cyclic strain hardening exponent, $n'$</td>
<td>0.112</td>
</tr>
<tr>
<td>Size of elastic stress increments</td>
<td>1 MPa</td>
</tr>
<tr>
<td>Number of hardening surfaces</td>
<td>28</td>
</tr>
<tr>
<td>Value of tol when calculating elastic-plastic stress and strain</td>
<td>0.02 MPa</td>
</tr>
</tbody>
</table>
5.1.1 Convergence study – size of stress increment

To investigate the size of the increment required to obtain a result as near ideal as possible, a convergence study is performed. The shear stress and strain were chosen as the quantity to use when calculating error. The values were extracted for a specific point in time from the implemented numerical model and compared to the result of the transient, elastic-plastic FEA.
The chosen load case used was that of section 5.1.7. The specific time was chosen to be the point of maximum equivalent stress. Extracting the shear stress and strain component at this point and inserting it into eq. (62), as

\[
\text{Error} = \left( \frac{\sigma_{23}^{\text{FEA}} - \sigma_{23}^{\text{MM}}}{\sigma_{23}^{\text{FEA}}} + \frac{\varepsilon_{23}^{\text{FEA}} - \varepsilon_{23}^{\text{MM}}}{\varepsilon_{23}^{\text{FEA}}} \right) / 2
\]  

(62)

where the upper index \textit{FEA} represents the point in time of maximum equivalent stress of the FEA, and the upper index \textit{MM} represents the implemented mathematical model. The CPU time of using varying increment sizes was calculated by the Matlab commands \texttt{tic} and \texttt{toc}. Increment sizes was varied from 1 to 60 MPa, and the results of error as well as CPU time of using a various increment size is shown in fig. 32. The error increases rapidly at increment sizes larger than 50 MPa. Smaller increments do not yield a significantly better result. However, CPU time does not significantly decrease further when using increments larger than 15 MPa. For the following numerical load-cases, an increment size of 1 MPa will still be used, in order to be certain that the increment size does not affect the results.

![Figure 32: Convergence study of size of stress increments.](image-url)
5.1.2 One cycle of axial loading

The model is loaded axially in one cycle of tension and compression. The input stress is shown in fig. 33. Compared to a transient elastic-plastic FEA, the results are acceptable but not perfect, see fig. 34. The stress and strain component calculated by the mathematical model are slightly higher than those found by a FEA.

![Figure 33: Elastic stress input for axial loading.](image1)

![Figure 34: Results for axial loading.](image2)
5.1.3 One cycle of torsional loading

The model is loaded by torsion, see fig. 35. The results compared to a transient elastic-plastic FEA are acceptable but not perfect, as shown in fig. 36.

Figure 35: Elastic stress input for torsional loading.

Figure 36: Results for torsional loading.
5.1.4 One cycle of proportional multiaxial loading

The FE-model is loaded biaxially in one cycle in-phase, i.e. proportionally. The elastic stress input to the mathematical model are shown in fig. 37. The results are displayed in fig. 38 for the three stress components and the four strain components. The stress and strain components that are not directly loaded, $\sigma_{22}$ and $\varepsilon_{22}$, but affected by contraction due to a different normal component being loaded, deviates somewhat. However, in absolute terms of stress and strain, the difference is small.

Figure 37: Elastic input stress for proportional multiaxial loading.

Figure 38: Results for proportional multiaxial loading.
5.1.5 One cycle of non-proportional multiaxial loading

The FE-model is loaded biaxially of one cycle out-of-phase, i.e. non-proportionally. The elastic stress input to the mathematical model are shown in fig. 39. The results are displayed in fig. 40 and does not appear to be worse than for the multiaxial proportional load case. The calculated values of the two components being directly loaded are fairly accurate, and acceptable. The same phenomenon as for the proportional case can be observed, where the results of the component not being primarily loaded deviates from the results of the FEA, however only in relation and not absolute terms.

![Figure 39: Elastic input stress for non-proportional multiaxial loading.](image)

![Figure 40: Results for non-proportional multiaxial loading.](image)
5.1.6 Cyclic proportional multiaxial loading

In this case, ten cycles of biaxial proportional loading were tested, and one of the cycles of the elastic stress input to the mathematical model are shown in fig. 41. The results are displayed in fig. 42, and accurate for the components being directly loaded. Similarly, to what have been observed for other cases, the component affected by contraction deviates. Although only in relation and not absolute terms compared to the FEA.

![Figure 41](image1.png)

Figure 41: One cycle of elastic input stress for proportional multiaxial cyclic loading.

![Figure 42](image2.png)

Figure 42: Results of proportional multiaxial cyclic loading.
5.1.7 Complex proportional multiaxial

A relatively complex biaxial, proportional load case is used in which several unloading and reloading events occur. The elastic input signal is shown in fig. 43 and the results in fig. 44. For this load case, the results are very similar to what has been observed before. However, the calculated stress and strain components is deviates larger to the corresponding FEA values compared to previous cases.

![Figure 43: Elastic input stress of complex proportional multiaxial loading.](image1)

![Figure 44: Results of complex proportional multiaxial loading.](image2)
5.2 Evaluation of $J'$- function

An experiment was performed to test the validity of the proposed $J'$- function, see (47) as opposed to the previously established $J$– function from [30] when identifying full cycles. The input uniaxial stress signal plotted in fig. 45 was used, where it should be possible to identify two cycles. The point in time where these two cycles are found are depicted as large dots in the figure.

![Graph showing stress vs time](image1)

**Figure 45**: Input stress to test of $J'$- function.

The result of using the method is shown in fig. 48, where the $J'$-function, is plotted as a function of time. The shape of the curve is indeed such that the cycles would be able to be found and at the correct point in time by the definitions explained in fig. 15-16 and the aforementioned text of the figures.

![Graph showing J' vs time](image2)

**Figure 46**: Resultant $J'$- function of the input stress.
5.3 Correction of active local reference system

To validate the proposed method of correcting the active local reference system, the numerical implementation is compared to a case in [30], where the reversals of local reference systems for the time signal of elastic stress is known, see fig. 47. The points where the local reference system should be reverted is here depicted as dots. At each point of reversal, the two latest peaks are removed as also described in section 2.4.5.

![Graph](image)

**Figure 47:** Input stress to test of LRS correction.

The result is shown in fig. 48, where the dashed line shows the current active LRS throughout the transient loading. The large dots show where a reversal of active LRS is performed. By comparing fig. 48 to fig. 47, the results of using the presented method is equal to the desired results.

![Graph](image)

**Figure 48:** Resultant LRS of using the proposed method.
5.4 Fatigue

The model using the critical plane methodology in combination with rainflow cycle counting to calculate fatigue life is compared to experiments of different load cases. From the experiments used, both stabilized stress and strain are included in their respective published articles. The results are plotted in log-log graphs including dashed lines representing an interval that within a factor of two from a perfect result, i.e. where experimental life is equal to predicted life. This interval should be viewed as relatively narrow, since two multiaxial fatigue experiments of equal loading will often show different life, in terms of cycles to failure [38]. The different strain paths of all used load cases are shown in fig. 49.

Two squares \( \gamma / \sqrt{3} \)  
Square \( \gamma / \sqrt{3} \)  
Torsion with mean stress \( \gamma / \sqrt{3} \)  
Proportional \( \gamma / \sqrt{3} \)  
30° out-of-phase \( \gamma / \sqrt{3} \)  
Butterfly \( \gamma / \sqrt{3} \)  
60° out-of-phase \( \gamma / \sqrt{3} \)  
90° out-of-phase \( \gamma / \sqrt{3} \)  
45° out-of-phase \( \gamma / \sqrt{3} \)  

Figure 49: Strain paths used in fatigue analysis.
5.4.2 Multiaxial loading of aluminium alloy 7075 T-651

The results of using the presented method for predicting fatigue life is compared to experimental cases from [39], where a thin-walled tubular specimen is subjected to repeated loading. The used parameters are shown in table 2. Several different multiaxial load cases are used. They include: in-phase loading, a so-called butterfly load path, torsion loading with a tension mean stress and 90° out-of-phase loading. The results are shown in the log-log plot in fig. 50. A comparison is made to the widely accepted damage parameter by Fatemi and Socie (FS) [24]. The FS parameter includes a constant, the so-called normal stress sensitivity parameter \( k \), which is ideally found by fitting the damage parameter equation to experimental results. This constant is dependent on both the material and the applied loading. Here, a value of 1.0 was chosen, which is often the case when no fitted value is available [24,40]. The results of using this damage parameter is shown in fig. 51. The predicted life for the different load cases are generally accurate or somewhat conservative. However, for the torsion with mean stress load case, the FS damage parameter appear to predict life inaccurately, compared to Ince’s parameter.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Young’s Modulus, ( E )</td>
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</tr>
<tr>
<td>Poisson’s ratio, ( \nu )</td>
<td>0.306</td>
</tr>
<tr>
<td>Yield limit, ( \sigma_y )</td>
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</tr>
<tr>
<td>Axial fatigue strength exponent, ( b )</td>
<td>-0.122</td>
</tr>
<tr>
<td>Axial fatigue ductility exponent, ( c )</td>
<td>-0.806</td>
</tr>
<tr>
<td>Axial Fatigue strength coefficient, ( \sigma' )</td>
<td>1231</td>
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<tr>
<td>Axial fatigue ductility coefficient, ( \varepsilon' )</td>
<td>0.263</td>
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<td>Shear fatigue strength exponent, ( b_\theta )</td>
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<tr>
<td>Shear fatigue ductility exponent, ( c_\theta )</td>
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<td>Shear Fatigue strength coefficient, ( \tau' )</td>
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<td>Shear fatigue ductility coefficient, ( \gamma' )</td>
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<tr>
<td>Normal stress sensitivity parameter, ( k )</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Figure 50: Aluminium fatigue results using the damage parameter by Ince.

Figure 51: Aluminium fatigue results using the FS damage parameter.
5.4.3 Multiaxial loading of steel alloy 30CrNiMo8H

The results of using the presented method for predicting fatigue life is compared to experimental cases from [41], also in this case a thin-walled tubular specimen is loaded. The different load paths include: in-phase/proportional, 30°-, 45°-, 60°- and 90° out-of-phase as well as so-called square, two squares and butterfly load path. The results are acceptable for in-phase, square, two square and butterfly load paths, however as the loading is more out-of-phase, the results, see fig. 52, tend to become more non-conservative. Using the damage parameter by Ince compared to Fatemi-Socie, shown in fig. 53, the results are almost equal.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tr>
<td>Young’s Modulus, $E$</td>
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<td>Poisson’s ratio, $\nu$</td>
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<td>Yield limit, $\sigma_y$</td>
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<td>Axial fatigue strength exponent, $b$</td>
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<tr>
<td>Axial fatigue ductility exponent, $c$</td>
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<td>Axial Fatigue strength coefficient, $\sigma^f$</td>
<td>951.16</td>
</tr>
<tr>
<td>Fatigue ductility coefficient, $\varepsilon^f$</td>
<td>1.064</td>
</tr>
<tr>
<td>Shear fatigue strength exponent, $b_\tau$</td>
<td>-0.041</td>
</tr>
<tr>
<td>Shear fatigue ductility exponent, $c_\tau$</td>
<td>-0.733</td>
</tr>
<tr>
<td>Shear Fatigue strength coefficient, $\tau^f$</td>
<td>549.15</td>
</tr>
<tr>
<td>Shear fatigue ductility coefficient, $\gamma^f$</td>
<td>1.843</td>
</tr>
<tr>
<td>Normal stress sensitivity parameter, $k$</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Figure 52: Steel fatigue results using the damage parameter by Ince.

Figure 53: Steel fatigue results using the FS damage parameter.
6. Discussion

This section discusses the validity and applicability of the presented model, as well as the numerical results. Potential factors of improvement for future work will also be discussed.

6.1 Calculation of elastic-plastic stress and strain

By comparing the results of the proposed mathematical model to an elastic-plastic FEA, the results may be reliably validated. The results show fair accuracy of both monotonic and cyclic loading, proportional and non-proportional. However, it is known that when loading a metal specimen either proportionally (in-phase) or non-proportionally (out-of-phase), the specimen loaded out-of-phase will show greater hardening [24,42]. The stress level of a given strain level is higher of non-proportional loadings than proportional. This effect is not considered in the presented mathematical model or in the FEA performed, so that the actual stress and strain of non-proportional load cases will differ from the results that may be achieved by the model or FEA.

The equations used in the assembled system of equations includes the incremental Neuber relations, which are only valid for components subjected to a load that leads to local plasticity. The field of plasticity should be small and enclosed by an elastic region. An FEA where these conditions are not fulfilled have been performed. The elastic input is shown in fig. 54, as well as the results in fig. 55. The results show an error of calculated stress and strain which is much larger than what is achieved in the cases presented in the previous chapter, where the conditions were fulfilled. Still, there is a possibility that this effect causes inaccuracies of the load cases used in section 5.1.2-5.1.7. If the loadings were to have been lower in the linear-elastic and elastic-plastic FEA’s performed, the plastic field would be smaller which could possibly lead to better results using the mathematical model. It is also important to note that the incremental Neuber rule is not solidly grounded in physics, which could possibly mean that using these equations provide a source of error in elastic-plastic stress and strain calculations.

Another factor that could affect the results is the re-location of the active local reference system. Complex or random loading requires that the active local reference system is moved in real time during elastic-plastic stress-strain calculation, which for multiaxial loading is not straightforward. This is largely due to that defining the current load direction, which is either negative or positive, is difficult but needed to distinguish full cycles from of half-cycles. The stress and strain values of the current active local reference systems is detrimental to the resultant elastic-plastic stress and strain. This entails that uncertainty of using the correct values greatly affects the reliability of the model.

Even though a few rather complex load-cases of two cycles have been tested, the results of employing a realistic, more random time signal might prove inaccurate and is yet to be used in validation.
One aim of this study is in turn to decrease the computational time when calculating the elastic-plastic response of long, real, load histories. Whether or not this is achieved thus far is not certain, the two alternatives have not been compared yet. The mathematical model presented should be compared to elastic-plastic FEA of a long load history. A benefit of using the model presented is that the result of unit load cases can be scalar multiplied and super positioned to obtain the linear-elastic (pseudo elastic) response of any load case. This entails that after a sufficient number of unit load cases has been performed, no more are ever necessary. Still, the presented model has to be faster than an elastic-plastic FEA for it to be viable to use. The model only gives the resultant stress-strain histories for one node, for which the linear-elastic stress and strain was used as input, whereas the elastic-plastic FEA after completion gives the result of all nodes in the structure. This entails that the program has to be performed for a large number of nodes, which could mean that the computational time advantage is reduced or even that the FEA will be faster. However, for structures with notches, the point or area of interest is very possibly small. The elastic-plastic stress and strain histories of interest are generally at a critical point, such that using the presented model would only be necessary for a few points or nodes in a small area. The numerical implementation of the algorithm has in this study been done in Matlab, which is not known to be state-of-the-art when it comes to computational efficiency. Compared to programming languages such as Python or Fortran, Matlab will most often require longer CPU time to execute a program, depending on the program used [43]. If the number of total flops in a program increases, which occurs when large loops are used, the relation between CPU time of Matlab compared to Fortran or Python will usually increase, such that Matlab requires even more time [44]. If this algorithm is to be implemented to be used in accordance with mechanical design of component, which is the intention, it should preferably be coded in a faster language.

6.2 Proposed method to avoid artefacts during loading

By inspecting the results of section 5.3, some points may be observed. The shape of the curve when using the J’-function, shown in fig. 48, instead allows for the two cycles to be identified, at the correct point in time. However, the curve appears flipped, so that the local maxima of the stress input, shown in fig. 47, are local minima in the J’-curve, and vice versa. Although, this should not affect the behaviour of the algorithm, since the cycles will still be found and then the corresponding peaks removed. Even though this method appears to work for the cases used in this study, there is no guarantee that it will work for any loading. From reviewing the available literature on this topic, it is evident that no tried proven and accepted method which corrects the active LRS in real-time during the plasticity model exists.

The J’-function may also be used to find peaks to use as LRS’s, although it should be equal to using the presented method, using the LC. It may be interpreted as a peak should be stored when the stress goes from moving away from the centre of the yield surface to moving into the yield surface, or vice versa. This method could prove inaccurate for complex, non-proportional loading which has not been tested as of yet. The reason why problems may arise is that when the loading is non-proportional, the J’-function i.e. equivalent stress and strain, and also the LC, may stay constant even though the stress components fluctuates. In other words, the out of phase fluctuation of loading components cancels out, so that the equivalent stress and strain is constant. A different method which might remedy this is described in [45], in which each stress and strain component is treated individually when finding peaks, i.e. LRS’s.
6.3 Fatigue

The resultant life estimation of the presented model was compared to two different studies where both stress and strain amplitudes for all cases was included. This should eliminate the effect of non-proportional hardening which has not been considered in the stress-strain correction model. Although, when reviewing the results of life prediction, the predictions tends to be more non-conservative for non-proportional loading. This behaviour is in line with the expected results of non-proportional fatigue, even when using up-to-date methodology [46]. Calculating life expectancy for multiaxial non-proportional load cases are a topic where no established and reliable method have been presented and verified.

The damage parameter used is relatively new but have since its presentation been verified to show acceptable accuracy [14]. However, it assumes that

\[ \tau'_f = \sqrt{3} \cdot \sigma'_f \]

which is not always true, although it is still a common and acceptable assumption to make.

No significant difference of the performance between the two used damage parameters may be observed from the cases studied here. The parameter by Ince appears to fare better in the cases studied in section 4.4.2, however the damage parameters performed equally in the cases of section 4.4.3. It is important to notice that the constant required in the FS parameter should ideally be fitted to fatigue data to obtain better predictions. This has not been done in this study. If fitted data is available or an approximation can be made, the FS damage parameter will likely produce more accurate fatigue life prediction than the parameter by Ince.

6.4 Socio-economic impact

If the presented mathematical model to calculate transient elastic-plastic stresses and strains is deemed accurate enough to be applied when performing strength calculations of components, the benefits could be large. The time required to; create FE-models of different load-cases as well as solving the FE-analysis could be greatly reduced. In turn, engineers could then focus on other issues and resources could also be used elsewhere. The described method to calculate fatigue life at notches can likely be used and would in practise reduce the number of failures of components while in operation. It could enable design engineers to make fewer conservative choices, which would reduce spent resources of manufacturing and decreasing its environmental impact.
6.5 Recommendations

The mathematical method to estimate the elastic-plastic response of a pseudo-elastic input presented show acceptable accuracy. It could be used to estimate fatigue life of components avoiding long, computationally intensive elastic-plastic FEA’s. However, there definitely exists areas of potential improvements of the model. A verified method for all types of loading that moves the active local reference system has not been implemented yet. One method is available in [30], but it does not incorporate the effects of load direction, which entails that it does not differentiate between full- or half-cycles when moving the local reference systems which could affect the results. Since the size of the plastic zone affects the calculated results, a study that determines the relation between size of the plastic zone, or amount of equivalent stress or strain, to resultant error would be useful.

The effects of non-proportional loading could be accounted for by implementing some available theory of the topics, regarding non-proportional hardening. A model that corrects the constant $K'$ in eq. (27) have been presented and reads [24]

$$K_{np}' = K'(1 + \alpha F)$$

where $\alpha$ is a material constant and $F$ is a value ranging from 0 to 1 depending on the non-proportionality of loading. For in-phase loading, $F = 0$, and for $90^\circ$ out-of-phase, $F = 1$. A method presented by Itoh et. al. [47,48] can be used to calculate $F$ for any loading. For constant amplitude and constant phases, it is straightforward to calculate $F$, since it will be constant. By correcting to value of $K'$ to $K_{np}'$, the additional hardening due to non-proportional loading can then be accounted for and, as a result, the calculated stress will be higher and life predictions will be lower. For complex, non-constant amplitude loading where the phase difference between the strain components are varying, the value of $F$ will vary throughout the loading. To be able to accurately model the non-proportional hardening, the method presented in [47] could be used in accordance with rainflow cycle counting. The rainflow cycle counting could be used with the elastic stress-strain input data to associate points in time to a specific value of $F$. In turn, the value of $K_{np}'$ will vary throughout the loading. Another method to incorporate the effect of non-proportional hardening is to calculate a mean value of $F$ prior to the elastic-plastic stress-strain correction. This method should be acceptable to use if the phase difference is somewhat constant.

However, since the studies from which the experimental fatigue test data was extracted included both stabilized stress and strain amplitudes, non-proportional hardening should be accounted for. The damaging effect of non-proportional loading on a material requires an additional method to account for this effect. As previously stated, this is a topic still being researched and where no firmly established method exists. Although, one simple method similar to the hardening model is presented in [49] which could be implemented in an eventual future algorithm.

The system of eight equations used to find elastic-plastic stress and strain components could be altered to possibly improve the accuracy compared to elastic-plastic FEA.
Figure 54: Elastic input stress causing non-local plasticity.

Figure 55: Results of loading causing non-local plasticity.
7. Conclusions

In this thesis, a mathematical model has been presented that can be used to estimate elastic-plastic stress and strain components at notches using stresses and strains from linear elastic FEAs. The results of using the model show fair albeit not ideal accuracy, compared to performing transient elastic-plastic FEA’s, when the multiaxial loading is proportional or non-proportional.

A method to estimate fatigue life of components has also been presented and tested versus experiments for many different load cases. The method shows acceptable results for proportional loading. As the loading is closer to completely out-of-phase (90°), the life predictions made by the model are less accurate. However, a method that incorporates the damaging effect of non-proportional loadings are available and could be implemented.

The full mathematical model presented to calculate elastic-plastic stress and strain and thereafter make life predictions could in its current state be used to estimate life of components, when loaded proportionally. If the loading of a component is non-proportional, the fatigue life predictions should be assumed to be non-conservative, i.e. that the actual life is shorter than the predicted.
Reference List


[6]. Singh MNK. Notch tip stress strain analysis in bodies subjected to non-proportional cyclic loads. 1998.

[7]. Ince, Development of Computational Multiaxial Fatigue Modelling For Notched Components, Ph. D, University of Waterloo, 2013.


[41]. Luiz De Rose, David Padua. Benchmarking FALCON’s MATLAB-to-Fortran 90 Compiler on an SGI Power Challenge.


Appendix A: Stress-strain correction system of equations in matrix form

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}G + K(S_{11}^a - \xi_{11})S_{11}^a & -K(S_{11}^a - \xi_{11})S_{22}^a & -K(S_{11}^a - \xi_{11})S_{33}^a & -K(S_{11}^a - \xi_{11})S_{23}^a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}G + K(S_{22}^a - \xi_{22})S_{22}^a & -K(S_{22}^a - \xi_{22})S_{11}^a & -K(S_{22}^a - \xi_{22})S_{33}^a & -K(S_{22}^a - \xi_{22})S_{23}^a & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{2}G + K(S_{33}^a - \xi_{33})S_{33}^a & -K(S_{33}^a - \xi_{33})S_{11}^a & -K(S_{33}^a - \xi_{33})S_{22}^a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}G + K(S_{23}^a - \xi_{23})S_{23}^a & -K(S_{23}^a - \xi_{23})S_{11}^a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}G + K(S_{23}^a - \xi_{23})S_{23}^a & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\mathbf{x} = [de_{11}^a \quad de_{22}^a \quad de_{33}^a \quad de_{23}^a \quad dS_{11}^a \quad dS_{22}^a \quad dS_{33}^a \quad dS_{23}^a]^T
\]

\[
\mathbf{b} = [0 \quad 0 \quad 0 \quad 0 \quad (S_{22}^e - S_{22}^{e0})de_{22}^e + (e_{22}^e - e_{22}^{e0})dS_{22}^e \quad (S_{33}^e - S_{33}^{e0})de_{33}^e + (e_{33}^e - e_{33}^{e0})dS_{33}^e \quad (S_{23}^e - S_{23}^{e0})de_{23}^e + (e_{23}^e - e_{23}^{e0})dS_{23}^e]^T
\]

\[
K = \frac{9}{4\sigma_{eq}^a\sigma_y}
\]
Appendix B: Numerical implementation of Mróz-Garud cyclic plasticity model

The following section describes how to apply the Mróz-Garud translation rule [6]. A current stress-state, \( \sigma \), is assumed to be active after some preceding loading. The stress state lies on an active nested surface, \( f_1 \), with a radius \( R_{f1} \) and position \( \alpha_{f1} \). This surface is located internally to a larger, inactive surface \( f_2 \), radius \( R_{f2} \) and position \( \alpha_{f2} \). At the start of loading, assuming the material is at a virgin state, the inner surface is equal to the yield surface with a radius corresponding to the yield limit and the position, i.e. back stress, is the zero tensor. However, after some loading that induces plastic flow, the active surface will gradually become a larger nested surface.

A load increment is applied, \( d\sigma \), so that the total load state is external to the active surface and plastic flow occurs. The consistency condition requires that the surface must be moved so that the resultant load after the increment is on the yield surface. The required equation that must hold is

\[
F(\sigma_{ij} + d\sigma_{ij}, \alpha_{ij} + d\alpha_{ij}, R_{f1}) = 0,
\]

where \( F \) is the yield criterion for kinematic hardening, \( \sigma \) is current stress, \( d\sigma \) is elastic-plastic stress increment \( \alpha \) is current backstress and \( \Delta\alpha \) is increment of backstress. \( R_{f1} \) is the size of the current active nested surface. This equation corresponds to the expression

\[
F(\sigma + d\sigma, \alpha + d\alpha, R_{f1}) = \sigma_e(\sigma_{ij} + d\sigma_{ij} - \alpha_{ij} - d\alpha_{ij}) - R_{f2} = 0,
\]

where \( \sigma_e \) is equivalent stress. To find the resultant change of backstress, \( d\alpha \), due to the stress increment, the following procedure is used. The stress increment is extended by a factor, \( x \), so that it intersects the target, non-active, surface, \( f_2 \), at the position denoted \( A_1 \). Which is described in mathematical terms as

\[
\|\sigma_{ij} + x \cdot d\sigma_{ij} - \alpha_{ij_{f2}}\| = \frac{2}{3} R_{f2}.
\]

Here the left-hand side is a Euclidian length/magnitude of a matrix and the coordinates of \( A_1 \) in stress space is identified as

\[
\sigma_{ij_{A1}} = \sigma_{ij} + x \cdot d\sigma_{ij}.
\]
Since only three components of the stress tensor is non-zero, equation (A. 3) can be developed to

\[
\left(\sigma_{22} - \alpha_{22f2} + x \cdot d\sigma_{22}\right)^2
- \left(\sigma_{22} - \alpha_{22f2} + x \cdot d\sigma_{22}\right)\left(\sigma_{33} - \alpha_{33f2} + x \cdot d\sigma_{33}\right) + (A. 5)
\]

\[
\left(\sigma_{33} - \alpha_{33f2} + x \cdot d\sigma_{33}\right)^2 + 3\left(\sigma_{23} - \alpha_{23f2} + x \cdot d\sigma_{23}\right)^2 = R_f^2
\]

which can be rewritten as

\[
Ax^2 + Bx + C = 0 \quad (A. 6)
\]

with the solution

\[
x = \frac{-B + \sqrt{\Delta}}{2A} \quad (A. 7)
\]

Where

\[
A = d\sigma_{22}^2 - d\sigma_{22}d\sigma_{33} + d\sigma_{33}^2 + 3(d\sigma_{23}^2) \quad (A. 8)
\]

\[
B = 2\left\{\left(\sigma_{22} - \alpha_{22f2}\right)d\sigma_{22} + \left(\sigma_{33} - \alpha_{33f2}\right)d\sigma_{33} + 3\left[\left(\sigma_{23} - \alpha_{23f2}\right)d\sigma_{23}\right]\right\}
- \left(\sigma_{22} - \alpha_{22f2}\right)d\sigma_{33} - \left(\sigma_{33} - \alpha_{33f2}\right)d\sigma_{22} \quad (A. 9)
\]

\[
C = \left(\sigma_{22} - \alpha_{22f2}\right)^2 - \left(\sigma_{22} - \alpha_{22f2}\right)\left(\sigma_{33} - \alpha_{33f2}\right) + \left(\sigma_{33} - \alpha_{33f2}\right)^2 + 3\left(\sigma_{23} - \alpha_{23f2}\right)^2 - R_f^2 \quad (A. 10)
\]

The now known point \(A_1\) is connected to the position of \(f_2\), i.e. \(\alpha_{22}\).
\[ \alpha_{ij} A^i_2 \sigma_{ij} = \sigma_{ij} A^i_1 - \alpha_{ij} f_2 \]  

A. 11

Parallel to this line, another line is created from the centre of \( f_1 \) to its surface, denoted \( A_G \).

\[ \sigma_{ij}^{A_G} = \frac{R_{f1}}{R_{f2}} \left( \sigma_{ij}^{A_1} - \alpha_{ij} f_2 \right) + \alpha_{ij} f_1 \]  

A. 12

The two points \( A_1 \) and \( A_G \) is connected to find the direction of surface translation. This direction tensor is denoted as \( \sigma^{AA} \).

\[ \sigma^{AA} = A_G A_1 = \sigma_{ij}^{A_1} - \sigma_{ij}^{A_G} \]  

A. 13

Next, the magnitude of translation is to be found. It is calculated by again applying the yield function, so that the active surface is translated until the updated stress after the stress increment is located on the surface. This condition is defined as

\[ \left\| \sigma_{ij} + d\sigma_{ij} - \alpha_{ij} f_1 - \Delta\alpha_{ij} f_1 \right\| = R_{f1} \]  

A. 14

where the increment of back stress tensor is expressed as

\[ d\alpha_{ij} f_1 = y \cdot \sigma^{AA} \]  

A. 15

Where \( y \) is a scalar multiplier. Substituting the following expression into eq. (A. 14),

\[ \sigma_{ij}^{\alpha} = \sigma_{ij} + d\sigma_{ij} - \alpha_{ij} f_1 \]  

A. 16
(A.14) can then be written as

$$\left| \sigma_{ij}^{\alpha} - y \cdot \sigma^{AA} \right| = R_{f_1}$$  \hspace{1cm} A. 17

which can be developed to

$$(\sigma_{22}^\alpha - y \cdot d\sigma_{22}^{AA})^2 - (\sigma_{22}^\alpha - y \cdot d\sigma_{33}^{AA})(\sigma_{33}^\alpha - y \cdot d\sigma_{33}^{AA}) + (\sigma_{33}^\alpha - y \cdot d\sigma_{22}^{AA})^2 = R_{f_1}^2.$$  \hspace{1cm} A. 18

Applying the pq-formula gives

$$y = \frac{-B - \sqrt{\Delta}}{2A}$$  \hspace{1cm} A. 19

Where the entities $\Delta, A, B, C$ is calculated by

$$\Delta = B^2 - 4AC$$  \hspace{1cm} A. 20

$$A = (\sigma_{22}^{AA})^2 + (\sigma_{33}^{AA})^2 + 3(\sigma_{22}^{AA})^2 - (\sigma_{22}^{AA})(\sigma_{33}^{AA})$$  \hspace{1cm} A. 21

$$B = -2[\sigma_{22}^\alpha \sigma_{22}^{AA} + \sigma_{33}^\alpha \sigma_{33}^{AA} + 3(\sigma_{23}^\alpha \sigma_{23}^{AA})] + \sigma_{22}^\alpha \sigma_{33}^{AA} + \sigma_{22}^\alpha \sigma_{33}^{AA}$$  \hspace{1cm} A. 22

$$C = (\sigma_{22}^\alpha)^2 + (\sigma_{33}^\alpha)^2 - \sigma_{22}^\alpha \sigma_{33}^\alpha + 3(\sigma_{23}^\alpha)^2 - R_{f_1}^2$$  \hspace{1cm} A. 23

The updated position of $f_1$ can then be expressed as

$$\alpha_{ij f_1}' = \alpha_{ij f_1} + d\alpha_{ij f_1}$$  \hspace{1cm} A. 24