Some Controllability Aspects for Iterative Learning Control

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SOME CONTROLLABILITY ASPECTS FOR ITERATIVE LEARNING CONTROL

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ABSTRACT

Some controllability aspects for iterative learning control (ILC) are discussed. Via a batch (lifted) description of the problem a state space model of the system to be controlled is formulated in the iteration domain. This model provides insight and enables analysis of the conditions for and relationships between controllability, output controllability and target path controllability. In addition, the property miminum lead target path controllability is introduced. This property, which is connected to the number of time delays, plays an important role in the design of ILC algorithms. The properties are illustrated by a numerical example.

Key Words: Iterative Learning Control, Controllability, Output controllability, Target path controllability

I. Introduction

ILC is a control method which improves the control of processes that perform the same task repeatedly [3, 15]. A classic example is an industrial robot performing e.g. arc welding or laser cutting, but it has also been used in other applications, such as in the hard disk drive industry [5]. The system where ILC is applied can be both in open loop as well as in closed loop, and in block form it can be illustrated by Figure 1.

Typically, the ILC input signal $\mathbf{u}_k(t) \in \mathbb{R}^{n_u}$ is updated according to an equation of the type

$$\mathbf{u}_{k+1}(t) = \mathcal{F}\left(t, \{\mathbf{u}_k(i)\}_{i=0}^{N}, \{\mathbf{e}_k(i)\}_{i=0}^{N}\right),$$

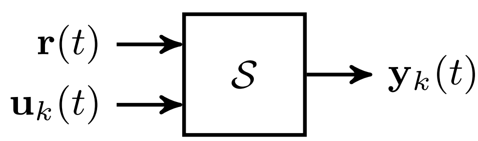

Fig. 1. System description. The reference signal is denoted by $\mathbf{r}(t)$, the ILC input signal by $\mathbf{u}_k(t)$ and the output signal by $\mathbf{y}_k(t)$, where k is the ILC iteration index and t the time index.

with $t = 0, \ldots, N$ and where $\mathbf{e}_k(t) = \mathbf{r}(t) - \mathbf{y}_k(t)$ is the control error, $\mathbf{r}(t) \in \mathbb{R}^{n_y}$ the reference signal, $\mathbf{y}_k(t) \in \mathbb{R}^{n_y}$ the measurement signal, k the iteration index, t the time index and $\mathcal{F}(\,\cdot\,)$ is an update function. The main task in ILC design is to find an update function that is able to drive the error to zero in some suitable norm, as the number of iterations tends to infinity, i.e.,

$$\|\mathbf{e}_k(t)\| \to 0, \quad k \to \infty, \quad t = 0, \ldots, N \qquad (1)$$

The controllability concept, covered in this contribution, provides general results concerning what information is necessary to fulfil (1). The results also provide insights in how to choose design parameters in a certain class of ILC algorithms.

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In this paper, the system depicted in Figure 1 is described using the following linear, discrete-time and time-invariant state space model

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B_u}\mathbf{u}(t), \quad (2a)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad (2b)$$

where $x(t) \in \mathbb{R}^{n_x}$. For simplicity, disturbances and the reference input are not considered.

Definition 1 (Controllability [19]) *A linear time-invariant state space model is called controllable if, given any state $\mathbf{x}_f$, there is a positive integer t_f and an input signal $\mathbf{u}(t)$ such that the corresponding response of the system, beginning at $\mathbf{x}(0) = \mathbf{0}$, satisfies $\mathbf{x}(t_f) = \mathbf{x}_f$.*

Theorem 1 (Controllability [19]) *An* LTI *system is controllable if and only if the rank of the controllability matrix $\mathscr{C}$ is equal to the state dimension, i.e.,*

$$\operatorname{rank}\begin{pmatrix}\mathbf{B_u} & \mathbf{AB_u} & \cdots & \mathbf{A}^{n_x-1}\mathbf{B_u}\end{pmatrix} = n_x. \quad (3)$$

A related property is the output controllability of a system. A formal definition of output controllability follows from Definition 1 with $\mathbf{x}$ replaced by $\mathbf{y}$. The requirement for output controllability is that the output controllability matrix, denoted by $\mathscr{C}^o$, has full rank [17], where

$$\mathscr{C}^o = \begin{pmatrix}\mathbf{CB_u} & \mathbf{CAB_u} & \cdots & \mathbf{CA}^{n_x-1}\mathbf{B_u}\end{pmatrix} \quad (4)$$

for a state space model parametrised as in (2).

By using a batch (iteration domain) description of the system in (2) a number of controllability aspects related to the design and analysis of ILC algorithms can be studied. In Section II the iteration domain description is introduced. Section III presents conditions for state and output controllability. The target path controllability (TPC) concept is recovered from [7]. An extension, the new concept of Minimum Lead Target Path Controllability (MLTPC) is also introduced. This concept is especially suitable for application to ILC. In the related area of repetitive control, controllability has been investigated in [8, 18, 10], where a batch formulation of the repetitive system is utilised and controllability conditions are derived. Another approach to controllability based on the z-transform of the repetitive system is presented in [6]. In Section IV aspects of controllability are illustrated and it is discussed how TPC can be used in ILC. The material presented below naturally extends and builds upon the work in [4].

II. System Description in the Iteration Domain

Consider the discrete-time state space model in (2). According to [19] the system has the following update formula for the state vector

$$\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0) + \sum_{j=0}^{t-1}\mathbf{A}^{t-j-1}\mathbf{B_u}\mathbf{u}(j)$$

for $t \geq 1$. By introducing the vectors

$$\bar{\mathbf{x}} = \begin{pmatrix}\mathbf{x}(1)^\mathsf{T} & \dots & \mathbf{x}(N)^\mathsf{T}\end{pmatrix}^\mathsf{T} \in \mathbb{R}^{Nn_x},$$
$$\bar{\mathbf{u}} = \begin{pmatrix}\mathbf{u}(0)^\mathsf{T} & \dots & \mathbf{u}(N-1)^\mathsf{T}\end{pmatrix}^\mathsf{T} \in \mathbb{R}^{Nn_u},$$
$$\bar{\mathbf{y}} = \begin{pmatrix}\mathbf{y}(1)^\mathsf{T} & \dots & \mathbf{y}(N)^\mathsf{T}\end{pmatrix}^\mathsf{T} \in \mathbb{R}^{Nn_y},$$

the model in (2) can be written more compactly for a batch of length N as

$$\bar{\mathbf{x}} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\bar{\mathbf{u}} \quad (5a)$$
$$\bar{\mathbf{y}} = \boldsymbol{\mathcal{C}}\bar{\mathbf{x}} \quad (5b)$$

where $\mathbf{x}(0)$ is the initial value, $\boldsymbol{\mathcal{C}} = \mathbf{I}_N \otimes \mathbf{C}$, and

$$\mathbf{\Phi} = \left(\mathbf{A}^T, (\mathbf{A}^2)^T, \dots, (\mathbf{A}^N)^T\right)^T, \quad (6a)$$

$$\mathbf{S_{xu}} = \begin{pmatrix}\mathbf{B}_u & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{AB}_u & \mathbf{B}_u & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}^{N-1}\mathbf{B}_u & \mathbf{A}^{N-2}\mathbf{B}_u & \cdots & \mathbf{B}_u\end{pmatrix}. \quad (6b)$$

The batch formulation is also known as the lifted system representation in the ILC community and it is used in the design of ILC update laws, e.g. norm-optimal ILC [1, 9], but also for analysis of stability and convergence. The batch formulation as introduced above takes into account that the discrete time state space model (2) contains (at least) one time delay from input to output since there is no direct term between the input and the output. In the batch formulation this is done by shifting the elements of the vector of outputs $\bar{\mathbf{y}}$ one time step relative to the elements of the input vector $\bar{\mathbf{u}}$. This differs from the treatment in [9] where the delay is not taken into account, and the time indices of the input and output vectors are the same. At iteration k and $k+1$ it holds that

$$\bar{\mathbf{x}}_k = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\bar{\mathbf{u}}_k, \quad (7a)$$
$$\bar{\mathbf{x}}_{k+1} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\bar{\mathbf{u}}_{k+1}, \quad (7b)$$

where it is assumed that the initial state $\mathbf{x}(0)$ is iteration independent. This is a common assumptions when applying ILC [2]. Subtracting (7a) from (7b) gives the following expression

$$\bar{\mathbf{x}}_{k+1} = \bar{\mathbf{x}}_k + \mathbf{S_{xu}}(\bar{\mathbf{u}}_{k+1} - \bar{\mathbf{u}}_k), \tag{8}$$

Let $\Delta_{\bar{\mathbf{u}}_k} \triangleq \bar{\mathbf{u}}_{k+1} - \bar{\mathbf{u}}_k$ be a new input signal then the state space model in the iteration domain can be expressed as,

$$\bar{\mathbf{x}}_{k+1} = \bar{\mathbf{x}}_k + \mathbf{S_{xu}}\Delta_{\bar{\mathbf{u}}_k}, \tag{9a}$$

$$\bar{\mathbf{y}}_k = \boldsymbol{\mathcal{C}}\bar{\mathbf{x}}_k. \tag{9b}$$

where the state dimension is Nn_x, and $\mathbf{S_{xu}}$ and $\boldsymbol{\mathcal{C}}$ have dimensions $Nn_x \times Nn_u$ and $Nn_y \times Nn_x$ respectively.

III. Controllability Aspects

3.1. State Controllability

Since the dynamics in (9) consist of a set of integrators, the controllability matrix $\mathscr{C}$ is given by $\mathbf{S_{xu}}$ repeated N times, i.e.,

$$\mathscr{C} = \begin{pmatrix} \mathbf{S_{xu}} & \cdots & \mathbf{S_{xu}} \end{pmatrix} \tag{10}$$

and the rank is simply given by

$$\operatorname{rank} \mathscr{C} = \operatorname{rank} \mathbf{S_{xu}}. \tag{11}$$

From Theorem 1 it follows that the batch system in (9) is controllable if and only if $\operatorname{rank} \mathscr{C} = \operatorname{rank} \mathbf{S_{xu}} = Nn_x$. A necessary and sufficient condition for controllability of (9) is presented in Theorem 2.

Theorem 2 *The batch system* (9) *is controllable according to Definition 1 if and only if* $\operatorname{rank} \mathbf{B_u} = n_x$.

Proof 1 *Exploiting the structure of* $\mathbf{S_{xu}}$ *gives*

$$\mathbf{S_{xu}} = \boldsymbol{\Psi}\boldsymbol{\mathcal{B}}$$

with

$$\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{I} & 0 & \cdots & 0 \\ \mathbf{A} & \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{N-1} & \mathbf{A}^{N-2} & \cdots & \mathbf{I} \end{pmatrix},$$

$$\boldsymbol{\mathcal{B}} = \begin{pmatrix} \mathbf{B_u} & 0 & \cdots & 0 \\ 0 & \mathbf{B_u} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B_u} \end{pmatrix}$$

where it can be noted that the matrix $\boldsymbol{\Psi}$ *is square and triangular with all diagonal elements equal to 1. The determinant of a square triangular matrix is equal to the product of the diagonal elements [14], hence* $\det \boldsymbol{\Psi} = 1$ *and* $\boldsymbol{\Psi}$ *is non-singular. It now follows that*

$$\operatorname{rank} \mathbf{S_{xu}} = \operatorname{rank} \boldsymbol{\Psi}\boldsymbol{\mathcal{B}} = \operatorname{rank} \boldsymbol{\mathcal{B}} = N \operatorname{rank} \mathbf{B_u}.$$

The system is therefore controllable if and only if $\operatorname{rank} \mathbf{B_u} = n_x$.

Corollary 1 *A necessary condition for the system in* (9) *to be controllable is that* $n_u \geq n_x$.

Proof 2 *It is given that* $\mathbf{B_u} \in \mathbb{R}^{n_x \times n_u}$, *hence* $\operatorname{rank} \mathbf{B_u} \leq \min\{n_x, n_u\}$. *It is therefore necessary to have* $n_u \geq n_x$ *to be able to obtain* $\operatorname{rank} \mathbf{B_u} = n_x$.

The interpretation is that the number of inputs has to be larger than or equal to the state dimension to achieve controllability. Controllability in the iteration domain is therefore a very strong requirement. By definition the elements of $\bar{\mathbf{x}}_k$ consist of the state variables of the system in the different time instances within one iteration.

3.2. Output Controllability

The requirement of state controllability is very strict, but in many cases it is only necessary that the output follows a desired trajectory. A condition for output controllability for the system in (9) is presented in Theorem 3.

Theorem 3 *The batch system in* (9) *is output controllable if and only if*

$$\operatorname{rank} \boldsymbol{\mathcal{C}}\mathbf{S_{xu}} = Nn_y. \tag{12}$$

Proof 3 *From* (10) *and* (4) *it can be concluded that* $\mathscr{C}^o = \boldsymbol{\mathcal{C}}\mathscr{C}$ *for the system in* (9). *Hence, the system is output controllable if and only if*

$$\mathscr{C}^o = \begin{pmatrix} \boldsymbol{\mathcal{C}}\mathbf{S_{xu}} & \cdots & \boldsymbol{\mathcal{C}}\mathbf{S_{xu}} \end{pmatrix} \tag{13}$$

has full rank, i.e., $\operatorname{rank} \mathscr{C}^o = Nn_y$. *The result follows directly from the fact that* $\operatorname{rank} \mathscr{C}^o = \operatorname{rank} \boldsymbol{\mathcal{C}}\mathbf{S_{xu}}$.

Remark 1 *In the SISO case the matrix* $\mathscr{C}^o$, *determining the output controllabilty, becomes the* *matrix of the Markov parameters g_k of the input-output representation*

$$y(t) = \sum_{k=1}^{\infty} g_k u(t-k) \tag{14}$$

i.e.

$$\mathscr{C}^o = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ g_2 & g_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g_N & \cdots & g_2 & g_1 \end{pmatrix}. \tag{15}$$

where it is assumed that $g_0 = 0$. In case the time domain system (2) contains additional delays, $g_1 = 0$ and the matrix $\mathscr{C}^o$ will loose rank. In this case the system will not be output controllable, which in practice means that the output $y(1)$ cannot be affected by the input $u(0)$.

Output controllability is closely connected to the assumption that the matrix of Markov parameters has full row rank, which is used to proof convergence of ILC algorithms in [13, 12, 11].

Note that a general controllable LTI system is not necessarily output controllable in a batch formulation, and a general output controllable batch system is not necessarily controllable in an LTI system sense. It follows from (12) that if the system is output controllable the matrix $\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}$ must have Nn_y independent rows. Hence, the measurements in the time domain model (2) must be independent, in the sense that $\operatorname{rank}\mathbf{C} = n_y$. Theorem 2 presents a necessary, but not sufficient, condition for output controllability.

Corollary 2 *Assume* $\operatorname{rank}\mathbf{C} = n_y$. *A necessary condition for system* (9) *to be output controllable is that* $\operatorname{rank}\mathbf{B_u} \geq n_y$.

Proof 4 *The rank of a product of two matrices is less than or equal to the minimum of the rank of each matrix [14], hence*

$$\begin{aligned} \operatorname{rank}\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}} &\leq \min\{\operatorname{rank}\boldsymbol{\mathcal{C}}, \operatorname{rank}\mathbf{S}_{\mathbf{xu}}\} \\ &= \min\{N\operatorname{rank}\mathbf{C}, N\operatorname{rank}\mathbf{B_u}\} \end{aligned} \tag{16}$$

From the assumption $\operatorname{rank}\mathbf{C} = n_y$ *it follows that a necessary condition to have* $\operatorname{rank}\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}} = Nn_y$ *is to have* $\operatorname{rank}\mathbf{B_u} \geq n_y$

3.3. Target Path Controllability

In ILC it is of interest to reach a desired trajectory, in as few steps as possible and then be able to follow that trajectory for the complete iteration. This is exactly what the concept of target path controllability (TPC) [7] is considering. Target path controllability can be used to investigate if it is possible to track any given reference trajectory over a time interval given any initial state. A formal definition can be found in Definition 2, where $\bar{\mathbf{r}}(l, m)$ symbolises the vectors $\mathbf{r}(t)$ for $t = l, \ldots, m$ stacked on top of each other. The same notation holds for $\bar{\mathbf{u}}(l, m)$.

Definition 2 (Target path controllability [7]) *Let p and q be positive integers. A linear time-varying system is said to be target path controllable at t_0 with lead p and lag q, if for any initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and for any reference output trajectory $\bar{\mathbf{r}}(t_0 + p, t_0 + p + q - 1)$, there exists a control sequence $\bar{\mathbf{u}}(t_0, t_0 + p + q - 2)$ such that $\mathbf{y}(t) = \mathbf{r}(t)$ for $t = t_0 + p, \ldots, t_0 + p + q - 1$.*

The target path controllability will be abbreviated as TPC$(t_0; p, q)$. For $q = \infty$ the system is said to be globally TPC at t_0 with lead p. In this paper only LTI systems are considered, therefore the starting time $t_0 = 0$ is used without loss of generality, and TPC$(p, q) \triangleq$ TPC$(0; p, q)$. Theorem 4 provides a necessary and sufficient condition for TPC of an LTI system.

Theorem 4 *A linear time-invariant system is* TPC(p, q) *if and only if* $\operatorname{rank}\mathbf{S_{yu}}(p, q) = qn_y$, *where*

$$\mathbf{S_{yu}}(p, q) = \begin{pmatrix} \mathbf{CA}^{p-1}\mathbf{B_u} & \cdots & \mathbf{CB_u} & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ \mathbf{CA}^{p+q-2}\mathbf{B_u} & & \cdots & & \mathbf{CB_u} \end{pmatrix}.$$

Proof 5 *The result follows directly from [7, Lemma 8] using the* LTI *model in* (2).

The relation between TPC and output controllability is presented in Corollary 3.

Corollary 3 *Output controllability of the system in* (9) *is equivalent to the system in* (2) *being* TPC$(1, N)$.

Proof 6 *From Theorem 3 it holds that the system in* (9) *is output controllable if and only if* $\operatorname{rank}\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}} = Nn_y$. *Using $p = 1$ and $q = N$ gives*

$\mathbf{S_{yu}}(1, N) = \mathcal{C}\mathbf{S_{xu}}$ *and from Theorem 4 it follows that the system is* TPC$(1, N)$ *if and only if* rank $\mathbf{S_{yu}}(1, N) = Nn_y$. *Hence, the two properties are equivalent.*

Remark 2 *In the SISO case the matrix determining if the system is TPC(p,q) becomes*

$$\mathbf{S_{yu}}(p, q) = \begin{pmatrix} g_p & \cdots & g_1 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ g_{q-1} & & \cdots & & g_1 \end{pmatrix}.$$

In some cases it is useful to stress that a system is TPC with as small lead as possible therefore an extention to TPC is defined.

Definition 3 (Minimum Lead TPC) *Given the definition of* TPC *in Definition 2, the Minimum lead target path controllability is given by* MLTPC(p, N) *where p is the smallest number that satisfy* TPC(p, N).

Minimum lead TPC is a system property, and if the system operates under feedback the analysis should be done for the closed loop system. In next section the concept MLTPC will be illustrated in an example.

IV. Illustration of the Controllability Aspects

Given the different results on controllability from Section III some aspects related to controllability of the dynamic system and the iterative learning control algorithm are given next.

4.1. Controllability in a Dynamic System

Consider a second order ($n_x = 2$) continuous time system where the states are position and velocity, and the input is the acceleration,

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t). \qquad (17)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} p(t) & v(t) \end{pmatrix}^{\mathsf{T}}$$

and $p(t)$ and $v(t)$ denote position and velocity respectively. Consider now discretization using two different approaches, where Case I uses zero order hold and Case II uses the Euler forward method.

Case I: Discretisation of (17) using zero order hold gives the discrete-time model

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} T_s^2/2 \\ T_s \end{pmatrix} u(t), \qquad (18)$$

where T_s is the sample time. The batch vector $\overline{\mathbf{x}}$ becomes

$$\overline{\mathbf{x}} = \begin{pmatrix} p(1) & v(1) & p(2) & v(2) & \cdots & p(N) & v(N) \end{pmatrix}^{\mathsf{T}}.$$

From Theorem 2 it follows that the system in (9) is controllable if and only if rank $\mathbf{B_u} = n_x = 2$. Here, rank $\mathbf{B_u} = 1$ hence the system is not state controllable. If the position or the velocity is considered as the output, i.e., $n_y = 1$, then the necessary condition for output controllability from Theorem 2 is satisfied, and hence the system can be output controllable. Investigation of the matrix $\mathcal{C}\mathbf{S_{xu}}$ shows that the system is output controllable in both cases, and this can be understood as follows. Discretization using zero order hold of the continuous time model, with no time delay, gives a discrete time model with one time delay, which is taken into account when forming the batch model. Hence $g_1 \neq 0$ in the resulting matrix of Markov parameters and the system is MLTPC$(1, N)$.

Case II: Using the Euler method to discretise (17), then the discrete-time model becomes

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ T_s \end{pmatrix} u(t). \qquad (19)$$

Also here rank $\mathbf{B_u} = 1$, and hence the system is not controllable. The output controllability depends on which signal that is considered as output. Considering the position as output, i.e. $\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ gives the first row in $\mathcal{C}\mathbf{S_{xu}}$ equal to zero, since $g_1 = \mathbf{CB_u} = 0$, and hence the rank condition for $\mathcal{C}\mathbf{S_{xu}}$ is not satisfied. It means that the input signal does not affect the position directly in the next time step. The reason is that using the Euler method for discretization introduces an extra time delay between the input and the position. Since $g_1 = 0$ it will not be possible to have TPC with lead $p = 1$, and the first row of $\mathbf{S_{yu}}(1, q)$ will be zero. However, removing the first row with only zeros in $\mathcal{C}\mathbf{S_{xu}}$ gives the matrix $\mathbf{S_{yu}}(2, N-1)$. The conditions in Theorem 4 are now satisfied, hence the system is MLTPC with lead 2, according to Definition 3. If instead the velocity is used as output the necessary condition for output controllability from Theorem 2 is still satisfied. Consider again that position is the output which gives a system MLTPC with lead 2.

Theorem 4 states a requirement for the system to be TPC, which means that any output or reference should be possible to track. For specific trajectories, Remark 3 presents a necessary and sufficient condition for tracking.

Remark 3 *(*Strongly admissible reference trajectory [7, Theorem 8]*) A reference trajectory* $\overline{\mathbf{r}}(p, p+q-1)$ *is strongly admissible if and only if*

$$\operatorname{rank}\left(\left(\mathbf{S_{yu}}(p,q), \overline{\mathbf{z}}(p,p+q-1)\right)\right) = \operatorname{rank}\mathbf{S_{yu}}(p,q)$$

where

$$\overline{\mathbf{z}}(p,p+q-1) = \begin{pmatrix} z(p)^T & \dots & z(p+q-1)^T \end{pmatrix}^T$$

and $\mathbf{z}(i) = \mathbf{r}(i) - \mathbf{CA}^i\mathbf{x}(0)$.

Remark 3 states that the reference should lie in the range space of $\mathbf{S_{yu}}(p,q)$ to be admissible.

Given a reference signal where the first element is zero (and assuming zero initial condition) enables $\overline{\mathbf{r}}(1, N)$ to be strongly admissible, even though the system is not TPC$(1, N)$. The property, strongly admissible, is related to the reference signal and not to the system in general. Instead, TPC relates to a system property and works for any reference.

4.2. TPC and Iterative Learning Control

Controllability, and especially MLTPC, are highly useful for the ILC design. Given a system, as in Figure 1, a natural first step in an ILC design is to check MLTPC. It is important to note that in a case where ILC is applied to a closed loop system it is the closed loop system that should be checked. From the definition of MLTPC it follows that for a system with MLTPC(p, N), an ILC algorithm should be able to reduce the error according to,

$$\|\mathbf{e}_k(t)\| \to 0, \quad k \to \infty, \quad t = 0, \dots, N-1.$$

From Definition 2 it also follows that the ILC control signal, in this case should be

$$u_k(t),\ t = -p, \dots, N-p-1. \tag{20}$$

Given the information about MLTPC it is possible to provide a starting point for a simple ILC algorithm of P-type, as shown in the next example.

Example 1 *Consider Case II from the previous section, whith position as output. Before applying ILC it is assumed that the system is stabilized using state feedback. Since the state feedback will not change the structure of* B_u *and* C *in the state space system and hence the system is still* MLTPC$(2, N)$*, as shown earlier. A general P-type ILC algorithm is given by,*

$$u_{k+1}(t) = u_k(t) + \kappa q^{\delta} e_k(t), \tag{21}$$

where the gain κ *and the lead term* δ *are the design parameters. In this case it is natural to choose* $\delta = 2$*, according to, e.g. [16], since lead here represents the system delay. The* κ *value has to be chosen to get a stable ILC system. Typically this is achieved for* $0 < \kappa \leq 1$ *if the closed loop system has static gain* 1.

Hence, it is important the analysis of MLTPC before the algorithm design. Given that one can apply an input signal for $t < 0$, an input of the type in (20) will enable the error to converge to zero. When, however, the input signal can be applied for $t \geq 0$, the error can only be guaranteed to approach zero from time $t \geq p$.

V. Conclusions

Controllability is a fundamental property of a control system and it applies naturally to ILC. Provided a batch formulation of the controlled system, state controllability, however, becomes a very hard requirement to fulfil, since the number of independent inputs must be at least as many as the states of the system. In general, output controllability is a more realistic requirement and in this case the number of inputs to the system must be at least as many as the outputs, which is satisfied for most systems. In practice also output controllability fails in many cases and therefore Target Path Controllability (TPC) can be used to further extends the controllability concept with a time, lead time, in which the input will affect the output. To be more useful and also consistent, the concept of minimal lead TPC is introduced and it is shown how this concept can be naturally used in the design process for ILC when a P-type ILC algorithm is used. The minimal lead TPC concept can be used with any ILC algorithm and results are valid for multivariable systems. The implications of the results for evaluation and algorithm design in a multi variable case is left for future work.

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