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A generic framework for monetary performance attribution

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Abstract

We propose a generic framework for performance attribution in monetary terms. Through a second-order Taylor approximation, the changes in portfolio value are attributed to a set of systematic risk factors. By considering two error terms arising from the Taylor approximation, combined with an exact definition of the carry term, we derive a residual-free performance attribution framework, where we exert control over the size of the error terms. The framework incorporates foreign exchange rates and transaction costs, which is illustrated by simulating a European investor acting on the U.S. fixed income market. For the out-of-sample period, we show that we can attribute almost all portfolio value differences and variance using six risk factors obtained from principal component analysis. The results show that our method, in combination with high-quality estimates of risk factors, outperforms other fixed-income attribution models from the literature.

Keywords: Performance Attribution, Performance Analysis, Fixed Income

2000 MSC: 91G30, 91G10, 91G99

1. Introduction

Performance attribution and risk measurement can be viewed as dual aspects of accumulation of wealth, as both rely on an accurate description of the systematic risk factors. In risk measurement, risk factors are studied as random variables, whereas performance attribution is carried out by studying their ex-post realizations. If the portfolio variance can be accurately described by a limited set of risk factors, it enables us to study both the ex-ante risks, as well as the ex-post returns of an investment, in terms of these risk factors. It is thus possible to determine the origin of the returns, and whether the returns stem from the intended investment strategy and decision-making process. Usually, this analysis is carried out relative to a benchmark portfolio, for which the historical returns and portfolio weights are known.

The foundation of equity performance attribution was established by Brinson and Fachler (1985) and Brinson et al. (1995). Here, the excess returns are divided into an asset allocation effect, a security selection effect, and an interaction term which ensures that the attribution terms sum up to the active return. Ankrim and Hensel (1994) extend this model to incorporate currency management effects by including a currency forward premium term and a term for currency surprise effects. These types of decomposition models are still being used, e.g. in the recent paper by Chen et al. (2018) where managerial skills are studied.

Even though fixed income prices are driven by changes in the yield curve, fixed income attribution was originally not distinguished from equity attribution. Ankrim (1992) proposed a risk-adjusted performance attribution model, where beta is used as a proxy for risk. He acknowledged that there are limitations when beta is used for other security classes than equities, such as bonds, and suggested that duration-driven risk adjustments could be used instead. Litterman and Scheinkman (1991) showed that changes in the yield curve to a large extent could be explained by three orthogonal risk factors named shift, twist and butterfly. Kuberek (1995) incorporated this idea into his framework, using duration with respect to these three factors. This idea was extended by the work of Colin (2006), Zambruno (2008) and Daul et al. (2016).

Performance attribution over multiple periods poses challenges related to the properties of aggregated returns relative to a benchmark. Arithmetic attribution accumulates additively over assets, but compounds multiplicatively over time. This causes problems when linking arithmetic attributions over time in a residual-free...
manner (Menchero, 2005). Geometric attribution, on the other hand, uses logarithmic returns, which accumulates additively over time, but poses problems when attribution effects are aggregated over assets within a single period. Arithmetic and geometric attribution simply measure relative performance differently and the choice is based on application and preference, where the arithmetic attribution is usually preferred due to its intuitive appeal (Ryan, 2006). To link single-period arithmetic results, several smoothing algorithms have been developed, see the work of Carino (1999), Frongello (2002) and Menchero (2004, 2005). Several desirable properties for multi-period attribution have been presented in the above-mentioned literature. The most important properties are for the attribution to be residual-free, robust and fully linkable over multiple periods, while maintaining its intuitiveness and transparency (Menchero, 2004). Furthermore, it is desirable for the attribution method to be commutative, non-acausal and sincere, meaning that the result is independent of the order of the periods, not dependent on future values, and that excess return should contribute only to the period in which it occurs (Frongello, 2002). The method should also be general, while its multi-period results should be familiar to (i.e. have the same interpretation as) the single-period results (Carino, 1999).

In this paper, we present a generic framework for monetary performance attribution together with empirical evidence from the fixed income market. The foundation of the framework is built on a second-order Taylor approximation of the price sensitivities to the risk factors. While performing a Taylor approximation of the appropriate risk factors has previously been carried out by Colin (2006), Zambruno (2008), Daul et al. (2016), we present a much more accurate, detailed and residual-free decomposition in monetary terms. This follows from introducing two additional factors. The first one is the result of omitting insignificant risk factors and the second one is the error from performing the Taylor approximation. By conducting the analysis in monetary terms, we circumvent the problems of multi-period attribution. All the desired properties for multi-period attribution mentioned earlier are more easily satisfied when earnings are measured in monetary terms. Performing an accurate analysis in monetary terms is also a prerequisite for an accurate analysis in terms of returns. Additionally, returns do not make much sense for assets such as derivatives, whose values may fluctuate around zero.

Our performance attribution framework is model-free and can be applied to any asset type, using any set of risk factors derived from any model. Therefore, we present the framework in a general form. In addition, we provide an illustration of fixed income attribution of interest rate derivatives. The reason for choosing the fixed income market is that we are able to identify high-quality systematic risk factors for interest rate risk (Blomvall, 2017). Another reason is that this allows us to mimic the models by Zambruno (2008) and Daul et al. (2016) within our framework, and thereby compare our results. We focus on plain money market interest rate derivatives instead of bonds since the additional risk factors arising from credit risk and possible liquidity risk are harder to measure with the desired quality. This poses no limitation to the framework, and ideally, we believe that credit risk should be dealt with in the same way as the risk-free interest rate risk, using principal component analysis of default intensities in a reduced form model setting, see for instance Bielecki and Rutkowski (2004). To illustrate the importance of high-quality risk factors, we also compare our results with the results obtained from bootstrapped term structures interpolated by natural cubic splines.

The paper is disposed as follows. First, we present the theory of the generic framework in section 2. In section 3, we illustrate the theory using a more detailed description of how to perform fixed income attribution. In section 4 we present a description of the data and the portfolio used in the simulation. The results are presented in section 5, followed by a brief discussion in section 6, before our conclusions in section 7.

2. Theory

Assume that the theoretical price $P_{t,i}$ of a financial asset $i \in I$, at time $t$, is a function of the state of $M$ financial quantities, $\tilde{\xi}_{j,t}, j = 1, \ldots, M$. This price is primarily affected by the movements, $\Delta \tilde{\xi}_{j,t}, j = 1, \ldots, m$, in $m$ significant risk factors, where usually $m \ll M$. The change in price between $t-1$, and $t$ can be approximated by the differences in the states of these significant risk factors, according to

$$\Delta P_{t,i} = P_{t,i}(\tilde{\xi}_{t}) + D_{t,i} - P_{t-1,i}(\tilde{\xi}_{t-1}) \approx \Delta P^S_{t,i}(\Delta \tilde{\xi}_{t}), \quad (1)$$

where $D_{t,i}$ represents possible dividend from instrument $i$ since the last time $t-1$ and $\Delta P^S$ is a function of the

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1Colin (2006) presents a performance attribution model based on how the yield of an asset relates to yield curve changes in terms of shift, twist and curvature. By evaluating the price given first shift, followed by shift/twist and finally shift/twist/curvature, the performance attribution will not be order independent, and the factors will also contain higher-order terms. Since the Colin (2006) model is set up in terms of yields for individual instruments, it does not fit into our performance attribution framework.
significant risk factors, $\Delta \xi_i \in \mathbb{R}^{m \times 1}$, hence the superscript $S$. Performing a second-order Taylor approximation of the price with respect to the risk factors allows us to study the changes in price for small perturbations to these risk factors

$$\Delta P_{t,i}^S(\Delta \xi_i) \approx g^T_{t-1,i} \Delta \xi_i + \frac{1}{2} \Delta \xi_i^T H_{t-1,i} \Delta \xi_i, \quad (2)$$

where the gradient and Hessian are given by

$$g_{t-1,i} = \nabla_P P_{t,i} |_{\Delta \xi_i = 0}, \quad (3)$$

$$H_{t-1,i} = \nabla^2_P P_{t,i} |_{\Delta \xi_i = 0}. \quad (4)$$

Note that the Taylor approximation of the pricing function at time $t-1$ is performed using the pricing function at time $t$ and evaluated using the information available at time $t-1$. So far, this is a rather crude approximation, since we do not account for the interest gained between time $t-1$ and $t$, known as carry effect. Introducing a carry term and two error terms will allow us to replace the approximations in (1) and (2) with equalities.

Without loss of generality, we assume that the risk factors are sorted in terms of decreasing significance. We can express the price change resulting from a move in risk factors as

$$\Delta P_{t,i}(\Delta \xi_i) \equiv P_{t,i}(\tilde{\xi}_{t-1} + \Delta \xi_i) + D_{t,i} - P_{t-1,i}(\tilde{\xi}_{t-1}), \quad (5)$$

where $\tilde{\xi}_i = \xi_{t-1} + \Delta \xi_i$. We define the error term arising from truncating the insignificant risk factors from the full set,

$$e_{t,i}^l \equiv \Delta P_{t,i}(\Delta \xi_i) - \Delta P_{t,i}(\Delta \xi_i ; 0) = P_{t,i}(\tilde{\xi}_{t-1} + \Delta \xi_i) - P_{t,i}(\tilde{\xi}_{t-1}; 0). \quad (6)$$

In addition to the Taylor approximation of risk factor sensitivity, we also add a term to account for the passage of time between $t-1$ and $t$. This term is usually calculated as an analytical derivative of the pricing function with respect to time, multiplied with the elapsed time $\Delta t$. Instead, we define this term as

$$\theta_{t-1,i} \Delta t \equiv P_{t,i}(\tilde{\xi}_{t-1}) + D_{t,i} - P_{t-1,i}(\tilde{\xi}_{t-1}), \quad (7)$$

where the first term involves pricing the instrument at time $t$ using the information available at $t-1$. Unlike an analytical time-derivative, this method does not introduce any approximation error caused by $\Delta t$ not being infinitesimal in practice. In fixed income performance attribution, this term is known as the carry. Some authors further split this term into accrued interest over the period and the roll-down effect caused by the instrument “rolling down” the slope of the yield curve. For a further discussion of the roll-down effect, see the work of Ilmanen (2011), Bolder (2016) and Daul et al. (2016).

Combining the Taylor approximation of risk factors in (2) with the time sensitivity in (7), we get an approximation of price change that includes the carry effect and the most significant risk factors

$$\Delta P_{t,i}^A(\Delta \xi_i) \equiv \theta_{t-1,i} \Delta t + g^T_{t-1,i} \Delta \xi_i + \frac{1}{2} \Delta \xi_i^T H_{t-1,i} \Delta \xi_i. \quad (8)$$

Using this definition, we can calculate the error term from the Taylor approximation as

$$\Delta P_{t,i}(\Delta \xi_i ; 0) \equiv \Delta P_{t,i}^A(\Delta \xi_i) + \Delta e_{t,i}^l \equiv \Delta \epsilon_{t,i}^p + \Delta \epsilon_{t,i}^c. \quad (9)$$

This error is caused by omission of higher-order terms in the Taylor series, plus the effect of taking a non-infinitesimal step time.

In cases when we can observe a quoted market price, $\bar{P}_{t,i}$, we can calculate the difference from our theoretical price, $P_{t,i}$, as

$$\epsilon_{t,i}^p \equiv \bar{P}_{t,i} - P_{t,i}(\tilde{\xi}_i). \quad (10)$$

Errors in this term is caused by noise in the observed market price, or from the theoretical valuation model used to price the instrument with the measured financial quantity $\tilde{\xi}_i$. Studying price changes $\Delta P_{t,i} = \bar{P}_{t,i} - \bar{P}_{t-1,i}$, in combination with theoretical price errors, $\Delta \epsilon_{t,i}^p = \epsilon_{t,i}^p - \epsilon_{t-1,i}^p$, yield

$$\Delta P_{t,i}(\Delta \xi_i) \equiv (5) P_{t,i}(\tilde{\xi}_{t-1} + \Delta \xi_i) + D_{t,i} - P_{t-1,i}(\tilde{\xi}_{t-1}) \equiv (10) \bar{P}_{t,i} - \epsilon_{t,i}^p + D_{t,i} - \bar{P}_{t-1,i} + \epsilon_{t-1,i}^p$$

$$\Delta \epsilon_{t,i}^p \equiv \Delta P_{t,i} - \Delta \epsilon_{t,i}^c. \quad (11)$$

This forms the foundation of our performance attribution framework where we use a second-order Taylor approximation to decompose value differences of each instrument into first- and second-order sensitivities to some appropriate set of risk factors, plus an interest component due to the passage of time. We have derived two error terms related to the Taylor approximation: $\epsilon_{t,i}^A$ is caused by the Taylor approximation itself and $\epsilon_{t,i}^c$. 

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2It would also be possible to replace $\Delta \xi_i = 0$ with the more general $E(\Delta \xi_i | F_{t-1})$, where this expectation is taken under the filtration $F_{t-1}$, representing all information available up until time $t-1$.

3It would also be possible to use the more general $P_{t,i}(E(\xi_i | F_{t-1}))$, where this expectation is taken under the filtration $F_{t-1}$, representing all information available up until time $t-1$. 


The first term arises from differences in asset value, for which we have made our Taylor approximation, seen in (12). The second term arises from differences in exchange rate, and the third term is a cross term between changes in value and foreign exchange rates.

We consider all trades performed at time \( t \) to be executed at the prices available at time \( t \). The holdings in this period are given by the holdings in the previous period, adjusted by the units bought, \( \Delta h_{t,i}^B \), and sold, \( \Delta h_{t,i}^S \), according to

\[
h_{t,i} = h_{t-1,i} + \Delta h_{t,i}^B - \Delta h_{t,i}^S.
\]

While these transactions affect the currency holdings, there are no attribution effects from the instruments bought at the end of the period, apart from possible theoretical price errors, \( \varepsilon_{t,i}^B \) and transaction costs. The change in portfolio value arising from changes in the values of our instrument holdings at time \( t \) thus becomes

\[
\Delta V_t^h = \sum_{i \in I} h_{t-1,i} \Delta P_{t,i}^L.
\]

Any currency holdings from the previous period are adjusted by the applicable local simple interest rate \( r_{e,t} \). Dividends from assets to which we are eligible, are paid out in the respective local currency. Money from trades affect the currency holdings in proportion to the quoted price, plus a transaction cost term, which includes fees and possible spreads. When conducting foreign exchange transactions, we add a transaction cost that reduces the amount of currency received among all traded currency pairs. We consider \( e_1 \) to be the base currency which is exchanged for \( f_{e_1,e} \) units of the term currency \( e_2 \), resulting in the updated currency holdings \( \Delta h_{t,e_1,e}^B \) and \( \Delta h_{t,e_1,e}^S \). The value difference for each currency holding measured in local currency can thus be calculated as

\[
h_{e,t} = h_{t-1,e} (1 + r_{e,t-1} \Delta t_{e,t-1})
\]

\[
+ \sum_{i \in I} h_{t-1,i} D_{i,t} f_{e_1,e(i)}
\]

\[
+ \sum_{i \in I} (P_{i,t} - P_{i,t-1}) \Delta h_{t,i}^B
\]

\[
- \sum_{i \in I} (P_{i,t} + \Delta S_{i,t}) \Delta h_{t,i}^S
\]

\[
+ \sum_{e \in E} \Delta h_{t,e,t,e}^B
\]

\[
- \sum_{e \in E} (f_{e,t,e} + \Delta S_{e,t,e}) \Delta h_{t,e,t,e}^B
\]

where \( \Delta t_{e,t-1} \) denotes the time measured with the day count convention in the local market of \( e \). When assets

\[
\Delta P_{t,i} = \theta_{t-1,i} \Delta t + g_{t-1,i}^L \Delta \xi_t + \frac{1}{2} \Delta \xi_t^2 \bar{H}_t \Delta \xi_t
\]

\[
+ \varepsilon_{t,i}^A + \varepsilon_{t,i}^B.
\]

Without an observed market price, (11) can be used to rewrite (12) into an expression of the differences in theoretical prices, \( \Delta P_{t,i}(\Delta \xi_t) \), instead of quoted prices, \( \Delta P_{t,i} \). The error terms introduced make the analysis of price change for each individual instrument completely residual-free, which is a desired property as explained in section 1.

The change in portfolio level incorporating currency effects and transaction costs. Suppose that we have a portfolio consisting of \( n \) assets, for which we know the number of units held in each asset \( i \) at time \( t \), denoted \( h_{t,i} \). Let \( E \) denote the set of exchange rates and \( e(i) \in E \) denote the index of the exchange rate belonging to asset \( i \). Its corresponding exchange rate is thus denoted by \( f_{e(i)} \). The portfolio value in local currency can be written as the value of all instrument holdings and foreign currency holdings, \( h_{t,e} \), multiplied by applicable exchange rates,

\[
V_t = \sum_{i \in I} h_{t,i} \bar{P}_{t,i} f_{e(i)} + \sum_{e \in E} h_{t,e} f_{e,1}.
\]

Denote the change in quoted price adjusted by possible dividends, and exchange rates by

\[
\Delta \bar{P}_{t,i} = g_{t-1,i} \Delta t + g_{t-1,i}^L \Delta \xi_t + \frac{1}{2} \Delta \xi_t^2 \bar{H}_t \Delta \xi_t
\]

\[
+ \varepsilon_{t,i}^A + \varepsilon_{t,i}^B.
\]

Incorporating exchange rate effects, the change in local price for asset \( i \) can be written as

\[
\Delta P_{t,i} = (\bar{P}_{t,i} + D_{t,i}) f_{e(i)} - \bar{P}_{t-1,i} f_{e(i)}
\]

\[
= (\bar{P}_{t,i} + D_{t,i} - \bar{P}_{t-1,i}) f_{e(i)} + \bar{P}_{t-1,i} (f_{e(i)} - f_{e(i)} - \varepsilon_{t-1,i})
\]

\[
= \Delta \bar{P}_{t,i} f_{e(i)} + \bar{P}_{t-1,i} \Delta f_{e(i)}.
\]

By adding and subtracting \( \Delta \bar{P}_{t,i} f_{e(i)} \) we get

\[
\Delta P_{t,i} = \Delta \bar{P}_{t,i} f_{e(i)} + \bar{P}_{t-1,i} \Delta f_{e(i)} + \Delta \bar{P}_{t,i} (f_{e(i)} - f_{e(i)})
\]

\[
= \Delta \bar{P}_{t,i} f_{e(i)} + \bar{P}_{t-1,i} \Delta f_{e(i)} + \Delta \bar{P}_{t,i} \Delta f_{e(i)}.
\]
are bought or sold, \( s_{t,j}^B \) and \( s_{t,j}^S \) denote the transaction costs in local currency. The last term accounts for FX-transactions in each currency pair, where \( s_{t,e,j}^B \) denotes the transaction costs when currency \( e \) is bought and currency \( j \) is sold. Using (19) and (20), the change in total portfolio value in domestic currency can be summed up as

\[
\Delta V_t = \Delta V_{t}^h + \sum_{e \in E} (\Delta h_{t,e} f_{t-1,e} + h_{t-1,e} \Delta f_{t,e} + \Delta h_{t,e} \Delta f_{t,e}),
\]

where \( \Delta h_{t,e} = h_{t,e} - h_{t-1,e} \). This completes the aggregation to portfolio level, incorporating foreign exchange transactions in each currency pair, where \( s_{t,e} \) is bought or sold, \( \Delta r_{t}^{f} \) denotes our discretized term structure of forward rates at time \( t \) and \( \Delta r_{t}^{s} \) denotes our discretized term structure of spot rates at time \( t \). The price of a risk-free fixed income instrument can be obtained through automatic differentiation

\[
\frac{\partial}{\partial \Delta \xi_t} \left( r_{t,0}^{f} - r_{t,0}^{s} \right) = E^T \left( r_{t}^{f} - r_{t}^{s} \right).
\]

The price of a risk-free fixed income instrument can be obtained by discounting its cash flows with the risk-free spot rate. Hence, we convert our forward rates to spot rates, and express the corresponding spot rate risk factor innovations in terms of forward rate innovations \( \Delta \xi_t \). To convert discretized term structures between continuously compounded forward rates and spot rates, we use

\[
r_{t,j}^{f} = r_{t,0,T}^{j} - r_{t,T-1}^{j},
\]

and

\[
r_{t,j}^{s} = \frac{r_{t,0,T}^{j} (\tau_T - \tau_{T-1}) + r_{t,T-1}^{j} \tau_{T-1}}{\tau_T},
\]

where \( \tau_T \) is the time from \( t \) to \( T \). Equivalently, we can introduce the integrating matrix, \( A \in \mathbb{R}^{M \times M} \), where

\[
A_{i,j} = \begin{cases} 1/i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
B_{i,j} = \begin{cases} 1 - i, & \text{if } j = i - 1 \\ i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}
\]

to transform a vector of daily discretized forward rates, \( r_{t}^{f} \), into spot rates by \( r_{t}^{s} = Ar_{t}^{f} \). Furthermore, we can introduce the differentiating matrix \( B \), which is the inverse of \( A \), where

\[
B_{i,j} = \begin{cases} 1/i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
P_t^S(\Delta \xi_t) = \sum_{j \in J_f} c_j \exp \left[ (r_{t-1,j}^{s} + a_j^T \Delta \xi_t) \tau_{j} - (r_{t-1,j}^{s} + a_j^T \Delta \xi_t) \tau_{j} \right] \]

Differentiating\(^4\) this expression with respect to \( \Delta \xi_t \) gives

\[^4\text{Derivatives can be obtained through automatic differentiation when analytical derivatives are not available.}\]
us the gradient
\[ \nabla \xi P^f_t (\Delta \xi^f_t) = \sum_{j \in J} \left( a_{j_0} \tau_{j_0} - a_j \tau_j \right) c_j \exp \left[ \left( r_{t-1,j_0}^f + \alpha_j^f \Delta \xi^f_t \right) \tau_{j_0} - \left( r_{t-1,j}^f + \alpha_j^T \Delta \xi^f_t \right) \tau_j \right], \]  
\[ (28) \]
and the Hessian
\[ \nabla^2 \xi P^f_t (\Delta \xi^f_t) = \sum_{j \in J} \left( a_{j_0} \tau_{j_0} - a_j \tau_j \right)^2 c_j \exp \left[ \left( r_{t-1,j_0}^f + \alpha_j^f \Delta \xi^f_t \right) \tau_{j_0} - \left( r_{t-1,j}^f + \alpha_j^T \Delta \xi^f_t \right) \tau_j \right]. \]  
\[ (29) \]
Setting \( \Delta \xi^f = 0 \) in accordance with the definitions in (3) and (4) provides us with the gradient and Hessian to be used in the Taylor approximation,
\[ g_{t-1,j} = \sum_{j \in J} \left( a_{j_0} \tau_{j_0} - a_j \tau_j \right) c_j \exp \left[ \left( r_{t-1,j_0}^f + \alpha_j^f \Delta \xi^f_t \right) \tau_{j_0} - \left( r_{t-1,j}^f + \alpha_j^T \Delta \xi^f_t \right) \tau_j \right] \]  
\[ (30) \]
and
\[ H_{t-1,j} = \sum_{j \in J} \left( a_{j_0} \tau_{j_0} - a_j \tau_j \right)^2 c_j \exp \left[ \left( r_{t-1,j_0}^f + \alpha_j^f \Delta \xi^f_t \right) \tau_{j_0} - \left( r_{t-1,j}^f + \alpha_j^T \Delta \xi^f_t \right) \tau_j \right]. \]  
\[ (31) \]
This completes the derivation of the components used in the Taylor approximation for a risk-free fixed income instrument.

3.1. Term structure measurement

In order to price interest rate derivatives, and extract risk factors, term structures must be estimated. The term structure of interest rates is not directly observable but has to be measured by solving an inverse problem based on observed market prices. An important property of a well-posed inverse problem is that small changes in input data should result in small changes in output data. Hence, a term structure measurement method should be insensitive to measurement noise in observed market prices. This is not the case for the popular cubic spline interpolation methods, which tend to be unstable (Hagan, 2015). To handle this shortcoming, new methods such as tension splines (Andersen, 2007) or kriging (Cousin et al., 2016) have been proposed.

It is of course desirable to decompose the returns into a few significant risk factors, without substantial risk factor truncation errors, \( \epsilon_j^f \). To make this possible, we require a term structure measurement method that provides realistic risk factors with high explanatory power. When conducting PCA, the explanatory power is the proportion of the in-sample covariance matrix that can be explained by approximating it with the reduced set of eigenvalues and eigenvectors. Using the eigenvalues, the explanatory power can be computed as
\[ \sum_{j=1}^m \lambda_j / \sum_{j=1}^M \lambda_j. \]  
\[ (32) \]

The full set of risk factors can explain any change in the term structure, even out of sample, since it merely constitutes an orthogonal decomposition of the covariance matrix. Since instruments are priced as functions of the term structure, changes in instrument prices would not introduce any truncation errors, \( \epsilon_j^f \), if the full set of risk factors were used. A reduced set of risk factors will introduce truncation errors that relate to the unexplained in-sample term structure variance. In this study we use the risk factor loadings, \( E \), out of sample, which certainly does not improve the explanatory power. Hence, high in-sample explanatory power is key to explain the differences in portfolio value out of sample.

To increase the explanatory power of PCA risk factors, spot rates are often used instead of forward rates when PCA is performed (Huij and Derwall, 2008). Laurini and Ohashi (2015) provide an excellent review of the problems caused by using forward rates when noise is present in market prices. They present a proposition that states that the number of risk factors should be the same regardless whether spot rates or forward rates are used. A difference in the number of risk factors that is needed in practice to achieve the same explanatory power for forward rates as for spot rates, is considered to be strong evidence of measurement noise in the market data. Blomvall (2017) addresses the problem with noise by allowing and penalizing market price deviations in an optimization model for measuring term structures. The measurements are based on a trade-off between smoothness and squared price errors and is accomplished by discretization and regularization of the optimization problem.⁢ This results in a finite dimensional optimization problem of measuring the term structure in a non-parametric setting. As stated by Blomvall (2017), the framework can be viewed as a generalization of earlier methods found in the literature by introducing various constraints to the optimization problem.

⁢In this study, the regularization is carried out by penalizing the discrete version of the second order derivative with exponential weights according to
\[ w^{(2)}_T = \begin{cases} \frac{1}{T^{1/4}}, & \text{if } T \leq 730 \text{ days}, \\ 1, & \text{if } T > 730 \text{ days}. \end{cases} \]  
\[ (33) \]
where these weights are scaled by a penalty parameter \( P = 10 \).
3.2. Risk factors

As mentioned in section 1, we compare our results to similar risk factors obtained from cubic spline term structures and the models by Zambruno (2008) and Daul et al. (2016). In figure 1 we illustrate the systematic risk factor loadings used in the different models. The risk factor loadings for the Blomvall (2017) and cubic spline methods are obtained through PCA using daily forward rate innovations $\Delta r^f_T = r^f_T - r^f_{T-1}$. The eigenvectors corresponding to the $m = 6$ largest eigenvalues constitutes the risk factor loadings used for this comparison, see the top panels of figure 1. To allow comparison of our method against the methods by Zambruno (2008) and Daul et al. (2016), we use their proposed risk factors in our framework to mimic the results. Zambruno (2008) proposes analytical formulas for shift, twist and butterfly in terms of forward rates. $\Delta r^f(T) = \kappa$ for shift, $\Delta r^f(T) = T - \phi$ for twist and $\Delta r^b(T) = (T - \phi)^2$ for butterfly. We choose $\kappa = 1$ and $\phi = 5$ years. To facilitate computation of risk factors, we use Gram-Schmidt to orthogonalize and normalize these risk factors. This allows us to compute risk factors the same way as for PCA. The resulting orthogonal risk factors are displayed in the bottom right panel of figure 1.

Daul et al. (2016) discuss using PCA to obtain risk factors, but instead propose key rate durations (see Ho, 1992; Nawalkha et al., 2005) at 6M, 2Y, 5Y, 10Y maturities, see the bottom left panel of figure 1. To handle convexity, Daul et al. (2016) propose a single second-order risk factor as a constant shift in spot rates. We have included this factor in the attribution results but omitted it from figure 1 since it is not an ordinary first-order risk factor and hence not belonging to the Taylor approximation. Even though Daul et al. (2016) suggest computing exposure through a numerical derivative, we use the analytical approach described earlier.

3.3. In-sample explanatory power

We illustrate the importance of high-quality term structure measurements when explaining term structure variance, particularly in forward rates. This is done by comparing Blomvall (2017) to the natural cubic spline method, commonly used by practitioners (Hagan and West, 2006). For the method developed in Blomvall (2017), the output of the model is daily spaced discrete forward rates. For the cubic spline method, spot rates are bootstrapped and interpolated using natural cubic splines. The spot rates are discretized into daily spot rates, which can be converted to forward rates. An equivalent expression of the explanatory power defined in (32) can be obtained by approximating the covariance matrix, $C$, by a subset of its largest eigenvalues and corresponding eigenvectors. Let $A$ be a diagonal matrix consisting of the $m$ largest eigenvalues of $C$, and $E$ a matrix containing their corresponding eigenvectors, the explanatory power can be computed as

$$\text{Tr}\left( E A E^T \right) / \text{Tr}(C) , \quad (34)$$

where $\text{Tr}$ denotes the trace of the matrix. Using (25) and (26), we can investigate the explanatory power of PCA for different term structure methods, in terms of both forward rates and spot rates. Many financial instruments, such as bonds and interest rate swaps, are priced in terms of spot rates. For these instruments, it may be sufficient to be able to explain the variance in spot rates in order to explain the variance of a portfolio. Other contracts, or specific portfolio strategies, are more sensitive to the variance in forward rates. For such portfolios, we must be able to explain the variance in forward rates in order to explain the portfolio variance. By performing the PCA on forward rates, we obtain a matrix, $E$, containing $M$ eigenvectors and corresponding eigenvalues in a diagonal matrix $A$. We can compute the explanatory power of spot rate variance using (32) or (34). By performing the PCA on spot rates instead, we obtain eigenvectors, $E'$ and eigenvalues $A'$. We can compute the explanatory power of $m$ spot rate risk factors, used for explaining forward rate variance, as

$$\text{Tr}\left( B E A E^T B^T \right) / \text{Tr}(C) . \quad (35)$$

The difference in the results of this computation for risk factors obtained from Blomvall (2017) and cubic splines, can be seen in figure 2. We clearly see that the Blomvall (2017) method does a better job than cubic splines to explain the forward rate variance with this reduced set of risk factors. For both methods, however, we see that we achieve a higher explanatory power of forward rate variance when the PCA is performed on forward rates. To achieve a similar explanatory power for cubic spline, 15 forward rate risk factors are needed. Using the same technique to illustrate the proportion of spot rate variance explained, we obtain the results in figure 3. Here, we see that the cubic spline with spot rate risk factors is able to explain 98.04% of the variance in spot rates using six spot rate risk factors, while the same factors only explain 43.63% of the variance in forward rates. For the Blomvall (2017) method, there is almost no difference in the amount of explained variance between forward or spot rate risk factors, indicating low sensitivity to noise.

The results in figures 2 and 3 indicate that in order to explain forward rate variance using a reduced set of risk
Since the risk factors are not data driven, and it is clear that they contradict the Blomvall (2017) data in this aspect. The Daul et al. (2016) key rate durations are not the systematic risk factors since each one of them can be avoided by investing in different maturities

4. Data

The data used for estimating term structures in this study has been retrieved from Thomson Reuters Eikon. We focus on the U.S. market due to its high liquidity and the availability of data. We split the data into an in-sample period (2002-01-04 to 2018-12-29), where we perform the performance attribution. The data consist of bid-ask yields for the U.S. 3-month LIBOR, all available forward rate agreements and the 1-10-year interest rate swaps. All these instruments are used as input to the Blomvall (2017) method for measuring term structures, while the bootstrapped curves only contain the 3-month LIBOR, the 3×6, 6×9, . . ., 18×21 months forward rate agreements and the 2-10 year swaps. Since acquired yields contain noise, we perform some basic cleaning of the data.

4.1. Cleaning yields

For each yield time series, we compute the daily differences Δyt = yt − yt−1. We investigate all dates where Δyt ≠ 0, which gives us N candidate dates t1, . . . , tN of unrealistic price movements in the form of spikes that last for one or more days (at a constant level). We assume that Δyt ∼ N(µ, σ) where µ is estimated as the sample average and σ is estimated using a centered rolling window one year back and ahead (in-sample and out-of-sample data are cleaned separately). We examine the probability of observing a change of at least ∆Yt and ∆Yt+1, where ∆Yt = yt − yt+1. If both are smaller than 10−4 and of opposite signs, we assume that we have encountered an unrealistic spike, which is removed from the yield time series. For the set of swaps, this proce-
Figure 2: The cumulative proportion of the forward rate variance explained by the first eigenvectors of the Blomvall (2017) and the cubic spline method.

4.2 Cleaning term structures and principal components

Some of the term structures estimated using cleaned data are still unrealistic. We perform PCA on forward rate innovations and compute the risk factors \( \xi_t = (\Delta r_f^t)^T \Sigma \). Utilizing, \( \Lambda \), which is the diagonal covariance matrix for \( \xi_t \), we obtain a multivariate Gaussian distribution, \( N(0, \Lambda) \), by assuming a zero mean. If two consecutive principal component observations have a probability density less than \( \alpha \), the term structure is removed from the data set. The process is then repeated with a new PCA until no additional anomalies can be found. For the out-of-sample data, we choose \( \alpha = 10^{-10} \), which removes two trading days. Since bootstrapping and cubic splines amplify noise, and we wish to compare all models over the same days, we remove the same two term structures.

Furthermore, we want to remove unrealistic term structure innovations when estimating the systematic risk factor loadings. Since PCA maximizes variance, it is sensitive to noisy observations because noise tends to increase variance. We thus remove them in the same iterative manner using a lower threshold for the in-sample data set. We use \( \alpha = 10^{-10} \), which removes a total of 14 + 13 + 8 + 4 = 39 innovations for the Blomvall (2017) method and 18 + 9 + 6 + 3 = 36 innovations for the cubic spline method.

4.3 Portfolio

To emphasize our ability to attribute value changes of a portfolio which is highly sensitive to changes in forward rates, we use forward rate agreements. To incorporate foreign exchange effects, we consider a European investor acting on the U.S. market, attributing differences in portfolio value in Euro. The portfolio consists of all forward rate agreements (FRA) in the U.S. market for which quotes are available. At our starting date 2002-01-04, this includes FRA up to 9 × 12 months. To illustrate the theoretical price errors and their origin, we systematically enter short positions for these instruments where quotes are available. We receive floating rate and pay fixed rate to a notional value of $1,000. To fill up the portfolio with instruments that are sensitive to forward rates of longer maturities, we also construct synthetic market-consistent forward rate agreements up to 117 × 120 months, spaced with three months in between. These contracts are initially traded at zero net present value (NPV), using a yield equal to the forward rate between the value date and maturity.
date. To maintain the maturity structure when the shortest contracts mature during the out-of-sample period, a new $117 \times 120$ contract is traded every three months. The paying or receiving directions of all the synthesized contracts are randomized together with the notional amount using a $N(0, 1)$-distributed random number multiplied by the base notional amount of $\$1,000$. This provides performance attribution for 112 different instruments over 4,166 days.

5. Results

In this section, we present the results from the simulated portfolio. In section 5.1, we wish to illustrate the type of information that can be extracted by the performance attribution framework. We also want to underline the importance of using high-quality risk factors by comparing the risk factors from the Blomvall (2017) method to the ones from cubic splines, their analytical counterparts from Zambruno (2008) and the key rate durations suggested by Daul et al. (2016). This is done in section 5.2.

5.1. Results with high-quality risk factors

By attributing the value changes of each instrument and each day to each component, we obtain a three-dimensional data set. Since this data is difficult to visualize all at once, we aggregate the results of all the components down to instrument level and date level respectively. Before aggregating the results, we start by studying the results for the of the shortest FRA contract, as seen in table 1. For this $1 \times 4$ month contract, a fixed cash flow is paid, and a floating three-month LIBOR is received. The floating rate is fixed at its fixing date, two business days before the value date, which in turn, is one month ahead of our trade date. Since we consider trades to be executed to close prices, there will be no risk factor contributions or FX-terms at the trade date. However, a theoretical price error, $\Delta \epsilon_{\text{P}}$, arises because our model computes a value different from the quoted market price. This value is often negative because we must cross the bid-ask spread. Exceptions may arise since our interest rate curves allow for prices to slightly deviate from the observed market prices. Usually, the deviations are within the bid-ask spread, but sometimes, as in the case with the $1 \times 4$ months FRA, the model considers the fair value of the paid fixed rate to be higher than the ask yield. Over the following month, the NPV differences are attributed to the different risk factors (some of which have been aggregated to save space), FX-terms and error terms, which we note are small. At the fixing date, the floating payment is fixed to the LIBOR rate, resulting in a difference from the fixing rate estimated.
from the term structure. Even though this technically is the realization of the risk in the LIBOR fixing, we attribute it to the carry term, in accordance with our definition in (7).

Next, we study the results over the lifetime of all instruments for which quotes were available. This allows us to study each trade in the light of the intended investment strategy. Since we have a total of 112 different trades, plus the results from our U.S. dollar holdings, we have to reduce the set of instruments to display in table 2. We consider our currency holdings to be a separate instrument. This instrument also includes cash flows separated from the instrument, yet to be paid. We consider cash flows from forward rate agreements to be separated from the instrument and transferred to the portfolio at the time of the 3M LIBOR fixing. Up until the payment date three months later, they are valued by discounting the amount to present value using the term structure. The interest accruing from these cash flows separated from the instrument, yet to be paid.

portfolio at the time of the 3M LIBOR fixing. Up until the payment date three months later, they are valued by discounting the amount to present value using the term structure. The interest accruing from these cash flows appear in the carry term for the U.S. dollar instrument. In this performance attribution, however, no interest is paid out for actual currency holdings. Since we consider a European investor, foreign exchange risk applies to the U.S. dollar holdings. We also note that we have theoretical price errors, \( \Delta \bar{P} \), for all instruments with quoted prices for instruments up to 9 \times 12 months. Since all of these instruments are short FRA contracts with the same notional value, as described in section 4.3, we can compare their sign and magnitude. For the 1 \times 4 contract we have \( \Delta \bar{P} > 0 \), as seen previously in table 1. This implies that crossing the spread to perform that trade provides immediate value to the portfolio. For the other instruments where quotes are available, the Blomvall (2017) model recognizes a cost for crossing the spread on this date.

For the last view of the performance attribution, we consider the attribution of the final portfolio value into all the factors previously described. The results can be seen in table 3, together with the results of a few individual days. We can see that the first-order risk factor components together with carry and the foreign exchange term \( \bar{P}_{t-1} \Delta \bar{f}_t \), account for a large proportion of the total value. Although all the first-order risk factors make a substantial contribution, the magnitude of the truncation error, \( \epsilon_t^f \), is small in comparison to their individual values. The second-order shift component also makes a sizable contribution, while the contribution of the sum of the remaining second-order risk factors is smaller. The error arising from the Taylor approximation, \( \epsilon_t^\Delta \), is small, as well as the FX-cross term, \( \Delta \bar{P} \Delta \bar{f}_t \). This is not always the case, since the value of the interest rate derivatives often is close to zero, as illustrated in figure 4. The theoretical price error term, \( \Delta \bar{P} \), is small by construction since it only contains pricing errors from nine different trades.

The study of the attribution for a few individual days together with the total value found in table 3 does not give us the full picture of the evolution of the attribution terms throughout the entire out-of-sample period. Figure 5 presents a clearer view of this through a graphical illustration of the accumulated decompositions in table 3. The error terms are too small to be studied in this figure, and therefore, we provide a closer look at the error terms \( \Delta \bar{P} \), \( \epsilon_t^f \) and \( \epsilon_t^\Delta \) in the left bottom panel of figure 5. It can be noted that although the contribution of the risk factor truncation error term, \( \epsilon_t^f \), has a negligible impact on the portfolio value, its daily contributions are often large, relative to their accumulated value, which can be seen in its noisy appearance. The attribution of the individual first-order risk factors, plus the second-order shift component is shown in the bottom right panel of figure 5.

The attribution of the individual factors to the final portfolio value, as seen in table 3, is similar to studying the expected contribution from each attribution factor to the portfolio value. We wish to perform the corresponding analysis for the portfolio variance to find out which factors are driving the portfolio variance. Starting from the full performance attribution in (12), we can remove 15 second-order factor duplicates by accounting for the symmetry in the Hessian. We then get a 33 \times 33 covariance matrix containing the contribution of 6 first-order risk factors, 21 second-order factors, carry, three error terms (\( \Delta \bar{P} \), \( \epsilon_t^f \) and \( \epsilon_t^\Delta \)) and two FX-terms (\( \bar{P}_{t-1} \Delta \bar{f}_t \) and \( \Delta \bar{P} \Delta \bar{f}_t \)). Accounting for symmetry in this covariance matrix...
<table>
<thead>
<tr>
<th>Date</th>
<th>Shift</th>
<th>Twist</th>
<th>Butterfly</th>
<th>4th-6th</th>
<th>∑ 2nd</th>
<th>carry</th>
<th>∆P</th>
<th>∆A</th>
<th>∆P,∆A</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/11</td>
<td>0.232</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>4/11</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>4/11</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>4/11</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>4/11</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
</tbody>
</table>

Table 1: Performance attribution of the 1-4 month FRA contract traded in the first out-of-sample day, January 4th, 2002.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Shift</th>
<th>Twist</th>
<th>Butterfly</th>
<th>4th-6th</th>
<th>∑ 2nd</th>
<th>carry</th>
<th>∆P</th>
<th>∆A</th>
<th>∆P,∆A</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD,(\text{X})=0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>USD,(\text{X})=0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>USD,(\text{X})=0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>USD,(\text{X})=0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
<tr>
<td>USD,(\text{X})=0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.033</td>
<td>0.334</td>
<td>0.0334</td>
<td>0.323</td>
</tr>
</tbody>
</table>

Table 2: Performance attribution over the lifespan of the subset of FRA contracts which had quotes available at the first day, January 4th, 2002.
Table 3: Performance attribution of all instruments displayed over a few days, and the total summed over the entire out-of-sample period. The terms are sorted by absolute contribution. Note that the most important second order risk factor Shift$^2$ contributes more to the final portfolio value than the sum all other second-order risk factors.

<table>
<thead>
<tr>
<th>Date</th>
<th>02-01-04</th>
<th>02-01-07</th>
<th>02-01-08</th>
<th>...</th>
<th>18-12-24</th>
<th>18-12-27</th>
<th>18-12-28</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butterfly</td>
<td>0</td>
<td>1.35</td>
<td>-0.842</td>
<td>...</td>
<td>0.0113</td>
<td>0.00318</td>
<td>-0.000774</td>
<td>-38.2</td>
</tr>
<tr>
<td>NPV</td>
<td>-1.1</td>
<td>3.32</td>
<td>-0.54</td>
<td>...</td>
<td>-1.43</td>
<td>0.963</td>
<td>-1.24</td>
<td>-26.6</td>
</tr>
<tr>
<td>Twist</td>
<td>0</td>
<td>1.26</td>
<td>-0.283</td>
<td>...</td>
<td>-0.0587</td>
<td>0.16</td>
<td>0.099</td>
<td>-25.6</td>
</tr>
<tr>
<td>$P_{t-1}\Delta f_t$</td>
<td>0</td>
<td>0.00172</td>
<td>-0.000961</td>
<td>...</td>
<td>-0.103</td>
<td>-0.1</td>
<td>-0.0228</td>
<td>22</td>
</tr>
<tr>
<td>carry</td>
<td>0</td>
<td>0.127</td>
<td>0.125</td>
<td>...</td>
<td>-0.00679</td>
<td>-0.0113</td>
<td>-0.00712</td>
<td>21.9</td>
</tr>
<tr>
<td>4th-6th</td>
<td>0</td>
<td>0.0585</td>
<td>0.125</td>
<td>...</td>
<td>0.12</td>
<td>-0.0749</td>
<td>0.0993</td>
<td>-11.7</td>
</tr>
<tr>
<td>Shift</td>
<td>0</td>
<td>0.507</td>
<td>0.354</td>
<td>...</td>
<td>-1.43</td>
<td>1.03</td>
<td>-1.41</td>
<td>8.49</td>
</tr>
<tr>
<td>Shift$^2$</td>
<td>0</td>
<td>-0.00151</td>
<td>-0.000747</td>
<td>...</td>
<td>-0.00615</td>
<td>-0.00317</td>
<td>-0.00593</td>
<td>-1.8</td>
</tr>
<tr>
<td>$\Delta P_t\Delta f_t$</td>
<td>0</td>
<td>-0.00508</td>
<td>0.000256</td>
<td>...</td>
<td>-0.00373</td>
<td>0.00278</td>
<td>-0.000746</td>
<td>-1.47</td>
</tr>
<tr>
<td>$\sum_{i} \text{rem. 2nd}$</td>
<td>0</td>
<td>0.00511</td>
<td>-0.000128</td>
<td>...</td>
<td>0.000227</td>
<td>4.76e-05</td>
<td>0.00427</td>
<td>-1.39</td>
</tr>
<tr>
<td>$\Delta \epsilon^1$</td>
<td>-1.1</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1.1</td>
</tr>
<tr>
<td>$\epsilon^1_t$</td>
<td>0</td>
<td>-0.00571</td>
<td>0.00521</td>
<td>...</td>
<td>0.00277</td>
<td>-0.00168</td>
<td>0.000565</td>
<td>-0.527</td>
</tr>
<tr>
<td>$\epsilon^2_t$</td>
<td>0</td>
<td>0.0325</td>
<td>-0.0231</td>
<td>...</td>
<td>0.0325</td>
<td>-0.0459</td>
<td>0.00205</td>
<td>-0.499</td>
</tr>
</tbody>
</table>

Figure 5: Attribution of the portfolio value during the entire out-of-sample period, using the Blomvall (2017) method where PCA is performed on forward rates.
matrix allows us to study \( \frac{33 \times 33}{2} = 561 \) unique terms, which we sort according to their absolute values. This gives us the covariances presented in table 4. The column of explained variance is calculated using absolute values of the covariance terms, but the total explained variance is not. The calculations performed are

\[
\text{Exp.}(i) = \frac{|c_i|}{\sum_{j=1}^{561} c_j}, \quad \text{Tot. Exp.}(i) = \frac{\sum_{j=1}^{561} c_i}{\sum_{j=1}^{561} |c_j|} \quad (36)
\]

where \( c_i \) is the total covariance of a factor pair, accounting for duplicates. This implies that the total explained variance of some of the factors will exceed 100% until negative covariance terms enter the sum. We only present the 20 most significant terms, but we clearly see that the variance of each term rapidly declines and becomes less than 0.40% for the excluded terms. The first-order risk factors, and the FX-terms make up the 14 most significant terms, while the first error term, \( \varepsilon_I \), follows by \( \theta_t \) risk factors, and the FX-terms make up the 14 most significant terms, while the first error term, \( \varepsilon_t^I \), in combination with twist accounts for merely 1.12% of the total portfolio variance. This illustrates our ability to explain the out-of-sample portfolio variance for this sensitive portfolio, without any substantial error terms, using only a few risk factors.

Table 4: Covariance/variance of the different factor pairs and their contribution to the total portfolio variance.

<table>
<thead>
<tr>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Cov.</th>
<th>Exp. (%)</th>
<th>Tot. Exp. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twist</td>
<td>Twist</td>
<td>0.44</td>
<td>43.82</td>
<td>43.82</td>
</tr>
<tr>
<td>Shift</td>
<td>Shift</td>
<td>0.32</td>
<td>31.84</td>
<td>75.66</td>
</tr>
<tr>
<td>Butterfly</td>
<td>Butterfly</td>
<td>0.12</td>
<td>11.73</td>
<td>87.39</td>
</tr>
<tr>
<td>( P_{t-1} \Delta f_t )</td>
<td>( P_{t-1} \Delta f_t )</td>
<td>0.05</td>
<td>5.09</td>
<td>92.48</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
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<td>Twist</td>
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<td>2.66</td>
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</tr>
<tr>
<td>4th</td>
<td>Butterfly</td>
<td>-0.03</td>
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<td>95.40</td>
</tr>
<tr>
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<td>2.50</td>
<td>97.90</td>
</tr>
<tr>
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<td>1.83</td>
<td>99.73</td>
</tr>
<tr>
<td>Twist</td>
<td>Shift</td>
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<td>1.71</td>
<td>101.44</td>
</tr>
<tr>
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<td>4th</td>
<td>-0.02</td>
<td>1.59</td>
<td>99.85</td>
</tr>
<tr>
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<td>0.01</td>
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<td>101.23</td>
</tr>
<tr>
<td>( P_{t-1} \Delta f_t )</td>
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<td>1.12</td>
<td>98.79</td>
</tr>
<tr>
<td>( P_{t-1} \Delta f_t )</td>
<td>Butterfly</td>
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<td>1.09</td>
<td>99.88</td>
</tr>
<tr>
<td>( P_{t-1} \Delta f_t )</td>
<td>Shift</td>
<td>-0.01</td>
<td>1.03</td>
<td>98.85</td>
</tr>
<tr>
<td>4th</td>
<td>Shift</td>
<td>0.01</td>
<td>0.49</td>
<td>99.35</td>
</tr>
<tr>
<td>( \varepsilon_t^I )</td>
<td>Butterfly</td>
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<td>0.40</td>
<td>99.75</td>
</tr>
<tr>
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<td>6th</td>
<td>-0.00</td>
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<td>99.35</td>
</tr>
<tr>
<td>( \varepsilon_t^I )</td>
<td>( \varepsilon_t^I )</td>
<td>( \varepsilon_t^I ) ( \varepsilon_t^I )</td>
<td>( \varepsilon_t^I ) ( \varepsilon_t^I )</td>
<td></td>
</tr>
<tr>
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<td>Twist/4th</td>
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</tr>
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</table>

5.2. Comparison to other models

Conducting the same analysis with the cubic spline model, using risk factors from forward rates, we obtain the results in figure 6. We see that the risk factor truncation error, \( \varepsilon_t^f \), explains the majority of the risk in the portfolio. Even though the first six forward rate risk factors explain 82.27% of the in-sample forward rate variance (see figure 2), the risk factors cannot be generalized to the out-of-sample period to the same extent as the risk factors from Blomvall (2017). Regarding the total portfolio variance, the 5:th risk factor makes the largest contribution to the portfolio variance, 25.06%, followed by \( \varepsilon_t^I \) at 19.56%. According to the results in figure 2 and 3, the cubic spline model should perform better using forward rate risk factors instead of spot rate risk factors. This is indeed the case as \( \varepsilon_t^f \) accounts for 970.55%, almost ten times the actual portfolio variance, when spot rate risk factors are used instead.

By using the analytical risk factors corresponding to shift, twist and butterfly, as suggested by Zambruno (2008), we obtain the results in figure 7. Zambruno (2008) assumes that the expected return of all assets is the instantaneous short rate. This assumption ignores the interest rate premium, captured by \( \theta_t \Delta t \) in (12), leaving us with an extra error term which we denote \( \varepsilon_t^\text{carry} \). As seen in figure 7, this error term becomes very large for this portfolio where the expected return from the short rate is negligible in comparison to the part of the carry resulting from interest rate exposure. Furthermore, the three analytical risk factors leave us with a substantial risk factor truncation error \( \varepsilon_t^f \). This is partly because these three risk factors model the short rate variance inaccurately. As can be seen in figure 5, the forth to sixth risk factors are important, and as seen in figure 2, these risk factors account for the vast majority of the variance in the short end of the term structure.

Daul et al. (2016) suggest using four key durations in combination with a factor to capture convexity. For the currency attribution, the term \( \Delta P_t \Delta f_t \) ends up in the \( P_t \Delta f_t \) term, which we have replicated. This gives us the results in figure 8. Instead of systematic changes in the yield curves, the key rate durations provide a decomposition in terms of specific key rates. These risk factors also fail to explain the systematic term structure movements, which causes substantial risk factor truncation errors \( \varepsilon_t^f \).

6. Discussion

In finance, second-order risk terms such as bond convexity or option gamma are commonly used. To use the full second-order Taylor approximation thus seems like a natural extension to first order models such as Zambruno (2008) and Daul et al. (2016). Performing a Taylor approximation usually introduces errors, \( \varepsilon_t^f \). These
Figure 6: Performance attribution of the portfolio value for the out-of-sample period using the cubic spline method where PCA is performed on forward rates.

Figure 7: Performance attribution of the portfolio value for the out-of-sample period by using the risk factors suggested by Zambruno (2008).
Figure 8: Performance attribution of the portfolio value for the out-of-sample period by using the risk factors suggested by Daul et al. (2016).

Large pricing errors, $\Delta \varepsilon_P$, are observed, the quality of the quoted prices should be investigated. If no obvious errors can be found, the theoretical pricing models should be improved. This includes improving the quality of the measurement of the financial quantities used in the pricing models. If large risk factor truncation errors, $\varepsilon_I$, are observed, one possibility is always to add more risk factors. If the risk factors are estimated using a data-driven process, such as PCA, another possibility is to improve the quality of the measurement of the financial quantities used to extract these risk factors. If large Taylor approximation errors, $\varepsilon_A$, are observed, taking smaller time steps, or including higher-order terms in the Taylor series, can be considered. For instruments with a positive definite Hessian, such as bonds, for which $c_i > 0$, large contributions from second-order terms might indicate problems with negative autocorrelation in the risk factors. We note that the term $\Delta \xi^T_t H_{i,j} \Delta \xi_t$ in (12) is a quadratic form, and that negative autocorrelation thus causes the sign of some elements in $\xi_t$ alternate over time. This leads to total contribution increasing over time if the magnitude of the elements in $\xi_t$ is large. The solution to this would also be to improve the quality of the measurement of the financial quantities in order to mitigate the autocorrelation problems, which are
present in noisy measurements.

7. Conclusion

In this paper, we present a generic framework for performance attribution in monetary terms. The framework is model-free and based on a second-order Taylor approximation of the analytical sensitivities to relevant risk factors. The formulation is made residual-free by introducing error terms caused by the Taylor approximation and the omission of insignificant risk factors. This allows us to study the size of the error terms and take appropriate action when they reach unsatisfactory levels. The framework also incorporates foreign exchange effects, transaction costs and pricing/model errors. The performance of the framework has been illustrated for a European investor acting on the U.S. fixed income market by simulating a portfolio consisting of forward rate agreements. The results are compared against the two first-order fixed income performance attribution models by Zambruno (2008) and Daul et al. (2016). We suggest using PCA of high-quality forward rate term structures to obtain risk factors, but we also conduct the corresponding analysis for cubic spline term structures. Using the method developed in Blomvall (2017), we are able to attribute the risk to six risk factors while maintaining negligible error terms with respect to contribution to both portfolio value and variance. For cubic splines, the corresponding analysis shows that risk factor truncation error dominates the portfolio variance when only six risk factors are taken into account. For the analytical risk factors proposed by Zambruno (2008) and the key rate durations proposed by Daul et al. (2016), the results show that the risk factor truncation errors become large. This exposes weaknesses in their proposed risk factors, as well as their performance attribution models.

References