Informative Path Planning in the Presence of Adversarial Observers

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***Abstract*—This paper considers the problem of gathering information about features of interest in adversarial environments using mobile robots equipped with sensors. The problem is formulated as an informative path planning problem where the objective is to maximize the gathered information while minimizing the tracking performance of the adversarial observer. The optimization problem, that at first glance seems intractable to solve to global optimality, is shown to be equivalent to a mixed-integer semidefinite program that can be solved to global optimality using off-the-shelf optimization tools.**

***Index Terms*—Informative path planning, risk minimization, global optimization**

I. Introduction

This paper addresses the problem of optimizing paths for mobile sensors that operate in adversarial environments. We consider the scenario illustrated in Fig. 1, where a mobile robot is required to gather information while avoiding being tracked by an adversarial observer. The robot is equipped with a sensor that can obtain noisy measurements of features of interest that are located within a limited sensing range. The problem is to plan a sensing path for the mobile robot that maximizes the information gathered about the features and minimizes the tracking performance of the adversarial observer.

The problem of planning sensing paths to maximize the information gathered about a given environment has been widely studied, and is commonly referred to as *informative path planning* (IPP) [1]. In its most general form, IPP is an instance of planning under uncertainty and can be formulated as a partially observable Markov decision process (POMDP) [2]. The computational complexity of solving POMDP problems is considerable, and approximations are required in practice. An example is [3], that formulates the problem of path planning for an unmanned aerial vehicle (UAV) to optimize tracking of ground vehicles as a POMDP, and proposes an approximate solution. Another application of IPP is environmental monitoring, of which [4] provides an extensive overview. A recent work in this area is [5], which develops a receding horizon approach for weed detection in precision agriculture using UAVs.

Numerical optimal control methods have been applied to solve IPP problems, *e.g.*, in [6, 7]. However, since the optimization problems in general are nonconvex, only locally

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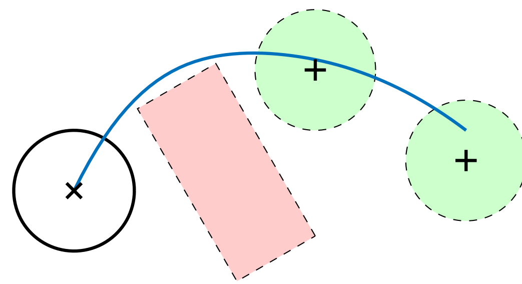

Fig. 1. A mobile robot equipped with a sensor can obtain noisy measurements of static features in the environment if they are within the sensor's sensing range. In the same environment, there is an adversarial observer with limited sensing region that tracks the mobile robot. The problem, which is illustrated in the figure, is to find a sensing path that is maximizing the information that the robot is gathering while minimizing the tracking performance of the adversarial observer. The adversarial sensing region is the red area, the initial position of the mobile robot is indicated by the ×, and the black circle illustrates its sensing range. A feature (marked with +) is visible to the mobile robot if the robot is located within the corresponding green region.

optimal solutions can be guaranteed and no certificate of the level of suboptimality is provided. Furthermore, the quality of the solution is dependent on the initial guess used by the optimization solver [8].

Our previous work [9] addresses IPP for environmental monitoring, and formulates the problem as a nonlinear mixed-integer program with a measure of the information matrix in the objective function. The problem, which at first glance seems intractable, is shown to be equivalent to a problem that robustly can be solved to global optimality using off-the-shelf optimization tools. In this paper, we extend the work on global optimization for IPP to also handle scenarios that include adversarial observers.

The problem of planning paths to minimize the robot's exposure to adversarial observers, *i.e.*, computing *stealthy* paths, has also received some attention, and a survey of different approaches can be found in [10]. The problem formulation of this paper makes use of binary variables to indicate whether the adversarial observer obtains measurements of the mobile robot. There is thus a connection to collision avoidance problems, that can be solved using mixed-integer optimization [11], but the adversarial sensing regions in this paper are not to be considered as obstacles, as the robot is allowed to travel through them, at a certain cost.

The optimal solution to a mixed-integer optimization problem can in theory be obtained by solving the corresponding continuous problem for each possible combination of the binary variables, and picking the solution that results in the smallest objective value. However, as the number of possible combinations increases exponentially with the number of binary variables, this approach quickly becomes computationally intractable. Several approaches to solve mixed-integer problems more efficiently have been proposed; this paper uses a branch-and-bound (BnB) method [12].

BnB methods are algorithms for global optimization in nonconvex problems. They maintain lower and upper bounds on the globally optimal objective function value by solving simplified subproblems of the original problem, usually obtained using convex relaxations. As a result of these bounds, the methods can be terminated at any time with a certificate that proves the current level of suboptimality.

This paper proceeds as follows. Sections II and III describe the stealthy path planning problem and show that it can be solved to global optimality using BnB. In Section IV, the stealthy path planning problem is combined with the IPP problem from [9] and we show that the combined problem can be solved to global optimality. Numerical illustrations are then presented in Section V, and finally, the conclusions are presented in Section VI.

II. Problem Formulation

The main focus of this paper is planning stealthy paths for a mobile robot, such that it avoids being tracked by an adversarial sensor, as illustrated in Fig. 2. The problem is formulated as a mixed-integer program where the models that are defined in this section are used as constraints.

A. Motion and measurement models

Consider a mobile robot with linear dynamics according to

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where x_k is the state of the robot and u_k is the control input, both at time k. The set of feasible states is denoted by $\mathcal{X}$ and the set of admissible control inputs is denoted by $\mathcal{U}$.

The mobile robot is operating in an environment where there is a static adversarial sensor that seeks to estimate the state of the robot. The adversarial sensor obtains measurements of the robot if it is within a limited sensing region, denoted Π. The measurement model is defined by

$$y_k = \begin{cases} Hx_k + e_k, & x_k \in \Pi, \\ \emptyset, & x_k \notin \Pi, \end{cases} \quad e_k \sim \mathcal{N}(0, R), \tag{2}$$

where y_k and e_k are the measurement output and measurement noise, respectively. The measurement noise is assumed to be white and Gaussian with covariance $R \succ 0$. The discrete nature of the measurement model is captured by a binary variable γ_k for each time step, that indicates whether the sensor can obtain a measurement or not. The variable $\gamma_k = \gamma(x_k, \Pi)$ is a function of the state of the mobile robot and takes the value one if the mobile robot is within the adversarial sensor's sensing region Π and the value zero otherwise:

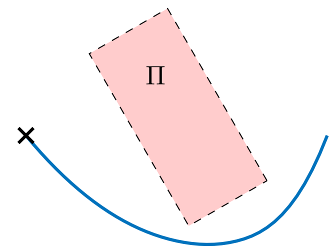

Fig. 2. A mobile robot is being tracked by an adversarial sensor, that obtains measurements if the mobile robot is within its sensing region (red area). The problem is to find a stealthy path for the mobile robot; the objective is to minimize the adversarial sensor's tracking performance.

$$\gamma_k = \begin{cases} 1, & x_k \in \Pi, \\ 0, & x_k \notin \Pi. \end{cases} \tag{3}$$

B. Information representation

The adversarial sensor is assumed to employ a Kalman filter [13] to estimate the state of the mobile robot. The true dynamics of the robot might be unknown to the adversarial sensor, and a general linear motion model is applied to represent its dynamics:

$$x_{k+1} = Fx_k + w_k, \quad w_k \sim \mathcal{N}(0, Q), \tag{4}$$

where w_k is the process noise which is assumed to be white and Gaussian with covariance Q.

Let $P_{k|k}$ be the covariance matrix corresponding to the estimate of x_k given all measurements up to time step k and let P_k denote the one-step predicted covariance matrix:

$$P_k = P_{k|k-1} = FP_{k-1|k-1}F^\intercal + Q. \tag{5}$$

The recursion for the covariance matrix of the filtered estimates can then be written as

$$\begin{aligned} P_{k|k} &= f_P(P_{k-1|k-1}, \gamma_k) \\ &= P_k - \gamma_k P_k H^\intercal (HP_kH^\intercal + R)^{-1} HP_k. \end{aligned} \tag{6}$$

Note that $\gamma_k = 0$ corresponds to propagating the previous covariance matrix when there is no measurement update available at time k.

Since a Kalman filter is used to track the mobile robot, an indication of the adversarial sensor's tracking performance can be obtained by a measure of the resulting covariance matrix $P_{k|k}$. The objective of the stealthy path planning problem is to minimize the tracking performance, which corresponds to making the covariance matrix as big as possible.

C. Stealthy path planning problem

We are now ready to state the optimization problem. For notational convenience, the variables for all time steps from zero to a given prediction horizon N are concatenated to form

$\mathbf{x} = \{x_k,\ k = 0, \ldots, N\}$, $\mathbf{u} = \{u_k,\ k = 0, \ldots, N-1\}$, and $\mathbf{P} = \{P_{k|k},\ k = 0, \ldots, N\}$. Furthermore, the binary variables $\boldsymbol{\gamma} = \{\gamma_k,\ k = 1, \ldots, N\}$ are considered as auxiliary decision variables connected to the state of the mobile robot via constraints. The stealthy path planning problem can then be formally stated as the following mixed-integer program:

$$\begin{array}{ll} \underset{\mathbf{x},\mathbf{u},\mathbf{P},\boldsymbol{\gamma}}{\text{minimize}} & J(\mathbf{x}, \mathbf{u}, \mathbf{P}) \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k \\ & P_{k|k} = f_P(P_{k-1|k-1}, \gamma_k) \\ & \gamma_k = \gamma(x_k, \Pi) \in \{0, 1\} \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \\ & x_0 = \bar{x}_0, \quad P_{0|0} = \overline{P}_{0|0}, \end{array} \tag{7}$$

where $J(\mathbf{x}, \mathbf{u}, \mathbf{P})$ is a function that encodes the objective of minimizing the adversarial sensor's tracking performance, $\bar{x}_0$ is the initial state of the mobile robot, and $\overline{P}_{0|0}$ is the covariance matrix corresponding to the adversarial observers's estimate of x_0. It should be noted that $\mathbf{x}$, $\boldsymbol{\gamma}$, and $\mathbf{P}$ can be computed directly from $\mathbf{u}$, $\bar{x}_0$, and $\overline{P}_{0|0}$. However, considering these variables as auxiliary decision variables connected by constraints yields a more favorable optimization problem structure.

III. Computing Globally Optimal Solutions

BnB methods make use of relaxations of the original problem when searching for the globally optimal solution to a mixed-integer program. The relaxations are obtained by relaxing the binary constraints $\gamma_k \in \{0, 1\}$, often to interval constraints $\gamma_k \in [0, 1]$. A prerequisite for the search to be correct is that the relaxations are convex problems, to guarantee that they can be solved to global optimality. Since the stealthy path planning problem (7) includes nonlinear equality constraints that correspond to the covariance matrix recursion (6), a straight-forward continuous relaxation of the problem is a nonconvex problem [14]. This section shows that if the objective function is separable in $(\mathbf{x}, \mathbf{u})$ and $\mathbf{P}$ and matrix decreasing in $\mathbf{P}$, the problem can be reformulated as an equivalent problem where the continuous relaxations are convex problems.

A. Reformulation of the nonlinear equality constraints

With reasonable assumptions on the objective function $J(\cdot)$, the nonlinear equality constraints of (7) can be equivalently replaced by convex inequality constraints. The main result of our previous work [9] will be used to prove the claim. The following definition is needed:

Definition 1: (Matrix increasing, extension of [14].) Assume $\mathcal{C} \subseteq \mathbb{S}^n$ and let $\preceq$ denote the generalized inequality associated with the positive semidefinite cone. A function $f : \mathbb{S}^n \to \mathbb{S}^m$ is called *matrix increasing* on $\mathcal{C}$ if

$$X \preceq Y,\ X \neq Y \implies f(X) \preceq f(Y),\ f(X) \neq f(Y),$$

for all $X, Y \in \mathcal{C}$. For $m = 1$, f is called *matrix increasing* on $\mathcal{C}$ if

$$X \preceq Y,\ X \neq Y \implies f(X) < f(Y).$$

Matrix decreasing is defined analogously.

The following theorem shows that nonlinear equality constraints in an optimization problem with a certain structure can be relaxed to convex inequality constraints, and that the obtained problem is equivalent to the original nonconvex problem.

Theorem 1: ([9]) Suppose that $f_0 : \mathbb{S}^n \to \mathbb{R}$ is matrix decreasing on $\mathbb{S}^n_{++}$, and that $f : \mathbb{S}^n \to \mathbb{S}^n$, is finite-valued and matrix increasing on $\mathbb{S}^n_{++}$, and that $\overline{X}_0 \in \mathbb{S}^n_{++}$. Then, the optimal solution to the problem

$$\begin{array}{ll} \underset{X_0,\ldots,X_N}{\text{minimize}} & \sum_{k=1}^{N} f_0(X_k) \\ \text{subject to} & X_{k+1} = f(X_k), \quad k = 0, \ldots, N-1 \\ & X_0 = \overline{X}_0, \end{array} \tag{8}$$

coincides with the optimal solution to the problem

$$\begin{array}{ll} \underset{X_0,\ldots,X_N}{\text{minimize}} & \sum_{k=1}^{N} f_0(X_k) \\ \text{subject to} & X_{k+1} \preceq f(X_k), \quad k = 0, \ldots, N-1 \\ & X_0 \preceq \overline{X}_0. \end{array} \tag{9}$$

For the sake of argument, consider a simplified version of the problem (7), where the binary variables $\boldsymbol{\gamma}$ are given and fixed. This problem involves continuous variables only, but is still nonconvex due to the nonlinear equality constraints. Reasonable assumptions are that the objective function is separable, *i.e.*, a weighted sum of functions of the individual variables

$$J(\mathbf{x}, \mathbf{u}, \mathbf{P}) = w_{x,u} J_{x,u}(\mathbf{x}, \mathbf{u}) + w_P J_P(\mathbf{P}), \tag{10}$$

and that $J_{x,u}(\mathbf{x}, \mathbf{u})$ is convex in $\mathbf{x}$ and $\mathbf{u}$. The simplified problem is then completely separable in $(\mathbf{x}, \mathbf{u})$ and $\mathbf{P}$, and it is thus possible to optimize over $\mathbf{P}$ separately from $(\mathbf{x}, \mathbf{u})$.

It can be shown [15] that the right-hand side of the covariance matrix recursion (6) is a matrix increasing function of the covariance matrix $P_{k-1|k-1}$, meaning that the corresponding equality constraints of the simplified problem satisfy the requirements of Theorem 1. Assuming that the part of the objective function that is a function of $\mathbf{P}$ is given by

$$J_P(\mathbf{P}) = \sum_{k=1}^{N} f_0(P_{k|k}), \tag{11}$$

where $f_0(P_{k|k})$ is a matrix decreasing function on $\mathbb{S}^n_{++}$, *e.g.*, the negative trace or the negative determinant of the covariance matrix $P_{k|k}$, all requirements of Theorem 1 are satisfied, which thus can be applied to the simplified problem.

Since the value of $\boldsymbol{\gamma}$ in the simplified problem is arbitrary, the reasoning holds for any given value of the binary variables $\boldsymbol{\gamma}$. The equality constraints representing the covariance matrix recursion in the original mixed-integer problem (7) can thus equivalently be replaced by inequality constraints according to

$$\begin{aligned} P_{k|k} &\preceq f_P(P_{k-1|k-1}, \gamma_k) \\ &= P_k - \gamma_k P_k H^\mathsf{T} (H P_k H^\mathsf{T} + R)^{-1} H P_k. \end{aligned} \tag{12}$$

Each inequality constraint in (12) can be further rewritten by introducing γ_k at appropriate places, which is possible since γ_k is binary, and then applying the Schur complement condition for positive semidefiniteness of a block matrix [14]:

$$
\begin{aligned}
P_{k|k} &\preceq P_k - \gamma_k P_k H^{\intercal}(H P_k H^{\intercal} + R)^{-1} H P_k \quad \iff \\
P_{k|k} &\preceq P_k - \gamma_k P_k H^{\intercal}(\gamma_k H P_k H^{\intercal} + R)^{-1} H P_k \gamma_k \quad \iff \\
0 &\preceq \begin{bmatrix} P_k - P_{k|k} & \gamma_k P_k H^{\intercal} \\ H P_k \gamma_k & \gamma_k H P_k H^{\intercal} + R \end{bmatrix}. \qquad (13)
\end{aligned}
$$

The last inequality involves products of the binary variable γ_k and the continuous variable P_k. Following [16], a linear representation of this product can be obtained by introducing an auxiliary real variable $\widetilde{P}_k \triangleq \gamma_k P_k$ that satisfies

$$
\widetilde{P}_k = \begin{cases} P_k, & \gamma_k = 1 \\ 0, & \gamma_k = 0. \end{cases} \qquad (14)
$$

By letting M_k be an over-estimate of the largest eigenvalue of P_k and m_k be an under-estimate of the smallest eigenvalue of P_k, the relation $\widetilde{P}_k = \gamma_k P_k$ is imposed by the following set of inequalities:

$$
\begin{aligned}
\widetilde{P}_k &\preceq \gamma_k M_k I, && (15a)\\
\widetilde{P}_k &\succeq \gamma_k m_k I, && (15b)\\
\widetilde{P}_k &\preceq P_k - (1-\gamma_k) m_k I, && (15c)\\
\widetilde{P}_k &\succeq P_k - (1-\gamma_k) M_k I. && (15d)
\end{aligned}
$$

Note that appropriate values for m_k and M_k can be obtained by propagating the initial covariance matrix $P_{0|0}$ through (6) with $\boldsymbol{\gamma} = 1$ and $\boldsymbol{\gamma} = 0$, respectively.

Inserting the new auxiliary variable $\widetilde{P}_k$ into (13) gives

$$
0 \preceq \begin{bmatrix} P_k - P_{k|k} & \widetilde{P}_k H^{\intercal} \\ H \widetilde{P}_k & H \widetilde{P}_k H^{\intercal} + R \end{bmatrix}, \qquad (16)
$$

which is a linear matrix inequality in $P_{k|k}$, P_k, and $\widetilde{P}_k$. An equivalent, linear, representation of the nonlinear constraint (13) is thus given by (15) and (16).

B. Adversarial sensing region

It is assumed that the adversarial sensing region can be described by a polyhedron in the form of the intersection of q halfspaces: $\Pi = \{x \mid a_i^{\intercal} x \leq b_i,\ i = 1, \ldots, q\}$. Let $\tilde{\gamma}_k$ be an auxiliary binary vector, where the value of the ith component, $\tilde{\gamma}_k^i$, is determined by

$$
\tilde{\gamma}_k^i = \begin{cases} 1, & a_i^{\intercal} x_k \leq b_i, \\ 0, & a_i^{\intercal} x_k > b_i. \end{cases} \qquad (17)
$$

The relation (3) between x_k and γ_k can then be modeled using a big-M formulation [17]. The following constraints:

$$
\begin{aligned}
a_i^{\intercal} x_k - b_i &> -\tilde{\gamma}_k^i M_\gamma, && i = 1, \ldots, q, && (18a)\\
a_i^{\intercal} x_k - b_i &\leq (1 - \tilde{\gamma}_k^i) M_\gamma, && i = 1, \ldots, q, && (18b)
\end{aligned}
$$

where M_γ is a sufficiently large constant coefficient [17], force all elements of the auxiliary binary vector $\tilde{\gamma}_k$ to take the value one when $x_k \in \Pi$ and at least one element of $\tilde{\gamma}_k$ to take the value zero when $x_k \notin \Pi$. This means that the condition $\gamma_k = 1$ if and only if $x_k \in \Pi$ can be achieved by letting the product of all elements in $\tilde{\gamma}_k$ represent γ_k, *i.e.*,

$$
\gamma_k = \prod_{i=1}^{q} \tilde{\gamma}_k^i, \qquad (19)
$$

The product of binary variables is equivalent to the following set of inequalities [16]:

$$
\begin{aligned}
\gamma_k - \tilde{\gamma}_k^i &\leq 0, \quad i = 1, \ldots, q, && (20a)\\
-\gamma_k + \sum_{i=1}^{q} \tilde{\gamma}_k^i &\leq q - 1. && (20b)
\end{aligned}
$$

Hence, the auxiliary binary vector $\tilde{\gamma}_k$ together with (18) and (20) can be used to represent the constraint $\gamma_k = \gamma(x_k, \Pi)$. Since both (18) and (20) are linear constraints, they are convex constraints when γ_k and $\tilde{\gamma}_k$ are considered as continuous variables.

C. Reformulated problem

Using the reformulations described in the previous two subsections, an equivalent formulation of (7) is given by the following mixed-integer semidefinite program (SDP):

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{u},\mathbf{P},\widetilde{\mathbf{P}},\boldsymbol{\gamma},\tilde{\boldsymbol{\gamma}}}{\text{minimize}} \quad & J(\mathbf{x}, \mathbf{u}, \mathbf{P}) \qquad (21)\\
\text{subject to} \quad & x_{k+1} = A x_k + B u_k \\
& 0 \preceq \begin{bmatrix} P_k - P_{k|k} & \widetilde{P}_k H^{\intercal} \\ H \widetilde{P}_k & H \widetilde{P}_k H^{\intercal} + R \end{bmatrix} \\
& \widetilde{P}_k \preceq \gamma_k M_k I \\
& \widetilde{P}_k \succeq \gamma_k m_k I \\
& \widetilde{P}_k \preceq P_k - (1-\gamma_k) m_k I \\
& \widetilde{P}_k \succeq P_k - (1-\gamma_k) M_k I \\
& P_k = F P_{k-1|k-1} F^{\intercal} + Q \\
& a_i^{\intercal} x_k - b_i > -\tilde{\gamma}_k^i M_\gamma \\
& a_i^{\intercal} x_k - b_i \leq (1 - \tilde{\gamma}_k^i) M_\gamma \\
& \gamma_k - \tilde{\gamma}_k^i \leq 0 \\
& -\gamma_k + \textstyle\sum_{i=1}^{q} \tilde{\gamma}_k^i \leq q - 1 \\
& x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \gamma_k, \tilde{\gamma}_k^i \in \{0, 1\} \\
& x_0 = \bar{x}_0, \quad P_{0|0} = \overline{P}_{0|0},
\end{aligned}
$$

where the concatenated variables $\widetilde{\mathbf{P}} = \{\widetilde{P}_k,\ k = 1, \ldots, N\}$ and $\tilde{\boldsymbol{\gamma}} = \{\tilde{\gamma}_k,\ k = 1, \ldots, N\}$ have been introduced for notational convenience.

The relaxation of (21) that is obtained by relaxing the binary constraints $\gamma_k, \tilde{\gamma}_k^i \in \{0, 1\}$ to interval constraints $\gamma_k, \tilde{\gamma}_k^i \in [0, 1]$ is a convex SDP problem. This means that a BnB method can be used to solve the problem (21) to global optimality, or to a user-defined certified level of suboptimality. Furthermore, the equivalence between (7) and (21) implies that the optimal solutions to both problems coincide, and that a globally optimal solution to the seemingly intractable problem (7) can be obtained by solving the tractable problem (21).

IV. Stealthy Informative Path Planning

In this section, the stealthy path planning problem is combined with the IPP problem considered in [9]. The combined problem, which is illustrated in Fig. 1, involves not only an adversarial sensor but also features of interest that are scattered in the environment, and a sensor mounted on the mobile robot. The task of the mobile robot is then twofold; maximizing the information that is gathered about the features while minimizing the adversarial sensor's tracking performance.

A. Information representation

The locations of the features are static and known, but each feature has an associated unknown quantity of interest that evolves according to a linear model. The mobile robot's sensor obtains measurements of features that are within a limited sensing range. As for the case with the adversarial sensor, the discrete nature of the sensing domain of the mobile robot's sensor is captured by binary variables. The variable $\delta_k^{(j)}$ takes the value one if a measurement of the jth feature is obtained at time step k, and the value zero otherwise.

An information filter [13] is employed to estimate the state of the features from the obtained measurements. The information filter maintains an information matrix to represent the accuracy of the state estimate, which can be used as a measure of the amount of information that has been gathered. By letting $\mathcal{I}_{k|k}^{(j)}$ denote the information matrix corresponding to the jth feature given all measurements up to time step k, the information matrix recursion can be written as

$$\mathcal{I}_{k|k}^{(j)} = f_{\mathcal{I}}(\mathcal{I}_{k-1|k-1}^{(j)}, \delta_k^{(j)}). \tag{22}$$

Hence, the current information matrix $\mathcal{I}_{k|k}^{(j)}$ depends only on the binary variable $\delta_k^{(j)}$ and the information matrix in the previous time step. For notational convenience, we concatenate the variables related to the information matrices to form $\boldsymbol{\delta} = \{\delta_k^{(j)},\ k = 1, \ldots, N,\ j = 1, \ldots, L\}$ and $\boldsymbol{\mathcal{I}} = \{\mathcal{I}_{k|k}^{(j)},\ k = 0, \ldots, N,\ j = 1, \ldots, L\}$, where L is the number of features.

B. Stealthy informative path planning problem

In [9], the IPP problem is formulated as a mixed-integer optimization problem with structure similar to the stealthy path planning problem (7). With the assumption of matrix decreasing objective function, it is shown that the problem can be reformulated and solved to global optimality. By combining the problem considered in [9] with the problem discussed in the previous sections of this paper, the stealthy IPP problem can be formulated as the following mixed-integer program:

$$\begin{aligned}
\underset{\mathbf{x},\mathbf{u},\mathbf{P},\boldsymbol{\gamma},\boldsymbol{\mathcal{I}},\boldsymbol{\delta}}{\text{minimize}} \quad & J(\mathbf{x}, \mathbf{u}, \mathbf{P}, \boldsymbol{\mathcal{I}}) \\
\text{subject to} \quad & x_{k+1} = Ax_k + Bu_k \\
& P_{k|k} = f_P(P_{k-1|k-1}, \gamma_k) \\
& \gamma_k = \gamma(x_k, \Pi) \in \{0, 1\} \\
& \mathcal{I}_{k|k}^{(j)} = f_{\mathcal{I}}(\mathcal{I}_{k-1|k-1}^{(j)}, \delta_k^{(j)}) \\
& \delta_k^{(j)} = \delta^{(j)}(x_k) \in \{0, 1\} \\
& x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \\
& x_0 = \bar{x}_0, \quad P_{0|0} = \overline{P}_{0|0}, \quad \mathcal{I}_{0|0}^{(j)} = \overline{\mathcal{I}}_{0|0}^{(j)},
\end{aligned} \tag{23}$$

where $J(\mathbf{x}, \mathbf{u}, \mathbf{P}, \boldsymbol{\mathcal{I}})$ is the objective function, $\bar{x}_0$ is the initial state of the mobile robot, $\overline{P}_{0|0}$ is the covariance matrix corresponding to the adversarial sensor's estimate of x_0, and $\overline{\mathcal{I}}_{0|0}^{(j)}$ is the initial information matrix corresponding to the jth feature.

C. Reformulations

Assuming that the objective function is separable, *i.e.*,

$$J(\mathbf{x}, \mathbf{u}, \mathbf{P}, \boldsymbol{\mathcal{I}}) = w_{x,u} J_{x,u}(\mathbf{x}, \mathbf{u}) + w_P J_P(\mathbf{P}) + w_{\mathcal{I}} J_{\mathcal{I}}(\boldsymbol{\mathcal{I}}),$$

where $w_{x,u}, w_P$, and $w_{\mathcal{I}}$ are weighting parameters, the problem (23) is, for given $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, completely separable in $(\mathbf{x}, \mathbf{u})$, $\mathbf{P}$, and $\boldsymbol{\mathcal{I}}$. This means that if $J_{x,u}(\mathbf{x}, \mathbf{u})$ is convex in $\mathbf{x}$ and $\mathbf{u}$, and

$$J_P(\mathbf{P}) = \sum_{k=1}^{N} f_0^P(P_{k|k}), \tag{24}$$

where f_0^P is matrix decreasing on $\mathbb{S}_{++}^n$, the reasoning in Section III holds also for this problem. Hence, the nonlinear equality constraints in (23) related to the covariance matrix recursion can be equivalently replaced by auxiliary decision variables and linear inequality constraints. Furthermore, the separability means that we can optimize over $\boldsymbol{\mathcal{I}}$ separately from the other variables and thus that the reasoning in [9] can be applied. Hence, if

$$J_{\mathcal{I}}(\boldsymbol{\mathcal{I}}) = \sum_{k=1}^{N} \sum_{j=1}^{L} f_0^{\mathcal{I}}(\mathcal{I}_{k|k}^{(j)}), \tag{25}$$

where $f_0^{\mathcal{I}}$ is matrix decreasing on $\mathbb{S}_{++}^n$, the constraints in (23) related to the information matrix recursion can be equivalently replaced by linear matrix inequalities. For details on the latter reformulation, the reader is referred to [9].

The stealthy IPP problem (23) is thus equivalent to a mixed-integer SDP problem that can be solved to global optimality using off-the-shelf BnB methods.

V. Numerical Illustrations

This section presents results from simulations where the proposed method has been used to plan the path for a mobile robot. The considered scenario involves a single feature of interest and one adversarial sensing region, and was selected to illustrate the concept and to demonstrate that intuitively sound trajectories are computed. The optimization problems are solved in Matlab 2017b using YALMIP's BnB solver [18], which uses MOSEK [19] to solve the convex relaxations.

In the simulations, the dynamics of the mobile robot are modeled as a discretized near-constant velocity model, where the state $x = [p_1\ p_2\ v_1\ v_2]^\intercal$ represents the robot's two-dimensional position and its velocity, and the control input u corresponds to its acceleration. The set of admissible control inputs is given by $\mathcal{U} = \{u \mid -1 \preceq u \preceq 1\}$. The state of the feature is governed by a random walk model subject to white Gaussian noise with unit covariance. The adversarial sensor estimates the state of the robot using measurements of its position, corrupted by white Gaussian measurement noise with unit covariance. A prediction horizon of $N = 6$ is used, and the objective function is a weighted sum of the following terms:

$$J_{x,u}(\mathbf{x}, \mathbf{u}) = \sum_{k=1}^{N} x_k^\intercal V_x x_k, \tag{26a}$$

$$J_P(\mathbf{P}) = -\sum_{k=1}^{N} \mathrm{tr}(P_{k|k}), \tag{26b}$$

$$J_{\mathcal{I}}(\boldsymbol{\mathcal{I}}) = -\sum_{k=1}^{N} \mathrm{tr}(\mathcal{I}_{k|k}), \tag{26c}$$

where the weighting matrix $V_x = \mathrm{blkdiag}(0, I)$ is used to encode a penalty on the velocity of the mobile robot.

In the first experiment, stealth and informativeness are weighted equally in the objective function, and there is a small penalty on velocity. The resulting path is shown in Fig. 3. The robot completely avoids entering the adversarial sensor's sensing region and is able to obtain three measurements of the feature of interest.

Results from a second experiment are shown in Fig. 4. Here, information gathering has been prioritized by lowering the weight parameter on stealth. This results in a path through the adversarial sensor's sensing region, and gives away one measurement to the adversary, but the mobile robot is able to obtain a fourth measurement of the feature of interest.

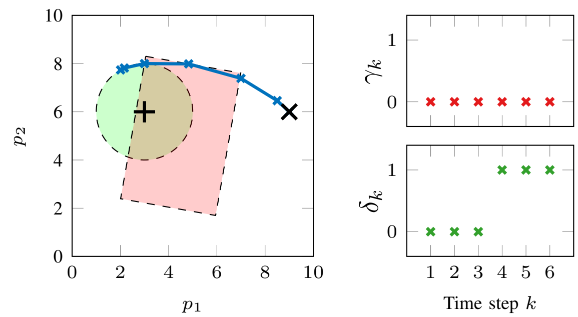

Fig. 3. The mobile robot starts at rest at (9,6). The feature of interest is located at (3,6) and is visible to the mobile robot if the robot is located in the green area. The red area is the sensing region of the adversarial sensor. The weight parameters of the objective function are set to equally prioritize stealth and information gathering, and a small penalty on velocity ($w_P = 1, w_{\mathcal{I}} = 1, w_{x,u} = 0.01$). Left: Computed path for the mobile robot. Upper right: Binary variables indicating whether or not the adversarial sensor obtains measurements. Lower right: Binary variables indicating whether or not the mobile robot obtains measurements.

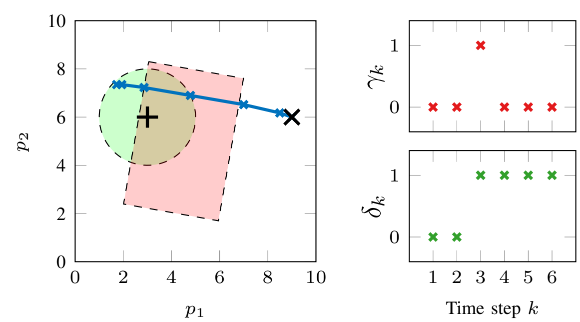

Fig. 4. The setup is the same as in Fig. 3, but the weight of stealth has been given a lower value ($w_{\mathcal{I}} = 1, w_P = 0.01, w_{x,u} = 0.01$). Left: Computed path for the mobile robot. Upper right: Binary variables indicating whether or not the adversarial sensor obtains measurements. Lower right: Binary variables indicating whether or not the mobile robot obtains measurements.

VI. Conclusions

A method to compute globally optimal sensing paths in adversarial environments has been proposed. The path planning problem was formulated as a mixed-integer program with matrix valued states, that at first seemed intractable to solve to global optimality. Using a sequence of reformulations, we have shown that the problem is in fact equivalent to a mixed-integer SDP problem, that can be solved to global optimality using off-the-shelf optimization tools. The proposed approach allows for a trade-off between stealth and path informativeness, and simulations have demonstrated that it generates intuitively reasonable trajectories.

References

[1] A. Singh, A. Krause, C. Guestrin, W. J. Kaiser, and M. A. Batalin, "Efficient planning of informative paths for multiple robots." in *Proceedings of AAAI International Joint Conference on Artificial Intelligence*, Hyderabad, India, 2007, pp. 2204–2211.

[2] L. P. Kaelbling, M. L. Littman, and A. R. Cassandra, "Planning and acting in partially observable stochastic domains," *Artificial intelligence*, vol. 101, no. 1-2, pp. 99–134, 1998.

[3] S. Ragi and E. K. Chong, "UAV path planning in a dynamic environment via partially observable Markov decision process," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 49, no. 4, pp. 2397–2412, 2013.

[4] M. Dunbabin and L. Marques, "Robots for environmental monitoring: Significant advancements and applications," *IEEE Robotics and Automation Magazine*, vol. 19, no. 1, pp. 24–39, 2012.

[5] M. Popović, G. Hitz, J. Nieto, I. Sa, R. Siegwart, and E. Galceran, "Online informative path planning for active classification using UAVs," in *Proceedings of IEEE International Conference on Robotics and Automation*, Singapore, Singapore, 2017, pp. 5753–5758.

[6] S. Patil, G. Kahn, M. Laskey, J. Schulman, K. Goldberg, and P. Abbeel, "Scaling up Gaussian belief space planning through covariance-free trajectory optimization and automatic differentiation," in *Algorithmic Foundations of Robotics XI*. Cham, Switzerland: Springer, 2015, pp. 515–533.

[7] C. Leung, S. Huang, N. Kwok, and G. Dissanayake, "Planning under uncertainty using model predictive control for information gathering," *Robotics and Autonomous Systems*, vol. 54, no. 11, pp. 898–910, 2006.

[8] J. Nocedal and S. J. Wright, *Numerical Optimization*. New York, USA: Springer, 2006.

[9] P. Boström-Rost, D. Axehill, and G. Hendeby, "On global optimization for informative path planning," *IEEE Control Systems Letters*, vol. 2, no. 4, pp. 833–838, 2018.

[10] M. Al Marzouqi and R. A. Jarvis, "Robotic covert path planning: A survey," in *Proceedings of IEEE Conference on Robotics, Automation and Mechatronics*, Beijing, China, 2011, pp. 77–82.

[11] A. Richards, T. Schouwenaars, J. P. How, and E. Feron, "Spacecraft trajectory planning with avoidance constraints using mixed-integer linear programming," *Journal of Guidance, Control, and Dynamics*, vol. 25, no. 4, pp. 755–764, 2002.

[12] C. A. Floudas, *Nonlinear and mixed-integer optimization: Fundamentals and applications*. London, UK: Oxford University Press, 1995.

[13] T. Kailath, A. Sayed, and B. Hassibi, *Linear Estimation*. Upper Saddle River, NJ, USA: Prentice Hall, 2000.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.

[15] G. Hendeby and F. Gustafsson, "Fundamental filtering limitations in linear non-Gaussian systems," in *Proceedings of the 16th Triennial IFAC World Congress*, Prague, Czech Republic, 2005, pp. 1209–1220.

[16] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407 – 427, 1999.

[17] H. P. Williams, *Model building in mathematical programming*. New York, USA: John Wiley & Sons, 2013.

[18] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[19] MOSEK ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 8.1.*, 2017.