RESEARCH ARTICLE

Mathematical analysis of complex SIR model with coinfection and density dependence

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An SIR model with the coinfection of the two infectious agents in a single host population is considered. The model includes the environmental carry capacity in each class of population. A special case of this model is analyzed, and several threshold conditions are obtained, which describes the establishment of diseases in the population. We prove that, for small carrying capacity K , there exists a globally stable disease-free equilibrium point. Furthermore, we establish the continuity of the transition dynamics of the stable equilibrium point, that is, we prove that, (1) for small values of K , there exists a *unique* globally stable equilibrium point, and (b) it moves continuously as K is growing (while its face type may change). This indicates that the carrying capacity is the crucial parameter and an increase in resources in terms of carrying capacity promotes the risk of infection.

KEYWORDS

carrying capacity, coinfection, global stability, SIR model

1 | INTRODUCTION

Coinfection means that a person is affected by more than one infectious agent at a time. Many pathogens that infect humans (eg, viruses, bacteria, protozoa, fungal parasites) often coexist within individuals.¹ Consequently, coinfection of individual hosts by multiple infectious agents is a phenomenon that is very frequently observed in natural populations. The study of complete dynamics of this phenomenon involves many complexities. By understanding the multiple interactions that cause coinfection, we will be able to understand and intelligently predict how a suite of coinfections will together respond to medical interventions as well as other environmental changes.²

Mathematical analysis of infectious diseases has significant importance in infectious disease epidemiology to study not only the dynamics of disease, but it is also very helpful to design the practical controlling strategies. Mathematical analysis and models have successfully explained previously mystifying observations and played a central part in public health strategies in many countries.^{3,4}

Many mathematical studies exist on interaction of multiple-strain and multiple-disease cointeractions.^{5–13} Some of the studies are about the general dynamics of coinfection.^{14–16}

Because coinfection includes a lots of complexities and therefore a lot of different classes and interactions between classes. This implies the necessity of advanced mathematical analysis to handle this problem. However, there is also a risk that one reaches the boarder of what is actually possible to analyze. Previously, Allen et al¹⁷ studied an SI model with density-dependent mortality and coinfection in a single host where one strain is vertical transmitted and the other is horizontally transmitted and the model has application on hantavirus and arenavirus, and Gao et al¹⁸ studied an

SIS model with dual infection. Simultaneous transmission of infection and no immunity has been considered. The study revealed that the coexistence of multiple agents caused coinfection and made the disease dynamics more complicated. It was observed that coexistence of two disease can only occur in the presence of coinfection. In the above models, they considered that the number of births per unit time is constant. Our approach in this paper is to show different degrees of complexities and to suggest a model that is an extension of the model studied by Ghersheen et al,¹⁹ where an SIR model was analyzed. That model describes the coinfection of the two infectious agents in a single host population with an addition of limited growth of susceptible in terms of carrying capacity. Previously, to diminish the complexity, it was considered that there is no interaction between two single infectious agents and that coinfection only occurs as a result of interaction between coinfecting class and single infected class, and coinfecting class and susceptible class. In this model, we add more complexity by relaxing all these assumptions and adding the density dependence in each class. The addition of two viruses with density dependence for human population is a new modeling perspective because death and birth rates change over time in human population. One billion out of eight billions of human population is facing the problem of hunger due to the lack of resources that can fluctuate the birth rates over time. Therefore, in contrast to the works of Allen et al¹⁷ and Gao et al,¹⁸ we first analyze the model with the same interaction terms, but only the growth of the susceptible class is limited in terms of carrying capacity, because an infected and recovered population is regulated by its death rate. We assume that infected and recovered individuals cannot reproduce.

We formulate an SIR model with coinfection and logistic-type population growth in each class population to study the effects of carrying capacity on disease dynamics. However, contrary to the work of Allen et al,¹⁷ to study the global behavior of the system, the reduction of the system is needed to some sense. Therefore, in the first place, we consider the relative simplified model and only limit the growth of susceptible populations in terms of carrying capacity. We assume that the infected population cannot reproduce due to infection. We carried out the local and global stability analysis using a generalized Volterra function for each stable point to study the complete dynamics of disease. We analyze an SIR model with complete cross immunity. In Section 2, we begin with the description of a full model, and in Section 2, we presented and analyzed a submodel. In Section 3, we recall some general facts about the SIR model (1), and in the remaining sections, we characterize all equilibrium points and give the results regarding local and global stability.

2 | FORMULATION OF THE MODEL

The first model is the relevant extension to the model presented in the work of Ghersheen et al¹⁹ to understand the complex dynamics of coinfection. Firstly, we assume that a susceptible individual can be infected with either one or both infectious agents as a result of contact with coinfecting individual. Secondly, coinfection occurs as a result of contact between two single infected individuals or between a coinfecting person and a single infected person. This process is illustrated in the compartmental diagram in Figure 1.

Following other works,^{17,19-21} we assume limited population growth by making the per capita reproduction rate depend on the density of population. We also consider the recovery of each infected class. The SIR model is then described by the

FIGURE 1 Flow diagram for the two-strains coinfection model

system of five ordinary differential equations as follows:

$$\begin{aligned}
 S' &= \left(b_1 \left(1 - \frac{N}{K_1} \right) - \alpha_1 I_1 - \alpha_2 I_2 - (\beta_1 + \beta_2 + \alpha_3) I_{12} - \mu_0 \right) S, \\
 I_1' &= \left(b_2 \left(1 - \frac{N}{K_2} \right) + \alpha_1 S - \eta_1 I_{12} - \gamma_1 I_2 - \mu_1 \right) I_1 + \beta_1 S I_{12}, \\
 I_2' &= \left(b_3 \left(1 - \frac{N}{K_3} \right) + \alpha_2 S - \eta_2 I_{12} - \gamma_2 I_1 - \mu_2 \right) I_2 + \beta_2 S I_{12}, \\
 I_{12}' &= \left(b_4 \left(1 - \frac{N}{K_4} \right) + \alpha_3 S + \eta_1 I_1 + \eta_2 I_2 - \mu_3 \right) I_{12} + (\gamma_1 + \gamma_2) I_1 I_2, \\
 R' &= \left(b_5 \left(1 - \frac{N}{K_5} \right) - \mu_4' \right) R + \rho_1 I_1 + \rho_2 I_2 + \rho_3 I_{12}.
 \end{aligned} \tag{1}$$

Here, S represents the susceptible class; I_1 and I_2 are infected classes from strain 1 and strain 2, respectively; I_{12} represents coinfecting class; and R represents the recovered class. Finally,

$$N = S + I_1 + I_2 + I_{12} + R$$

is the total population. Here,

- b_i is the birthrate of class $i = 1, 2, 3, 4$;
- K_i is the carrying capacity of class $i = 1, 2, 3, 4$;
- ρ_i is the recovery rate from each infected class ($i = 1, 2, 3$);
- β_i is the rate of transmission of single infection from coinfecting class ($i = 1, 2$);
- γ_i is the rate at which one infected with one strain gets infected with the other strain and moves to the coinfecting class ($i = 1, 2$);
- μ_i' is the death rate of each class, ($i = 1, 2, 3, 4$);
- $\alpha_1, \alpha_2, \alpha_3$ are rates of transmission of strain 1, strain 2, and both strains (in the case of coinfection);
- η_i is the rate at which one infected from one strain gets infection from the coinfecting class ($i = 1, 2$) and $\mu_i = \rho_i + \mu_i'$, $i = 1, 2, 3$.

In this paper, we address a certain specialization of (1) (an SIR model with limited growth of susceptible population). More precisely, we assume that the per capita reproduction of susceptible population is limited by carrying capacity $K < \infty$ while infected and recovered individuals cannot reproduce, ie,

$$b_i = 0 \quad \text{for } i \geq 2.$$

The corresponding SIR model reduces to the following system:

$$\begin{aligned}
 S' &= \left(b \left(1 - \frac{S}{K} \right) - \alpha_1 I_1 - \alpha_2 I_2 - (\beta_1 + \beta_2 + \alpha_3) I_{12} - \mu_0 \right) S, \\
 I_1' &= (\alpha_1 S - \eta_1 I_{12} - \gamma_1 I_2 - \mu_1) I_1 + \beta_1 S I_{12}, \\
 I_2' &= (\alpha_2 S - \eta_2 I_{12} - \gamma_2 I_1 - \mu_2) I_2 + \beta_2 S I_{12}, \\
 I_{12}' &= (\alpha_3 S + \eta_1 I_1 + \eta_2 I_2 - \mu_3) I_{12} + (\gamma_1 + \gamma_2) I_1 I_2, \\
 R' &= \rho_1 I_1 + \rho_2 I_2 + \rho_3 I_{12} - \mu_4' R.
 \end{aligned} \tag{2}$$

We consider some natural assumptions on the fundamental parameters of the system and the initial data. First note that if the reproduction rate of susceptible is less than their death rate, then the population will die out quickly. Therefore, we shall always assume that

$$b > \mu_0.$$

The system is considered under the natural initial conditions

$$S(0) > 0, \quad I_1(0) \geq 0, \quad I_2(0) \geq 0, \quad I_{12}(0) \geq 0. \tag{3}$$

Then, it follows from the standard theory (see, for example, Proposition 2.1 in the work of Haddad et al²²) that any integral curve with (3) is staying in the nonnegative cone for all $t \geq 0$. Note also that, because the variable R is not present

in the first four equations, we may consider only the first four equations of system (1). In the work of Gherseen et al,¹⁹ the particular case of (2) was completely treated, when the parameters β_i and γ_j vanish, ie,

$$\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0. \quad (4)$$

The corresponding system has the Lotka-Volterra type, and an approach based on the linear complimentary problem was suggested. The latter allows to obtain an effective description of the transition dynamics of (2) for any admissible values of the fundamental parameters. The present model is more involved and is no longer a Lotka-Volterra system. However, thinking of (2) as a perturbation of the Lotka-Volterra case, it is reasonable to believe that some basic properties can be extended for positive small values β_i and γ_j . Below, we shall see that this is indeed the case.

Let us introduce the reproduction/threshold numbers of the system (2) for strains 1 and 2, respectively, by

$$\sigma_i := \frac{\mu_i}{\alpha_i}, \quad 1 \leq i \leq 2.$$

Then, by change of the indices (if needed), we may assume that $\sigma_1 < \sigma_2$, ie,

$$\sigma_1 < \sigma_2.$$

The letter also means that strain 1 is more aggressive than strain 2.

Because the interaction between coinfectd and susceptible classes results a single infection transmission or a simultaneous transmission of two infections,

$$\begin{aligned} \beta_1 SI_{12} &\rightarrow I_1 \\ \beta_2 SI_{12} &\rightarrow I_2 \\ \alpha_3 SI_{12} &\rightarrow I_{12}, \end{aligned}$$

it follows from Figure 1 that the corresponding reproduction/threshold number of the coinfectd class is determined by

$$\sigma_3 = \frac{\mu_3}{\hat{\alpha}_3} = \frac{\mu_3}{\alpha_3 + \beta_1 + \beta_2}, \quad (5)$$

where

$$\hat{\alpha}_3 := \alpha_3 + \beta_1 + \beta_2 \quad (6)$$

is the total transmission rate of infection from coinfectd class to susceptible class. Note that in the Lotka-Volterra case (4) $\sigma_3 = \frac{\mu_3}{\alpha_3}$, which is completely consistent with the notation of Gherseen et al.¹⁹

For biological reasons, the latter total transmission rate should be less than other transmission rates comparable with the corresponding death rates. On the other hand, it is natural to assume that the death rates μ_i are almost the same for different classes. This makes it natural to assume the following hypotheses:

$$\sigma_1 < \sigma_2 < \sigma_3. \quad (7)$$

Finally, let us introduce an important parameter of the above system, the so-called *modified carrying capacity* defined by

$$S^{**} := K \left(1 - \frac{\mu_0}{b} \right) > 0. \quad (8)$$

Note that S^{**} is always less than K , but it is proportional to K whenever b and μ_0 are fixed. We study the transition dynamics of stable equilibrium states depending on the value of K in Section 6. The vector of fundamental parameters

$$p = (b, K, \mu_i, \alpha_j, \eta_k, \gamma_k, \beta_k) \in R_+^{11}, \quad 0 \leq i \leq 3, 1 \leq j \leq 3, 1 \leq k \leq 2,$$

is said to be *admissible* if (7) holds.

3 | GENERAL FACTS ON THE SIR MODEL

In this section, we study some basic properties for the system (2), which are essential in our analysis of stability results. We follow the approach given in the work of Gherseen et al¹⁹ for the Lotka-Volterra case (4).

Proposition 1. If $(S, I_1, I_2, I_{12})(t)$ is a solution of (2) with $S(0)$ positive, then

$$S(t) \leq \frac{1}{\frac{1}{S^{**}}(1 - e^{-(b-\mu_0)t}) + \frac{1}{S(0)}e^{-(b-\mu_0)t}}. \quad (9)$$

In particular,

$$S(t) \leq \max\{S^{**}, S(0)\} \quad (10)$$

and

$$\limsup_{t \rightarrow \infty} S(t) \leq S^{**}. \quad (11)$$

Proof. It follows from the first equation of (2) that

$$S' - (b - \mu_0)S \leq -\frac{bS^2}{K},$$

which can be written as

$$(Se^{-(b-\mu_0)t})' \leq -\frac{b}{K}e^{-(b-\mu_0)t}S^2.$$

Dividing both sides by $(S'e^{-(b-\mu_0)t})^2$ and integrating from 0 to t give

$$\frac{e^{(b-\mu_0)t}}{S} \geq \frac{b}{K(b-\mu_0)}(e^{(b-\mu_0)t} - 1) + \frac{1}{S(0)},$$

which yields (9). Then, relations (10) and (11) follow immediately from (9). $\square$

Another important property of the general system (2) is the following.

Proposition 2 (Global estimates).

If $(S, I_1, I_2, I_{12})(t)$ is a solution of (2) with positive initial data, then

$$S(t) + I_1(t) + I_2(t) + I_{12}(t) \leq \max \left\{ S(0) + I_1(0) + I_2(0) + I_{12}(0), \frac{bS_m}{\mu} \right\} \quad (12)$$

for $t \geq 0$, where $\mu = \min_{0 \leq i \leq 3} \mu_i$ and $S_m = \max\{S^{**}, S(0)\}$.

Proof. Summing up the first four equations of (2), we obtain

$$S' + I_1' + I_2' + I_{12}' \leq \left(b - \frac{bS}{K}\right)S - \mu(S + I_1 + I_2 + I_{12}).$$

Let $y(t) = S + I_1 + I_2 + I_{12}$; then,

$$y' \leq bS_m - \mu y.$$

Multiplying both sides by $e^{\mu t}$ and integrating the above equation from 0 to t give

$$y(t) \leq e^{-\mu t}y(0) + \frac{bS_m}{\mu}(1 - e^{-\mu t}).$$

By (10), we have $y(t) \leq \max\{y(0), \frac{bS_m}{\mu}\}$, which proves our claim. $\square$

Finally, in the global stability analysis given below in Section 5, we shall need the following result established recently in the work of Ghersheen et al.¹⁹

Proposition 3. Suppose that $f(t) \in L^p([0, \infty)) \cap C^1([0, \infty))$, where $p \geq 1$, and the first k higher derivatives are bounded:

$f', \dots, f^{(k)} \in L^\infty([0, \infty))$. Then,

$$\lim_{t \rightarrow \infty} f(t) = \dots = \lim_{t \rightarrow \infty} f^{(k-1)}(t) = 0.$$

4 | EQUILIBRIUM POINTS: THE LOCAL STABILITY ANALYSIS

4.1 | Basic properties

In this section, we identify all equilibria of the system (2) in the case when

$$\beta_1 > 0, \quad \beta_2 > 0, \quad \gamma_1 + \gamma_2 > 0 \quad (13)$$

and determine their local stability properties. First, let us remark some useful balance relations, which hold for any equilibrium point $Y = (Y_0, Y_1, Y_2, Y_3)$ of (2). Because we are only interested in nonnegative equilibrium states, we always assume that

$$Y = (Y_0, Y_1, Y_2, Y_3) \geq 0.$$

Then, we have

$$\begin{aligned} \left(b \left(1 - \frac{Y_0}{K} \right) - \alpha_1 Y_1 - \alpha_2 Y_2 - (\beta_1 + \beta_2 + \alpha_3) Y_3 - \mu_0 \right) Y_0 &= 0 \\ (\alpha_1 Y_0 - \eta_1 Y_3 - \gamma_1 Y_2 - \mu_1) Y_1 + \beta_1 Y_0 Y_3 &= 0 \\ (\alpha_2 Y_0 - \eta_2 Y_3 - \gamma_2 Y_1 - \mu_2) Y_2 + \beta_2 Y_0 Y_3 &= 0 \\ (\alpha_3 Y_0 + \eta_1 Y_1 + \eta_2 Y_2 - \mu_3) Y_3 + (\gamma_1 + \gamma_2) Y_1 Y_2 &= 0. \end{aligned} \quad (14)$$

Denote by $G_1 := (0, 0, 0, 0)$ the trivial equilibrium state. Then,

$$Y_0 \neq 0 \quad \text{unless } Y = G_1. \quad (15)$$

Indeed, if $Y_0 = 0$, then we have from the second equation of (14) that $(\eta_1 Y_3 + \gamma_1 Y_2 + \mu_1) Y_1 = 0$. However, by the positivity assumption, $\eta_1 Y_3 + \gamma_1 Y_2 + \mu_1 \geq \mu_1 > 0$; hence, $Y_1 = 0$. For the same reason, we have $Y_2 = 0$; thus, the last equation in (14) yields

$$\mu_3 Y_3 = (\gamma_1 + \gamma_2) Y_1 Y_2 = 0;$$

hence, $Y_3 = 0$ too. This proves that $Y = G_1$ and proves (15).

It follows that any nontrivial equilibrium state $Y \neq G_1$ must satisfy

$$b \left(1 - \frac{Y_0}{K} \right) - \alpha_1 Y_1 - \alpha_2 Y_2 - (\beta_1 + \beta_2 + \alpha_3) Y_3 - \mu_0 = 0,$$

which implies by (8) and (6) the balance equation

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \hat{\alpha}_3 Y_3 = \frac{b}{K} (S^{**} - Y_0). \quad (16)$$

In addition, summing up equations in (14), we obtain

$$\mu_1 Y_1 + \mu_2 Y_2 + \mu_3 Y_3 = \frac{b}{K} (S^{**} - Y_0) Y_0. \quad (17)$$

Taking into account (15), the latter identities imply a priori bounds for Y_0 .

Lemma 1. *Let $Y \neq G_1$ be a nontrivial equilibrium point of (2). Then,*

$$0 < Y_0 \leq S^{**}, \quad (18)$$

and the equality holds if and only if $Y_1 = Y_2 = Y_3 = 0$. Furthermore,

$$\sigma_1 \leq Y_0 \leq \min\{S^{**}, \sigma_3\}, \quad (19)$$

unless $Y_0 = S^{**}$.

Proof. Indeed, the first claim follows immediately from (16). Next, assuming that $Y_0 \neq S^{**}$ and dividing (17) by (16), we get $Y_0 = \frac{\mu_1 Y_1 + \mu_2 Y_2 + \mu_3 Y_3}{\alpha_1 Y_1 + \alpha_2 Y_2 + \hat{\alpha}_3 Y_3}$, which readily yields (19). $\square$

Combining the above estimates, we obtain from (16) the following a priori bound on the coordinates of an arbitrary equilibrium point Y distinct from $G_2 := (S^{**}, 0, 0, 0)$ that

$$\|Y\|_\infty := \max_{0 \leq i \leq 3} Y_i \leq \max \left\{ \frac{b - \mu_0}{\alpha_1}, \frac{b - \mu_0}{\alpha_2}, \frac{b - \mu_0}{\hat{\alpha}_3}, \sigma_3 \right\}. \quad (20)$$

4.2 | Equilibrium points of (2)

Note that (2) always has the trivial equilibrium state

$$G_1 = (0, 0, 0, 0).$$

The first nontrivial equilibrium point is the disease-free equilibrium (or a healthy equilibrium), which is an equilibrium such that the disease is absent in all the patches. In the present notation, the disease-free equilibrium corresponds to $I_1 = I_2 = I_{12} = 0$, and it follows from (2) that

$$G_2 = (S^{**}, 0, 0, 0).$$

Then, Lemma 1 shows that the value of the susceptible class for the healthy equilibrium state G_2 is *the largest possible among all equilibrium points*.

Suppose that $Y_3 = 0$. Then, it follows from (14) that, besides G_2 , there exist exactly two equilibrium points with the presence of the first or the second strains, given respectively by

$$G_3 = \left(\sigma_1, \frac{b}{K\alpha_1}(S^{**} - \sigma_1), 0, 0 \right) \quad \text{when } S^{**} > \sigma_1 \quad (21)$$

and

$$G_4 = \left(\sigma_2, 0, \frac{b}{K\alpha_2}(S^{**} - \sigma_2), 0 \right) \quad \text{when } S^{**} > \sigma_2. \quad (22)$$

Finally, suppose that Y is a nontrivial equilibrium point such that $Y_3 \neq 0$. Because $Y_0 \neq 0$, the second and the third equations in (14) immediately imply that $Y_1 \neq 0$ and $Y_2 \neq 0$ as well. Thus, $Y_3 \neq 0$ implies that Y must have all positive coordinates. This equilibrium point is related to the coexistence of both strains with coinfection and is called the coexistence equilibrium point. We shall denote it by

$$G_5 = (S^*, I_1^*, I_2^*, I_{12}^*).$$

Note that, due to the complexity of our model, it is difficult to find the coordinates of the G_5 -type equilibrium points explicitly. It is also a priori unclear how many such coexistence equilibrium points can exist. We address this issue in a forthcoming paper.

4.3 | The trivial equilibrium point G_1

The Jacobian matrix for $G_1 = (0, 0, 0, 0)$ is

$$J = \begin{bmatrix} b - \mu_0 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & -\mu_2 & 0 \\ 0 & 0 & 0 & -\mu_3 \end{bmatrix}.$$

Because $b - \mu_0 > 0$, the trivial equilibrium point G_1 is always locally unstable. In fact, we have a stronger observation.

Proposition 4. *If*

$$\kappa := \limsup_{t \rightarrow \infty} (\alpha_1 I_1 + \alpha_2 I_2 + \hat{\alpha}_3 I_{12}) < (b - \mu_0), \quad (23)$$

then

$$\liminf_{t \rightarrow \infty} S(t) \geq \frac{K}{b}(b - \mu_0 - \kappa). \quad (24)$$

In particular, if $I_1, I_2, I_{12} \rightarrow 0$ as $t \rightarrow \infty$, then $S(t)$ is separated from zero.

Proof. By virtue of (23), we have for every $\kappa_1 \in (\kappa, (b - \mu_0))$ that there exist $t_1 > 0$ such that

$$\alpha_1 I_1 + \alpha_2 I_2 + \hat{\alpha}_3 I_{12} \leq \kappa_1$$

holds for any $t \geq t_1$. It follows from the first equation of (2) that

$$S'(t) - (b - \mu_0 - \kappa_1)S(t) \geq -\frac{bS(t)^2}{K},$$

for $t \geq t_1$; hence,

$$(S(t)e^{-(b-\mu_0-\kappa_1)t})' \geq -\frac{b}{K}e^{-(b-\mu_0-\kappa_1)t}S(t)^2.$$

Dividing both sides by $(S'e^{-(b-\mu_0-\kappa_1)t})^2$ and integrating from t_1 to t give

$$\frac{e^{(b-\mu_0-\kappa_1)(t-t_1)}}{S(t)} \leq \frac{b}{K(b-\mu_0-\kappa_1)}(e^{(b-\mu_0-\kappa_1)(t-t_1)} - 1) + \frac{1}{S(t_1)},$$

which leads to

$$S(t) \geq \frac{S(t_1)}{\frac{b}{K(b-\mu_0-\kappa_1)}S(t_1)(1 - e^{-(b-\mu_0-\kappa_1)(t-t_1)}) + e^{-(b-\mu_0-\kappa_1)(t-t_1)}}, \quad \text{for } t \geq t_1,$$

and gives

$$\liminf_{t \rightarrow \infty} S(t) \geq \frac{K}{b}(b - \mu_0 - \kappa_1).$$

Because κ_1 is an arbitrary number from $(\kappa, b - \mu_0)$, this implies (24). $\square$

4.4 | The disease-free equilibrium G_2

As we know by Lemma 1, the disease-free equilibrium $G_2 = (S^{**}, 0, 0, 0)$ has the largest possible among all equilibrium points. The Jacobian matrix for $G_2 = (S^{**}, 0, 0, 0)$ is

$$J = \begin{bmatrix} -(b - \mu_0) & -\alpha_1 S^{**} & -\alpha_2 S^{**} & -\hat{\alpha}_3 S^{**} \\ 0 & \alpha_1 S^{**} - \mu_1 & 0 & \beta_1 S^{**} \\ 0 & 0 & \alpha_2 S^{**} - \mu_2 & \beta_2 S^{**} \\ 0 & 0 & 0 & \alpha_3 S^{**} - \mu_3 \end{bmatrix},$$

and has all negative eigenvalues if $S^{**} \leq \sigma_1$ and (7) holds. This implies the following proposition.

Proposition 5. *The disease-free equilibrium point G_2 is locally stable whenever $S^{**} \leq \sigma_1$ holds.*

4.5 | The equilibrium point with the presence of the first strain G_3

The local stability analysis of equilibrium points G_3 and G_4 is more involved. First note that it follows from (21) that G_3 is nonnegative if and only if

$$I_1^* := \frac{b}{K\alpha_1}(S^{**} - \sigma_1) \geq 0,$$

and, moreover, $G_3 = G_2$ when $I_1^* = 0$. Using (21), we find the corresponding Jacobian matrix evaluated at G_3 :

$$J = \begin{bmatrix} A & \star \\ 0 & B \end{bmatrix} = \begin{bmatrix} -b\frac{\sigma_1}{K} - \alpha_1 \sigma_1 & -\alpha_2 \sigma_1 & -\hat{\alpha}_3 \sigma_1 \\ \alpha_1 I_1^* & 0 & -\gamma_1 I_1^* & -\eta_1 I_1^* + \beta_1 \sigma_1 \\ 0 & 0 & -\alpha_2(\sigma_2 - \sigma_1) - \gamma_2 I_1^* & \beta_2 \sigma_1 \\ 0 & 0 & (\gamma_1 + \gamma_2)I_1^* & \alpha_3 \sigma_1 + \eta_1 I_1^* - \mu_3 \end{bmatrix},$$

where we partitioned the Jacobian matrix into 2×2 blocks with

$$A = \begin{bmatrix} -b\frac{\sigma_1}{K} & -\alpha_1\sigma_1 \\ \alpha_1 I_1^* & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\alpha_2(\sigma_2 - \sigma_1) - \gamma_2 I_1^* & \beta_2\sigma_1 \\ (\gamma_1 + \gamma_2)I_1^* & \alpha_3\sigma_1 + \eta_1 I_1^* - \mu_3 \end{bmatrix}.$$

It follows that the equilibrium point G_3 is stable if and only if both A and B are stable. We have for the first block matrix

$$\text{tr} A = -b\frac{\sigma_1}{K} < 0, \quad \det A = \alpha_1^2 I_1^* \sigma_1 > 0;$$

therefore, A is stable for any choice of parameters provided that G_3 exists and is distinct of G_2 (ie, $I_1^* > 0$).

Next, notice that the matrix B is stable if and only if its trace is negative and the determinant is positive. Because the first diagonal element in B is negative by (7) and the anti-diagonal elements are positive, the positivity of the determinant implies that

$$B_{22} = \alpha_3\sigma_1 + \eta_1 I_1^* - \mu_3 < 0. \quad (25)$$

Therefore, $\det B > 0$ implies that $\text{tr} B = B_{11} + B_{22} < 0$, ie, B , is stable. This shows that the block B is stable if and only if $\det B > 0$ holds. In summary, we have

Proposition 6. *The equilibrium point with the presence of the first strain G_3 is locally stable if and only if $I_1^* > 0$ and*

$$\det B = \det \begin{bmatrix} -\alpha_2(\sigma_2 - \sigma_1) - \gamma_2 I_1^* & \beta_2\sigma_1 \\ (\gamma_1 + \gamma_2)I_1^* & \alpha_3\sigma_1 + \eta_1 I_1^* - \mu_3 \end{bmatrix} > 0. \quad (26)$$

Let us consider the determinantal condition (26) in more detail. In the Lotka-Volterra case (4) treated in the work of Ghersheen et al,¹⁹ one has $\gamma_i = \beta_j = 0$; hence, the corresponding determinant condition

$$\det B_0 = \det \begin{bmatrix} -\alpha_2(\sigma_2 - \sigma_1) & 0 \\ 0 & \alpha_3\sigma_1 + \eta_1 I_1^* - \mu_3 \end{bmatrix} > 0$$

becomes equivalent to a simpler inequality (cf (25)),

$$I_1^* = \frac{b}{K\alpha_1}(S^{**} - \sigma_1) < \frac{\mu_3 - \alpha_3\sigma_1}{\eta_1}.$$

Coming back to the general case (13), the determinant

$$\Delta(\lambda) := \det \begin{bmatrix} -\alpha_2(\sigma_2 - \sigma_1) - \gamma_2\lambda & \beta_2\sigma_1 \\ (\gamma_1 + \gamma_2)\lambda & \alpha_3\sigma_1 + \eta_1\lambda - \mu_3 \end{bmatrix}$$

is a quadratic polynomial in λ with a negative leading coefficient. Further, by virtue of (5) and (7), we have

$$\Delta(0) = \alpha_2\alpha_3(\sigma_2 - \sigma_1)(\sigma_3 - \sigma_1) > 0$$

hence, the equation $\Delta(\lambda)$ has a unique positive root. We denote it by Λ . Then, one can easily see that (26) is equivalent to the inequality

$$0 < I_1^* = \frac{b}{K\alpha_1}(S^{**} - \sigma_1) < \Lambda, \quad (27)$$

which is equivalent to

$$\sigma_1 < S^{**} < \sigma_1 + \frac{K\alpha_1}{b}\Lambda, \quad (28)$$

and under this condition, the equilibrium point G_3 is stable.

When $\Delta(I_1^*) \approx 0$ and negative, it is plausible to expect that G_3 bifurcates into an inner point of G_5 -type. We shall consider this question in short in Section 6 below.

4.6 | The equilibrium point with the presence of the second strain G_4

Similar to the above, G_4 is nonnegative if and only if

$$I_2^* := \frac{b}{K\alpha_2}(S^{**} - \sigma_2) \geq 0.$$

Note that if G_4 is nonnegative, then $S^{**} \geq \sigma_2$; hence, by virtue of (7), G_3 is nonnegative too. The Jacobian matrix evaluated at G_4 is

$$J = \begin{bmatrix} -b \frac{\sigma_2}{K} & -\alpha_1 \sigma_2 & -\alpha_2 \sigma_2 & -\hat{\alpha}_3 \sigma_2 \\ 0 & \alpha_1(\sigma_2 - \sigma_1) - \gamma_1 I_2^* & 0 & \beta_1 \sigma_2 \\ \alpha_2 I_2^* & -\gamma_2 I_2^* & 0 & -\eta_2 I_2^* + \beta_2 \sigma_2 \\ 0 & (\gamma_1 + \gamma_2) I_2^* & 0 & -\mu_3 + \alpha_3 \sigma_2 + \eta_2 I_2^* \end{bmatrix}. \quad (29)$$

Note that the elementary row operation of the matrix (29) does not affect the eigenvalues of this matrix. Therefore, after an obvious rearrangement, the stability of J is equivalent to that of the following matrix:

$$\tilde{J} = \begin{bmatrix} C & \star \\ 0 & D \end{bmatrix}$$

with the diagonal 2×2 -blocks

$$C = \begin{bmatrix} -b \frac{S}{K} & -\alpha_2 \sigma_2 \\ \alpha_2 I_2^* & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \alpha_1(\sigma_2 - \sigma_1) - \gamma_1 I_2^* & \beta_1 \sigma_2 \\ (\gamma_1 + \gamma_2) I_2^* & -\mu_3 + \alpha_3 \sigma_2 + \eta_2 I_2^* \end{bmatrix}. \quad (30)$$

Therefore, J is stable if and only if the blocks (30) are stable. The first block C is stable provided G_4 is nonnegative. Thus, the stability of G_4 is equivalent to that of D . Similar to the previous case, we have the following proposition.

Proposition 7. *The equilibrium point with the presence of the second strain G_4 is locally stable if and only if $S^{**} > \sigma_2$, and*

$$\det D = \det \begin{bmatrix} \alpha_1(\sigma_2 - \sigma_1) - \gamma_1 I_2^* & \beta_1 \sigma_2 \\ (\gamma_1 + \gamma_2) I_2^* & -\mu_3 + \alpha_3 \sigma_2 + \eta_2 I_2^* \end{bmatrix} > 0 \quad (31)$$

and

$$\text{tr} D = \alpha_1(\sigma_2 - \sigma_1) - \gamma_1 I_2^* - \mu_3 + \alpha_3 \sigma_2 + \eta_2 I_2^* < 0. \quad (32)$$

Remark 1. Note that if (4) is satisfied, then D is upper triangular and has an eigenvalue $\alpha_1(\sigma_2 - \sigma_1) > 0$, thus unstable. This shows that for the local stability of G_4 , γ_1 must be a larger a priori lower bound. Because we consider (2) as a suitable modification of the Lotka-Volterra case (4), G_4 is an unstable point for small perturbations of the parameters γ_i and β_j . The latter is completely consistent with the results in the work of Gherseen et al.¹⁹

5 | GLOBAL STABILITY OF EQUILIBRIUM POINTS

5.1 | The Lyapunov function

Let us denote

$$Y = \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} S \\ I_1 \\ I_2 \\ I_{12} \end{pmatrix}$$

and rewrite our system (2) in notation

$$\frac{dY_k}{dt} = F_k(Y) \cdot Y_k + H_k, \quad k = 0, 1, 2, 3, \quad (33)$$

where we denote

$$F(Y) = -q + AY, \quad (34)$$

with

$$F(Y) = \begin{pmatrix} F_0(Y) \\ F_1(Y) \\ F_2(Y) \\ F_3(Y) \end{pmatrix}, \quad q = \begin{pmatrix} -b + \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{b}{K} & -\alpha_1 & -\alpha_2 & -\hat{\alpha}_3 \\ \alpha_1 & 0 & -\gamma_2 & -\eta_1 \\ \alpha_2 & -\gamma_1 & 0 & -\eta_2 \\ \alpha_3 & \eta_1 & \eta_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ \beta_1 Y_0 Y_3 \\ \beta_2 Y_0 Y_3 \\ (\gamma_1 + \gamma_2) Y_1 Y_2 \end{pmatrix}. \quad (35)$$

Then, $Y^* = (Y_0^*, Y_1^*, Y_2^*, Y_3^*)$ is an equilibrium point of (2) if and only if

$$Y_i^* F_i(Y^*) = -H_i(Y^*), \quad 0 \leq i \leq 3. \quad (36)$$

We associate with Y^* the Lyapunov function

$$v_{Y^*}(Y_0, Y_1, Y_2, Y_3) = \sum_{i=0}^3 (Y_i - Y_i^* \ln Y_i).$$

The derivative computations are slightly different from the Lotka-Volterra case considered previously and the resulting function contains both the γ_i , β_j and the H -terms. More precisely, using consequently (34), (35), and (36), we obtain for the time derivative of v_{Y^*} along any integral trajectory of (33)

$$\begin{aligned} \frac{d}{dt} v_{Y^*} &:= (\nabla v_{Y^*})^T \frac{dy}{dt} = \sum_{i=0}^3 \frac{Y_i - Y_i^*}{Y_i} (F_i(Y) Y_i + H_i(Y)) \\ &= \sum_{i=0}^3 \frac{Y_i - Y_i^*}{Y_i} ((F_i(Y) - F_i(Y^*)) Y_i + F_i(Y^*) Y_i + H_i(Y)) \\ &= \sum_{i,j=0}^3 A_{ij} (Y_i - Y_i^*) (Y_j - Y_j^*) + \sum_{i=0}^3 (Y_i - Y_i^*) F_i(Y^*) + \sum_{i=1}^3 \frac{Y_i - Y_i^*}{Y_i} H_i(Y) \\ &= -\frac{b}{K} (Y_0 - Y_0^*)^2 - (\gamma_1 + \gamma_2) (Y_1 - Y_1^*) (Y_2 - Y_2^*) - (\beta_1 + \beta_2) (Y_0 - Y_0^*) (Y_3 - Y_3^*) \\ &\quad + \sum_{i=0}^3 Y_i F_i(Y^*) + \sum_{i=1}^3 H_i(Y^*) + \frac{Y_i - Y_i^*}{Y_i} H_i(Y). \end{aligned} \quad (37)$$

Using (35), we find further that

$$\frac{d}{dt} v_{Y^*} = -\frac{b}{K} (Y_0 - Y_0^*)^2 + \sum_{i=0}^3 Y_i F_i(Y^*) + \Phi, \quad (38)$$

where

$$\Phi = (\gamma_1 + \gamma_2) \left(Y_1 Y_2^* + Y_1^* Y_2 - \frac{Y_3^* Y_1 Y_2}{Y_3} - Y_1^* Y_2^* \right) - Y_0 Y_3 \left(\beta_1 \frac{Y_1^*}{Y_1} + \beta_2 \frac{Y_2^*}{Y_2} \right) + (\beta_1 + \beta_2) (Y_3 Y_0^* + Y_0 Y_3^* - Y_0^* Y_3^*). \quad (39)$$

Note that, in our notation (see (35)), all functions $H_i(Y)$ are nonnegative for any nonnegative Y ; hence, it follows from (36) that, for any equilibrium point,

$$F_i(Y^*) \leq 0. \quad (40)$$

Therefore, the sign of the derivative of the Lyapunov function depends on the sign of Φ . Below, we apply the above computations to global stability results for the first two equilibrium points.

5.2 | Global stability of equilibrium point G_2

First, we consider the disease-free equilibrium point. Recall that, by Proposition 5, the point G_2 is *locally* stable if and only if

$$S^{**} \leq \sigma_1 \quad (41)$$

holds. Remarkably, the latter condition also implies the global stability. Furthermore, in the G_2 case, we are able to establish a global stability result that guarantees that the disease cannot invade and go extinct in small populations. In particular, the following result shows that a disease cannot persist in a small population.

Proposition 8. Let (41) be satisfied. Then, the equilibrium point G_2 is globally asymptotically stable, ie,

$$\begin{aligned}\lim_{t \rightarrow \infty} I_1(t) &= \lim_{t \rightarrow \infty} I_2(t) = \lim_{t \rightarrow \infty} I_{12}(t) = 0, \\ \lim_{t \rightarrow \infty} S(t) &= S^{**}.\end{aligned}$$

Proof. We have for the Lyapunov function of $Y^* = G_2 = (S^{**}, 0, 0, 0)$ the following:

$$v(t) := S - S^{**} \ln S + I_1 + I_2 + I_{12}. \quad (42)$$

We have $F_0(G_2) = 0$ and $F_i(G_2) = \alpha_i S^{**} - \mu_i$, $1 \leq i \leq 3$, and $H_i(G_2) = 0$ for $1 \leq i \leq 3$. Substituting this into (38) and (39), we obtain

$$\begin{aligned}v'(t) &= -\frac{b}{K}(S^{**} - S)^2 - (\mu_1 - \alpha_1 S^{**})I_1 - (\mu_2 - \alpha_2 S^{**})I_2 - (\mu_3 - \hat{\alpha}_3 S^{**})I_{12} \\ &= -\frac{b}{K}(S^{**} - S)^2 - \alpha_1(\sigma_1 - S^{**})I_1 - \alpha_2(\sigma_2 - S^{**})I_2 - \hat{\alpha}_3(\sigma_3 - S^{**})I_{12} \\ &\leq 0.\end{aligned} \quad (43)$$

First, suppose that we have the strong inequality $S^{**} > \sigma_1$. Then, $\sigma_i - S^{**}$ are nonzero and strongly positive for any i . Integrating (43) over $[0, \infty]$ and using the fact that by Proposition 2 all S, Y_1, Y_2, Y_{12} are bounded, we obtain

$$\int_0^\infty (S - S^{**})^2 d\tau < \infty, \quad \text{and} \quad \int_0^\infty |I_k| d\tau = \int_0^\infty I_k d\tau < \infty \quad \text{for } I_k = I_1, I_2, I_{12}.$$

Using again the boundedness of S, Y_1, Y_2, Y_{12} , in virtue of the system (2), their derivatives are also bounded. Applying Proposition 3 we conclude that S converges to S^{**} and I_1, I_2, I_{12} converge to zero.

Now suppose that $S^{**} = \sigma_1$. Because $\sigma_i - S^{**} > 0$ for $i = 2, 3$, the above argument implies that S converges to S^{**} and I_2, I_{12} converge to zero. Let us show that $\lim_{t \rightarrow \infty} I_1 = 0$. Because $\lim_{t \rightarrow \infty} S(t)$ exist, then by Proposition 3, $\lim_{t \rightarrow \infty} S'(t) \rightarrow 0$. Therefore, the first equation in (2) implies by virtue of $\lim_{t \rightarrow \infty} S(t) = S^{**} \neq 0$ that

$$\frac{b}{K}(S(t) - S^{**}) - \alpha_1 I_1(t) - \alpha_2 I_2(t) - \hat{\alpha}_3 I_{12}(t) \rightarrow 0,$$

which implies $\lim_{t \rightarrow \infty} I_1(t) = 0$ and finishes the proof. $\square$

Remark 2. It is interesting to note that when $S^{**} = \sigma_1$, we have no local stability because one eigenvalue is equal to zero. However, despite of this, due to a nonlinear character of the problem, a linear term gives the global asymptotic stability in this case.

5.3 | Global stability of equilibrium point G_3

Proposition 9. Let

$$S^{**} > \sigma_1 \quad (44)$$

and

$$I_1^* = \frac{b}{K\alpha_1}(S^{**} - \sigma_1) \leq \min \left\{ \frac{\alpha_2(\sigma_2 - \sigma_1)}{\gamma_1}, \frac{\hat{\alpha}_3(\sigma_3 - \sigma_1)}{\eta_1} \right\}; \quad (45)$$

then, the equilibrium point G_3 is globally stable, ie,

$$\lim_{t \rightarrow \infty} I_2(t) = \lim_{t \rightarrow \infty} I_{12}(t) = 0, \quad \lim_{t \rightarrow \infty} S(t) = S^{**}, \quad \lim_{t \rightarrow \infty} I_1(t) = I_1^*.$$

Remark 3. Note that the global stability condition (45) coincides with the local stability conditions (26) when (4) is satisfied. In the general case, (45) implies (26). Indeed, note that, in the notation of Section 4.5, we have by (45)

$$B_{11} + B_{21} = -\alpha_2(\sigma_2 - \sigma_1) - \gamma_2 I_1^* + (\gamma_1 + \gamma_2) I_1^* = -\alpha_2(\sigma_2 - \sigma_1) + \gamma_1 I_1^* \leq 0,$$

and similarly,

$$B_{12} + B_{22} = \beta_2 \sigma_1 + \alpha_3 \sigma_1 + \eta_1 I_1^* - \mu_3 \leq (\beta_2 + \alpha_3) \sigma_1 - \hat{\alpha}_3 \sigma_1 = -\beta_1 \sigma_1 < 0;$$

hence, $B_{22} < -B_{12} < 0$, and $B_{11} < -B_{21} < 0$, which implies $B_{22}B_{11} > B_{12}B_{21}$. Therefore, $\det B > 0$. The latter implies by Proposition 6 that G_3 is locally stable.

Proof. In this case, $Y^* = G_3 = (\sigma_1, I_1^*, 0, 0)$, where $I_1^* := \frac{b}{K\alpha_1}(S^{**} - \sigma_1)$, and the corresponding Lyapunov function is given by

$$v(t) := S - \sigma_1 \ln S + I_1 - I_1^* \ln I_1 + I_2 + I_{12}. \quad (46)$$

Substituting this into (38) and (39) and using (45), we obtain

$$v'(t) = -\frac{b}{K}(\sigma_1 - S)^2 - (\alpha_2(\sigma_2 - \sigma_1) - \gamma_1 I_1^*) I_2 - (\hat{\alpha}_3(\sigma_3 - \sigma_1) - \eta_1 I_1^*) I_{12} - \beta_1 \frac{I_1^*}{I_1} S I_{12} \leq 0. \quad (47)$$

We suppose first that the strong inequality in (45) holds. Then, as above, integrating (47) over $[0, \infty]$, we obtain

$$\int_0^\infty (S - \sigma_1)^2 d\tau < \infty, \quad \int_0^\infty I_2 d\tau < \infty, \quad \int_0^\infty I_{12} d\tau < \infty.$$

Therefore, by Proposition 3, S converges to σ_1 , and I_2, I_{12} converge to zero. Now, we consider the convergence of I_1 . Arguing as in the proof of Proposition 8, we obtain that $\lim_{t \rightarrow \infty} S(t) = \sigma_1$ and because the latter limit is nonzero, we find from the first equation in (2) that

$$0 = \lim_{t \rightarrow \infty} \left(\frac{b}{K}(S^{**} - S(t)) - \alpha_1 I_1(t) - \alpha_2 I_2(t) - \hat{\alpha}_3 I_{12}(t) \right) = \frac{b}{K}(S^{**} - \sigma_1) - \alpha_1 \lim_{t \rightarrow \infty} I_1(t).$$

This proves that $\lim_{t \rightarrow \infty} I_1 = I_1^*$.

The case when the equality in (45) attains is studied similar to that in the proof of Proposition 8. $\square$

Remark 4. Finally remark, that a similar analysis in the case for G_4 shows that the corresponding Lyapunov function does not have a negative derivative because by assumption (7) one has

$$(\alpha_1(\sigma_2 - \sigma_1) + \gamma_2 I_2^*) > 0.$$

This indirectly shows that two infectious agents may not coexist together in the absence of coinfection.

6 | TRANSITION DYNAMICS

As it was already mentioned in the Introduction, in the work of Ghersheen et al,¹⁹ the Lotka-Volterra version of our present model (2) corresponds to the vanishing of the transmission parameters (4).

In the work of Ghersheen et al,¹⁹ a very striking result has been established asserting that, for any $K > 0$ and each admissible choice of the fundamental parameter,

$$\hat{p} = (b, \mu_0, \mu_1, \mu_2, \mu_3, \alpha_1, \alpha_2, \alpha_3, \eta_1, \eta_2) \in R_+^{10},$$

there exists exactly one stable equilibrium point $E = E(K, \hat{p})$, which depends continuously on the data $(K, \hat{p})$. It is natural to consider the transition dynamics of the stable equilibrium point $E(K, \hat{p})$ as a function of the carrying capacity K keeping other parameters in $\hat{p}$ fixed. This transition dynamics has a clear biological meaning establishing the relationship between the infection transition rates $\hat{p}$ and the character of the corresponding equilibrium state. Furthermore, this also implies all possible scenarios how equilibrium states depend on the carrying capacity of the system. More precisely, in the work of Ghersheen et al,¹⁹ there were only *three* possible scenarios, each starting with the healthy equilibrium state for small values of K and ending up at a certain equilibrium state when K becomes sufficiently large, and it was referred as the *continuity of the transition dynamics* of stable equilibrium points.

It is natural to expect that the continuity of the transition dynamics will hold true for small positive β_i and γ_j , as a perturbation of (4). The numerical simulations support this conjecture. It is, however, unreasonable to believe that the latter property should hold for any positive data $(\beta_1, \beta_2, \gamma_1, \gamma_2)$. Below, we prove a part of our conjecture.

It is convenient to formulate our result for the relative carrying capacity S^{**} instead of K (note that S^{**} differs from K by a multiplicative constant only). Let us also introduce the threshold in (45). Namely, let σ_0 denote the solution of the following equation:

$$\frac{b}{K\alpha_1}(\sigma_0 - \sigma_1) = \min \left\{ \frac{\alpha_2(\sigma_2 - \sigma_1)}{\gamma_1}, \frac{\hat{\alpha}_3(\sigma_3 - \sigma_1)}{\eta_1} \right\}. \quad (48)$$

Then,

$$\sigma_0 = \sigma_0(K) > \sigma_1,$$

and the global stability of G_3 holds whenever

$$\sigma_1 \leq S^{**} \leq \sigma_0.$$

Combining the latter with Proposition 8 and Proposition 9, we arrive at the following continuity of the transition dynamics.

Theorem 1. *Let $\hat{p} > 0$ be fixed. Then,*

- (i) *if S^{**} is increasing and $0 < S^{**} < \sigma_1$, then the only possible stable equilibrium state of (2) is G_2 ;*
- (i') *when $S^{**} = \sigma_1$, G_2 coincides with G_3 and it is the only possible stable equilibrium state of (2);*
- (ii) *there exists $\sigma_0 > \sigma_1$ such that G_3 is the only possible stable equilibrium state of (2) for $\sigma_1 < S^{**} < \sigma_0$.*

Proof. The claims immediately follow from the global stability of the corresponding equilibria (indeed, by virtue of the global stability there can exist at most one locally stable point!) $\square$

The case $S^{**} > \sigma_0$ is more subtle, but we have at least some local information near $\hat{\sigma} = \sigma_1 + \frac{K\alpha_1}{b}\Lambda$ and it follows from remark 3, $\hat{\sigma} > \sigma_0$.

The following theorem describes the bifurcation of equilibrium point G_3 in the case when β_i and γ_j are small and S^{**} cross the threshold $\hat{\sigma}$.

Theorem 2. *Let $\hat{p} > 0$ be fixed and $\beta_1, \beta_2, \gamma_1$, and γ_2 are small; then, there exists $\epsilon > 0$ such that, if $S^* \in (\hat{\sigma}, \hat{\sigma} - \epsilon)$, then the equilibrium point G_3 is unstable and the equilibrium point G_5 is stable and located near G_3 .*

Proof. We have a sketch of the proof. Let

$$D := \det B = (\alpha_2(\sigma_2 - \sigma_1) + \gamma_2 I_1^*)(\mu_3 - \alpha_3 \sigma_1 - \eta_1 I_1^*) - \beta_2(\gamma_1 + \gamma_2)\sigma_1 I_1^*, \quad (49)$$

where we keep the notation

$$I_1^* = \frac{b}{K\alpha_1}(S^{**} - \sigma_1).$$

Let us consider the evolution of the equilibrium point G_3 for small values of the determinant D under an additional natural condition that $S^{**} > \sigma_1$. Certainly, the equilibrium point G_3 exists for any such admissible values. When $D > 0$ is small, by Proposition 6, G_3 is locally stable, and when $D = 0$, it loses the local stability. It turns out that, at this moment, G_3 bifurcates into a pair of points: (a) it continues as G_3 for small negative values of D , and (b) it appears one more equilibrium point of type G_5 in a neighborhood of G_3 . Indeed, to describe the latter, we will seek it by perturbation of G_3 in the form

$$Y_0 = \sigma_1 + \xi_0, \quad Y_1 = I_1^* + \xi_1,$$

where ξ_0, ξ_1 are small real parameters such that

$$\lim_{D \rightarrow 0} \xi_0 = \lim_{D \rightarrow 0} \xi_1 = 0$$

and

$$Y_2 = \lambda Y_3, \quad Y_3 \neq 0, \quad \lim_{D \rightarrow 0} Y_3 = 0.$$

Then, the third equation in (14) yields

$$\lambda = \frac{\beta_2 Y_0}{\mu_2 + \gamma_2 Y_1 + \eta_2 Y_3 - \alpha_2 Y_0}.$$

For $\beta_2 = 0$, we have $\lambda = 0$, and for $\beta_2 > 0$, we find

$$\lambda = \lambda^* + o(1), \quad \lambda^* = \frac{\beta_2 \sigma_1}{\mu_2 + \gamma_2 I_1^* - \alpha_2 \sigma_1}$$

is positive. Here, $o(1)$ denotes the rest term when $D \rightarrow 0$. From the fourth equation in (14), we obtain by elimination of Y_2

$$(\mu_3 - \alpha_3 Y_0 - \eta_1 Y_1 - \lambda \eta_2 Y_3)(\mu_2 + \gamma_2 Y_1 + \eta_2 Y_3 - \alpha_2 Y_0) - \beta_2(\gamma_1 + \gamma_2)Y_0 Y_1 = 0. \quad (50)$$

Now, let us express ξ_0 and ξ_1 through Y_3 . From the first two equations in (14) (with vanishing derivatives), we get

$$\xi_0 = \frac{1}{\alpha_1} \left(\eta_1 + \lambda^* \gamma_1 - \beta_1 \frac{\sigma_1}{I_1^*} \right) Y_3 + O(Y_3^2) \quad (51)$$

and

$$\xi_1 = -\frac{1}{\alpha_1} (\lambda^* \alpha_2 + \delta) Y_3 - \frac{b \xi_0}{\alpha_1 K} + O(Y_3^2). \quad (52)$$

Using the notation (49) and setting

$$A = \mu_3 - \alpha_3 \sigma_1 - \eta_1 I_1^*, \quad B = \alpha_2(\sigma_2 - \sigma_1) + \gamma_2 I_1^*,$$

we can write (50) as

$$D + A(\gamma_2 \xi_1 + \eta_2 Y_3 - \alpha_2 \xi_0) - B(\alpha_3 \xi_0 + \eta_1 \xi_1 + \lambda \eta_2 Y_3) - \beta_2(\gamma_1 + \gamma_2)(\sigma_1 \xi_1 + I_1^* \xi_0) + O(Y_3^2).$$

This yields

$$Y_3 = cD + O(D^2), \quad (53)$$

where the coefficient c can be easily evaluated in the case $\beta_1 = \beta_2 = 0$ and $\gamma_1 = \gamma_2 = 0$. Indeed, in this case $\lambda = 0$,

$$A = O(D) \quad \text{and} \quad \alpha_3 \xi_0 + \eta_1 \xi_1 = -\frac{b \eta_1^2}{K \alpha_1^2}.$$

This yields

$$B \frac{b \eta_1^2}{K \alpha_1^2} Y_3 = -D + O(D^2);$$

hence, we obtain (53) with

$$c = -\frac{K \alpha_1^2}{B b \eta_1^2}.$$

It follows from (53) that, for small negative D , the perturbation of G_3 bifurcates in an equilibrium point of type G_5 with *positive* coordinates. By the continuity argument, the latter property still holds true for small β_i and γ_j . $\square$

7 | DISCUSSION

In this paper, we designed an SIR model as an extension of the model presented in the work of Gherseen et al¹⁹ and observed the effect of density dependence population regulation on disease dynamics. The complete local and global stability analysis of two boundary equilibrium points revealed that, for small carrying capacity, the disease-free equilibrium point is always stable so a disease cannot persist in a small population, but for relatively large carrying capacity under some conditions, we have one globally stable endemic equilibrium point. The existence of an endemic equilibrium point

guarantees the persistence of the disease with a possible future threat of any outbreak in the population. In the future, similar to the work of Ghersheen et al,¹⁹ we would like to investigate the existence and uniqueness of the equilibrium point G_5 . Because the equilibrium point G_3 loses its stability and bifurcates to the coexistence equilibrium point, it is worthwhile to show the existence of the coexistence equilibrium in that case to understand the complete dynamics of the disease. We would like to see the dynamical behavior of the system for significantly large K and the case when $K = \infty$.

ACKNOWLEDGEMENTS

A PhD scholarship to Samia Ghersheen from the Higher Education Commission of Pakistan by SBK Women's University Quetta, Pakistan, is gratefully acknowledged.

CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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How to cite this article: Ghersheen S, Kozlov V, Tkachev V, Wennergren U. Mathematical analysis of complex SIR model with coinfection and density dependence. *Comp and Math Methods*. 2019;1:e1042. <https://doi.org/10.1002/cmm4.1042>