Time of Flight Estimation for Radio Network Positioning

Kamiar Radnosrati

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Kamiar Radnosrati

kamiar.radnosrati@liu.se

www.control.isy.liu.se

Division of Automatic Control

Department of Electrical Engineering

Linköping University

SE-581 83 Linköping

Sweden

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Abstract

Trilateration is the mathematical theory of computing the intersection of circles. These circles may be obtained by time of flight (ToF) measurements in radio systems, as well as laser, radar and sonar systems. A first purpose of this thesis is to survey recent efforts in the area and their potential for localization. The rest of the thesis then concerns selected problems in new cellular radio standards as well as fundamental challenges caused by propagation delays in the ToF measurements, which cannot travel faster than the speed of light. We denote the measurement uncertainty stemming from propagation delays for positive noise, and develop a general theory with optimal estimators for selected distributions, which can be applied to trilateration but also a much wider class of estimation problems.

The first contribution concerns a narrow-band mode in the long-term evolution (LTE) standard intended for internet of things (IoT) devices. This LTE standard includes a special position reference signal sent synchronized by all base stations (BS) to all IoT devices. Each device can then compute several pair-wise time differences that correspond to hyperbolic functions. The simulation-based performance evaluation indicates that decent position accuracy can be achieved despite the narrow bandwidth of the channel.

The second contribution is a study of how timing measurements in LTE can be combined. Round trip time (RTT) to the serving BS and time difference of arrival (TDOA) to the neighboring BS are used as measurements. We propose a filtering framework to deal with the existing uncertainty in the solution and evaluate with both simulated and experimental test data. The results indicate that the position accuracy is better than 40 meters 95% of the time.

The third contribution is a comprehensive theory of how to estimate the signal observed in positive noise, that is, random variables with positive support. It is well known from the literature that order statistics give one order of magnitude lower estimation variance compared to the best linear unbiased estimator (BLUE). We provide a systematic survey of some common distributions with positive support, and provide derivations and summaries of estimators based on order statistics, including the BLUE one for comparison. An iterative global navigation satellite system (GNSS) localization algorithm, based on the derived estimators, is introduced to jointly estimate the receiver's position and clock bias.

The fourth contribution is an extension of the third contribution to a particular approach to utilize positive noise in nonlinear models. That is, order statistics have been employed to derive estimators for a generic nonlinear model with positive noise. The proposed method further enables the estimation of the hyperparameters of the underlying noise distribution. The performance of the proposed estimator is then compared with the maximum likelihood estimator when the underlying noise follows either a uniform or exponential distribution.

Populärvetenskaplig sammanfattning

Om man kan mäta avståndet till tre kända punkter, så kan man bestämma sin position genom att rita en cirkel kring varje punkt med respektive uppmätta radie. Den unika skärningspunkten svarar då mot positionen. Denna princip fungerar i både två och tre dimensioner, med tre eller fler cirklar, och används i en rad olika positionerings- och navigations-system idag. Avståndet kan mätas genom att mäta hur lång tid det tar för en radar, laser, radio eller akustisk signal att gå till och/eller från de kända punkterna. Satellitnavigering som GPS bygger på denna princip.

Det matematiska problemet att räkna ut skärningspunkten för flera cirklar kallas för trilaterering. Problem uppstår om radierna inte kan mätas exakt utan är behäftade med mätfel. Denna avhandling behandlar en rad i praktiken förekommande utmaningar, både för moderna cellulära system såväl som mer generella problem som uppkommer av att signaler inte kan färdas snabbare än den gräns som signalens medium sätter (ljushastighet eller ljudhastighet).

I 5G finns en speciell mod för Internet of Things (IoT). Denna ska vara sparsam med signaleringen för att klara miljarder av uppkopplade prylar, och därför får man i praktiken nöja sig med två cirklar som svarar mot den basstation man är uppkopplad mot, samt en till närliggande basstation. De mätningar man får svarar mot en cirkel och en hyperbel snarare än två cirklar. Detta problem kallas för multilaterering, och avhandlingen analyserar hur man bäst kan utnyttja denna minimala information för att positionera IoT-prylar.

En radiosignal eller laserljus kan inte färdas fortare än ljuset, likaså kan en akustisk signal, ultraljud eller sonar inte färdas fortare än ljudet i sitt medium (luft eller vatten). Däremot är det troligt att den fördröjs, så att den radie som uppmäts alltid överskattas. Detta kallar vi positivt brus, mätfelen på radierna kan inte vara negativa. Ett exempel är när stora byggnader blockerar signaler mellan satellit och GPS-mottagare, och det är reflektioner från andra byggnader som uppfångas av GPS-mottagaren. Avhandlingen ger en grundlig matematisk analys för skattningsproblem när mätningarna är behäftade med positivt brus. Motivet kommer från lokalisering, men klassen av problem som studeras är betydligt större. För en rad olika fördelningar av det positiva bruset presenteras optimala skattningar av den okända variabeln (t.ex. positionen), där optimalt definieras som minsta möjliga standardavvikelse i skattningsfelet.

Två exempel illustrerar hur positivt brus kan användas inom GPS-tillämpningar. Det första beskriver hur tidmätningen för en stationär GPS-mottagare med känd position kan bli en storleksordning noggrannare än med standardmetoder. Det andra exemplet visar hur en ny metod för positionsskattning baserat på positivt brus ger mycket noggrannare resultat än standardmetoder.

Acknowledgments

The quest for pursuing a doctoral degree -if I consider my master's studies as its starting point- took me to two beautiful Nordic countries over the past eight years. If I were to summarize these past years in chronological order, I would say that I left my beloved family in Iran, made a handful of amazing friends in Finland, successfully managed to turn my friendship with Parinaz into a loving relationship in a nerdy, yet sweet way, and lastly, got my PhD from Linköping University.

Five years ago, Fredrik Gustafsson accepted to supervise me during my doctoral studies and since then, funded my research through various sources. Merely judging from Fredrik's impressive academic record and without having any prior knowledge about his personality, the first few meetings with him, were among the most stressful moments of my life! Now, reflecting based on the pleasure of being in his company for the past five years, my posterior belief is nothing but respect and gratitude for his brilliant ideas, kindness and support.

In an ideal world, one could only wish for a co-supervisor who pushes them towards success, is always there when one needs help, have inspiring ideas, and at times, is simply a good friend. Gustaf Hendeby, there is a long list of things to be thankful to you for, and I only mentioned a few. I would also like to thank Fredrik Gunnarsson and Carsten Fritsche for their support and ideas.

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*Stockholm, February 2020
Kamiar Radnosrati*

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Notation

MATHEMATICAL SYMBOLS AND OPERATIONS

Notation	Meaning
a	Scalar variable
$\mathbf{a}$	Column vector variable
$\mathbf{A}$	Matrix variable
$\mathbf{0}$	Column zero vector of appropriate size
$\mathbf{I}_N$	Identity matrix of size $N \times N$
$[\cdot]^T$	Vector/Matrix transpose
$[\cdot]^{-1}$	Non-singular square matrix inverse
$\text{tr}(\cdot)$	Trace of square matrix
$\ \cdot\ $	Euclidean norm
$ \cdot $	Cardinality of a set
$\arg \max$	Maximizing argument
$\arg \min$	Minimizing argument

PROBABILITY SYMBOLS AND OPERATIONS

Notation	Meaning
$p(\cdot)$	Probability density function
$p(\cdot \cdot)$	Conditional probability density function
$p(\cdot; a)$	Probability density function parameterized by variable or expression a
$\mathbb{E}_a$	Expectation with respect to stochastic variable $\mathbf{a}$
$\text{Cov}(\mathbf{a})$	Covariance of the stochastic variable $\mathbf{a}$
$\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$

ABBREVIATION

Abbreviation	Meaning
3GPP	3rd Generation Partnership Project
BS	Base Station
BLUE	Best Linear Unbiased Estimator
CDF	Cumulative Distribution Function
CP	Constant Position
CRLB	Cramér-Rao Lower Bound
CV	Constant Velocity
EKF	Extended Kalman Filter
EPA	Extended Pedestrian A
ETU	Extended Typical Urban
GNSS	Global Navigation Satellite System
IOT	Internet of Things
JMM	Jump Markov Model
KF	Kalman Filter
LTE	Long Term Evolution
MAP	Maximum A Posteriori
MLE	Maximum Likelihood Estimator
MMSE	Minimum Mean Squared Error
MVU	Minimum Variance Unbiased
NBIOT	Narrowband Internet of Things
NPRS	Narrowband Positioning Reference Signal
OTDOA	Observed Time Difference Of Arrival
PDF	Probability Density Function
PRS	Positioning Reference Signal
RMSE	Root Mean Square Error
RSTD	Reference Signal Time Difference
RTT	Round Trip Time
SSM	State-Space Model
TDOA	Time Difference Of Arrival
TOA	Time Of Arrival
UE	User Equipment
UT	Unscented Transform
WLS	Weighted Least Squares

Part I

Background

1

Introduction

Positioning capability in devices and gadgets is currently in transformation from “nice to have” to “a must” feature. First, we have safety legislations giving tough specifications on the position information in emergency calls. Then, we have the rapid development of location based services which require positioning in situations where satellite navigation systems do not work (indoors, underground, etc.). Further, there is a rapid increment in the number of devices connected to the cellular network that might not be operated by humans. We have the trends of internet of things (IoT), machine to machine communication, autonomous vehicles and systems, etc., where communication and positioning will be the key enabler for future functions and services.

While cellular radio networks were traditionally designed for communication purposes, their importance for positioning was soon recognized. In the early stages of realizing the potential of cellular systems for positioning, the achievable accuracy was rather poor. The performance degradation was mainly due to the fact that the used signals were not designed for positioning purposes. Hence, in recent years there have been tremendous efforts to increase this accuracy.

In addition to accuracy, other important aspects of positioning algorithms are scalability and reliability. For instance, radio network standardization entities have been trying to introduce new positioning specific signaling schemes to address scalability and reliability. Additionally, researchers have been trying to introduce better system models, more robust estimators, information fusion techniques, etc. This thesis strives to address some of the existing challenges in radio network positioning.

1.1 Background

Dedicated positioning solutions such as global navigation satellite systems (GNSS) have been, and are being, used by end users for a long time. However, mutual benefits of more reliable, yet accurate, source of information for users and cellular network operators has emerged as a new research direction; combining positioning and communication systems into a single system.

Positioning in cellular communication networks can be based on indirect observations of the user equipment's (UE's) position, measured from various properties of the wireless communication channel between the transceivers. For example, given a number of directional antenna arrays, it is possible to measure the angle at which the signal is received at the receiver. One major problem with this method is that antenna arrays are still not so common in standards today.

Additionally, the received version of a transmitted signal can be used to estimate the distance between the receiver and transmitter. For example, the range can be estimated from the changes in the transmitted and the received signal strength (RSS). Modeling the energy degradation in the wireless channel as a function of the distance between transceivers, it is possible to estimate the range from the measured RSS. The major problem with RSS-based ranging systems is that the channel model can be quite complex and inaccurate in different environments.

Alternatively, the distance can be estimated from the time it takes for the transmitted signal to travel between the transmitter and the receiver. Assuming that the speed of signal in the transmission medium is known, the measured times can be used to estimate the traveled distance. Timing-based positioning is mostly based on time of arrival (TOA), round trip time (RTT), and time difference of arrival (TDOA) measurements.

TOA gives the absolute distance between the transmitter and the receiver and is estimated by cross-correlating the received signal with a replica of the transmitted signal. Hence, TOA requires accurate synchronization between UE and BS which is not always possible. One solution is to use RTT which is twice the absolute distance computed from the time it takes for the transmitted signal to travel to the receiver and return back to the transmitter. TDOA is the relative distance computed by taking the difference between two absolute distances. Both RTT and TDOA are based on TOA measurements at the UE as well as the base stations (BS). TOA is used to estimate RTT by combining TOA estimated at BS and UE, while TDOA is estimated using TOA associated to two different BSs.

Positioning using absolute range measurements can be performed using trilateration. The distances from TOA measurements can be illustrated in 2D as circles. Given three circles, corresponding to distances to three nodes with known positions, the unknown position can be estimated uniquely. Figure 1.1a illustrates an example of trilateration using three perfect TOAs. However, since measurements are always corrupted with random measurement noise, we need to formulate the positioning problem into a stochastic estimation problem.

In case of TDOA measurements, multilateration, a technique also known as hyperbolic positioning, can be used. Multilateration, in principle, is similar to tri-

Figure 1.1: Trilateration and multilateration positioning techniques using TOA and TDOA measurements.

lateralation except that instead of circles we need to find the intersection of hyperbolas in 2D scenarios. Figure 1.1b illustrates hyperbolas formed from three perfect TDOAs. Since, in practice, TDOA measurements are also imperfect, stochastic estimation methods are required.

1.2 Challenges

The localization accuracy in all timing-based methods is influenced by the accuracy of the estimated TOA and the number of available TOAs, among other factors. In this section, we try to briefly discuss different limiting factors existing in radio networks.

1.2.1 Physics

The accuracy of TOA estimation is affected by different quantities such as the characteristics of the wireless channel, the transmitted signal and the measurement noise. The signal to noise ratio (SNR), defined as the ratio of signal power to the noise power, the communication channel bandwidth and signal carrier frequency are the main components that affect the TOA estimation accuracy.

Theoretically, a lower bound on the TOA estimation accuracy can be defined using the Cramér–Rao lower bound (CRLB). Let the center frequency of the signal be denoted by f_c and the bandwidth of the communication channel be denoted by B . As shown in (Carter and Knapp, 1976) and (Patwari et al., 2003), the variance of the estimated TOA, $\hat{\tau}$, is bounded by

$$\text{Var}(\hat{\tau}) \geq \text{CRLB}(\hat{\tau}) = \frac{1}{8\pi^2 B T f_c^2 \text{SNR}}, \quad (1.1)$$

where T is the signal duration. As an example, consider a scenario in which we are interested in estimating the distance to a 1-second long rescue scream with bandwidth 5 kHz and transmitted with 5 kHz carrier frequency. For simplicity, assume that SNR is equal to one. Using (1.1), the lower bound of the estimated distance is $c_0 \times \text{CRLB}(\hat{\tau}) = 0.1 \text{ mm}$ with c_0 denoting the speed of sound. That shows, in theory, TOA can be estimated very accurately.

1.2.2 Standardization

As mentioned earlier, the main objective of the cellular networks used to be solely to provide communication infrastructure. Hence, the signals and their transmission schemes were not optimal for positioning purposes. In fact, in the very early stages, positioning was only available using proximity identification methods. That is, once a user entered the supporting range of a base stations, the cell ID of the BS was used to identify its position. Given no other information, the user's position was crudely estimated as the position of one of the cells covered by one BS.

Today's cellular radio network standards enable the configuration of positioning reference signals (PRS) from BSs which enable UE to estimate TOA measurements. Since the 3rd generation partnership project (3GPP) LTE Release 9, PRS can be defined based on orthogonal patterns, as well as muting schemes, where some BSs transmit a PRS, while other BSs are muted, in order to suppress interference and ensure a wide detectability of signals.

Since Release 14, the immense number of use cases inspired by IoT motivated 3GPP to introduce narrowband IoT (NB-IoT). Contrary to the broadband services in which high data rates are required, in most cases, lower data rates are acceptable for IoT applications (Lin et al., 2017) but with higher requirements on better coverage, lower power consumption, and cheaper devices. Thus, 3GPP developed two machine type communication (MTC) technologies, LTE MTC (LTE-M) and NB-IoT, introduced in its Release 13 for low power wide area IoT connectivity. Limited positioning support for both LTE-M and NB-IoT in Release 13, motivated 3GPP for improvements in Release 14 (Lin et al., 2017).

LTE-M is based on LTE and operates on a minimum system bandwidth of 1.4 MHz but with additional features resulting in better support for IoT services (Rico-Alvarino et al., 2016). NB-IoT, on the other hand, is a new radio access technology that requires 180 KHz system bandwidth allowing for more deployment flexibility (Wang et al., 2017).

1.2.3 Communication constraints

In LTE systems, positioning is traditionally considered to be enabled, in 2D coordinates, when the UE provides measurements of at least three different BSs. However, we do not always get all TOAs in the system, but only a small subset of possible TOAs. The reason for the limited number of TOAs can be one of the following:

Figure 1.2: NLOS condition affecting the distance estimation. The transmitted signal is received at the receiver from different environments denoted by *Env.*

- Poor neighbor cell PRS SNR due to interference or low transmission power (in LTE, downlink power is essentially not adjusted/controlled).
- Information about a limited number of neighbor cells provided to the UE. The main reason is to limit the downlink (DL) signaling load.
- Imposed restriction to the UE regarding the number of cells to report. In NB-IoT, two restrictions are standardized, either a limited number of neighbors to be reported or a limited report message size. The main reason is to limit the signaling load from the UEs.

1.2.4 Environment

Trilateration using TOA relies on the line of sight (LOS) path. In practice, however, non-line of sight (NLOS) conditions, if not treated properly, add a bias to the estimated TOA. NLOS occurs if the LOS path between the communicating entities is blocked by obstacles. Figure 1.2 illustrates such a case. Let d_{LOS} denote the true distance between the transmitter and the receiver. The estimated distance under NLOS conditions is $\hat{d} = d_{LOS} + b + e$ with b denoting the NLOS bias caused by NLOS and e standing for other sources of error.

NLOS mitigation techniques have been widely studied in the literature and the research is still ongoing. The introduced timing measurement error models in (Gustafsson and Gunnarsson, 2005) assume that LOS errors belong to a zero-mean Gaussian distribution while NLOS errors belong to a shifted Gaussian distribution. The authors in (Fritsche and Klein, 2009; Fritsche et al., 2009;

Figure 1.3: The error histograms of time-of-arrival measurements collected from three separate cellular antennas described in (Medbo et al., 2009).

Hammes and Zoubir, 2011; Liao and Chen, 2006) also model NLOS errors using shifted Gaussian densities and introduce robust timing-based position estimation methods. In (Hammes et al., 2009), the second component in the mixture distribution corresponding to the NLOS errors is modeled using the convolution of the probability distribution function (PDF) of a positive random variable and the zero-mean Gaussian density of LOS errors. The authors in (Cong and Zhuang, 2005) consider the Gaussian-distributed NLOS error mitigation problem and consider three different cases in which NLOS are assumed to have known statistics, limited prior information, or totally unknown statistics.

Due to NLOS and other propagation effects, TOA measurements, typically, have noise with positive support. For example, the error histograms of time-of-arrival measurements collected from three separate cellular antennas are given in Figure 1.3. For detailed description of hardware and the measurement campaign see (Medbo et al., 2009).

1.3 Problem formulation

This thesis studies the problem of positioning in radio network. As indicated by the challenges described above, this is far from a solved research problem. In this thesis, the different challenges are addressed with following problem formulations:

- How accurate localization of UE can be achieved in NB-IoT?

The positioning performance in NB-IoT systems that use observed time difference of arrival (OTDO) measurements is studied using realistic simulations. OTDOA is a downlink positioning method in LTE systems based on the PRSs transmitted by the BS. The downlink TDOA measurements estimated from narrowband positioning reference signals are used in the evaluations.

- How can the position of a UE be tracked using measurements reported from only two base stations?

The focus is on positioning in LTE with communication constraint. This boils down to positioning based on time series of timing measurements gathered from two BSs with known positions. The set of two measured BSs changes as the UE moves, as the reports are constrained to only the RTT for the serving base station, and a TDOA measurement for the most favorable neighboring BS relative to the serving BS. This leads to multiple solutions that must be somehow resolved.

- How can the positive noise that naturally occurs in, e.g., TOA measurements be used to improve tracking performance?

Methods are studied for both linear and nonlinear problems where the additive random variable, representing the measurement noise, has a positive value. Multiple noise distributions with positive support are considered. For each problem, knowing the functional form of the distribution, an iterative estimation framework is proposed. The derived estimators are based on order statistics of the collected measurements.

1.4 Contributions

The main contributions of this thesis can be divided into two main categories: (i) Evaluating the range-based localization performance considering the communication constraints in LTE as well as the new NB-IoT standard. (ii) Deriving estimators tailored for the considered ranging based localization problems. These contributions can be summarized as:

- A. Timing-based positioning in LTE, when enough TOAs are available, is widely studied in the literature while the performance evaluation for the newly released NB-IoT systems is not treated with the same level of detail. This thesis addresses this gap by evaluating the positioning performance in NB-IoT systems using the observed TDOA measurements. The OTDOA positioning method uses the UE estimation of the relative distance between a reference base station and a number of neighboring base stations. The estimated reference signal time differences (RSTD) are then reported by the UE to a positioning center to estimate the unknown location of the UE. The possibility of optimizing the number of such reports while maintaining the final horizontal position estimation accuracy within an acceptable range in a simulated network is investigated. It is shown that the new positioning reference signals, introduced in NB-IoT, can be used to achieve a good trade-off between horizontal positioning accuracy and resource consumption [Paper A].
- B. A filter bank approach for hybrid RTT and TDOA positioning in LTE systems when only two base stations are detected by the user equipment has been developed. To deal with the ambiguity caused by multiple solutions, a multi-modal jump Markov system is introduced in which each mode of the system contains a possible position of the UE. The proposed method can be employed

to deal with the communication constraints in timing-based positioning methods in LTE introduced in Section 1.2.3. Additionally, the lower bound on the positioning error using the proposed method is computed. Performance evaluations performed on simulated and real data indicates that the proposed filtering framework can solve the ambiguity in position estimation at the cost of some additional delay [Paper B].

- C. The fact that TOA measurements are mainly corrupted by positive noise motivates us to focus on estimation problems with additive noise with positive support. Order statistics are used to derive unbiased estimators of a signal observed in additive noise. It is shown that the estimation variance of the proposed estimators are less than best linear unbiased estimator (BLUE) even for small sample sizes. The proposed estimators are used to develop a GNSS localization framework to estimate the receiver's clock bias and its static position. The estimators are derived with or without knowledge of the hyperparameters of the underlying noise distribution. Both simulated and real data are used to evaluate the performance of the proposed estimators. The results show that the derived estimators outperform BLUE even when the hyperparameters of the underlying noise are unknown [Paper C].
- D. An approach for utilizing positive noise in parameter estimation problems, with nonlinear measurement model, is proposed. Taking the positiveness of the noise into account, it is shown that the parameters of interest can be estimated in an iterative manner. The performance of the proposed approach, for two selected noise distributions, is evaluated in terms of the estimation mean squared error and compared to the maximum likelihood estimator and shown to have comparable accuracy with lower computational complexity [Paper D].

1.5 Thesis outline

The thesis consists of two parts. Part I provides a background of the four papers presented in Part II. In the rest of this chapter, a summary of each paper together with the author's contributions are provided. Chapter 2 provides a survey of general positioning framework in which three main levels of information flow in positioning systems are first highlighted. This chapter is an edited version of

K. Radnosrati, F. Gunnarsson, and F. Gustafsson. New trends in radio network positioning. In *Proc. of 18th International Conference on Information Fusion (Fusion)*, pages 492–498, Washington, D.C., USA, July 2015.

Chapter 3 is devoted to the estimation problem in non-Gaussian noise scenarios with positive support. Finally, Chapter 4 provides some concluding remarks on the contributions of this thesis.

Paper A

Paper A is an edited version of

K. Radnosrati, G. Hendeby, C. Fritsche, F. Gunnarsson, and F. Gustafsson. Performance of OTDOA positioning in narrowband IoT systems. In *Proc. of 28th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC)*, pages 1–7, Montreal, Canada, October 2017.

Summary:

NB-IoT is an emerging cellular technology designed to target low-cost devices, high coverage, long device battery life (more than ten years), and massive capacity. We investigate opportunities for device tracking in NB-IoT systems using OTDOA measurements. RSTD reports are sent to the mobile location center periodically or on-demand basis. We investigate the possibility of optimizing the number of reports per minute budget on horizontal positioning accuracy using an on-demand reporting method based on the SNR of the measured cells received by the UE. Wireless channels are modeled considering multipath fading propagation conditions. Extended pedestrian A (EPA) and extended typical urban (ETU) delay profiles corresponding to low and high delay spread environments, respectively, are simulated for this purpose. To increase the robustness of the filtering method, measurement noise outliers are detected using confidence bounds estimated from filter innovations. The results obtained for the EPA channel indicate that the reporting budget can be limited to less than 45 reports per minute while the horizontal positioning error do not exceed 18 m, 67% of the time. The accuracy for the ETU channel with the same budget increases to around 120 m.

Contributions:

The idea of this paper originated from Fredrik Gunnarsson and the author of this thesis and was further refined by discussion among all authors. The author of this thesis wrote the majority of the paper and conducted the simulation analysis.

Paper B

Paper B is an edited version of

K. Radnosrati, C. Fritsche, F. Gunnarsson, F. Gustafsson, and G. Hendeby. Localization in 3GPP LTE based on one RTT and one TDOA observation. *Accepted for publication in IEEE Transactions on Vehicular Technology*, December 2019a.

which is an extension of the earlier contribution

K. Radnosrati, C. Fritsche, G. Hendeby, F. Gunnarsson, and F. Gustafsson. Fusion of TOF and TDOA for 3GPP positioning. In *Proc. of 19th International Conference on Information Fusion (FUSION)*, pages 1454–1460, Heidelberg, Germany, July 2016.

Summary:

We study the fundamental problem of fusing one RTT observation associated with a serving base station with one TDOA observation associated to the serving base station and a neighbor base station to localize a 2D mobile station (MS). This situation can arise in 3GPP LTE when the number of reported cells of the mobile station is reduced to a minimum in order to minimize the signaling costs and to support a large number of devices. The studied problem corresponds geometrically to computing the intersection of a circle with a hyperbola, both with measurement uncertainty, which generally has two solutions. We derive an analytical representation of these two solutions that fits a filter bank framework that can keep track of different hypothesis until potential ambiguities can be resolved. Further, a performance bound for the filter bank is derived. The proposed filter bank is first evaluated in a simulated scenario, where the set of serving and neighbor base stations is changing in a challenging way. The filter bank is then evaluated on real data from a field test, where the result shows a precision better than 40 m, 95 % of the time.

Contributions:

The idea of this paper originated from Gustaf Hendeby and Fredrik Gustafsson and was further refined in discussions among all authors. The majority of the paper was written by the author of this thesis, who has also performed simulation analysis and provided the experimental results.

Paper C

Paper C is an edited version of

K. Radnosrati, G. Hendeby, and F. Gustafsson. Exploring positive noise in estimation theory. *Submitted to IEEE Transactions on Signal Processing*, December 2019b.

Summary:

Estimation of the mean of a stochastic variable observed in noise with positive support is considered. It is well known from literature that order statistics gives one order of magnitude better estimation variance compared to the BLUE. We provide a systematic survey of some common distributions with positive support, and provide derivations of estimators based on order statistics, including BLUE for comparison. The estimators are derived with or without knowledge of the hyperparameters of the underlying noise distribution. In addition to additive noise with positive support, we also consider the mixture of uniform and normal noise distribution for which an order statistics-based unbiased estimator is derived. Finally, an iterative GNSS localization algorithm with uncertain pseudorange measurements is proposed which relies on the derived estimators for receiver clock bias estimation. Simulation data for GNSS time estimation and

experimental GNSS data for joint clock bias and position estimation are used to evaluate the performance of the proposed methods. The obtained results indicate that if the functional form of the underlying noise distribution is known, the derived estimators can be employed to estimate the unknown parameter accurately. For instance, the 95% percentile of the horizontal and vertical root mean squared error (RMSE), when the receiver's clock bias is estimated using the proposed estimator, is around 7 m.

Contributions:

The idea of this paper originated from Fredrik Gustafsson. The paper was written by the author of this thesis, who has also performed simulation analysis and provided the experimental results.

Paper D

Paper D is an edited version of

K. Radnosrati, G. Hendeby, and F. Gustafsson. Order statistics in non-linear parameter estimation with positive noise. *Submitted to IEEE Transactions on Signal Processing*, January 2020.

Summary:

The nonlinear parameter estimation problem in the presence of additive noise with positive support is considered. Given N independent measurements with unknown measurement noise covariance, the most intuitive approach is to use a batch nonlinear least squares (NLS) estimator. It is shown that the batch NLS estimate can be improved by taking knowledge of the positive support of noise into consideration. A two-stage iterative estimation framework is presented where, at each iteration, a subset of measurements, selected from the order statistics of the residuals, are used to re-estimate the parameter. The estimated parameters are then used to update the noise hyperparameter. Noting that the CRLB cannot be computed, the asymptotically efficient maximum likelihood (ML) estimator is used for performance evaluations. Simulation analysis verifies that the iterative estimator's performance is comparable to that of the ML estimator.

Contributions:

The idea of this paper originated from the author of this thesis and was further refined by discussion among all authors. The paper was written by the author of this thesis, who has also performed simulation analysis.

1.6 Publications

Published work of the author is listed below in chronological order. Publications indicated by ★ are included in part II of this thesis. The content of all publications

is reused in this thesis courtesy of IEEE.

K. Radnosrati, D. Moltchanov, and Y. Koucheryavy. Trade-offs between compression, energy and quality of video streaming applications in wireless networks. In *Proc. of IEEE International Conference on Communications (ICC)*, pages 1100–1105, Sydney, Australia, June 2014.

★K. Radnosrati, F. Gunnarsson, and F. Gustafsson. New trends in radio network positioning. In *Proc. of 18th International Conference on Information Fusion (Fusion)*, pages 492–498, Washington, D.C., USA, July 2015.

K. Radnosrati, C. Fritsche, G. Hendeby, F. Gunnarsson, and F. Gustafsson. Fusion of TOF and TDOA for 3GPP positioning. In *Proc. of 19th International Conference on Information Fusion (FUSION)*, pages 1454–1460, Heidelberg, Germany, July 2016.

★K. Radnosrati, G. Hendeby, C. Fritsche, F. Gunnarsson, and F. Gustafsson. Performance of OTDOA positioning in narrowband IoT systems. In *Proc. of 28th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC)*, pages 1–7, Montreal, Canada, October 2017.

★K. Radnosrati, C. Fritsche, F. Gunnarsson, F. Gustafsson, and G. Hendeby. Localization in 3GPP LTE based on one RTT and one TDOA observation. *Accepted for publication in IEEE Transactions on Vehicular Technology*, December 2019a.

P. Kasebzadeh, K. Radnosrati, G. Hendeby, and F. Gustafsson. Joint pedestrian motion state and device pose classification. *Accepted for publication in IEEE Transactions on Instrumentation and Measurement*, November 2019.

★K. Radnosrati, G. Hendeby, and F. Gustafsson. Exploring positive noise in estimation theory. *Submitted to IEEE Transactions on Signal Processing*, December 2019b.

★K. Radnosrati, G. Hendeby, and F. Gustafsson. Order statistics in nonlinear parameter estimation with positive noise. *Submitted to IEEE Transactions on Signal Processing*, January 2020.

2

Radio network positioning

The major effort in radio network positioning has been to address different trends existing in this research area. The first trend is the accuracy demand that might go beyond what can be achieved with today's measurements. Another trend is that measurements and positioning algorithms are approaching each other, so some parts of the positioning are performed on the chip-sets (lowest layer) and low-level measurements are available to the operating system (highest level). This chapter describes the over-all picture of how state of the art is organized today, advances in how the fundamental measurements are computed in recent standards, and pointing out new trends.

2.1 Introduction

Awareness of the position of a device, either in absolute terms or relative to a reference location, is becoming increasingly important. Use cases include emergency calls positioning, navigation, gaming, autonomous vessels, logistics, fleet management, proximity services, location-based services, network management to mention a few. Up to date, it has mainly been emergency call positioning that has driven much of the work in cellular networks due to regulatory requirements. However, some use cases can also be addressed via crude positioning such as cell ID association.

The emergency call positioning requirements by the Federal Communications Commission (FCC) in the United States have been refined several times, initially with requirements on network-based positioning, and subsequently with tighter requirements on mobile-assisted positioning (FCC, 2015; Hatfield, 2002; Razavi et al., 2018). In February 2015, FCC has refined the requirements to give particular attention to requirements for positioning of indoor devices. These requirements are presented as a roadmap with stricter requirements over time, and con-

sidering all mobiles, both outdoors and indoors. The requirement is a horizontal accuracy corresponding to a dispatchable address or within a radius of 50 meters for 40 percent of all wireless 911 calls within two years, gradually tightened to 80 percent of the wireless 911 calls within six years. Furthermore, for vertical positioning information, compatible mobiles shall deliver barometric pressure information within three years. In addition, operators commit to develop a specific vertical location accuracy metric that would be used as the standard for any future deployment, and to be generally adopted within eight years. An alternative, or a complement to pressure reports, is a plausible nationwide National Emergency Address Database (NEAD) containing locations of WiFi access points and Bluetooth beacons.

Positioning in wireless networks is based on the measurements collected either at the UE and reported to the network, at the BS, or a combination thereof. All the measurements, despite the large variety of positioning systems, are essentially either based on identity labels of involved BSs, commonly referred to as cell identity, or properties of the communication link between the UE and BS. The positioning is then either based on snapshot measurements or a time series of measurements. The survey research articles (Caffery and Stuber, 1998; Drane et al., 1998; Gustafsson and Gunnarsson, 2005; Sayed et al., 2005; Sun et al., 2005; Zhao, 2002) report extensive information about wireless network positioning together with their associated accuracies. This chapter describes the information flow of current positioning algorithms and discusses existing trends aiming to enhance the achievable accuracy.

2.2 Positioning framework

The information flow in current positioning algorithms can be categorized in different levels as presented in Figure 2.1. Throughout this thesis, both the UE and the involved reference points are restricted to two dimensional scenarios. Let $\theta_t = (\theta_{x_t}, \theta_{y_t})^T$ denote the unknown position of the UE at time t and $\ell_t^{(i)} = (\ell_{x_t}^{(i)}, \ell_{y_t}^{(i)})$ denote the known position of the reference point i .

The generic measurement $y_t^{(i)}$ relative to the reference point i at time t is a function of both θ_t and $\ell_t^{(i)}$, subject to measurement noise $e_t^{(i)}$. Under additive measurement noise assumption, the generic model is given by

$$y_t^{(i)} = h_t(\theta_t, \ell_t^{(i)}) + e_t^{(i)}. \quad (2.1)$$

The measurement model (2.1) is in the most generic form where the reference points can also move in time, as in some ad-hoc network problems. However, in case of snapshot measurements, or time series of measurements with fixed reference points, the time subscripts may be ignored.

Figure 2.1: Levels of Information fusion for radio network positioning.

2.2.1 Level 1: Radio measurement principles

Radio measurement, in the lowest layer of the system, is based on the received pilot signal which is transmitted over the communication channel for different purposes including referencing. The transmitted pilot symbol $s^{(i)}(t)$, in the physical layer, is sampled at the receiver

$$z^{(i)}(t) = \sum_{k=0}^n \alpha_k^{(i)} s\left(\beta_k^{(i)}(t - \tau_k^{(i)})\right) + e_k^{(i)}(t), \quad (2.2)$$

where $\alpha_k^{(i)}$ is the impulse response of the multi-path channel, $\tau_k^{(i)}$ is the time delay per incoming path, and $\beta_k^{(i)}$ is the Doppler shift that scales time. Assuming that the receiver can estimate these parameters, different higher layer position-related measurements can be defined based on the parameters α , τ , or β as described in the following. The generic function $h(\cdot)$ introduced in (2.1) can then be defined for each position-related measurement.

Measurements based on τ

Three different higher layer measurements can be defined corresponding to $\tau_k^{(i)}$:

1. Time of Arrival corresponds to the absolute distance between the emitting and receiving nodes using the travel time of the signal transmitted between the two

$$y_t^{(i),\text{TOA}} = \frac{1}{c} \|\theta_t - \ell_t^{(i)}\| + e_t^{(i),\text{TOA}}, \quad (2.3)$$

where $\|\cdot\|$ is the norm operator, c is the speed of radio waves and the measurement error $e_t^{(i),\text{TOA}}$ captures both the estimation error and the model error due to multipath assuming that the emitter and receiver are perfectly synchronized. Otherwise, an additional error emerges from the clock offset between transceivers.

Figure 2.2: RTT measurement in LTE systems. In the uplink, random access (RA) or demodulation reference signal (DMRS) is transmitted. In the downlink, primary synchronization signal (PSS) or secondary synchronization signal (SSS) is transmitted.

2. Time Difference of Arrival is the timing difference between two TOA measurements estimated from signals that are sent at the same time. This yields

$$y_t^{(ij),\text{TDOA}} = \frac{1}{c} \|\theta_t - \ell_t^{(i)}\| - \frac{1}{c} \|\theta_t - \ell_t^{(j)}\| + e_t^{(i),\text{TOA}} - e_t^{(j),\text{TOA}}. \quad (2.4)$$

TDOA measurements can be obtained in both uplink and downlink directions. In the former, the UE transmits a signal to a pair of receiving BSs, hence the network is responsible for estimating the uplink TDOA. In the downlink mode, a pair of BSs will instead send reference signals to the receiving UE that is responsible for estimating the observed TDOA, known as OTDOA. Since the emission time of the signal is exactly the same, the synchronization between receiver and transmitter is no longer required. Instead, in both cases, the involved BSs need to be synchronized.

3. Round trip time ¹ corresponds to the sum of the TOA measurements in both uplink and downlink directions. Figure 2.2 illustrates how RTT is estimated in LTE systems. In LTE, $T_s \approx 32$ ns is the basic time unit (3GPP TS 36.211), hence only N_T , in steps of $16 T_s$, depends on the channel quality and is updated by the network.

At the uplink transmission time Tx_{UL} , the UE transmits either a random access or demodulation reference signal and the BS measures the uplink TOA (TOA_{UL}). The BS then sends a first N_T to the UE to be used when deciding when to send the next uplink transmission in relation to the downlink TOA

¹or time of flight (TOF).

(TOA_{DL}). For subsequent uplink transmissions, the BS regularly sends relative corrections to N_T in steps of $16 T_s$ which means that the UE as well as the network maintains an updated N_T . In addition, the BS tries to match a certain arrival time of uplink signals in relation to the downlink transmission time (start of DL frame), Tx_{DL} , and this is represented by ΔT . The RTT measurement is thus given by

$$\begin{aligned} y_t^{(i),\text{RTT}} &= N_T T_s - \Delta T + e_t^{(i),\text{RTT}} \\ &= \frac{2}{c} \|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\| + e_t^{(i),\text{TOA}_{DL}} + e_t^{(i),\text{TOA}_{UL}}. \end{aligned} \quad (2.5)$$

4. Angle of arrival (AOA) can be computed by comparing delays τ of the received signal to multiple antennas or by using directional antennas. The high-level measurement is

$$y_t^{(i),\text{AOA}} = \arctan\left(\theta_{y_t} - \ell_{y_t}^{(i)}, \theta_{x_t} - \ell_{x_t}^{(i)}\right) + e_t^{(i),\text{AOA}}. \quad (2.6)$$

The angle of the received signal could either be computed using directional antennas in which the main drawback is implementation cost of such antennas, if their sizes need to be rather small. Using an array of antennas is yet another alternative in which AOA is inferred indirectly from TOA measurement. Sophisticated algorithms are defined for array processing problems, see (Krim and Viberg, 1996). Additionally, AOA estimation can be performed using the antenna lobe diagram, see (Gunnarsson et al., 2014) for example.

Measurements based on α

Received signal strength is a ranging measurement that corresponds to the total energy of the received signal, $\sum_{k=0}^n \alpha_{i,k}^2$. The generic model for RSS measurement is given by

$$y_t^{(i),\text{RSS}} = h_{\text{RSS}}(\|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\|) + e_t^{(i),\text{RSS}}, \quad (2.7a)$$

where $h_{\text{RSS}}(\|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\|)$ is a deterministic function denoting the received signal strength due to path loss. Let $P_0^{(i)}$ denote the measured RSS of the i th BS at a reference distance d_0 . The deterministic function for RSS due to path loss can be written as

$$h_{\text{RSS}}(\|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\|) = P_0^{(i)} + 10\eta \log\left(\frac{\|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\|}{d_0}\right), \quad (2.7b)$$

where η is the path-loss exponent.

Measurements based on β

The estimated parameter β can be interpreted as a measure of the relative speed between the UE and BS. Thus the measurement model is

$$y_t^{(i),\text{Doppler}} = \frac{\partial \|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t\|}{\partial t} + e_t^{(i),\text{Doppler}}. \quad (2.8)$$

2.2.2 Level 2: Spatial fusion

The information obtained from multiple, spatially distributed, sensors is fused at the second level. Let N denote the number of transmitters from which measurements corresponding to the ones introduced in Section 2.2.1 are obtained. The set of equations are given by

$$y_t^{(i),\text{type}} = h^{\text{type}} \left(\|\boldsymbol{\theta}_t - \boldsymbol{\ell}_t^{(i)}\| \right) + e_t^{(i),\text{type}}, \quad i = 1, \dots, N \quad (2.9)$$

where type is either TOA, TDOA, AOA, RSS, or Doppler. Basic methods of position estimation using the first four types of measurements are briefly explained in the remainder of this section. To see more advanced optimization-based position estimation methods in static systems, see Appendix A.

It must be noted that in addition to the wireless positioning methods other alternatives also exist. For instance there are some frameworks that do not use wireless communication infrastructures but rather depend on, for example, image processing techniques or dead reckoning approaches (Kasebzadeh et al., 2016). To maintain the focus of this thesis, they are not discussed further.

Using the range or angle measurements, the known position of BSs, and the trigonometry properties, it is possible to estimate the unknown position of the UE. Since no temporal dependency is considered in these methods, and to simplify the notation, the time subscript t is dropped in the derivations. Additionally, the measurement noises of N involved BSs are assumed to be normally distributed with zero mean and covariance $\mathbf{R}$, $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{N \times N})$. In this Section, we use different tricks to linearize the system to obtain the matrices $\mathbf{H}_{N \times 2}$ and $\mathbf{Y}_{N \times 1}$ such that $\mathbf{Y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{e}$. Thus, $\hat{\boldsymbol{\theta}}$ can be computed as weighted least squares (WLS) estimator, $\hat{\boldsymbol{\theta}} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{W} \mathbf{Y}$ where the weighting matrix $\mathbf{W} = \mathbf{R}^{-1}$. The only difference is how the $\mathbf{H}$ and $\mathbf{Y}$ are formed using either of TOA, TDOA, or AOA measurements. In the following, different methods are briefly introduced, see (Frattasi and Rosa, 2017) for more details.

TOA

The absolute distances between the UE and measured BSs are used in lateration techniques to localize the UE. In a noise-free situation, the TOA circles of $N \geq 3$ BSs intersect in a single location in 2D. However, in case of noisy measurements, the circles do not intersect in a single point and thus data fusion techniques are required to estimate the best possible position. In order to combine the available observations collected from N BSs, and to linearize the equations, one trick is

to subtract the distances between the UE and BSs⁽ⁱ⁾, $i = 2, \dots, N$ from a reference BS⁽¹⁾. Let $r_i = y^{(i),\text{TOA}}$, as shown in (Frattasi and Rosa, 2017), expanding (2.3) gives

$$r_i^2 - r_1^2 = \|\ell^{(i)}\|^2 - \|\ell^{(1)}\|^2 - 2\theta_x(\ell_x^{(i)} - \ell_x^{(1)}) - 2\theta_y(\ell_y^{(i)} - \ell_y^{(1)}), \quad (2.10a)$$

the matrices are thus given by

$$H = \begin{bmatrix} \ell_x^{(2)} - \ell_x^{(1)} & \ell_y^{(2)} - \ell_y^{(1)} \\ \ell_x^{(3)} - \ell_x^{(1)} & \ell_y^{(3)} - \ell_y^{(1)} \\ \vdots & \vdots \\ \ell_x^{(N)} - \ell_x^{(1)} & \ell_y^{(N)} - \ell_y^{(1)} \end{bmatrix}, \quad (2.10b)$$

$$Y = \frac{1}{2} \begin{bmatrix} r_1^2 - r_2^2 + \|\ell^{(2)}\|^2 - \|\ell^{(1)}\|^2 \\ r_1^2 - r_3^2 + \|\ell^{(3)}\|^2 - \|\ell^{(1)}\|^2 \\ \vdots \\ r_1^2 - r_N^2 + \|\ell^{(N)}\|^2 - \|\ell^{(1)}\|^2 \end{bmatrix}. \quad (2.10c)$$

TDOA

To localize the UE using relative distances given by TDOA measurements, hyperbolic localization techniques can be used. Using the same notation as in lateration, the relative distances $r_{i1} = r_i - r_1$. Following the method introduced in (Frattasi and Rosa, 2017), one can get

$$r_{i1}^2 + 2r_{i1}r_1 = r_i^2 - r_1^2, \quad (2.11a)$$

that can be expanded as

$$r_{i1}^2 + 2r_{i1}r_1 = \|\ell^{(i)}\|^2 - \|\ell^{(1)}\|^2 - 2\theta_x(\ell_x^{(i)} - \ell_x^{(1)}) - 2\theta_y(\ell_y^{(i)} - \ell_y^{(1)}). \quad (2.11b)$$

Since the TOA measurement r_1 is unknown, it should be added to the parameter vector as well. Thus, $\tilde{\theta} = (\theta_x, \theta_y, r_1)^T$ and we solve $Y = H\tilde{\theta}$ for $\tilde{\theta}$ where

$$H = \begin{bmatrix} \ell_x^{(2)} - \ell_x^{(1)} & \ell_y^{(2)} - \ell_y^{(1)} & r_{21} \\ \ell_x^{(3)} - \ell_x^{(1)} & \ell_y^{(3)} - \ell_y^{(1)} & r_{31} \\ \vdots & \vdots & \vdots \\ \ell_x^{(N)} - \ell_x^{(1)} & \ell_y^{(N)} - \ell_y^{(1)} & r_{N1} \end{bmatrix}, \quad (2.11c)$$

$$Y = \frac{1}{2} \begin{bmatrix} \|\ell^{(2)}\|^2 - \|\ell^{(1)}\|^2 - r_{21}^2 \\ \|\ell^{(3)}\|^2 - \|\ell^{(1)}\|^2 - r_{31}^2 \\ \vdots \\ \|\ell^{(N)}\|^2 - \|\ell^{(1)}\|^2 - r_{N1}^2 \end{bmatrix}. \quad (2.11d)$$

AOA

The position of the UE can be estimated from AOA measurements using angulation technique. Let α_i denote the measured angle of the received signal transmitted by the BS⁽ⁱ⁾. As discussed in (Frattasi and Rosa, 2017), equation (2.6) gives

$$\left(\ell_x^{(i)} - \theta_x\right) \sin(\alpha_i) = \left(\ell_y^{(i)} - \theta_y\right) \cos(\alpha_i), \quad (2.12a)$$

with

$$\mathbf{H} = \begin{bmatrix} -\sin(\alpha_1) & \cos(\alpha_1) \\ -\sin(\alpha_2) & \cos(\alpha_2) \\ \vdots & \vdots \\ -\sin(\alpha_N) & \cos(\alpha_N) \end{bmatrix}, \quad (2.12b)$$

$$\mathbf{Y} = \frac{1}{2} \begin{bmatrix} \ell_y^{(1)} \cos(\alpha_1) - \ell_x^{(1)} \sin(\alpha_1) \\ \ell_y^{(2)} \cos(\alpha_2) - \ell_x^{(2)} \sin(\alpha_2) \\ \vdots \\ \ell_y^{(N)} \cos(\alpha_N) - \ell_x^{(N)} \sin(\alpha_N) \end{bmatrix}. \quad (2.12c)$$

2.2.3 Level 3: Modality fusion and temporal filtering

The so called hybrid positioning techniques are based on a combination of different methods introduced in Section 2.2.2 aiming to improve reliability, accuracy, and wireless resource consumption, among other performance characteristics.

Using measurements of different modality (kind) is not a problem and is covered in the same nonlinear set of equation framework as (2.1). The only difference is that other sensor information can be included. The inertial sensor unit in smartphones is today used to compute various motion related parameters. These can be used on the device for positioning, but also transmitted to the network. For instance, inertial sensor measurements can be combined with the global positioning system (GPS) for classification of the user's motion modes, see (Kasebzadeh et al., 2017). Fusion of RTT and TDOA measurements is another example of this type.

The key idea with filtering is to include temporal correlation in a dynamic model, so that a prediction of the next position can be computed in a state-space model (SSM) framework. The unknown, unobserved states $\mathbf{x}$ of the system in a SSM framework are inferred from the measurement function $\mathbf{h}(\cdot)$ and evolved in time using the transition function $\mathbf{f}(\cdot)$. Although for the linear class of SSM in white Gaussian noise a closed-form solution exists, nonlinear SSM require approximative approaches to compute the recursions. Further discussions on Bayesian filtering and corresponding solutions are provided in Appendix B.

2.3 Practical considerations

This section continues the brief overview of radio network measurements given in Section 2.2.1, and provides a practical survey similar to (Gustafsson and Gunnarsson, 2005), extended with recent measurements and standards. Lower layer techniques for providing these measurements are not addressed, and instead we refer to (Drane et al., 1998; Zhao, 2001) for 2nd generation (2G), (Caffery, 1999; Zhao, 2002) for 3rd generation (3G) and (del Peral-Rosado et al., 2012; Keunecke and Scholl, 2013) for 4th generation (4G) cellular systems.

2.3.1 Received signal strength

In the RSS measurement (2.7a), in addition to the measurement noise $e_t^{(i),\text{RSS}}$, one might also consider the diffraction factor. This way, (2.7a) can be re-written as

$$y_t^{(i),\text{RSS}} = P_0^{(i)} + 10\eta \log \left(\frac{\|\theta_t - \ell_t^{(i)}\|}{d_0} \right) + e_t^{(i),\text{RSS}} + d_t^{(i),\text{RSS}}, \quad (2.13)$$

where $d_t^{(i),\text{RSS}}$ is the diffraction. Propagation also features diffraction effects which resembles shadow fading that is a lowpass spatial process. A number of methods exists to deal with the diffraction error. One way is to lump them together with the measurement error, see (Zanella, 2016) for more details. Another approach is to capture these variations in a model/database which essentially forms the fingerprinting method. A third way is to assume that the shadow fading is only present in the intermediate to far field from the antenna, but not in the near field. This way, in the near field, the only source of error is the measurement noise.

2.3.2 TOA and TDOA

Both RTT and TDOA are based on TOA measurements at the UE as well as the BS. TOA is estimated by cross-correlating the received signal with a replica of the transmitted signal waveform. TOA is used to estimate RTT by combining TOA estimated at BS and UE, while TDOA is estimated using TOA associated to two different BSs. (Gunnarsson et al., 2014) provides a novel method for RTT calculations in LTE systems using the uplink timing alignment mechanism.

The performance analysis performed by (Xu et al., 2016), indicates different levels of accuracy based on the pilots used, as well as the bandwidth of the LTE system. Let $\sigma_{\text{LB}}^{\text{TOA}}$ denote the lower bound (LB) on achievable TOA estimation with 68% confidence interval when the pilot signal is received at UE with signal to noise ratio (SNR) = -13 dB. For an ideal additive white Gaussian noise (AWGN) channel, for a 20 MHz LTE system using PRS, the lower bound on TOA standard deviation is $\sigma_{\text{LB}}^{\text{TOA}} = 2.4$ ns. Assuming the signal is transmitted at 3×10^8 m/s, this translates to about 0.7 m. Using the same pilot signal but reducing the system bandwidth to 1.4 MHz, the accuracy degrades to $\sigma_{\text{LB}}^{\text{TOA}} = 66$ ns or about 20 m.

Assuming that two independent TOA measurements, with $\sigma_{\text{LB}}^{1,\text{TOA}}$ and $\sigma_{\text{LB}}^{2,\text{TOA}}$, are used to estimate the TDOA, the lower bound on the achievable accuracy is

Figure 2.3: The measured elevation of the UE using known pressure at a reference point.

given by $\sigma_{\text{LB}}^{\text{TDOA}} = \sqrt{(\sigma_{\text{LB}}^{1,\text{TOA}})^2 + (\sigma_{\text{LB}}^{2,\text{TOA}})^2}$. For the 20 MHz and 1.4 MHz LTE systems, with the same setup as above, the accuracy levels are 1 m and 22 m , respectively, see (Xu et al., 2016) for more details.

2.3.3 Barometric pressure

All indoor navigation systems, require a reliable source of vertical measurement (along the z -axis) in multi-story environments to operate with an acceptable level of accuracy. This information can be obtained for example from GPS-based elevation estimation techniques. However, lack of accuracy and reliability on top of limited availability to outdoor environments motivates more reliable source of information. One complementary sensor that solves the tricky vertical position problems is barometric pressure sensors that are based on barometric formula stating that atmospheric pressure decreases with increasing altitude.

Given a reference point at which the height above the see level $\ell_{z_t}^{(i)}$, standard air temperature T_r , and air pressure p_r are known, see Figure 2.3, θ_{z_t} can be found by

$$\theta_{z_t} = p_r + \frac{T_r}{L} \left(\frac{p_\theta}{p_r}^{-\tilde{c}} - 1 \right), \quad (2.14)$$

where $\tilde{c}$ is the constant in barometric formula, L is temperature lapse rate, and p_θ is the known air pressure at location of the UE. Generic measurement of the altitude of UE relative to the reference point is thus given by

$$y_t^{i,\text{baro}} = \|\theta_{z,t} - \ell_{z,t}^{(i)}\| + e_t^{i,\text{baro}}. \quad (2.15)$$

An example of a possible use of a barometer in vertically oriented activities is presented in (Muralidharan et al., 2014). Three types of reference points exist:

- Meteorological stations for weather forecast already deployed by the national meteorological agencies. These stations have coarse spatial density

on the amplitude of tens of kilometers and low update frequency of almost once an hour.

- The elevation of a person with a smartphone in outdoor environment taken from Digital Elevation Model (DEM)-map based on his current location is called a "DEM reference".
- The third reference point is based on an ad-hoc fashion of smartphones within the system.

For the case a reference pressure is unavailable, (Liu et al., 2014) presents a framework that does not depend on any special infrastructure and provides accurate elevation measurements using only smartphones. The final accuracy obtained by applying the system presented in (Liu et al., 2014) is less than 5 m in 90% of the cases and less than 3 m in 75% of times.

2.4 Trends

So far, we described the area of positioning in radio networks followed by practical consideration. However, there are some important trends that are expected to further improve the achievable accuracy.

2.4.1 New and better information

New timing measurements

The timing measurement protocol was first introduced in the institute of electrical and electronics engineers (IEEE) 802.11v standard as an optional management for stations. Those stations who do not support this procedure, shall ignore a received timing measurement frame. Reference (IEEE Std 802.11-2016, 2016) presents the workflow of various wireless network management procedures of the IEEE 802.11v standard including timing measurements. Initiation of or stopping an ongoing procedure takes place by a "Request frame" sent by the receiving station. The value of the trigger field dictates if it is an initiative frame or a stop one.

Indoor environments, however, require more sophisticated measurement procedures to deal with practical challenges. For instance, while GNSS systems in outdoor environments are equipped with atomic clocks providing precise synchronization between all satellites, WiFi access points are not necessarily synchronized. The synchronization issue is compensated by measuring round-trip delays (L. Banin, 2016). A yet newer protocol, fine timing measurement (FTM) (IEEE Std 802.11-2016, 2016), enables indoor TOF positioning. Measurements in the FTM protocol can be performed for different bandwidths. Additionally, the number of measurement frames is configurable and can take a value between 1 and 32. Figure 2.4 illustrates a generic implementation of FTM initiated with FTM request.

Figure 2.4: Fine timing measurement protocol with FTM burst size N where $1 \leq N \leq 32$. Time of departure (TOD) and TOA are marked in the figure.

Massive MIMO

Classic array processing with multiple input multiple output (MIMO) antennas as surveyed in (Krim and Viberg, 1996), enables accurate direction of arrival estimation. Massive MIMO, where the number of antenna elements is on order of magnitude larger than the number of communication links they serve, scales very favorably. This and many other advantages are described in (Rusek et al., 2013).

In addition to the research on communications perspectives, as shown in (Guerra et al., 2015), massive MIMO is also an enabler for accurate localization. Authors in (Savic and Larsson, 2015) studied multiple users localization using fingerprinting solution by means of massive MIMO. Using the concept of direct localization, first introduced in (Wax et al., 1982), authors in (Garcia et al., 2017) studied direct localization for massive MIMO.

Ad-hoc networks

Localization services that are applicable to these networks must meet different demands such as low power consumption, availability, and reliability. That is why some existing services such as GPS cannot be employed on wireless ad-hoc networks. To address this issue, one alternative is to use short-range single-hop localization systems. However, there are cases in which reference nodes are not in the range of unknown ones. Then, multi-hop techniques must be taken into account. In these scenarios, beacon positions are broadcasted over multiple hops. This allows estimation of the distance to beacon nodes by calculating hop sizes

and number of hops.

An ad-hoc positioning system based on AOA measurements is reported in (Niculescu and Nath, 2003). Authors in (Xiao et al., 2017) use a distance vector hop algorithm for wireless sensor networks based on the received signal strength indicator for positioning. Experimental results on ad-hoc networks that are self-organized by means of flying robots are studied in (Rosati et al., 2016). An extensive survey on position-based routing in vehicular ad-hoc networks is performed in (Jiang et al., 2016).

2.4.2 New infrastructure

The infrastructure contains different entities that each of them can affect the measurement resolution drastically. All the devices at the lowest layer are connected to their upper layer devices via a short-range technology such as Bluetooth, Zig-Bee, etc. In the meantime, devices in the middle layer could vary from a simple UE acting as a gateway to a MTC device (Astely et al., 2013). Different types of access of the middle devices could be an IP-connectivity to another gateway, cellular access to the access point (AP) or even an intra-connection to another device of the same layer via a short range technology.

BLE beacons

Bluetooth low energy (BLE) beacons can be low cost tiny computers equipped with Bluetooth radios. More complex hand-held devices such as smartphones can also provide the same functionality. The general idea is that these devices emit short-range signals that can be decoded by another BLE-enabled device. The distance to the receiving beacon can then be estimated. The possibility of identification of multiple beacons simultaneously in parallel with relative distance calculations of each beacon, location awareness of the device becomes possible.

IoT

IoT can be seen as a great potential in many lines of research and development. However, massive signaling traffic produced by numerous objects that update their locations, causes new challenges that need to be addressed. Thus, there is a need for appropriate solutions that provide accurate location information while keeping the signaling level low.

M2M

Machine to machine (M2M) networks contain a number of devices such as radio-frequency identification, sensors, tags, etc. This type of network is employed in different location-based applications ranging from health monitoring to battle-field surveillance. M2M communication networks are self-configurable with the feature of being accessed remotely. The efficiency of approaches for location estimation of M2M network devices can be defined by scalability, whether or not they

depend on GPS systems, range-based or range free property, and error handling capabilities.

Fifth generation (5G)

Positioning is one of the most important design specifications for next generation 5G systems. Particularly, the millimeter wave technology operating in carrier frequencies beyond 30 GHz band (Agiwal et al., 2016) has specific properties that make it of great interest for radio-based positioning (Talvitie et al., 2017). The millimeter wave technology allows for packing massive arrays into a small area. For example, authors in (Kaouach et al., 2011) studied the problem of realizing millimeter-wave massive arrays with dimensions of a tablet.

The possibility of integrating a massive array in small areas, enabled by millimeter wave technology in 5G systems, motivated authors in (Guidi et al., 2016), to study the concept of personal mobile radar operating at millimeter-waves. Positioning for vehicular networks using millimeter wave technology in 5G systems is studied in (Wymeersch et al., 2017). More recently, by taking advantage of large MIMO and millimeter-wave technologies, authors in (Shahmansoori et al., 2018) study the problem of positioning and orientation estimation using only one BS.

Narrowband IoT

Although GNSS solutions are capable of determining the position of an object with a few meters accuracy in outdoor environments, the robustness of GNSS-based methods is always restricted by the availability of GNSS signals. Indoor environments and dense urban areas are examples where these solutions fail.

As a response, 3GPP LTE standard features positioning support since 3GPP Release 9. The subsequent releases, as explained in (3GPP TR 36.355, Release 12), further extended capabilities of positioning by introducing specific signaling infrastructures. For more information on positioning in LTE systems see (Chen and Wu, 2017; Driusso et al., 2017; Kangas et al., 2011).

The immense number of use cases inspired by IoT, however, motivated 3GPP to introduce Release 14, NB-IoT. Wearable technologies, asset tracking, environmental monitoring are examples of ‘things’ addressed by IoT. Low power consumption and the possibility to communicate in the most challenging locations, in terms of coverage, are among shared requirements in all these scenarios.

NB-IoT aims to offer deployment flexibility allowing an operator to allocate a small portion of its current available spectrum to NB-IoT. Co-existence performance with global system for mobile communications (GSM), general packet radio service (GPRS) and LTE technologies are primary design criterion for NB-IoT. As reported in (Wang et al., 2017), NB-IoT requires a minimum of 180 kHz system bandwidth for both downlink and uplink. A GSM operator can replace one GSM carrier (200 kHz) with NB-IoT. An LTE operator can deploy NB-IoT inside an LTE carrier by allocating one of the physical resource blocks (PRB) of 180 kHz to NB-IoT.

3

Non-Gaussian measurement noise with positive support

In practice, TOA measurements, defined by the measurement model (2.3), have inevitable delays. This implies that the distribution of $e_t^{(i),\text{TOA}}$ has a positive support. Hence, it is of great interest to study the estimation problems in which the random variable has positive values only. In this chapter, we visit the problem of estimating deterministic quantity observed in non-Gaussian additive noise.

More specifically, we study the estimation problem when measurement noises either have positive supports or follow a mixture of normal and uniform distribution. This is a problem of great interest specially in cellular positioning systems where the wireless signal is prone to multiple sources of noises which generally have a positive support. Multiple noise distributions are investigated and, if possible, minimum variance unbiased (MVU) estimators are derived. In case of uniform, exponential and Rayleigh noise distributions, unbiased estimators without any knowledge of the hyperparameters of the noise distributions are also given. For each noise distribution, the proposed order statistic-based estimator's performance, in terms of mean squared error, is compared to the BLUE, as a function of sample size, in a simulation study.

3.1 Motivation and related work

A bias compensated linear estimator as the sample mean has a variance that decays as $1/N$, while it is well-known from the statistical literature, see for example (Kay, 1993; Lehmann and Casella, 1998), that the minimum has a variance that decays as $1/N^2$. The minimum is the simplest example of order statistics. Certain care has to be taken for the cases where the parameters in the distributions are unknown, in which case bias compensation becomes tricky.

To deal with the estimation's performance degradation in non-Gaussian error conditions, conventional estimation techniques which are developed based on

Gaussian assumptions need to be adjusted properly. As discussed in (Yin et al., 2013), “identify and discard”, “mathematical programming”, and “robust estimation” are the three broad categories of estimation methods which are robust against non-Gaussian errors. Robustness of the estimator has been a concern for many years in both research (Stigler, 1973) and different engineering topics (Arce, 2004; Kassam, 1988; Kassam and Poor, 1985; Stewart, 1999) for a long time now. A more recent survey on this topic containing more references can be found in (Zoubir et al., 2012).

Robust estimation is crucial in a variety of applications where the main objective is to infer a parameter of interest from uncertain observations. While optimal estimators mainly rely on strong assumptions on error probability distribution, typically to have normal distribution (Kim and Shevlyakov, 2008), in practice we deal with noises whose densities are highly non-Gaussian.

In the estimation of location problems if the Gaussian noise assumption is fulfilled, the BLUE is given by sample mean which gives the same weight to all observations. In these scenarios, the sample mean estimator coincides with the MLE and is optimal in the Fisher’s sense. In many applications, however, noises are non-symmetric, skewed and non-Gaussian (Chen et al., 2009; Eling, 2012; Gustafsson and Gunnarsson, 2005; Kok et al., 2015), hence sample mean estimator gets a bias that needs to be compensated for.

Empirical analysis of the real data is performed in (Huerta and Vidal, 2005) to determine the error PDF of a reference signal propagated in outdoor environments. The errors are then modeled as a mixture of Gaussian, for measurement noise, and Rayleigh, for propagation effects. The authors in (Fritsche and Klein, 2009; Fritsche et al., 2009; Hammes and Zoubir, 2011; Liao and Chen, 2006) also model propagation errors using shifted Gaussian densities and introduce robust timing-based position estimation methods. In (Hammes et al., 2009), the second component in the mixture distribution corresponding to the propagation errors is modeled using the convolution of the probability distribution function PDF of a positive random variable and the zero-mean Gaussian density of measurement noise.

The MLE, developed under Gaussian assumptions, can be modified to become robust in presence of non-Gaussian noises. The authors in (Eskin, 2000) first detect and then reject the outliers by learning the PDF of the measurements and develop a mixture model of outliers and clean data. A similar idea to k-nearest-neighbor approach is used in (Chawla et al., 2010) to classify outliers as the data points that are less similar to a number of neighboring measurements. Surveys of advances in clustering the data into outliers and clean data can be found in (Fritsche et al., 2009; Hodge and Austin, 2004; Yin et al., 2013). While these approaches might result in high estimation accuracy, they typically require large datasets (Zoubir et al., 2012).

In this chapter, we strive to find MVU estimators for the location of estimation problems for non-Gaussian noise distributions where a number of distributions with positive support are considered. In case where MVU is not found, we introduce unbiased order-statistic-based estimators and compare their variances against the BLUE. The MVU estimators without any knowledge of the hyperpa-

rameters of the noise distributions are also derived, if possible. Finally, we derive an estimator for the case in which the noise follows a mixture of normal and uniform distribution.

3.2 Marginal distribution of order statistics

The marginal distribution of order statistics, in this work is computed by differentiating the corresponding cumulative distribution function (CDF). In this section, we first introduce the minimum, also known as first or extreme, order statistic and then give the generalization to any statistics of order k . Let F denote the common CDF of N independent and identically distributed sample of random variables $y_1, \dots, y_N$. We let $y_{(k)}$ denote the k :th order statistic of the sample, defined as the k :th smallest value of the set $\{y_i\}_{i=1}^N$. We define $f_{(k,N)}(y)$ as the marginal PDF of the k :th order statistics corresponding to a sample of size N . The PDF $f_{(k,N)}(y)$ is then calculated by differentiating $F_{(k,N)}(y)$ with respect to y .

3.2.1 Marginal distribution of minimum order statistic

To further illustrate the problem, consider first an example in which we have drawn $N = 5$ independent random variables $\{y_i\}_{i=1}^5$ each from a common distribution with PDF $f(y)$. Assume that we are interested in the PDF of the first order statistic, $f_{(1,5)}(y)$. The CDF $F_{(1,5)}(y)$ is defined as $P(y_{(1)} < y)$. We note that the minimum order statistic $Y_{(1)}$ would be less than y if at least 1 of the random variables y_1, y_2, y_3, y_4, y_5 are less than y . In other words, we need to count the number of ways that can happen such that at least one random variable is less than y . This leads to a binomial probability calculation. The 'success' is considered to be the event $\{y_i < y\}$, $i = 1$ and we let ζ denote the number of successes in five trials, then

$$\begin{aligned} F_{(1,5)}(y) &= P(y_{(1)} < y) = P(\zeta = 1) + \dots + P(\zeta = 5), \\ f_{(1,5)}(y) &= \frac{d}{dy} F_{(1,5)}(y). \end{aligned}$$

To generalize the example, let $y_{(1)} < y_{(2)} < \dots < y_{(N)}$ be the order statistics of N independent observations from a continuous distribution with cumulative distribution function $F(y)$ and probability density function $f(y) = F'(y)$. The marginal PDF $f_{(1,N)}(y)$ of the minimum order statistic can be obtained by considering the event $\{Y_i \leq y\}$, $i = 1$ as a "success," and letting ζ = the number of such successes in N mutually independent trials. ζ is a binomial random variable with N trials and probability of success $P(y_i \leq y)$. Hence, the CDF of the minimum order statistic is given by,

$$F_{(1,N)}(y) = \sum_{n=1}^N P(\zeta = n). \quad (3.1a)$$

Noting that the probability mass function of this binomial distribution is given by,

$$P(\zeta = n) = \binom{N}{n} [F(y)]^n [1 - F(y)]^{N-n}. \quad (3.1b)$$

Substituting (3.1b) into (3.1a) and taking the last term out of the sum, we get

$$F_{(1,N)}(y) = \sum_{n=1}^{N-1} \binom{N}{n} [F(y)]^n [1 - F(y)]^{N-n} + [F(y)]^N. \quad (3.1c)$$

Differentiating (3.1c) with respect to y gives a telescoping sum of the form,

$$\begin{aligned} f_{(1,N)}(y) &= \sum_{n=1}^{N-1} \frac{N!}{(n-1)!(N-n)!} [F(y)]^{n-1} f(y) [1 - F(y)]^{N-n} \\ &\quad + \sum_{n=1}^{N-1} \frac{N!}{n!(N-n-1)!} [F(y)]^n [1 - F(y)]^{N-n-1} (-f(y)) \\ &\quad + N[F(y)]^{N-1} f(y), \end{aligned} \quad (3.2)$$

in which, except the first term, all other terms cancel each other out. Hence, the marginal probability density function of the minimum order statistic of a set of N independent and identically random variables with common CDF $F(y)$ and PDF $f(y)$ is given by,

$$f_{(1,N)}(y) = N f(y) (1 - F(y))^{N-1}. \quad (3.3)$$

3.2.2 Marginal distribution of general order statistic

The marginal PDF $f_{(k,N)}(y)$ of the general order statistic k can be obtained by generalizing the results of the previous section, and considering the event $\{y_i \leq y\}$, $i = 1, 2, \dots, k$ as a "success," and letting ζ = the number of such successes in N mutually independent trials,

$$F_{(k,N)}(y) = \sum_{n=k}^{N-1} \binom{N}{n} [F(y)]^n [1 - F(y)]^{N-n} + [F(y)]^N. \quad (3.4)$$

Differentiating (3.4) with respect to y gives a telescoping sum of the form,

$$\begin{aligned} f_{(k,N)}(y) &= \sum_{n=k}^{N-1} \frac{N!}{(n-1)!(N-n)!} [F(y)]^{n-1} f(y) [1 - F(y)]^{N-n} \\ &\quad + \sum_{n=k}^{N-1} \frac{N!}{n!(N-n-1)!} [F(y)]^n [1 - F(y)]^{N-n-1} (-f(y)) \\ &\quad + N[F(y)]^{N-1} f(y), \end{aligned} \quad (3.5)$$

in which, except the first term, all other terms cancel each other. Hence, the marginal probability density function of the k :th order statistic of a set of N independent and identically random variables with common CDF $F(y)$ and PDF $f(y)$ is given by,

$$f_{(k,N)}(y) = N f(y) \binom{N-1}{k-1} F(y)^{k-1} (1 - F(y))^{N-k}. \quad (3.6)$$

3.3 Location estimation problem

Consider the location estimation problem in which we have measurements y_k , $k = 1, \dots, N$ of the unknown parameter x . Assuming that the measurements are corrupted with additive noise $e_k \sim p_e(\theta)$, where θ denotes the parameter(s) of the noise distribution, the measurement model is given by

$$y_k = x + e_k, \quad k = 1, \dots, N. \quad (3.7)$$

The BLUE for the estimation problem (3.7) is given by

$$\hat{x}_{\text{BLUE}}^{p_e(\theta)}(y_{1:N}, \theta) = \frac{1}{N} \sum_{k=1}^N y_k - \delta(\theta), \quad (3.8)$$

where $y_{1:N} = \{y_k\}_{k=1}^N$ and $\delta(\theta) = \mathbb{E}(e_k)$ is the bias compensation term.

In the following sections, closed-form expressions for the mean squared error (MSE) of the BLUE estimator for multiple noise distributions with positive support are provided. Given hyperparameter θ , the MVU estimator for each noise distribution p_e is denoted by $\hat{x}_{\text{MVU}}^{p_e}(y_{1:N}, \theta)$. MVU estimators with unknown hyperparameter are denoted by $\hat{x}_{\text{MVU}}^{p_e}(y_{1:N})$. If the MVU cannot be found, an unbiased order-statistics-based estimator is derived and denoted by $\hat{x}^{p_e}(y_{1:N}, \theta)$ and $\hat{x}^{p_e}(y_{1:N})$ for known and unknown hyperparameter cases, respectively. For example, $\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}, \beta)$ denotes the MVU estimator when $e_k \sim \mathcal{U}[0, \beta]$ and β is known. $\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N})$, on the other hand, corresponds to the MVU estimator of uniform noise with unknown hyperparameters of the distribution. Table 3.1 summarizes the notation used throughout this work.

For each noise distribution, we also consider the minimum order statistic estimator, denoted by $\hat{x}_{\min}^{p_e}(y_{1:N})$. Let $(y_{(m)})_{m=1}^N$ denote the ordered sequence obtained from sorting $y_{1:N}$ in an ascending order, $\hat{x}_{\min}^{p_e}(y_{1:N})$ is defined as

$$\hat{x}_{\min}^{p_e}(y_{1:N}) = y_{(1)} \triangleq \min_k y_k. \quad (3.9)$$

Noting that for any generic estimator $\hat{x}$ the MSE is given by

$$\text{MSE}(\hat{x}) = \text{Var}(\hat{x}) + b^2(\hat{x}). \quad (3.10)$$

The MSE for BLUE and MVU or any other bias compensated estimator coincides with the estimator's variance. In case of $\hat{x}_{\min}^{p_e}$, the existing bias enters the MSE.

Table 3.1: Notation.

$\{y_k\}_{k=1}^N$	noisy measurements of the unknown parameter x
$(y_{(m)})_{m=1}^N$	ordered measurement sequence
θ	parameters of the noise distribution
$\delta(\theta)$	bias compensation term
$\hat{x}_{\text{BLUE}}^{p_e}(y_{1:N}, \theta)$	BLUE when $e_k \sim p_e$ for known θ
$\hat{x}_{\text{MVU}}^{p_e}(y_{1:N}, \theta)$	MVU estimator when $e_k \sim p_e$ for known θ
$\hat{x}_{\text{MVU}}^{p_e}(y_{1:N})$	MVU estimator when $e_k \sim p_e$ for unknown θ
$\hat{x}^{p_e}(y_{1:N}, \theta)$	unbiased estimator when $e_k \sim p_e$ for known θ
$\hat{x}^{p_e}(y_{1:N})$	unbiased estimator when $e_k \sim p_e$ for unknown θ

In order to find the MVU estimator, the first step is to find the PDF $f(y_{1:N}; \theta)$ with θ denoting the parameters of the distribution. If the PDF satisfies regularity conditions, the Cramér-Rao lower bound (CRLB) can be determined. Any unbiased estimator that satisfies CRLB is thus the MVU estimator. However, the considered PDFs do not satisfy the regularity conditions,

$$\mathbb{E} \left[\frac{\partial \ln f(y_k; \theta)}{\partial \theta} \right] \neq 0. \quad (3.11)$$

Hence, the CRLB approach is not applicable. Instead, we rely on the Rao-Blackwell-Lehman-Scheffe (RBLS) theorem (Kay, 1993; Lehmann and Scheffé, 1950, 1955), to find the MVU estimator. The RBLS theorem (Lehmann and Scheffé, 1950) states that for any unbiased estimator $\hat{x}$ and sufficient statistics $T(y_{1:N})$, $\hat{x} = \mathbb{E}(\hat{x} | T(y_{1:N}))$ is unbiased and $\text{Var}(\hat{x}) \leq \text{Var}(\hat{x})$. Additionally, if $T(y_{1:N})$ is complete, then $\hat{x}$ is MVU.

As shown in (Kay, 1993), if the dimension of the sufficient statistics is equal to the dimension of the parameter, then the MVU estimator is given by $\hat{x} = g(T(y_{1:N}))$ for any function $g(\cdot)$ that satisfies

$$\mathbb{E}(g(T)) = x. \quad (3.12)$$

Hence, the problem of MVU estimator turns into the problem of finding a complete sufficient statistic. The Neyman-Fisher theorem (Fisher, 1922; Halmos and Savage, 1949) gives the sufficient statistic $T(y_{1:N})$, if the PDF can be factorized as follows

$$f(y_{1:N}; \Psi) = g(T(y_{1:N}), \Psi)h(y_{1:N}), \quad (3.13)$$

where ψ is the union of the noise hyperparameter(s) θ and x . The estimators in this work are derived in the order statistics framework.

3.4 Uniform distribution

As the first scenario, consider the case in which the additive noise e_k has a uniform distribution with a positive support, $p_e(\theta) = \mathcal{U}[0, \beta]$, $\beta > 0$ and $\theta = \beta$. The BLUE is given by

$$\hat{x}_{\text{BLUE}}^{\mathcal{U}}(y_{1:N}, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \frac{\beta}{2}. \quad (3.14a)$$

The MSE of BLUE for this case is given by

$$\text{MSE}(\hat{x}_{\text{BLUE}}^{\mathcal{U}}(y_{1:N}, \beta)) = \frac{1}{N^2} \sum_{k=1}^N \text{Var}\left(y_k - \frac{\beta}{2}\right) = \frac{\beta^2}{12N}. \quad (3.14b)$$

In order to find the MSE of the minimum order statistics estimator, $\hat{x}_{\min}^{\mathcal{U}}(y_{1:N})$, we need to find the first two moments of the estimator. Let $\tilde{y}_k = \frac{1}{\beta} y_k$. Since $y_k \sim \mathcal{U}[x, x + b]$, then for any constant $\beta > 0$, $\tilde{y}_k \sim \mathcal{U}[\frac{x}{\beta}, \frac{x}{\beta} + 1]$. Hence, $f(\tilde{y}_k) = 1$ and $F(\tilde{y}_k) = \frac{1}{\beta}(y_k - x)$. From (3.6) we get,

$$\begin{aligned} f_{(k,N)}^{\mathcal{U}[0,\beta]}(\tilde{y}) &= N \binom{N-1}{k-1} \left(\frac{\tilde{y}-x}{\beta}\right)^{k-1} \left(\frac{\beta-(\tilde{y}-x)}{\beta}\right)^{N-k} \\ &= \frac{(N)!}{(k-1)!(N-k)!} \left(\frac{\tilde{y}-x}{\beta}\right)^{k-1} \left(\frac{\beta-(\tilde{y}-x)}{\beta}\right)^{N-k}. \end{aligned} \quad (3.15a)$$

since $N \in \mathbb{N}^+$, $k \in \mathbb{N}^+$, and $k \in [1, N]$ we can change the factorials to gamma functions,

$$f_{(k,N)}^{\mathcal{U}[0,\beta]}(\tilde{y}) = \frac{\Gamma(N+1)}{\Gamma(k)\Gamma(N-k+1)} \left(\frac{\tilde{y}-x}{\beta}\right)^{k-1} \left(\frac{\beta-(\tilde{y}-x)}{\beta}\right)^{N-k}. \quad (3.15b)$$

The marginal distribution (3.15b) is a generalized beta distribution, also known as four parameters beta distribution (McDonald and Xu, 1995). The support of this distribution is from 0 to $\beta > 0$ and $f_{(k,N)}^{\mathcal{U}[0,\beta]}(\cdot) = \frac{1}{\beta} f_{(k,N)}^{\mathcal{U}[0,1]}(\cdot)$. The bias and variance of the general k :th order statistic estimator $\hat{x}_{(k)}^{\mathcal{U}}(y_{1:N})$ in case of uniform noise with support on $[0, \beta]$ are given by

$$\mathbb{E}(\hat{x}_{(k)}^{\mathcal{U}}(y_{1:N})) = \frac{\beta k}{N+1}, \quad (3.16a)$$

$$\text{Var}(\hat{x}_{(k)}^{\mathcal{U}}(y_{1:N})) = \frac{k(N-k+1)\beta^2}{(N+1)^2(N+2)}. \quad (3.16b)$$

The first two moments of the minimum order statistic estimator are obtained by

letting $k = 1$ in (3.16)

$$b\left(\hat{x}_{\min}^{\mathcal{U}}(y_{1:N})\right) = \frac{\beta}{N+1} \quad (3.17a)$$

$$\text{Var}\left(\hat{x}_{\min}^{\mathcal{U}}(y_{1:N})\right) = \frac{N\beta^2}{(N+1)^2(N+2)}. \quad (3.17b)$$

The MSE of $\hat{x}_{\min}^{\mathcal{U}}(y_{1:N})$ is then given by

$$\text{MSE}\left(\hat{x}_{\min}^{\mathcal{U}}(y_{1:N})\right) = \frac{2\beta^2}{(N+1)(N+2)}. \quad (3.18)$$

3.4.1 MVU estimator

In order to find the MVU estimator, we note that the PDF can be written in a compact form using the step function $\sigma(\cdot)$ as

$$f(y_k; x, \beta) = \frac{1}{\beta} [\sigma(y_k - x) - \sigma(y_k - x - \beta)]. \quad (3.19a)$$

which gives

$$\begin{aligned} f(y_{1:N}; x, \beta) &= \frac{1}{\beta^N} \prod_{k=1}^N [\sigma(y_k - x) - \sigma(y_k - x - \beta)] \\ &= \frac{1}{\beta^N} [\sigma(y_{(1)} - x) - \sigma(y_{(N)} - x - \beta)], \end{aligned} \quad (3.19b)$$

where $y_{(N)} \triangleq \max_k y_k$, $k = 1, \dots, N$. The expressions for the MVU estimator is derived for two different scenarios. We first assume that the hyperparameter β of the noise distribution is known and then further discuss the unknown hyperparameter case. In the general case, let $\Psi = [x, \beta]^\top$ denote the unknown parameter vector, the Neyman-Fisher factorization gives $h(y_{1:N}) = 1$ and

$$T(y_{1:N}) = \begin{bmatrix} y_{(1)} \\ y_{(N)} \end{bmatrix} = \begin{bmatrix} T_1(y_{1:N}) \\ T_2(y_{1:N}) \end{bmatrix}. \quad (3.20)$$

Known hyperparameter β

When the maximum support of the uniform noise β is known, the dimensionality of the sufficient statistic is larger than that of the unknown parameter x . As discussed in (Kay, 1993), the RBLS theorem can be extended to address this case if the form of a function $g(T_1(y_{1:N}), T_2(y_{1:N}))$ can be found that combines T_1 and T_2 into a single unbiased estimator of x .

Let $Z = T_1(y_{1:N}) + T_2(y_{1:N}) = u + v$. Since T_1 and T_2 are dependent,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{y_{(1)}, y_{(N)}}(u, z - u) d_u, \quad (3.21a)$$

where $f_{y_{(1)}, y_{(N)}}(u, z - u)$ is the joint density of minimum and maximum order statistics. As shown in (David and Nagaraja, 2004), for $-\infty < u < v < \infty$, the joint density of two order statistics $y_{(i)}$ and $y_{(j)}$ is given by

$$\begin{aligned} f_{y_{(i)}, y_{(j)}}(u, v) &= \frac{N!}{(i-1)!(j-1-i)!(N-j)!} \\ &\quad \times f_Y(u) f_Y(v) [F_Y(u)]^{i-1} \\ &\quad \times [F_Y(v) - F_Y(u)]^{j-1-i} [1 - F_Y(v)]^{N-j}, \end{aligned} \quad (3.21b)$$

that for the extreme orders, $i = 1$ and $j = N$ can be simplified such that for $u < v$

$$f_{y_{(1)}, y_{(N)}}(u, v) = N(N-1) [F_Y(v) - F_Y(u)]^{N-2} f_Y(u) f_Y(v). \quad (3.21c)$$

and zero otherwise. Substituting (3.21c) into (3.21a), we get

$$f_Z(z) = \frac{1}{2} N \beta^{-N} (2x + 2\beta - z)^{N-1}, \quad (3.21d)$$

for $2x + \beta < z < 2(x + \beta)$ and

$$f_Z(z) = -\frac{1}{2} N \beta^{-N} \frac{(z - 2x)^N}{2x - z}, \quad (3.21e)$$

for $2x < z \leq 2x + \beta$ and zero otherwise. It can be shown that

$$\mathbb{E}(f_Z(z)) = 2x + \beta. \quad (3.21f)$$

Hence, noting that β is known, the function $g(T_1(y_{1:N}), T_2(y_{1:N}))$ that gives an unbiased estimator should be of the form of

$$\begin{aligned} \hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}, \beta) &= g(T_1(y_{1:N}), T_2(y_{1:N})) \\ &= \frac{1}{2}(y_{(1)} + y_{(N)}) - \frac{\beta}{2}. \end{aligned} \quad (3.22a)$$

The MSE of the MVU estimator is given by

$$\text{MSE}(\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}, \beta)) = \frac{\beta^2}{2N(N+3)+4}. \quad (3.22b)$$

Comparing to (3.14b), the order statistics based MVU estimator outperforms the BLUE one order of magnitude.

Unknown hyperparameter β

In this case, the MVU estimators for the parameter vector $\Psi = [x, \beta]^\top$ can be derived from sufficient statistics (3.20),

$$\hat{\Psi} = g(T(y_{1:N})), \quad \text{s.t.} \quad \mathbb{E}(g(T(y_{1:N}))) = \Psi. \quad (3.23)$$

In this case, we have

$$\mathbb{E}(T(y_{1:N})) = \begin{bmatrix} x + \frac{\beta}{N+1} \\ x + \frac{N\beta}{N+1} \end{bmatrix} \quad (3.24)$$

To find the transformation that makes (3.24) unbiased, we define

$$g(T(y_{1:N})) = \begin{bmatrix} \frac{1}{N-1} (N T_1(y_{1:N}) - T_2(y_{1:N})) \\ \frac{N+1}{N-1} (T_2(y_{1:N}) - T_1(y_{1:N})) \end{bmatrix} \quad (3.25a)$$

that gives

$$\mathbb{E}(g(T(y_{1:N}))) = \begin{bmatrix} x \\ \beta \end{bmatrix}. \quad (3.25b)$$

Finally, the MVU estimator of x when the hyperparameter β is unknown is given by

$$\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}) = \frac{N}{N-1} y_{(1)} - \frac{1}{N-1} y_{(N)}. \quad (3.26a)$$

and its MSE is

$$\text{MSE}(\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N})) = \frac{N\beta^2}{(N+2)(N^2-1)}. \quad (3.26b)$$

This is naturally slightly larger than (3.22b) for finite N .

3.5 Distributions in the exponential family

The exponential family of probability distributions, in their most general form, is defined by

$$f(y; \theta) = h(y)g(\theta) \exp \{A(\theta) \cdot T(y)\}, \quad (3.27)$$

where θ is the parameter of the distribution, and $h(y)$, $g(\theta)$, $A(\theta)$, and $T(y)$ are all known functions. In this section, we only consider some example distributions of this family and show that the minimum order statistic estimator gets the same form of distribution as the noise distribution but with modified parameters. For the selected distributions, if possible, MVU estimators for both cases of known and unknown hyperparameter are derived. Otherwise, unbiased estimators with less variance than BLUE are proposed.

3.5.1 Exponential distribution

Exponential distributions are members of the gamma family with shape parameter 1; strongly skewed with no left sided tail ($y_k \in [x, \infty]$). Let $\beta > 0$ denote the scale parameter, the PDF of an exponential distribution is then given by

$$f^{\text{Exp}}(y_k; x, \beta) = \begin{cases} \frac{1}{\beta} \exp\left(-\frac{(y_k - x)}{\beta}\right) & y_k \geq x, \\ 0 & y_k < x. \end{cases} \quad (3.28a)$$

and the CDF, for $y \geq x$, is given by

$$F^{\text{Exp}}(y_k; x, \beta) = 1 - \exp\left(-\frac{(y_k - x)}{\beta}\right). \quad (3.28b)$$

For the BLUE estimator (3.8), from the properties of exponential distribution, we have

$$\hat{x}_{\text{BLUE}}^{\text{Exp}}(y_{1:N}, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \beta, \quad \text{MSE}(\hat{x}_{\text{BLUE}}^{\text{Exp}}) = \frac{\beta^2}{N}. \quad (3.29)$$

Substituting (3.28) into (3.6), the marginal density of the k :th order statistic is given by

$$f_{(k,N)}^{\text{Exp}}(y; x, \beta) = \frac{N}{\beta} \binom{N-1}{k-1} \left(1 - \exp\left(-\frac{(y-x)}{\beta}\right)\right)^{k-1} \exp\left(-\frac{(N-k+1)(y-x)}{\beta}\right). \quad (3.30)$$

The first order statistic density is then given by letting $k = 1$ in (3.30) that results in another exponential distribution,

$$f_{(1,N)}^{\text{Exp}}(y; x, \beta) = f^{\text{Exp}}(y; x, \bar{\beta}), \quad (3.31)$$

where $\bar{\beta} = \frac{\beta}{N}$. Hence, the MSE of the minimum order statistics estimator is given by

$$\text{MSE}\left(\hat{x}_{\min}^{\text{Exp}}(y_{1:N})\right) = \frac{2\beta^2}{N^2}. \quad (3.32)$$

In order to find the MVU estimator, we re-write the PDF as

$$f(y_{1:N}; x, \beta) = \frac{1}{\beta^N} \exp\left[-\frac{1}{\beta} \sum_{k=1}^N y_k\right] \exp\left[-\frac{N}{\beta} x\right] \times \sigma(y_{(1)} - x). \quad (3.33)$$

Known hyperparameter β

In case of the known hyperparameter β , the Neyman-Fisher factorization of PDF (3.33) gives

$$T(y_{1:N}) = y_{(1)} \quad (3.34a)$$

$$h(y_{1:N}) = \frac{1}{\beta^N} \exp\left[-\frac{1}{\beta} \sum_{k=1}^N y_k\right]. \quad (3.34b)$$

The MVU estimator can then be obtained from a transformation of the minimum order statistic that makes it an unbiased estimator. Finally, in case of exponential noise with known hyperparameter of the distribution, the MVU estimator and its MSE are given by

$$\hat{x}_{MVU}^{\text{Exp}}(y_{1:N}, \beta) = y_{(1)} - \frac{\beta}{N} \quad (3.35a)$$

$$\text{MSE}(\hat{x}_{MVU}^{\text{Exp}}(y_{1:N}, \beta)) = \frac{\beta^2}{N^2}. \quad (3.35b)$$

Unknown hyperparameter β

If the hyperparameter β is unknown, the factorization gives

$$T(y_{1:N}) = \begin{bmatrix} y_{(1)} \\ \sum_{k=1}^N y_k \end{bmatrix} = \begin{bmatrix} T_1(y_{1:N}) \\ T_2(y_{1:N}) \end{bmatrix}. \quad (3.36a)$$

Noting that sum of exponential random variables results in a Gamma distribution, we have $T_2(y_{1:N}) \sim \Gamma(N, \beta)$. Hence,

$$\mathbb{E}(T(y_{1:N})) = \begin{bmatrix} x + \frac{\beta}{N} \\ N(x + \beta) \end{bmatrix}. \quad (3.36b)$$

Following the same line of reasoning as in Section 3.4.1, the unbiased estimator is given by the transformation

$$g(T(y_{1:N})) = \begin{bmatrix} \frac{1}{N-1} (N T_1(y_{1:N}) - \frac{1}{N} T_2(y_{1:N})) \\ \frac{1}{N-1} (T_2(y_{1:N}) - N T_1(y_{1:N})) \end{bmatrix}, \quad (3.37a)$$

that gives

$$\mathbb{E}(g(T(y_{1:N}))) = \begin{bmatrix} x \\ \beta \end{bmatrix}. \quad (3.37b)$$

Finally, the MVU estimator when the hyperparameter β is unknown, is given by

$$\begin{aligned} \hat{x}_{MVU}^{\text{Exp}}(y_{1:N}) &= \frac{N}{N-1} y_{(1)} - \frac{1}{N(N-1)} \sum_{k=1}^N y_k \\ &= \frac{N}{N-1} y_{(1)} - \frac{1}{N-1} \bar{y}, \end{aligned} \quad (3.38a)$$

where $\bar{y}$ is the sample mean. Assuming that N is large $\min_k y_k$ and $\bar{y}$ are independent and the MSE of the estimator, asymptotically, is given by

$$\text{MSE}(\hat{x}_{MVU}^{\text{Exp}}(y_{1:N})) = \frac{\beta^2(N+1)}{N(N-1)^2}. \quad (3.38b)$$

3.5.2 Rayleigh distribution

One generalization of the exponential distribution is obtained by parameterizing in terms of both a scale parameter β and a shape parameter α . Rayleigh distribution is a special case obtained by setting $\alpha = 2$

$$f^{\text{Rayleigh}}(y_k; x, \beta) = \begin{cases} \frac{y_k - x}{\beta^2} \exp\left(-\frac{(y_k - x)^2}{2\beta^2}\right) & y_k > x, \\ 0 & y_k \leq x. \end{cases} \quad (3.39a)$$

and the CDF, for $y_k > x$ is given by

$$F^{\text{Rayleigh}}(y_k; x, \beta) = 1 - \exp\left(-\frac{(y_k - x)^2}{2\beta^2}\right). \quad (3.39b)$$

Hence, the BLUE estimator (3.8), becomes

$$\hat{x}_{\text{BLUE}}^{\text{Rayleigh}}(y_{1:N}, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \sqrt{\frac{\pi}{2}} \beta, \quad (3.40a)$$

$$\text{MSE}\left(\hat{x}_{\text{BLUE}}^{\text{Rayleigh}}(y_{1:N}, \beta)\right) = \frac{(4 - \pi)\beta^2}{2N}. \quad (3.40b)$$

The marginal density of the k :th order statistic is given by

$$f_{(k,N)}^{\text{Rayleigh}}(y; x, \beta) = \begin{cases} \frac{Ny}{\beta^2} \binom{N-1}{k-1} \left(1 - \exp\left(-\frac{(y-x)^2}{2\beta^2}\right)\right)^{k-1} \exp\left(-\frac{(N-k+1)(y-x)^2}{2\beta^2}\right) & y > x, \\ 0 & y \leq x. \end{cases} \quad (3.41)$$

Hence, the minimum order statistics density also is Rayleigh distributed

$$f_{(1,N)}^{\text{Rayleigh}}(y; x, \beta) = f^{\text{Rayleigh}}(y; x, \bar{\beta}), \quad (3.42)$$

where $\bar{\beta} = \frac{\beta}{\sqrt{N}}$. The MSE of the minimum order statistics is given by

$$\text{MSE}\left(\hat{x}_{\min}^{\text{Rayleigh}}(y_{1:N})\right) = \frac{2\beta^2}{N}. \quad (3.43)$$

The joint PDF of N independent observations $y_{1:N}$ is given by

$$f(y_{1:N}; x, \beta) = \frac{\prod_{k=1}^N (y_k - x)}{\beta^{2N}} \exp\left[\sum_{k=1}^N -\frac{(y_k - x)^2}{2\beta^2}\right] \sigma(y_{(1)} - x). \quad (3.44a)$$

Noting that

$$\sum_{k=1}^N (y_k - x)^2 = \sum_{k=1}^N y_k^2 - 2x \sum_{k=1}^N y_k + Nx^2, \quad (3.44b)$$

the PDF becomes

$$f(y_{1:N}; x, \beta) = \beta^{-2N} \prod_{k=1}^N (y_k - x) \exp \left[\frac{-1}{2\beta^2} \sum_{k=1}^N y_k^2 \right] \\ \times \exp \left[-\frac{Nx^2}{2\beta^2} \right] \exp \left[\frac{x}{\beta^2} \sum_{k=1}^N y_k \right] \sigma(y_{(1)} - x). \quad (3.44c)$$

Known hyperparameter β

Since (3.44c) cannot be factorized in the form of $f(y_{1:N}; x, \beta) = g(T(y_{1:N}), x)h(y_{1:N})$, the RBLs theorem cannot be used. Hence, even if an MVU estimator exists for this problem, we may not be able to find it. Thus, in case of Rayleigh-distributed measurement noise, we propose unbiased estimators based on order statistics.

If the hyperparameter of the distribution is known, the unbiased order statistic based estimator $\hat{x}^{\text{Rayleigh}}(y_{1:N}, \beta)$ is then given by,

$$\hat{x}^{\text{Rayleigh}}(y_{1:N}, \beta) = y_{(1)} - \frac{\sqrt{\pi}\beta}{\sqrt{2N}}, \quad (3.45a)$$

$$\text{MSE} \left(\hat{x}^{\text{Rayleigh}}(y_{1:N}, \beta) \right) = \frac{(4 - \pi)\beta^2}{2N}. \quad (3.45b)$$

which has the same variance as the BLUE estimator.

Unknown hyperparameter β

In case of unknown hyperparameters, as for the known case, no factorization that enables us to use the RBLs theorem can be found. In this case, we propose the following unbiased estimator

$$\hat{x}^{\text{Rayleigh}}(y_{1:N}) = \frac{\sqrt{N}}{\sqrt{N} - 1} y_{(1)} - \frac{1}{N(\sqrt{N} - 1)} \sum_{k=1}^N y_k \\ = \frac{1}{\sqrt{N} - 1} (\sqrt{N} y_{(1)} - \bar{y}). \quad (3.46)$$

Asymptotically, for large N , the sample mean and minimum order statistic are independent and the estimator MSE is given by

$$\text{MSE} \left(\hat{x}^{\text{Rayleigh}}(y_{1:N}) \right) = \frac{(1 + N)(4 - \pi)\beta^2}{2N(\sqrt{N} - 1)^2}. \quad (3.47)$$

3.5.3 Weibull distribution

Weibull distribution is a generalization of the Rayleigh, distribution that is parameterized by two parameters—scale parameter β and shape parameter $\alpha > 0$.

In fact Weibull distribution is obtained by relaxing the assumption $\alpha = 2$ in the Rayleigh distribution and its density function is given by

$$f^{\text{Weibull}}(y_k; x, \beta, \alpha) = \begin{cases} \frac{\alpha}{\beta} \left(\frac{y_k - x}{\beta} \right)^{\alpha-1} \exp\left(-\left(\frac{y_k - x}{\beta}\right)^\alpha\right) & y_k > x, \\ 0 & y_k \leq x. \end{cases} \quad (3.48a)$$

and the CDF, for $y_k \geq x$ is given by

$$F^{\text{Weibull}}(y_k; x, \beta, \alpha) = 1 - \exp\left(-\left(\frac{y_k - x}{\beta}\right)^\alpha\right). \quad (3.48b)$$

The BLUE estimator, in case of Weibull-distributed measurement noises is given by

$$\hat{x}_{\text{BLUE}}^{\text{Weibull}}(y_{1:N}, \beta, \alpha) = \frac{1}{N} \sum_{k=1}^N y_k - \beta \Gamma\left(1 + \frac{1}{\alpha}\right), \quad (3.48c)$$

$$\text{MSE}(\hat{x}_{\text{BLUE}}^{\text{Weibull}}(y_{1:N}, \beta, \alpha)) = \frac{\beta^2}{N} \left[\Gamma\left(\frac{\alpha+2}{\alpha}\right) - \left(\Gamma\left(\frac{\alpha+1}{\alpha}\right) \right)^2 \right]. \quad (3.48d)$$

The marginal density of the k :th order statistic is given by

$$f_{(k,N)}^{\text{Weibull}}(y; x, \beta, \alpha) = \frac{N\alpha}{\beta} \binom{N-1}{k-1} \left(\frac{y-x}{\beta} \right)^{\alpha-1} \left(1 - \exp\left(-\left(\frac{y-x}{\beta}\right)^\alpha\right) \right)^{k-1} \\ \times \exp\left(-\left(\frac{y-x}{\beta}\right)^\alpha\right). \quad (3.49)$$

Hence, the first order statistic density in case of $e_k \sim \text{Weibull}(\beta, \alpha)$, is another Weibull distribution,

$$f_{(1,N)}^{\text{Weibull}}(y; x, \beta, \alpha) = f^{\text{Weibull}}(y; x, \bar{\beta}, \alpha), \quad (3.50)$$

where $\bar{\beta} = \sqrt[N]{N}\beta$. This gives the MSE of the minimum order statistic estimator as

$$\text{MSE}(\hat{x}_{\min}^{\text{Weibull}}(y_{1:N})) = \beta^2 N^{\frac{-2}{\alpha}} \Gamma\left(\frac{\alpha+2}{\alpha}\right) \quad (3.51)$$

Given N independent observations, the joint density is given by

$$f^{\text{Weibull}}(y_{1:N}; x, \beta, \alpha) = \left(\frac{\alpha}{\beta}\right)^N \prod_{k=1}^N \left(\frac{y_k - x}{\beta}\right)^{\alpha-1} \exp\left(-\sum_{k=1}^N \left(\frac{y_k - x}{\beta}\right)^\alpha\right) \sigma(y_{(1)} - x) \quad (3.52)$$

Since (3.52) cannot be factorized using Neyman-Fisher factorization, RBLS is not applicable. Additionally, in this case, it is not possible to find an unbiased estimator when the hyperparameters α and β are unknown. In case of known hyperparameters, the unbiased minimum order statistic estimator, however, can be

computed. The unbiased estimator based on minimum order statistic is given by,

$$\begin{aligned}\hat{x}^{\text{Weibull}}(y_{1:N}, \beta, \alpha) &= y_{(1)} - \beta N^{-\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha}), \\ \text{MSE}(\hat{x}^{\text{Weibull}}(y_{1:N}, \beta, \alpha)) &= \beta^2 N^{-\frac{2}{\alpha}} \left[\Gamma(\frac{\alpha+2}{\alpha}) - \left(\Gamma(\frac{\alpha+1}{\alpha}) \right)^2 \right].\end{aligned}\quad (3.53)$$

An order-statistics-based unbiased estimator with unknown hyperparameters of the distribution could not be obtained.

3.6 Other distributions

In this section, we further study the location estimation problem for two other noise distributions. In the rest, the Pareto distribution with positive support is first studied followed by the mixture of uniform and normal distribution.

3.6.1 Pareto distribution

Let the scale parameter β (necessarily positive) denote the minimum possible value of y_k , and $\alpha > 0$ denote the shape parameter. The Pareto Type I distribution is characterized by β and α

$$f^{\text{Pareto}}(y_k; x, \beta, \alpha) = \begin{cases} \alpha \beta^\alpha (y_k - x)^{-(\alpha+1)} & y_k \geq x + \beta, \\ 0 & y_k < x + \beta. \end{cases} \quad (3.54a)$$

and the CDF is given by

$$F^{\text{Pareto}}(y_k; x, \beta, \alpha) = 1 - \left(\frac{\beta}{y - x} \right)^\alpha. \quad (3.54b)$$

For the BLUE we get,

$$\hat{x}_{\text{BLUE}}^{\text{Pareto}}(y_{1:N}, \beta, \alpha) = \frac{1}{N} \sum_{k=1}^N y_k - \frac{\alpha \beta}{\alpha - 1}, \quad \alpha > 1 \quad (3.55a)$$

$$\text{MSE}(\hat{x}_{\text{BLUE}}^{\text{Pareto}}(y_{1:N}, \beta, \alpha)) = \frac{\alpha \beta^2}{N(\alpha - 1)^2(\alpha - 2)}, \quad \alpha > 2 \quad (3.55b)$$

The RBLs theorem cannot be used in case Pareto-distributed noises. We provide an unbiased estimator using minimum order statistics and its variance. The marginal density of the k :th order statistic, for $y \geq x + \beta$ is given by

$$f_{(k,N)}^{\text{Pareto}}(y; x, \beta, \alpha) = N \alpha \beta^\alpha (y - x)^{-(\alpha+1)} \binom{N-1}{k-1} \left(1 - \left(\frac{\beta}{y-x} \right)^\alpha \right)^{k-1} \left(\frac{\beta}{y-x} \right)^\alpha \alpha (N-k). \quad (3.56)$$

The minimum order statistic has the same form of distribution

$$f_{(1,N)}^{\text{Pareto}}(y; \beta, \alpha) = f^{\text{Pareto}}(y; \beta, \bar{\alpha}), \quad (3.57)$$

where $\bar{\alpha} = N\alpha$. The MSE of the minimum order statistic estimator is

$$\text{MSE}(\hat{x}_{\min}^{\text{Pareto}}(y_{1:N})) = \frac{N\alpha\beta^2}{N\alpha - 2} \quad (3.58)$$

The unbiased estimator is thus given by,

$$\hat{x}^{\text{Pareto}}(y_{1:N}, \beta, \alpha) = y_{(1)} - \frac{N\alpha\beta}{N\alpha - 1}, \quad N\alpha > 1 \quad (3.59a)$$

$$\text{MSE}(\hat{x}^{\text{Pareto}}(y_{1:N}, \beta, \alpha)) = \frac{N\alpha\beta^2}{(N\alpha - 1)^2(N\alpha - 2)}, \quad N\alpha > 2 \quad (3.59b)$$

No unbiased estimator for unknown hyperparameter case could be found for Pareto distribution.

3.7 Mixture of normal and uniform noise distribution

Suppose the error is distributed as

$$e_k \sim \alpha \mathcal{N}(0, \sigma^2) + (1 - \alpha) \mathcal{U}[0, \beta],$$

where α is the mixing probability of the mixture distribution. Define $f^{\mathcal{U}, \mathcal{N}}(y_k)$ as the probability density function of the considered mixture distribution given by

$$f^{\mathcal{U}, \mathcal{N}}(y_k; x, \alpha, \sigma^2, \beta) = \begin{cases} \frac{a}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_k - x)^2}{2\sigma^2}\right] + \frac{1 - a}{\beta} & 0 \leq y_k - x \leq \beta \\ \frac{a}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_k - x)^2}{2\sigma^2}\right] & \text{Otherwise.} \end{cases} \quad (3.60)$$

The BLUE, in case of the mixture of normal and uniform measurement noises is given by

$$\hat{x}_{\text{BLUE}}^{\mathcal{U}, \mathcal{N}}(y_{1:N}, \alpha, \beta, \sigma^2) = \frac{1}{N} \sum_{k=1}^N y_k - \frac{\beta(1 - \alpha)}{2}, \quad (3.61a)$$

$$\text{MSE}(\hat{x}_{\text{BLUE}}^{\mathcal{U}, \mathcal{N}}(y_{1:N}, \alpha, \beta, \sigma^2)) = \frac{\beta^2(1 + (2 - 3\alpha)\alpha) + 12\alpha\sigma^2}{12N}. \quad (3.61b)$$

Noting that at $y_k - x = 0$ contributions of the uniform distribution and the mean (mode) of the normal distribution are added together, (3.60) is maximized

at this point. The order statistics PDF for $0 \leq y - x \leq \beta$ is given by

$$\begin{aligned}
 f_{(k,N)}^{\mathcal{U},\mathcal{N}}(y; \alpha, \beta, \sigma^2, x, k) = & \\
 & N \binom{N-1}{k-1} \left(\frac{\alpha \exp(-\frac{(y-x)^2}{2\sigma^2})}{\sqrt{2\pi}\sigma} + \frac{1-\alpha}{\beta} \right) \\
 & \times \left(\frac{(1-\alpha)(y-x)}{\beta} + \frac{\alpha}{2} (1 + \text{Erf} \left[\frac{y-x}{\sqrt{2}\sigma} \right]) \right)^{k-1} \\
 & \times \left(1 + \frac{(\alpha-1)(y-x)}{\beta} - \frac{\alpha}{2} (1 + \text{Erf} \left[\frac{y-x}{\sqrt{2}\sigma} \right]) \right)^{N-k}, \quad (3.62)
 \end{aligned}$$

where $\text{Erf}(\cdot) = \frac{2}{\sqrt{\pi}} \int_0^\cdot e^{-t^2} dt$ is the error function. In order to find the best order statistic estimator, we maximize the likelihood function $\ell(k | y = x, \alpha, \beta, \sigma^2)$

$$\begin{aligned}
 \ell(k | y = x, \alpha, \beta, \sigma^2) = & N \binom{N-1}{k-1} 2(2-\alpha)^{-k} \left(1 - \frac{\alpha}{2}\right)^N \alpha^{k-1} \\
 & \times \left(\frac{1-\alpha}{\beta} + \frac{\alpha}{\sqrt{2\pi}\sigma} \right). \quad (3.63a)
 \end{aligned}$$

Noting that $\left(\frac{1-\alpha}{\beta} + \frac{\alpha}{\sqrt{2\pi}\sigma}\right)$ is always positive and independent of k , we extract it from the likelihood function. Simplifying (3.63a) by means of manipulating the terms, we get

$$2(2-\alpha)^{-k} = 2^{1-k} \left(1 - \frac{\alpha}{2}\right)^{-k}, \quad (3.63b)$$

$$\alpha^{k-1} = \left(2\frac{\alpha}{2}\right)^{k-1} = 2^{k-1} \left(\frac{\alpha}{2}\right)^{k-1}. \quad (3.63c)$$

the likelihood function to be maximized can be re-written as

$$\ell(k | y = x, \alpha, \sigma^2) \propto \binom{N-1}{k-1} \left(\frac{\alpha}{2}\right)^{k-1} \left(1 - \frac{\alpha}{2}\right)^{N-k}. \quad (3.63d)$$

In order to find the maximum likelihood estimate $\hat{k} = \arg \max_k \ell(k | y - x = 0)$, we note that (3.63d) is a binomial distribution with probability of success $\frac{\alpha}{2}$. Hence, the maximum of the function is given at the mode of the distribution,

$$\hat{k} = \lfloor \frac{N\alpha}{2} \rfloor + 1 \quad \text{or} \quad \lceil \frac{N\alpha}{2} \rceil. \quad (3.64)$$

This gives the best order statistic estimator for the case when noise is a mixture of normal and uniform distribution as

$$\hat{x}^{\mathcal{U},\mathcal{N}}(y_{1:N}, \alpha) = y_{(\hat{k})}. \quad (3.65)$$

Table 3.2: Estimators and their MSEs derived for multiple noise distributions.

noise distribution	estimator	MSE
$\mathcal{U}[0, \beta]$	$\hat{x}_{\text{BLUE}}^{\mathcal{U}}(y_{1:N}, a_n) = \frac{1}{N} \sum_{k=1}^N y_k - \frac{\beta}{2}$	$\frac{\beta^2}{12N}$
	$\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}, a_n) = \frac{1}{2}(y_{(1)} + y_{(N)}) - \frac{\beta}{2}$	$\frac{\beta^2}{2N(N+3)+4}$
	$\hat{x}_{\text{MVU}}^{\mathcal{U}}(y_{1:N}) = \frac{N}{N-1}y_{(1)} - \frac{1}{N-1}y_{(N)}$	$\frac{N\beta^2}{(N+2)(N^2-1)}$
$\text{Exp}(\beta)$	$\hat{x}_{\text{BLUE}}^{\text{Exp}}(y_{1:N}, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \beta$	$\frac{\beta^2}{N}$
	$\hat{x}_{\text{MVU}}^{\text{Exp}}(y_{1:N}, \beta) = y_{(1)} - \frac{\beta}{N}$	$\frac{\beta^2}{N^2}$
	$\hat{x}_{\text{MVU}}^{\text{Exp}}(y_{1:N}) = \frac{N}{N-1}y_{(1)} - \frac{\sum_{k=1}^N y_k}{N(N-1)}$	$\frac{(N+1)\beta^2}{N(N-1)^2}$
$\text{Rayleigh}(\beta)$	$\hat{x}_{\text{BLUE}}^{\text{Rayleigh}}(y_{1:N}, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \sqrt{\frac{\pi}{2}}\beta$	$\frac{(4-\pi)\beta^2}{2N}$
	$\hat{x}_{\text{Rayleigh}}^{\text{Rayleigh}}(y_{1:N}, \beta) = y_{(1)} - \frac{\sqrt{\pi}\beta}{\sqrt{2N}}$	$\frac{(4-\pi)\beta^2}{2N}$
	$\hat{x}_{\text{Rayleigh}}^{\text{Rayleigh}}(y_{1:N}) = \frac{\sqrt{N}}{\sqrt{N-1}}y_{(1)} - \frac{\sum_{k=1}^N y_k}{N(\sqrt{N-1})}$	$\frac{(1+N)(4-\pi)\beta^2}{2N(\sqrt{N-1})^2}$
$\text{Weibull}(\beta, \alpha)$	$\hat{x}_{\text{BLUE}}^{\text{Weibull}}(y_{1:N}, \beta, \alpha) = \frac{1}{N} \sum y_k - \beta\Gamma(1 + \frac{1}{\alpha})$	$\beta^2 N^{-1} \left[\Gamma(1 + \frac{2}{\alpha}) - \left(\Gamma(1 + \frac{1}{\alpha}) \right)^2 \right]$
	$\hat{x}_{\text{Weibull}}^{\text{Weibull}}(y_{1:N}, \beta, \alpha) = y_{(1)} - \beta N^{-\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha})$	$\beta^2 N^{-\frac{2}{\alpha}} \left[\Gamma(1 + \frac{2}{\alpha}) - \left(\Gamma(1 + \frac{1}{\alpha}) \right)^2 \right]$
$\text{Pareto}(\beta, \alpha)$	$\hat{x}_{\text{BLUE}}^{\text{Pareto}}(y_{1:N}, \beta, \alpha) = \frac{1}{N} \sum y_k - \frac{\alpha\beta}{\alpha-1}, \alpha > 1$	$\frac{\alpha\beta^2}{N(\alpha-1)^2(\alpha-2)}, \alpha > 2$
	$\hat{x}_{\text{Pareto}}^{\text{Pareto}}(y_{1:N}, \beta, \alpha) = y_{(1)} - \frac{N\alpha\beta}{N\alpha-1}, N\alpha > 1$	$\frac{N\alpha\beta^2}{(N\alpha-1)^2(N\alpha-2)}$
$\alpha\mathcal{N}(0, \sigma^2) + (1-\alpha)\mathcal{U}[0, \beta]$	$\hat{x}_{\text{BLUE}}^{\mathcal{U}, \mathcal{N}}(y_{1:N}, \alpha, \beta) = \frac{1}{N} \sum_{k=1}^N y_k - \frac{\beta(1-\alpha)}{2}$	$\frac{\beta^2(1+(2-3\alpha)\alpha)+12\alpha\sigma^2}{12N}$
	$\hat{x}_{\mathcal{U}, \mathcal{N}}^{\mathcal{U}, \mathcal{N}}(y_{1:N}, \alpha, \beta) = y_{(\lfloor \frac{N\alpha}{2} \rfloor + 1)}$	Unknown

Table 3.3: Bias and MSE of minimum order statistics estimators $\hat{x}_{\min}^{P_e}$.

distribution	bias	MSE
$\mathcal{U}[0, \beta]$	$\frac{\beta}{N+1}$	$\frac{2\beta^2}{(N+1)(N+2)}$
$\text{Exp}(\beta)$	$\frac{\beta}{N}$	$\frac{2\beta^2}{N^2}$
$\text{Rayleigh}(\beta)$	$\frac{\sqrt{\pi}\beta}{\sqrt{2N}}$	$\frac{2\beta^2}{N}$
$\text{Weibull}(\beta, \alpha)$	$\beta N^{-\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha})$	$\beta^2 N^{-\frac{2}{\alpha}} \Gamma(1 + \frac{2}{\alpha})$
$\text{Pareto}(\beta, \alpha)$	$\frac{N\alpha\beta}{N\alpha-1}$	$\frac{N\alpha\beta^2}{N\alpha-2}$

Figure 3.1: Empirical CDF of measurement errors computed from noise realizations used in the simulations.

Figure 3.2: Analytical (marked with solid lines) and numerical (marked with crosses) MSE for uniform noise distribution as a function of the sample size N .

3.8 Performance evaluation

The estimators (both unbiased and the ones without bias compensation) derived in sections 3.5–3.6 for different noise distributions together with their MSE are summarized in Tables 3.2 and 3.3. The biased minimum order statistics based estimators and their MSE are also. The estimators derived for each noise distribution are compared against each other as a function of the sample size $N \in [2, \dots, 2000]$. Additionally, in order to verify the analytical derivations of the estimator variances, they are compared against the numerical variances obtained from $M = 5000$ Monte Carlo runs.

3.8.1 Simulation setup

For each sample size, N noisy measurements of the unknown parameter x are generated. The hyperparameters of the noise distributions are randomly selected in each repetition. In order to have a fair comparison, the hyperparameters are randomly drawn such that the error densities are mostly in the same range for all scenarios. The noise realizations are generated from the six considered distributions with the following hyperparameters

- Uniform noise: $\beta \sim \mathcal{U}[6, 50]$
- Exponential noise: $\beta \sim \mathcal{U}[5, 14]$
- Rayleigh noise: $\beta \sim \mathcal{U}[5, 12]$
- Weibull noise: $\beta = 1, \alpha \sim \mathcal{U}[5, 10]$

- Pareto noise: $\beta = 6$, $\alpha \sim \mathcal{U}[2.1, 2.5]$
- Mixture noise: $\sigma \sim \mathcal{U}[1, 9]$, $\beta \sim \mathcal{U}[1, 50]$

The empirical CDF of the error values used in the simulations are presented in Figure 3.1. The support of the noise values, as can be read from the figure, is $e_m \in [0, 60]$ unit.

Let $\hat{x}_N^{(m)}$ denote the estimated value of the unknown parameter x in the m :th repetition obtained from a sample of size N . For each noise distribution, the estimators' performances are evaluated in terms of the obtained MSEs. The theoretical MSE of each estimator, as defined in Table 3.2 and Table 3.3, is compared against the numerical MSE obtained in simulations.

We let $\mathbb{E}[\hat{x}_N] = \frac{1}{M} \sum_{m=1}^M \hat{x}_N^{(m)}$ and define

$$\hat{b}_N = \mathbb{E}[\hat{x}_N] - x \quad (3.66a)$$

$$\hat{\sigma}_N^2 = \frac{1}{M} \sum_{m=1}^M (\hat{x}_N^{(m)} - \mathbb{E}[\hat{x}_N])^2. \quad (3.66b)$$

The numerical MSE for each sample size N is then computed by

$$\hat{\text{MSE}}(\hat{x}_N) = \hat{\sigma}_N^2 + \hat{b}_N^2. \quad (3.66c)$$

3.8.2 Simulation results

Figure 3.2 presents the performance of the four estimators when the noise is uniformly distributed. The solid lines correspond to the theoretical MSEs and the crosses are the numerical MSEs obtained from $M = 5000$ repetitions. Both MVU estimators, with and without any knowledge of the hyperparameters of the underlying noise, result in noticeably less MSE compared to the BLUE estimator. The minimum order statistics estimator also outperforms BLUE when measurements are corrupted with additive, uniformly distributed, noise. It can be further observed that if the hyperparameter β is unknown, the MSE of the proposed estimator is negligibly larger than the case with known β .

For the exponential noise distribution, as shown in Figure 3.3a, there is still a non-negligible difference between BLUE and the other three estimators in terms of estimators' MSE. However, the two MVU estimators, specially for large values of N , behave similarly. In order to verify their performance for smaller sample sizes, Figure 3.3b illustrates the variances of all estimators for $N \leq 20$. At the beginning, $N \in [2, 4]$ the estimator with unknown hyperparameter has the largest MSE. However, for larger sample sizes, the two MVU estimators are almost equal and both have less MSE than the BLUE estimator. As in case of uniformly distributed measurement noise, the minimum order statistics estimator outperforms BLUE specially for large sample sizes.

In case of Rayleigh noise distribution, as given in Table 3.2, the minimum order statistics estimator has the largest MSE while the BLUE and the proposed unbiased estimator with known hyperparameter, result in similar estimation variance. This can be verified also in the simulation results presented in Figure 3.4a.

Figure 3.3: Analytical (marked with solid lines) and numerical (marked with crosses) MSE for exponential noise distribution as a function of the sample size N . The blue and the proposed estimator with known hyperparameters have equal variances, hence the blue line is invisible in these plots. Since the MVU estimators have similar results for large sample sizes, the MSE of the four estimators for smaller sample sizes are presented separately.

For large sample sizes, $N > 20$, these two estimators and the proposed estimator with unknown hyperparameter have similar values. However, for the smaller sample sizes, as illustrated in Figure 3.4b, the BLUE (and order statistic with known hyperparameter) estimator has smaller variance compared to the case with unknown hyperparameter. The minimum order statistics estimator results in larger MSE compared to the other three estimators in case of Rayleigh noise distribution.

As Table 3.2 suggests, for Pareto and Weibull noise distributions, we only derived BLUE and an unbiased order statistics based estimators when the two hyperparameters of the distributions are known. For both noise distributions, the MSE of the two unbiased estimators as well as the MSE of the minimum order statistics estimator are compared and the results are presented in Figure 3.5. In both cases, the proposed estimators outperform the BLUE in terms of variance. The minimum order statistics estimator results in a lower MSE than the BLUE for Weibull noise distributions. However, in case of Pareto noise, the BLUE has a better performance compared to the minimum order statistics estimator.

In case of mixture noise distribution, we consider three different scenarios based on the mixing probabilities; two extreme cases with dominant contribution from uniform noise, $\alpha = 0.01$, and dominant contribution from normal noise, $\alpha = 0.99$, and the case with $\alpha = 0.5$. Figure 3.6a illustrates the histogram of the noise realizations of the considered mixture noise distributions $e_k \sim \alpha \mathcal{N}(0, 8^2) +$

Figure 3.4: Analytical (marked with solid lines) and numerical (marked with crosses) MSE for Rayleigh noise distribution as a function of the sample size N . Since the BLUE and the proposed estimators have similar results for large sample sizes, the MSE of the four estimators for smaller sample sizes are presented separately.

Figure 3.5: Analytical (marked with solid lines) and numerical (marked with crosses) MSE for Weibull and Pareto noise distributions as a function of the sample size N .

Figure 3.6: Noise realizations, PDF, and empirical CDF of the mixture noise distribution $e_k \sim \alpha \mathcal{N}(0, 8^2) + (1 - \alpha)\mathcal{U}(0, 60)$ for three different values of α .

$(1 - \alpha)\mathcal{U}(0, 60)$ and the fitted densities. The empirical CDFs of the errors for the three cases are presented in Figure 3.6b.

In order to estimate the unknown parameter x , in each Monte Carlo run, we sort the measurements and then find the $(\lfloor \frac{N\alpha}{2} \rfloor + 1)$:th component. Figure 3.7 presents the estimation MSE for the three different scenarios with different mixing probabilities. As the results indicate, when the main contribution of the noise is from uniform distribution, $\alpha = 0.01$, BLUE outperforms the proposed estimator. In this case, a periodic behavior for the MSE can be observed. The jumps in the MSE occur exactly at points where $\lfloor \frac{N\alpha}{2} \rfloor + 1$ switches from the k :th measurement to the $k + 1$:th measurement. For instance, for $N \in [1, 199]$, $\lfloor \frac{N\alpha}{2} \rfloor = 0$, hence $\hat{x} = y_{(1)}$. However, at $N = 200$, $\lfloor \frac{N\alpha}{2} \rfloor = 1$, resulting in $\hat{x} = y_{(2)}$.

The proposed estimator and the BLUE result in similar estimation MSE for $\alpha = 0.99$, as shown in Figure 3.7b, in which the normal component is the dominant source of error. However, the most interesting results are obtained when both distributions have equal contributions in the measurement noise, i.e $\alpha = 0.5$. In this case, as Figure 3.7c suggests, the proposed estimator outperforms the BLUE.

Figure 3.7: Analytical (marked with solid lines) and numerical (marked with crosses) MSE for three different values of mixing probability α , when $e_k \sim \alpha \mathcal{N}(0, 8^2) + (1 - \alpha) \mathcal{U}(0, 60)$.

4

Concluding remarks

The main objective of this thesis is to address some of the existing challenges in positioning using ranging measurements in radio networks. The considered challenges stem from either the uncertainty in the measurements or from communication constraints and standards. Papers A and B study positioning in narrowband Internet of things (NB-IoT) and long-term evolution (LTE) systems using ranging measurements. Noting that the considered range measurements are likely to be corrupted by random variables with positive values, in papers C and D, we try to solve estimation problems in systems with positive noise.

4.1 Contributions

New systems are typically designed to address the challenges that are introduced by new applications. The emerging applications enabled by internet of things (IoT), for instance, brought upon a set of new requirements on the underlying communication service. Tracking, logistics and wearables are all exemplary IoT use cases in which location awareness is required. As a response, the NB-IoT system has been developed by 3rd generation partnership project (3GPP) for low power wide area IoT connectivity. We evaluate the performance of localization in NB-IoT systems in terms of the horizontal positioning accuracy in a simulation study.

In Paper A, the user equipment (UE) assisted Observed time difference of arrival (OTDOA) positioning method is evaluated using the enhanced narrowband positioning reference signal (NPRS) introduced in NB-IoT systems. Simulations account for imperfections in wireless channels by considering tapped-delay channel models; extended pedestrian A (EPA) and extended typical urban (ETU). The positioning performance of both channel models is evaluated for both static and dynamic cases. In the static case, it is assumed that the UE detects at least three

and up to six unique base stations (BSs). In the dynamic case, a reference signal time difference (RSTD) reporting budget is defined and the effect of the budget on the device tracking performance in NB-IoT systems is evaluated. The evaluations are to answer what position accuracy IoT can achieve, helping IoT producers tell what they can and cannot do without changes to the positioning scheme. A simulation analysis shows that in EPA channel models, it is possible to limit the RSTD reporting budget to 45 reports per minute while maintaining the horizontal positioning accuracy around 18 m, 67% of the times. In the ETU channel models, however, more RSTDs should be reported per minute to achieve an acceptable accuracy.

NB-IoT systems and the legacy LTE might impose some limitations to control the downlink signaling load in the network. For example, in LTE, information about a limited number of neighbor cells might be provided to the UE. Hence, we might have access to a limited number of range measurements in the system. In 2D positioning scenarios, measuring two BSs results in ambiguity in the position.

While different aspects of timing-based 2D positioning in LTE systems when at least three BSs are detected by the UE have already been studied in the literature, there is not an extensive solution in the extreme case of only two available range measurements. We consider this extreme scenario in which only two BSs, one serving and one neighboring, can be detected. Fusion of the round trip time (RTT) measurement of the serving and time difference of arrival (TDOA) measurement of the serving relative to the neighboring BSs is studied.

In Paper B, a framework based on a bank of Kalman filters is proposed to deal with the ambiguity in the estimates. Lower bounds on the estimation error are then compared to the obtained root mean squared error (RMSE) in a simulation study. To further evaluate the developed filtering method, it is tested on real time of arrival (TOA) measurements. It is shown that the proposed framework can resolve the ambiguity in the UE's position when measurements from only two BSs are available. The achieved results for both the simulated and real data indicate good performance of the introduced method. The evaluations, performed on real data from a field test, indicate that a UE can be tracked using measurements reported from only BSs with a precision better than 40 m 95 % of the time.

A factor affecting the positioning accuracy, in addition to the communication constraints, corresponds to assumptions made to model the uncertainty in measurements. We note that TOA measurements are typically corrupted by random variables with positive values. In order to utilize this additional information, we perform a systematic survey on some common distributions with positive support.

The problem of estimating a uni-dimensional unknown parameter from observations of the parameter corrupted by additive noise, known as the "estimation of location" problem, is studied. The additive noise terms are assumed to follow non-symmetric distributions with positive support. The estimators are unbiased and are derived using order statistics of the collected samples. While compensating for the bias might be trivial when the noise distribution hyperparameters are known, extra care has to be taken for the cases with unknown parameters in the distributions. Paper C derives all combinations of known/unknown parameters

for order statistics/ best linear unbiased estimator (BLUE) for some selected and common distributions.

The derived estimators are used to propose an iterative approach for joint clock bias and receiver's position estimation using raw global navigation satellite systems (GNSS) pseudorange measurements. The receiver is assumed to be static and its initial position is estimated using a separable least squares method. In each iteration, the receiver's clock bias is estimated using the proposed estimators and is further used to update its position. The real-data tests with GNSS positioning data indicate the merit of the proposed localization method. For instance, the 95% percentile of the horizontal and vertical RMSE, when the receiver's clock bias is estimated using the proposed estimator, is around 7 m.

We further extend the linear "estimation of location" problem to a parameter estimation problem with nonlinear measurement model in Paper D. Nonlinearity of the measurement model implies that the minimum value does not have the same significance as in Paper C. The considered problem is static and a deterministic signal model relates the unknown parameter to the observed measurements which are corrupted with additive random noise. The functional form of the distribution of the random variable is assumed to be known and to have positive support.

Uniform and exponential noise distributions are selected and closed-form expressions for the estimators are derived. The performance of the proposed estimators, in terms of the squared error, is compared against the asymptotically efficient ML estimator in a simulation study using ranging measurements collected from a single sensor. The simulation analysis indicates that the estimation error of the order statistics iterative estimator is comparable to the maximum likelihood (ML) estimator which is known to be asymptotically efficient. The evaluation results of Paper C and Paper D verify the merit of deriving estimators tailored for problems in which the underlying noise is inherently positive, as in TOA measurements. This allows for more accurate position estimation using ranging measurements in static problems.

4.2 Future work

One potential future research direction is to develop algorithms more robust against measurement outliers to be compared against the theoretical lower bounds. For example, in a filtering context with a decent prior, gating can be applied. A second alternative is to formulate the RSTD reporting budget in terms of a mixed integer optimization problem. For instance, given a certain lowest acceptable accuracy threshold, the developed algorithm should adaptively change the number of reported BSs and the reporting interval.

Another potential future work can be to improve the position estimation accuracy and to develop a more realistic testing environment of the proposed method when only two BSs are measured. For example, extending the filter bank solution with a smoothing stage can increase the estimation accuracy. Additionally, using the geometrical distance for finding the serving BS could be improved by

defining a more relevant handover criterion. The simulation study could also be further extended to be more realistic by adding Non-line-of sight imperfections to the measurement errors.

A third possible future research direction is to extend the applicability of the estimators, derived for problems with positive noise, so that a wider range of timing-based positioning applications can benefit from. For example, it might be possible to find closed-form estimators for a wider range of distributions with positive support. Additionally, it might be of great interest to extend the static estimation problem to a dynamic one and find the correspondence of order statistics in filtering applications.

Appendix

A

Optimization-based position estimation

The UE can be localized by finding the position that minimizes a specific cost function $\mathcal{J}(\theta)$ that in general can be presented by

$$\hat{\theta} = \arg \min_{\theta} \mathcal{J}(\theta), \quad (\text{A.1})$$

where $\mathcal{J}(\theta) = \|\mathbf{y} - \mathbf{h}(\theta, \ell)\|$ is one possibility. Different types of cost functions can be defined to estimate θ from known measurements $\mathbf{y}$ corrupted by the stochastic unknown error $\mathbf{e}$. For instance, if the probability distribution of the error, $p_e(\mathbf{e})$ is known, the maximum likelihood estimator (MLE) corresponding to the cost function

$$\mathcal{J}^{ML}(\theta) = \log p_e(\mathbf{y} - \mathbf{h}(\theta)), \quad (\text{A.2a})$$

can be used. The MLE is known to be asymptotically efficient (Kay, 1993). In the special case where measurement noises are independent, identically Gaussian distributed, $p_e(\mathbf{e}) \sim \mathcal{N}(\mathbf{e}; \mathbf{0}, \sigma_e^2 \mathbf{I})$, the MLE turns into a NLS problem

$$\mathcal{J}^{NLS}(\theta) = (\mathbf{y} - \mathbf{h}(\theta))^T (\mathbf{y} - \mathbf{h}(\theta)). \quad (\text{A.2b})$$

In case of spatial correlations in measurement noise terms $p_e(\mathbf{e}) \sim \mathcal{N}(\mathbf{e}; \mathbf{0}, \mathbf{R}(\theta))$, the weighted version of NLS, WNLS, should be used instead

$$\mathcal{J}^{WNLS}(\theta) = (\mathbf{y} - \mathbf{h}(\theta))^T \mathbf{R}^{-1}(\theta) (\mathbf{y} - \mathbf{h}(\theta)). \quad (\text{A.2c})$$

Optimization methods

Irrespective of which optimization criteria in (A.2) is considered, in general, there is no closed form solution to (A.1) and numerical optimization methods are required (Gustafsson, 2012). Initialized at $\hat{\theta}_0$, these methods produce a

sequence of iterative updates $\{\hat{\theta}_k\}_{k=0}^\infty$. In each iteration k , the algorithm computes the cost function at $\hat{\theta}_k$ to decide how to move to the next iterate such that $\mathcal{J}(\hat{\theta}_k) < \mathcal{J}(\hat{\theta}_{k-1})$. The iterations terminate either at a given threshold or when no more progress is made.

Two fundamental strategies for iterative updates of the unknown parameter are line search and trust region approaches. In this thesis we consider the line search method in which the iteration is defined by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \alpha_k \mathbf{p}_k, \quad (\text{A.3})$$

where $\mathbf{p}_k$ is the search direction and α_k is the step length that defines how far to move along the direction $\mathbf{p}_k$. Given an appropriate step length α_k at each iteration, any direction with an angle of less than $\pi/2$ radian with $-\nabla_{\theta} \mathcal{J}(\hat{\theta})_k$ is called a descent direction. Mainly, all line search algorithms consider a descent direction $\mathbf{p}_k$ that satisfies $\mathbf{p}_k^T \nabla_{\theta} \mathcal{J}(\hat{\theta}_k) < 0$. This guarantees the reduction in the cost function if we move along this direction (Nocedal and Wright, 2006). The general form of the search direction in line search methods is given by

$$\mathbf{p}_k = -\mathbf{Y}_k^{-1} \nabla_{\theta} \mathcal{J}(\hat{\theta}_k), \quad (\text{A.4})$$

where $\mathbf{Y}_k$ is a symmetric and non-singular matrix. Two local search algorithms that are widely used in positioning applications are steepest descent and Gauss-Newton (Dennis and Schnabel, 1996) as defined in the rest of this section.

Define $\mathbf{H}(\theta) = \nabla_{\theta} \mathbf{h}(\theta)$ for the least squares cost functions and $\mathbf{H}(\theta) = \nabla_{\theta} \log p_e(\mathbf{y} - \mathbf{h}(\theta))$ for MLE. It must be noted that in what follows, $\mathbf{R}$ should be set to an identity matrix of appropriate size if either of cost functions (A.2a) or (A.2b) are to be optimized.

Steepest descent

The most obvious choice of the search direction is the one used in steepest descent where $\mathbf{Y}_k$ is the identity matrix, hence $\mathbf{p}_k = -\nabla_{\theta} \mathcal{J}(\hat{\theta}_k)$. The steepest descent iterations are given by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \alpha_k \mathbf{H}^T(\hat{\theta}_{k-1}) \mathbf{R}^{-1} (\mathbf{y} - \mathbf{h}(\hat{\theta}_{k-1})). \quad (\text{A.5})$$

As discussed in (Nocedal and Wright, 2006), the steepest descent algorithm is globally convergent but might be very slow in practice.

Gauss-Newton

Newton's method uses information of the exact Hessian matrix to find the search direction. This guarantees a descent direction given that the exact Hessian matrix is positive definite. Otherwise, a search direction might not even exist and should be approximated using Hessian modification methods that might result in a direction that is not descent (Nocedal and Wright, 2006). Gauss-Newton is an approximated version of the Newton's method that uses an approximate Hessian (as in Levenberg-Marquardt method), $\nabla_{\theta}^2 \mathcal{J}(\hat{\theta}_k) \approx 2\mathbf{J}(\hat{\theta}_k)^T \mathbf{J}(\hat{\theta}_k)$ and

exact gradient (Dennis and Schnabel, 1996), where $J(\hat{\theta})$ is the Jacobian matrix. The Gauss-Newton iterations are given by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \alpha_k \left(H^T(\hat{\theta}_{k-1}) R^{-1} H(\hat{\theta}_{k-1}) \right)^{-1} H^T(\hat{\theta}_{k-1}) R^{-1} (y - h(\hat{\theta}_{k-1}))^T. \quad (\text{A.6})$$

Note that in the exact Newton's method, Υ_k is the exact Hessian $\nabla_{\theta}^2 \mathcal{J}(\hat{\theta}_k)$ giving the search direction $p_k = -\left(\nabla_{\theta}^2 \mathcal{J}(\hat{\theta}_k)\right)^{-1} \nabla_{\theta} \mathcal{J}(\hat{\theta}_k)$.

Gauss-Newton with modified step lengths

Possibility of convergence to a local optima is a well-known problem with all line search methods that might occur in case of poor initialization of the iterations. Rate of convergence is yet another important factor in all these optimization algorithms that need to be addressed while designing them. These two criteria can sometimes conflict with each other, hence extra care needs to be taken.

Additive updates in each iteration, α_k are damped by using $\alpha_k \in (0, 1]$

$$\mathcal{J}_k(\alpha_k) := \mathcal{J}(\hat{\theta}_{k-1} + \alpha_k p_k) \quad (\text{A.7a})$$

$$M := \left\{ \alpha_k \in \left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \right\} : \mathcal{J}_k(\alpha_k) < \mathcal{J}_k(0) \right\} \quad (\text{A.7b})$$

$$\alpha_k = \begin{cases} \frac{1}{8} & M = \emptyset \\ \max(M) & \text{otherwise.} \end{cases} \quad (\text{A.7c})$$

B

Bayesian filtering

Bayesian filters are recursive algorithms used to infer the states of dynamic systems from noisy observations. In localization context, the state can be the UE's location while indirect observations of the state are given by e.g. timing measurements. In each recursion, the previous state estimate together with a priori known dynamics of the system and current observations are used to update the state estimate.

Statistical system models

The statistical dynamic model explains how the states evolve in time and is given by

$$\mathbf{x}_{t+1} \sim p(\mathbf{x}_{t+1} | \mathbf{x}_t). \quad (\text{B.1a})$$

The statistical measurement models are used to relate the collected observations to the system states as

$$\mathbf{y}_t \sim p(\mathbf{y}_t | \mathbf{x}_t). \quad (\text{B.1b})$$

Assuming that the states of the system, $\mathbf{x}_t$, follow a Markovian property

$$p(\mathbf{x}_{t+1} | \mathbf{x}_1 \dots \mathbf{x}_t) = p(\mathbf{x}_{t+1} | \mathbf{x}_t), \quad (\text{B.2a})$$

$$p(\mathbf{y}_t | \mathbf{x}_1 \dots \mathbf{x}_t, \mathbf{y}_1 \dots \mathbf{y}_{t-1}) = p(\mathbf{y}_t | \mathbf{x}_t, \mathbf{y}_1 \dots \mathbf{y}_{t-1}). \quad (\text{B.2b})$$

Furthermore, noting that given $\mathbf{x}_t$, $\mathbf{y}_t$ is conditionally independent of $\mathbf{y}_1, \dots, \mathbf{y}_{t-1}$

$$p(\mathbf{y}_t | \mathbf{x}_t, \mathbf{y}_1 \dots \mathbf{y}_{t-1}) = p(\mathbf{y}_t | \mathbf{x}_t). \quad (\text{B.2c})$$

The posterior densities are estimated recursively in the Bayesian framework, initialized at the prior distribution of states $p(\mathbf{x}_0)$, see (Jazwinski, 2007) and (Särkkä, 2013).

Recursive state estimate

Given the set of measurements $\mathbf{Y}_{1:t} = [\mathbf{y}_1, \dots, \mathbf{y}_t]$, the state estimation problem is the problem of finding the predictive $p(\mathbf{x}_{t+1} | \mathbf{Y}_{1:t})$ and filtering $p(\mathbf{x}_t | \mathbf{Y}_{1:t})$ posterior distributions. The one-step ahead predictive distribution for a given dynamic model is given by

$$p(\mathbf{x}_{t+1} | \mathbf{Y}_{1:t}) = \int p(\mathbf{x}_{t+1} | \mathbf{x}_t) p(\mathbf{x}_t | \mathbf{Y}_{1:t}) d\mathbf{x}_t. \quad (\text{B.3})$$

The filtering distribution then further includes the information obtained from measurements $\mathbf{y}_t$ resulting in

$$p(\mathbf{x}_t | \mathbf{Y}_{1:t}) = \frac{p(\mathbf{y}_t | \mathbf{x}_t) p(\mathbf{x}_{t+1} | \mathbf{Y}_{1:t})}{p(\mathbf{y}_t | \mathbf{Y}_{1:t-1})}, \quad (\text{B.4a})$$

where the normalization constant in (B.4a) is defined as

$$p(\mathbf{y}_t | \mathbf{Y}_{1:t-1}) = \int p(\mathbf{y}_t | \mathbf{x}_t) p(\mathbf{x}_t | \mathbf{Y}_{1:t-1}) d\mathbf{x}_t. \quad (\text{B.4b})$$

In this thesis, we restrict our analysis to discrete-time filtering of sampled signals denoted by subscript k . The snapshot positions $\boldsymbol{\theta}$ are then indexed by k , stacked in the states of the system dynamics $\mathbf{x}_k$. Bayesian filters can then be applied to derive the filtering distribution of the state of the system at time instance k given all the measurements up to k .

Point estimates of the state vector at each time instance k , if desired, can then be computed from the marginal posterior distribution $p(\mathbf{x}_k | \mathbf{Y}_{1:k})$. For example, choosing the point estimate that gives the minimum mean squared error (MMSE) gives the conditional mean of the state $\mathbf{x}_k$ given the measurements $\mathbf{Y}_{1:k}$. MMSE is in fact the optimal solution to the optimization of the expected loss function $\mathbb{E}\{\mathcal{J}(\mathbf{x}) | \mathbf{Y}_{1:k}\}$ when a weighted quadratic loss function is to be optimized. Maximum a posteriori (MAP) is another alternative that gives the most probable value of the states as the point estimate. In the special case of Gaussian posterior distribution, the two solutions coincide,

$$\hat{\mathbf{x}}_{k|k}^{\text{MMSE}} = \mathbb{E}_{\mathbf{x}_k}\{\mathbf{x}_k | \mathbf{Y}_{1:k}\}, \quad (\text{B.5})$$

$$\hat{\mathbf{x}}_{k|k}^{\text{MAP}} = \arg \max_{\mathbf{x}_k} p(\mathbf{x}_k | \mathbf{Y}_{1:k}). \quad (\text{B.6})$$

For the general statistical model (B.1), closed-form solutions to the recursions (B.3) and (B.4a) might not always exist. In the rest of this section we first consider the special case in which (B.1) corresponds to a linear Gaussian state space model (SSM) for which analytical solutions exist. Then, one alternative for approximating nonlinear SSM is introduced.

Linear models; closed form solution

In the special case when both the dynamic and the measurement model are linear, we obtain the class of linear SSM,

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \boldsymbol{\omega}_{k+1}, \quad (\text{B.7a})$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{e}_k, \quad (\text{B.7b})$$

where $\boldsymbol{\omega}_k \sim p(\boldsymbol{\omega}_k) = \mathcal{N}(\boldsymbol{\omega}_k \mid \boldsymbol{\mu}_k^\omega, \mathbf{Q}_k) \in \mathcal{R}^{n_\omega}$ denote the disturbances entering the system as input and the vector $\mathbf{e}_k \sim p(\mathbf{e}_k) = \mathcal{N}(\mathbf{e}_k \mid \boldsymbol{\mu}_k^e, \mathbf{R}_k) \in \mathcal{R}^{n_e}$ is the measurement noise.

The well-known Kalman filter (KF) provides an optimal closed form solution to the estimation problem. Assuming zero-mean noise terms, $\boldsymbol{\mu}_K^e = \boldsymbol{\mu}_K^\omega = \mathbf{0}$, Algorithm 1 presents KF steps starting the recursions from initial state $\mathbf{x}_0$ with the prior uncertainty $\mathbf{P}_0$. See (Gustafsson, 2012), and (Särkkä, 2013) for proof. The algorithm can be used to obtain parameters of the following distributions

- Predictive distribution:

$$p(\mathbf{x}_{k+1} \mid \mathbf{Y}_{1:k}) = \mathcal{N}(\mathbf{x}_{k+1} \mid \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}). \quad (\text{B.8a})$$

- Filtering distribution:

$$p(\mathbf{x}_k \mid \mathbf{Y}_{1:k}) = \mathcal{N}(\mathbf{x}_k \mid \mathbf{x}_{k|k}, \mathbf{P}_{k|k}). \quad (\text{B.8b})$$

Nonlinear models; approximate solutions

In many practical applications, as the ones introduced in Section 2.2.1, the measurement model is not linear. In cases when either one or both of the dynamic and measurement models are nonlinear, approximations are required. Assuming that the noise terms enter the system additively, the nonlinear SSM is given by

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \boldsymbol{\omega}_{k+1} \quad (\text{B.11a})$$

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{e}_k. \quad (\text{B.11b})$$

The solution to the nonlinear filtering problem can be approximated in different ways. One general framework is to solve the problem by approximating non-Gaussian densities, resulting from nonlinearities in the model, by Gaussian densities. See (Wu et al., 2006), (Arasaratnam et al., 2007), (Arasaratnam et al., 2010), and (Arasaratnam and Haykin, 2009) for more details on generalized Gaussian filters applications.

Special cases of general Gaussian filters can also be derived using Taylor series approximations of nonlinear models as in the extended Kalman filter (EKF), see, (Smith et al., 1962) and (Jazwinski, 2007).

Algorithm 1 Kalman filter

Input: $\mathbf{x}_0, \mathbf{P}_0, \mathbf{y}_k$ **Output:** $\mathbf{x}_{k|k}, \mathbf{P}_{k|k}, \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}$.*Initialization :* $\mathbf{x}_{1|0} = \mathbf{x}_0, \mathbf{P}_{1|0} = \mathbf{P}_0$.1: **for** $k = 1$ to N **do**2: *State measurement update:*

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k, \quad (\text{B.9a})$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}, \quad (\text{B.9b})$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k (\mathbf{y}_k - \mathbf{H}_k \mathbf{x}_{k|k-1}), \quad (\text{B.9c})$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{H}_k \mathbf{P}_{k|k-1}. \quad (\text{B.9d})$$

3: *State time update:*

$$\mathbf{x}_{k+1|k} = \mathbf{F}_k \mathbf{x}_{k|k}, \quad (\text{B.10a})$$

$$\mathbf{P}_{k+1|k} = \mathbf{F}_k \mathbf{P}_{k|k} \mathbf{F}_k^T + \mathbf{Q}_k. \quad (\text{B.10b})$$

4: **end for**5: **return** $\mathbf{x}_{k|k}, \mathbf{P}_{k|k}, \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}$.

The EKF assumes that the posterior distribution is Gaussian $p(\mathbf{x}_k | \mathbf{Y}_{1:k}) \simeq \mathcal{N}(\mathbf{x}_k | \bar{\mathbf{x}}_{k|k}, \bar{\mathbf{P}}_{k|k})$ and approximates the first two moments. Algorithm 2 summarizes the steps of the EKF filtering method for the nonlinear SSM model (B.11) with additive noise.

The EKF estimated distributions, and their corresponding point estimates, are no longer optimal but the algorithm is simple and performs fairly well in many applications (Julier and Uhlmann, 2004). However in cases with severe nonlinearity, the algorithm might have poor performance as it is based on a local linear approximation. Performance improvements might be possible for example by employing information in the Hessian of the nonlinear models using a second order Taylor expansion. However, this further restricts the algorithm to models for which both first and second derivatives are available. In cases where the dynamic model is linear, and the measurement model is nonlinear the iterated EKF can provide performance improvements (Bell and Cathey, 1993). The idea with the iterated EKF is to better approximate the measurement model by iteratively repeating the measurement update phase (B.12) until a certain criteria is satisfied.

Algorithm 2 Extended Kalman filter

Input: $\mathbf{x}_0, \mathbf{P}_0, \mathbf{y}_k$
Output: $\mathbf{x}_{k|k}, \mathbf{P}_{k|k}, \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}$.

Initialization :
 $\mathbf{x}_{1|0} = \mathbf{x}_0, \mathbf{P}_{1|0} = \mathbf{P}_0$.

1: **for** $k = 1$ to N **do**

2: *State measurement update:*

$$\tilde{\mathbf{H}}_k = \left. \frac{\partial}{\partial \mathbf{x}} \mathbf{h}_k(\mathbf{x}) \right|_{\mathbf{x}=\mathbf{x}_{k|k-1}} \quad (\text{B.12a})$$

$$\mathbf{S}_k = \tilde{\mathbf{H}}_k \mathbf{P}_{k|k-1} \tilde{\mathbf{H}}_k^T + \mathbf{R}_k, \quad (\text{B.12b})$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \tilde{\mathbf{H}}_k^T \mathbf{S}_k^{-1}, \quad (\text{B.12c})$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{h}_k(\mathbf{x}_{k|k-1}) \right), \quad (\text{B.12d})$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \tilde{\mathbf{H}}_k \mathbf{P}_{k|k-1}. \quad (\text{B.12e})$$

3: *State time update:*

$$\tilde{\mathbf{F}}_k = \left. \frac{\partial}{\partial \mathbf{x}} \mathbf{f}_k(\mathbf{x}) \right|_{\mathbf{x}=\mathbf{x}_{k|k}} \quad (\text{B.13a})$$

$$\mathbf{x}_{k+1|k} = \mathbf{f}_k(\mathbf{x}_{k|k}), \quad (\text{B.13b})$$

$$\mathbf{P}_{k+1|k} = \tilde{\mathbf{F}}_k \mathbf{P}_{k|k} \tilde{\mathbf{F}}_k^T + \mathbf{Q}_k. \quad (\text{B.13c})$$

4: **end for**

5: **return** $\mathbf{x}_{k|k}, \mathbf{P}_{k|k}, \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}$.

Jump Markov models

In cases where the system has multiple operational modes, the general statistical model (B.1) needs to be extended accordingly. For example, consider the simplified tracking problem based on the measurement model $\tilde{\mathbf{y}}(t) = \tilde{\mathbf{x}}^2(t) + \tilde{\mathbf{e}}(t)$, where $\tilde{\mathbf{x}}(t)$ is a slowly varying scalar process. We could in principle solve for $\tilde{\mathbf{x}}(t)$ and get $\tilde{\mathbf{x}}(t) = \pm \sqrt{\tilde{\mathbf{y}}(t) - \tilde{\mathbf{e}}(t)}$, and the core problem is which branch to select, the positive or negative one.

One alternative to handle multi-modality in the positioning problem is to derive the filtering distribution of the states of a jump Markov model (JMM). In general, JMM is a state-space model where both the motion and measurement models can depend on the mode of the system, δ_t . The general class of JMMs

without any assumptions on the system and noise models are defined as

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \boldsymbol{\omega}_k, \delta_k), \quad (\text{B.14a})$$

$$\mathbf{y}_k = h(\mathbf{x}_k, \mathbf{e}_k, \delta_k), \quad (\text{B.14b})$$

where the set of possible modes of the system is given by $\delta_k \in \mathbf{S}$. In this thesis, the set of modes $\mathbf{S}$ is assumed to contain integers. Additionally, the mode exhibits Markov property. Switching between the modes can either be deterministic or be given by an additional model $\Pi_k^{a,b}$ denoting the probability of transition from mode a to mode b .

The filtering distribution of JMM can be computed in a filter bank framework. A filter bank can then be applied to ensure keeping track of all possible modes of the system at each time instant. Positioning using a bank of KFs is further investigated in Paper B for linear JMM with additive noise terms.

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