SUPPLEMENTARY MATERIALS: EIGENVALUE ANALYSIS FOR SUMMATION-BY-PARTS FINITE DIFFERENCE TIME DISCRETIZATIONS*

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Appendix A. Coefficients and auxiliary polynomials for the sixth order approximation.

The coefficients of S in (5.1) are [SM1]

$$s_{0,1} = x - \frac{953}{16200}, \quad s_{0,2} = -4x + \frac{715489}{259200}, \quad s_{0,3} = 6x - \frac{62639}{14400}$$

$$s_{0,4} = -4x + \frac{147127}{51840}, \quad s_{0,5} = x - \frac{89387}{129600}, \quad s_{1,2} = 10x - \frac{57139}{8640},$$

$$s_{1,3} = -20x + \frac{745733}{51840}, \quad s_{1,4} = 15x - \frac{18343}{1728}, \quad s_{1,5} = -4x + \frac{240569}{86400},$$

$$s_{2,3} = 20x - \frac{176839}{12960}, \quad s_{2,4} = -20x + \frac{242111}{17280}, \quad s_{2,5} = 6x - \frac{182261}{43200},$$

$$s_{3,4} = 10x - \frac{165041}{25920}, \quad s_{3,5} = -4x + \frac{710473}{259200}, \quad s_{4,5} = x,$$

The centrosymmetric matrix $P = \Delta t \cdot \mathrm{diag}\left(p_{0,0}, p_{1,1}, p_{2,2}, p_{3,3}, p_{4,4}, p_{5,5}, 1, \dots\right)$ have coefficients

$$p_{0,0} = \frac{13649}{43200}, \quad p_{1,1} = \frac{12013}{8640}, \quad p_{2,2} = \frac{2711}{4320},$$

$$p_{3,3} = \frac{5359}{4320}, \quad p_{4,4} = \frac{7877}{8640}, \quad p_{5,5} = \frac{43801}{43200}.$$

The polynomials $w_j\left(\xi, x\right)$ and $z_j\left(\xi, x\right)$ with $j = 6, \dots, 11$ are given by

$$\begin{aligned} w_6\left(\xi, x\right) = & \frac{4}{\psi_x}\big(859681646493696000000ix^2\xi \\ & +346015230514790400000x^2\xi^2 - 1309257042103044864000i\xi x \\ & -345146424717614954 29 - 527609483374341312000x\xi^2 \\ & +495218128344680806 61i\xi - 600454088117452800000x^2 \\ & +199851641924641638000\xi^2 + 913474253432711808000x\big), \end{aligned}$$

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$$\begin{aligned}
w_7\left(\xi,x\right) = & \frac{4}{\psi_x}\big(1065056293548195840000ix^2\xi \\
& +413132758434201600000x^2\xi^2 - 1614594755100447129600i\xi x \\
& -41212066581959253105 7 - 627263956804040582400x\xi^2 \\
& +608395474600054186485i\xi - 717030630940999680000x^2 \\
& +236709838471838695412\xi^2 + 1090612145284761715200x\big)\,,
\end{aligned}$$

$$\begin{aligned}
w_8\left(\xi,x\right) = & \frac{4}{\psi_x}\big(592062328418402304000 0ix^2\xi \\
& +244393675904802816000 0x^2\xi^2 - 895169242783792258560 0i\xi x \\
& -211049923369758132151 9 - 369339040835792542080 0x\xi^2 \\
& +336581628957804065449 3i\xi - 367190797301710848000 0x^2 \\
& +138816660259178574448 6\xi^2 + 558445325824509085440 0x\big)\,,
\end{aligned}$$

$$\begin{aligned}
w_9\left(\xi,x\right) = & \frac{24}{\psi_x}\big(346015230514790400000ix^2\xi^3 \\
& -527609483374341312000ix\xi^3 + 2604042075871641600000ix^2\xi \\
& +199851641924641638000i\xi^3 - 2922721961781657600000x^2\xi^2 \\
& -3566071054479712896000i\xi x + 4736511055964611008000x\xi^2 \\
& +1227565965369431421080i\xi - 980234496648806400000x^2 \\
& -1882920129957924054 7\xi^2 + 1482332521159732608000x \\
& -563608037840962150 47\big)\,,
\end{aligned}$$

$$\begin{aligned}
w_{10}\left(\xi,x\right) = & \frac{48}{\psi_x}\big(3622732329487564800000ix^2\xi^3 \\
& -5510562459204738816000ix\xi^3 - 24289959255094149120000ix^2\xi \\
& +2082881581020080848060i\xi^3 - 15675513485529784320000x^2\xi^2 \\
& +36921230363424734726400i\xi x + 23856764225763152937600x\xi^2 \\
& -13942452595064941719821i\xi + 13100048842656890880000x^2 \\
& -9019971825478792684617\xi^2 - 19927030424828770771200x \\
& +7529401904282397341322\big)\,,
\end{aligned}$$

$$\begin{aligned}
w_{11}\left(\xi,x\right) = & \frac{24}{\psi_x}\big(7407786414809088000000ix^2\xi^3 \\
& -11238160159741919385600 0ix\xi^3 - 47779834051416784896000 0ix^2\xi \\
& +42380210597670438999940i\xi^3 - 32516663334967959552000 0x^2\xi^2 \\
& +72456208360569213569280 0i\xi x + 49299904233675107602560 0x\xi^2 \\
& -27310493298180669733454 2i\xi + 23492504834726363136000 0x^2 \\
& -18579916912733792265195 1\xi^2 - 35731565392857359950080 0x \\
& +13502678370231750577508 7\big)\,,
\end{aligned}$$

$$
\begin{aligned}
z_6\left(\xi,x\right) =& \frac{1}{\psi_x}\left(436953677207961600000ix^2\xi\right.\\
&+156172028623257600000x^2\xi^2 - 668302669477158912000i\xi x\\
&-232963844218576215130 - 238781729628186624000x\xi^2\\
&+253871587928198729680i\xi - 402702120021196800000x^2\\
&\left.+90714867106402336800\xi^2 + 614816854811829504000x\right),
\end{aligned}
$$

$$
\begin{aligned}
z_7\left(\xi,x\right) =& \frac{1}{\psi_x}\left(601561273785384960000ix^2\xi\right.\\
&+186465147442790400000x^2\xi^2 - 905927045191010611200i\xi x\\
&-348335310089525885509 - 282923497528708377600x\xi^2\\
&+339545385968289956060i\xi - 627123377100718080000x^2\\
&\left.+106779856825842827824\xi^2 + 936638354766497356800x\right),
\end{aligned}
$$

$$
\begin{aligned}
z_8\left(\xi,x\right) =& \frac{1}{\psi_x}\left(377840438829416448000 0ix^2\xi\right.\\
&+110305711375663104000 0x^2\xi^2 - 562713749135321794560 0i\xi x\\
&-203404700030698537717 9 - 165862556783315182080 0x\xi^2\\
&+208834901172040679830 2i\xi - 373336903529054208000 0x^2\\
&\left.+621071144802388085816\xi^2 + 551886344824171230720 0x\right),
\end{aligned}
$$

$$
\begin{aligned}
z_9\left(\xi,x\right) =& \frac{6}{\psi_x}\left(156172028623257600000ix^2\xi^3\right.\\
&-238781729628186624000ix\xi^3 + 738682188325355520000ix^2\xi\\
&+90714867106402336800i\xi^3 - 180856612393943040000x^2\xi^2\\
&-101021475011077739520 0i\xi x + 302290549192744012800x\xi^2\\
&+345221672882333601665i\xi - 854541521679974400000x^2\\
&-123113796918437809636\xi^2 + 119220871989216199680 0x\\
&\left.-416258564887890370659\right),
\end{aligned}
$$

$$
\begin{aligned}
z_{10}\left(\xi,x\right) =& \frac{12}{\psi_x}\left(163509986601861120000 0ix^2\xi^3\right.\\
&-248913527097038169600 0ix\xi^3 - 138152021375021875200 00ix^2\xi\\
&+942116186108024654720i\xi^3 - 789956079055331328000 0x^2\xi^2\\
&+209861777589009520896 00i\xi x + 120202454280861504576 00x\xi^2\\
&-792348260691118136627 9i\xi + 999933524458389504000 0x^2\\
&-454673773381847436917 2\xi^2 - 150939075453332821632 00x\\
&\left.+566484661810819086603 0i\right),
\end{aligned}
$$

$$
\begin{aligned}
z_{11}(\xi,x) =& \frac{6}{\psi_x}\big(33434627437854720000000ix^2\xi^3\\
&-50645512722529990656000ix\xi^3 - 314802210504889958400000ix^2\xi\\
&+19086633778180219487120i\xi^3 - 170462527753646407680000x^2\xi^2\\
&+471705802315568322969600i\xi x + 256946608933388322278400x\xi^2\\
&-175989809585607907698739i\xi + 213800389798992445440000x^2\\
&-963974067684790443363336\xi^2 - 318408630433391268403200x\\
&+118121838276427015819707\big).
\end{aligned}
$$

Appendix B. Proof of Theorem 5.5 for $x = x_2 = 89387/129600$.

We follow the steps of Theorem 5.4. To start with, we observe that also in this case for $N \geq 17$ the first term of $D(\xi, x_2, N)$ in (5.3) is dominant, since

$$
|\tau_{1,2}(\xi,x_2)| \gtrapprox 7.576\cdot 10^{77}, \quad |\tau_{1,5}(\xi,x_2)| \lessapprox 6.459\cdot 10^{73},
$$

$$
|\tau_{1,6}(\xi,x_2)| \lessapprox 4.615\cdot 10^{73}, \quad |\tau_{2,5}(\xi,x_2)| \lessapprox 7.648\cdot 10^{73},
$$

$$
|\tau_{2,6}(\xi,x_2)| \lessapprox 6.459\cdot 10^{73}, \quad |\tau_{5,6}(\xi,x_2)| \lessapprox 1.210\cdot 10^{54},
$$

for $\xi \in I_\xi = [0, 4.21]$. This results in $D(\xi, x_2, N) \neq 0$ for any $N \geq 17$, and for these values the statement is proved due to Lemma 2.5.

However, differently from Theorem 5.4, by using (5.4) it can be verified that the first term of (5.3) ceases to be dominant for $N \leq 15$. For such values of N cancellation errors may occur in the computation of the determinant leading to untrustworthy results. On the other hand, for $N \in \{12, 13, 14, 15\}$ the claim can be verified numerically. Figure SM1 shows the spectrum of $\mathcal{D}$ for the sixth order approximation with

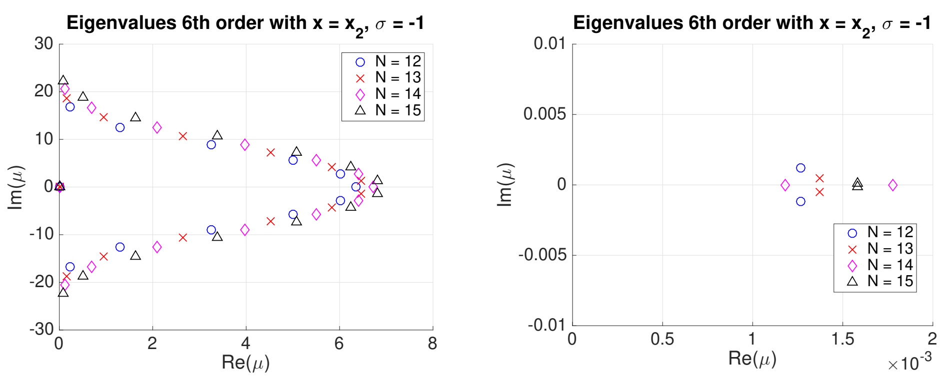

Fig. SM1. *The spectrum of the sixth order approximation $\mathcal{D}$ with $x = x_2 = 89387/129600$, $\sigma = -1$, $N \in \{12, 13, 14, 15\}$ and $\Delta t = 1/N$. The right figure is a magnification of the left figure.*

$x = x_2$, $\Delta t = 1/N$ and $\sigma = -1$. Note that the minimum value of $\mathrm{Re}(\mu)$ is positive for each N, but extremely close to the imaginary axis. This phenomenon appears to be peculiar for the parameter choice $x = x_2$ when N is small.

Appendix C. Proof of Theorem Theorem 5.5 for $x = x_3 = 0.70127127\ldots$.

As for Theorem 5.4, for $N \geq 17$ the term with $1/\left(r_1 r_2\right)^{N-17}$ in (5.3) is dominant, since

$$|\tau_{1,2}(\xi, x_3)| \gtrapprox 1.576 \cdot 10^{83}, \quad |\tau_{1,5}(\xi, x_3)| \lessapprox 7.559 \cdot 10^{71},$$

$$|\tau_{1,6}(\xi, x_3)| \lessapprox 6.121 \cdot 10^{71}, \quad |\tau_{2,5}(\xi, x_3)| \lessapprox 9.728 \cdot 10^{71},$$

$$|\tau_{2,6}(\xi, x_3)| \lessapprox 7.559 \cdot 10^{71}, \quad |\tau_{5,6}(\xi, x_3)| \lessapprox 2.219 \cdot 10^{52},$$

for $\xi \in I_\xi = [0, 4.21]$. This implies that $D(\xi, x_3, N) \neq 0$ for any $N \geq 17$.

The determinant can not be zero also for smaller values of $N \geq 12$. Indeed, the term with $1/\left(r_1 r_2\right)^{N-17}$ dominates even for $N = 12$, value for which we find

$$\left|\frac{\tau_{1,2}(\xi, x_3)}{(r_1 r_2)^{-5}}\right| \gtrapprox 9.446 \cdot 10^{73}, \quad \left|\frac{\tau_{1,5}(\xi, x_3)}{(r_1 r_5)^{-5}}\right| \lessapprox 7.559 \cdot 10^{71},$$

$$\left|\frac{\tau_{1,6}(\xi, x_3)}{(r_1 r_6)^{-5}}\right| \lessapprox 7.828 \cdot 10^{71}, \quad \left|\frac{\tau_{2,5}(\xi, x_3)}{(r_2 r_5)^{-5}}\right| \lessapprox 9.728 \cdot 10^{71},$$

$$\left|\frac{\tau_{2,6}(\xi, x_3)}{(r_2 r_6)^{-5}}\right| \lessapprox 7.559 \cdot 10^{71}, \quad \left|\frac{\tau_{5,6}(\xi, x_3)}{(r_5 r_6)^{-5}}\right| \lessapprox 4.126 \cdot 10^{61},$$

for $\xi \in I_\xi$.

As a result, the determinant $D(\xi, x_3, N) \neq 0$ for $N \geq 12$, $\xi \in I_\xi$. Hence, no eigenvectors of $P^{-1}S$ in (5.1) have first and last component equal to zero, if $x = x_3 = 0.70127127127\ldots$. As a result, the eigenvalues μ of $\mathcal{D}$ have strictly positive real parts due to Lemma 2.5.

REFERENCE

[SM1] B. Strand, *Summation by parts for finite difference approximations for d/dx*, Journal of Computational Physics, 110 (1994), pp. 47–67, https://doi.org/10.1006/jcph.1994.1005.