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Abstract

In this thesis we study the Cauchy problem for elliptic equations. It arises in many areas of application in science and engineering as a problem of reconstruction of solutions to elliptic equations in a domain from boundary measurements taken on a part of the boundary of this domain. The Cauchy problem for elliptic equations is known to be ill-posed.

We use an iterative regularization method based on alternatively solving a sequence of well-posed mixed boundary value problems for the same elliptic equation. This method, based on iterations between Dirichlet-Neumann and Neumann-Dirichlet mixed boundary value problems was first proposed by Kozlov and Maz'ya [13] for Laplace equation and Lame' system but not Helmholtz-type equations. As a result different modifications of this original regularization method have been proposed in literature. We consider the Robin-Dirichlet iterative method proposed by Mpinganzima et.al [3] for the Cauchy problem for the Helmholtz equation in bounded domains.

We demonstrate that the Robin-Dirichlet iterative procedure is convergent for second order elliptic equations with variable coefficients provided the parameter in the Robin condition is appropriately chosen.

We further investigate the convergence of the Robin-Dirichlet iterative procedure for the Cauchy problem for the Helmholtz equation in a an unbounded domain. We derive and analyse the necessary conditions needed for the convergence of the procedure.

In the numerical experiments, the precise behaviour of the procedure for different values of k^2 in the Helmholtz equation is investigated and the results show that the speed of convergence depends on the choice of the Robin parameters, μ_0 and μ_1. In the unbounded domain case, the numerical experiments demonstrate that the procedure is convergent provided that the domain is truncated appropriately and the Robin parameters, μ_0 and μ_1 are also chosen appropriately.

To my family,
for their continual support and love.

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Contents

1 – Introduction

Inverse and ill-posed problems have been widely and extensively studied since early 20th century. Hadamard in [10], was the first to define the notion of a well-posed problem for a differential equation. He further demonstrated this notion using the Cauchy problem for the Laplace equation. He however believed that ill-posed problems did not have physical applications.

The need to study ill-posed problems quickly grew over the years because of the increase in the number of ill-posed problems in science and industry. There was need to theoretically understand and find solutions to these problems. Significant and tremendious growth took place in the 1960s. Tikhonov, is among the many scientists in literature known for their substantial contributions in the progress and growth of the theory of ill-posed problems. In [20, 19], he proposed stable methods for solving these incorrectly formulated problems.

Due to the invention of powerful computers in the 1960s, many researchers, especially mathematicians, turned their attention to the study of inverse problems. Many problems in classical mathematics are ill-posed and prior to the invention of powerful computers it was difficult to solve them. Therefore, driven by the inventions, the need for solutions and theoretical understanding, much studies and different techniques for tackling inverse problems have been developed in literature. See [17, 4, 15, 18] for more details on inverse and ill-posed problems and the different regularization techniques developed for solving these problems.

Consequently, inverse problems arising in other fields of science like medicine, physics, geology etc. can be formulated mathematically as Cauchy problems for partial differential equations. For example in medicine, application arises in Computerized Tomography (CT) where density is recovered from X-ray measurements taken from a cross-section of the human body [8]. In physics applications arise in acoustics and electromagnetic waves i.e detection of source of acoustical noise inside the cabin of a midsize aircraft from measurements of acoustical pressure field inside the cabin [6, 7]. More information on applications in inverse acoustics and electromagnetic scattering can be found in [5, 12].

Let us recall Hadamard's definition of well-posed problems.

1.1 Ill-posed problems

Following Hadamard's definition [9], a well-posed problem must satisfy the following properties:

1. The problem must have a solution (existence)
2. The solution of the problem must be unique (uniqueness), and
3. The problem must depend continuously on the given data (stability)

In order for this definition to be mathematically precise, the function space for the solution and the notion of continuity must be well defined. In the event that any of these three conditions is violated, then the problem is said to be ill-posed.

Violation of the first condition implies that the problem does not have a solution within the desired space. This problem can be fixed. In the case of exact Cauchy data, existence of the solution can be imposed by redefining the notion of a solution. For example, if one cannot find a classical solution to a problem modelled by a partial differential equation, then one can seek for a solution in a weak sense.

Violation of the second condition implies that a problem has more than one solution in the space and the challenge is to choose the appropriate one. However this can be fixed by implementing a priori information about the solution or in the case of incomplete model, adding additional information to the model.

Violation of the third condition implies that the solution procedure is unstable. Unstable means that small perturbations of the Cauchy data can lead to huge deviation of the numerical solution from the exact solution. This condition is the most difficult to deal with because measurement errors and model errors are impossible to avoid. Therefore problems that violate this condition cannot be solved using classical numerical methods.

Cauchy problems for elliptic equations are known to be ill-posed. In the next section, we give a brief description of these problems, give an example and analyse the regularization methods used to approximate their solutions.

1.1.1 Cauchy problems for elliptic equations

An elliptic Cauchy problem is a boundary value problem that constitutes a second order linear partial differential equation that satisfy certain conditions given on the boundary of the domain (bounded or unbounded). Let us consider the general Cauchy problem for an elliptic equation in a domain Ω.

Let Ω be a bounded domain in R^n with a Lipschitz boundary Γ and Γ_0 be an open part of the boundary Γ. The Cauchy problem is to find a solution u which satisfies:

$$\begin{cases} Lu = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma_0, \\ \partial_\nu u = g & \text{on} \quad \Gamma_0, \end{cases} \tag{1.1}$$

where L is a second order elliptic operator in Ω, ν is the outward unit normal to Γ_0 and ∂_ν is the normal derivative of u on Γ_0. f and g are the Dirichlet and Neumann data on Γ_0.

For elliptic equations defined on unbounded (infinite) domains, conditions at infinity are usually imposed. For example a-priori bound for the solution or requiring that the solution decay exponentially to zero.

The Laplace operator: $Lu = \Delta u$, the Helmholtz operator: $Lu = (\Delta + k^2)u$ and the modified Helmholtz operator: $Lu = (\Delta - k^2)u$, where k is a real scalar are examples of the elliptic operator L. The Cauchy problem (1.1) is ill-posed in the sense that small pertubations of the boundary data f and g produces large errors in the solution, see [19].

Let us consider the following example to illustrate the concept of ill-posedness. Consider the following Cauchy problem for the 2-dimensional Laplace equation.

$$\begin{cases} \Delta u_n(x,y) = 0 & (x,y) \in (0,1) \times (0,1), \\ u_n(x,0) = 0 & x \in (0,1), \\ \partial_y u_n(x,0) = \phi_n(x) & x \in (0,1) \\ u_n(0,y) = u_n(1,y) = 0 & y \in (0,1) \end{cases} \tag{1.2}$$

Let $\phi_n(x) = \frac{\sin n\pi x}{n\pi}$ with n a positive integer. Using the method of separation of variables, the solution to (1.2) is given by

$$u_n(x,y) = \frac{\sin n\pi x \sinh n\pi y}{(n\pi)^2}$$

The sequence ϕ_n tends to zero as n tends to infinity while for a fixed $y > 0$, the solution $u_n(x,y)$ of (1.2) tends to infinity. Therefore the requirement that the solution should depend continuously on the given data is not fulfilled and the problem is ill-posed.

Failure of the solution to depend continuously on the Cauchy data is the main challenge in the numerical solution of Cauchy problems for elliptic equations. Classical numerical methods cannot stably approximate solutions to such problems. Regularization methods are applied instead. Regularization methods involves reformulation of the problem so that solution to the regularized problem is less sensitive to pertubations. A number of regularization methods have been proposed for solving Cauchy problems for elliptic equations. These include Tikhonov-type regularization methods and iterative type regularization methods. See [20, 19, 4, 17, 8] for more details on ill-posed problems and regularization methods.

Our motivation for using an iterative method is that apart from the fact that it is easy to implement, it can also be applied to general geometries and to elliptic equations with constant coefficients as well as elliptic equations with variable coefficients.

We also note that, discrete approximation of solutions to Cauchy problems for elliptic equations in unbounded domains are computed on bounded domains which are obtained from unbounded domains by an appropriate truncation.

In the next section we present the iterative method proposed by Kozlov and Maz'ya[13] for ill-posed elliptic problems in bounded domains, which forms the basis of our work in this thesis.

1.2 Alternating iterative procedure

The alternating iterative procedure is an iterative regularization method introduced by Kozlov and Maz'ya [13] for solving ill-posed partial differential equations in bounded domains. This method involve alternatively solving a sequence of well-posed mixed boundary value problems for the same equation. The regularizing character is achieved by appropriate choice of boundary conditions at each iteration.

In [14], Kozlov et.al demonstrated that the alternating iterative procedure converges for Cauchy problems associated to linear, elliptic and positive-definite operators. They considered the Cauchy problem for the Laplace equation and Lame' system on a bounded domain Ω with a smooth boundary Γ divided into two disjoint boundaries Γ_0 and Γ_1 with smooth common boundary. We use this example to illustrate how the alternating iterative procedure works.

Let u be the exact solution to the following Cauchy problem for the Laplace equation:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \Gamma_0, \quad \partial_\nu u = g \quad \text{on} \quad \Gamma_0 \tag{1.3}$$

where f, g are the specified Cauchy data.

The alternating iterative procedure for solving (1.3) consists of the following steps:

(1) Specify the initial approximation $\psi^{(0)}$ of the normal derivative on Γ_1 and solve the following well-posed mixed boundary problem to obtain the first approximation $u^{(0)}$

$$\begin{cases} \Delta u^{(0)} = 0 & \text{in} \quad \Omega \\ u^{(0)} = f & \text{on} \quad \Gamma_0 \\ \partial_\nu u^{(0)} = \psi^{(0)} & \text{on} \quad \Gamma_1 \end{cases} \tag{1.4}$$

(2) Having constructed the approximation $u^{(2n)}$, the following well-posed mixed boundary value problem is solved to obtain the approximation $u^{(2n+1)}$

$$\begin{cases} \Delta u^{(2n+1)} = 0 & \text{in} \quad \Omega \\ \partial_\nu u^{(2n+1)} = g & \text{on} \quad \Gamma_0 \\ u^{(2n+1)} = u^{(2n)} & \text{on} \quad \Gamma_1 \end{cases} \tag{1.5}$$

(3) Having constructed the approximation $u^{(2n+1)}$, the following well-posed mixed boundary value problem problem is solved to obtain approximation $u^{(2n+2)}$

$$\begin{cases} \Delta u^{(2n+2)} = 0 & \text{in} \quad \Omega \\ u^{(2n+2)} = f & \text{on} \quad \Gamma_0 \\ \partial_\nu u^{(2n+2)} = \partial_\nu u^{(2n+1)} & \text{on} \quad \Gamma_1 \end{cases} \tag{1.6}$$

The above mixed boundary value problems are well-posed and solvable in $H^1(\Omega)$ for appropriate function spaces for the Cauchy data (f, g) on Γ_0 and the approximate normal derivative ψ on Γ_1.

This original alternating iterative procedure introduced by Kozlov and Maz'ya includes iterations between Dirichlet-Neumann and Neumann-Dirichlet of the mixed boundary value problems. They used Dirichlet-Neumann mixed boundary conditions in their algorithm. It however does not necessarily converge if the elliptic operator is not positive-definite. Helmholtz-type operators are example of such operators.

Consider the following Cauchy problem for the Helmholtz equation.

$$(\Delta + k^2)u = 0 \text{ in } \Omega \quad u = f \text{ on } \Gamma_0, \quad \partial_\nu u = g \text{ on } \Gamma_0 \tag{1.7}$$

where k^2 is the wave number. The problem is to reconstruct the solution to the Helmholtz equation from Cauchy data (f, g) given on Γ_0.

It has been shown by Marin et.al [16] that in the case of the modified Helmholtz equation, that is when k is purely imaginary, the Kozlov-Maz'ya alternating iterative procedure [14] always converges. However, if k is real, the Kozlov-Maz'ya alternating iterative procedure does not converge for large values of k^2 in the Helmholtz equation, see [2].

In order to solve this problem of non-convergence for large values of k^2 in the Helmholtz equation, several variants of the Kozlov-Maz'ya alternating iterative procedure have been considered. See for examples Mpinganzima et.al [2, 3] and Johansson et.al [11] modifications. Johansson et.al [11], presented a modification of the Kozlov-Maz'ya alternating iterative procedure for Cauchy problems associated with elliptic operators which are symmetric but not positive and proved convergence of the modified algorithm. However, this algorithm by Johansson et.al is not easy to implement numerically.

Mpinganzima et.al [2] also presented a modification where they introduced an artificial interior boundary in such a way that convergence was restored. They also presented in [3], a simpler modification of the Kozlov-Maz'ya alternating iterative procedure for solving the Cauchy problem for the Helmholtz equation which is convergent even for large values of k^2 in the Helmholtz equation and easier to implement numerically.

The iterations consists of replacing the Dirichlet-Neumann iterations on Γ_1 by the Dirichlet-Robin iterations in the sequence solution of the following mixed boundary value problems:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \Gamma_0 \\ \partial_\nu u + \mu u = \eta & \text{in} \quad \Gamma_1 \end{cases} \tag{1.8}$$

and

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega \\ \partial_\nu u = g & \text{in} \quad \Gamma_0 \\ u = \phi & \text{in} \quad \Gamma_1 \end{cases} \tag{1.9}$$

For $(f, g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ if $\eta \in H^{-1/2}(\Gamma_1)$ and $\phi \in H^{\frac{1}{2}}(\Gamma_1)$ then the problems (1.8) and (1.9) are well-posed. Moreover, for appropriate choice of μ such that the quadratic form associated with the Helmholtz equation with Robin boundary conditions is positive, the Robin-Dirichlet alternating iterative procedure convergences in $H^1(\Omega)$ to the solution u of the Cauchy problem (1.7) for any initial approximation $\eta \in H^{-1/2}(\Gamma_1)$.

This thesis is based on this alternating iterative procedure proposed by Mpinganzima et.al [3]. In the next section we give a summary of the two papers in the thesis.

2 – Summary of Papers

Paper I

In this paper, we analyse the Robin-Dirichlet alternating iterative procedure for Cauchy problem for general elliptic equations of second order. We consider the following Cauchy problem for an elliptic equation

$$\begin{cases} Lu = D_j a^{ji}(x) D_i u + a(x)u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma_0, \\ Nu = g & \text{on} \quad \Gamma_0, \end{cases} \tag{2.1}$$

Here Ω is a bounded domain in R^d with a Lipschitz boundary Γ divided into two disjoint parts Γ_0 and Γ_1 with a common Lipschitz boundary in Γ. $D_j = \partial/\partial x_j$, a^{ji} and a are measurable real valued functions such that a is bounded, $a^{ij} = a^{ji}$ and

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \quad x \in \Omega, \quad \xi \in R^d, \quad \lambda = const > 0$$

The conormal derivative N is defined as

$$Nu = \nu a^{ij} D_i u$$

where $\nu = (\nu_1, \cdots, \nu_d)$. The Cauchy data $(f, g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$.
We make one of the following two equivalent assumptions.

$$\int_\Omega (a^{ji} D_i u D_j u - au^2)\, dx > 0 \quad for\ all \quad u \in H^1(\Omega, \Gamma)\backslash\{0\} \tag{2.2}$$

where $H^1(\Omega, \Gamma)$ is the space of all functions from $H^1(\Omega)$ which vanish on Γ and for two real valued measurable bounded functions μ_0 and μ_1 defined on Γ_0 and Γ_1 respectively,

$$\int_\Omega (a^{ji} D_i u D_j u - au^2)\, dx + \int_{\Gamma_0} \mu_0 u^2\, dS + \int_{\Gamma_1} \mu_1 u^2\, dS > 0 \tag{2.3}$$

for all $u \in H^1(\Omega)\backslash\{0\}$. We note that in (2.2), we require that functions are equal to zero on the boundary while (2.3), we do not require that.
With the assumptions (2.3) in place, the Robin-Dirichlet alternating iterative procedure is used to solve problem (2.1). See Section 2 for a complete description of the iterative procedure. Well-posedness of the mixed boundary value problems used in the Robin-Dirichlet alternating iterative procedure are proved in the space $H^1(\Omega)$, see Preposition 3.7. Convergence of the Robin-Dirichlet alternating iterative procedure to the solution of (2.1) is proved in Theorem 4.1. Numerical experiments are conducted using the Cauchy problem for the Helmholtz equation. In the experiments, the precise behaviour of the Robin-Dirichlet

alternating iterative procedure for different values of k^2 in the Helmholtz equation, is investigated. Also investigated is how the choice of the Robin parameters influence the convergence of the iterations.

Paper II

In this paper, we consider the case where L in (2.1) is the Helmholtz operator and the problem is prescribed in an unbounded domain described as follows. Let Ω be a domain in R^d, $d \geq 2$, with C^2 boundary and with N cylindrical outlets to infinity, i.e. for sufficiently large $|x|$ the domain Ω coincides with the union of N disjoint cylinders $\mathcal{C}^{(j)}$, $j = 1, \dots, N$, which can be described in a certain cartesian coordinates $x^{(j)} = (y^{(j)}, z^{(j)})$, as

$$\mathcal{C}^{(j)} = \{x^{(j)} \,:\, y^{(j)} \in \omega^{(j)},\, z^{(j)} \in R\},$$

where the cross-sections $\omega^{(j)}$ are bounded domains in R^{d-1} with C^2 boundaries. We denote the boundary of Ω by Γ. We assume that a certain bounded[1] open set Γ_0 is chosen on the boundary Γ and the boundary of this set is of class C^2 also. Let also Γ_1 be the interior of $\Gamma \setminus \Gamma_0$.
We then seek a real valued solution $u \in H^1(\Omega)$ to the following Cauchy problem for the Helmholtz equation

$$(\Delta + k^2)u = 0 \ \text{ in } \Omega \tag{2.4}$$

and

$$u = f_0 \quad \text{on} \quad \Gamma_0, \quad \partial_\nu u = g_0 \quad \text{on} \quad \Gamma_0 \tag{2.5}$$

where k is a non-negative number, ν is the outward unit normal to Γ, ∂_ν is the normal derivative. The Cauchy data $(f_0, g_0) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$.
One of the following two equivalent assumption concerning the parameter k are made. There exist a positive constant ϵ such that

$$\int_\Omega (|\nabla u|^2 - k^2|u|^2)dx \geq \epsilon||u||^2_{H^1(\Omega)} \ \text{ for all } u \in H^1(\Omega, \Gamma). \tag{2.6}$$

and there exist positive constants μ_0, μ_1 and δ such that

$$\int_\Omega (|\nabla u|^2 - k^2|u|^2)dx + \mu_0 \int_{\Gamma_0} |u|^2 dS + \mu_1 \int_{\Gamma_1} |u|^2 dS \geq \delta||u||^2_{H^1(\Omega)} \tag{2.7}$$

for all $u \in H^1(\Omega)$.
As in Paper 1, the Robin-Dirichlet alternating iterative procedure is proposed to solve problem (2.4) and (2.5). Complete description of the procedure is presented in section 1.1. Condition (2.6) is analysed and explicit estimates for k^2 in terms of eigenvalues of certain auxiliary problems are presented, see Lemma 2.2. Equivalence of condition (2.6) and (2.7) are also proved, see Lemma 2.3. The relationship between the first eigenvalue of the Robin-Laplacian and the first eigenvalue of the Dirichlet-Laplacian are established, see Example 2.5. An

[1]This is a set where measurements are taken and it is reasonable to assume it bounded

example illustrating how to explicitly calculate the first eigenvalue of the Robin-Laplacian in a domain Ω in R^2 is presented in Example 2.4. Also included is a table showing linear dependence of the first eigenvalue of the Robin-Laplacian on the Robin parameters μ_0 and μ_1, see Table 1. In the numerical experiments we demonstrate that by appropriate truncation of the domain and with appropriate choice of the Robin parameters μ_0 and μ_1, the Robin-Dirichlet alternating iterative procedure is convergent.

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Papers

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