Accelerated Dirichlet-Robin Alternating Algorithm for Solving the Cauchy Problem for an Elliptic Equation using Krylov Subspaces

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Linköping Studies in Science and Technology.
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Abstract

In this thesis, we study the Cauchy problem for an elliptic equation. We use Dirichlet-Robin iterations for solving the Cauchy problem. This allows us to include in our consideration elliptic equations with variable coefficient as well as Helmholtz type equations. The algorithm consists of solving mixed boundary value problems, which include the Dirichlet and Robin boundary conditions. Convergence is achieved by choice of parameters in the Robin conditions.

We have also reformulated the Cauchy problem for the Helmholtz equation as an operator equation. We investigate the conditions under which this operator equation is well-defined. Furthermore, we have also discussed possible extensions to the case where the Helmholtz operator is replaced by non-symmetric differential operators by using similar operator equations and model problems which are used for symmetric differential operators. We have observed that the Dirichlet–Robin iterations are equivalent to the classical Landweber iterations. Having formulated the problem in terms of an operator equation is an advantage since it lets us to implement more sophisticated iterative methods based on Krylov subspaces. In particular, we consider the Conjugate gradient method (CG) and the Generalized minimal residual method (GMRES). The numerical results shows that all the methods work well.

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1 – Introduction

Inverse problems has vast areas of application in science, engineering and technology. These problems are often formulated in terms of the Cauchy problem for elliptic equations of second order. For example, in medical imaging [29], underwater acoustics [12], nondestructive testing techniques [2], and the detection of corrosion [3], etc. These problems are severely ill-posed which implies that the solutions do not depend continuously on the Cauchy data given, see [14].

In relation to Hadamard's definition of well-posedness, one considers a problem to be well-posed if the following three conditions are fulfilled [14]:

1. There exists a solution to the problem.
2. There is at most one solution to the problem.
3. The solution depends continuously on the data.

Failure to fulfil one or more of the above conditions means the problem is ill-posed, see [13].

Let us now consider an examples of an ill-posed problem, that is, the Cauchy problem for the Helmholtz equation.

Example 1.0.1 Consider the following Cauchy problem for the Helmholtz equation in the rectangle $\Omega = (0, 1) \times (0, b)$:

$$\begin{cases} \Delta u(x, y) + k^2 u(x, y) = 0, & 0 < x < 1, \ 0 < y < b, \\ u(x, 0) = f_0(x), & 0 \leq x \leq 1, \\ u_y(x, 0) = g_0(x), & 0 \leq x \leq 1, \\ u(0, y) = u(1, y) = 0, & 0 \leq y \leq b, \end{cases} \quad (1.1)$$

where k is referred to as a wave number and $f_0, g_0 \in L^2(0, 1)$ are the available Cauchy data. By using the method of separation of variables the solution to this problem is as follows:

$$u(x, y) = \sum_{n=1}^{\infty} \sin(n\pi x) (C_n \cosh \mu_n y + \mu_n^{-1} D_n \sinh \mu_n y), \quad (1.2)$$

where $\mu_n = \sqrt{n^2 \pi^2 - k^2}$, and C_n and D_n are the Fourier sine coefficients.

Let now consider the case when $f_0(x) = n^{-2} \sin(n\pi x)$ and $g_0(x) = 0$. The solution to Cauchy problem (1.1) with this particular choice of $f_0(x)$ is given by

$$u(x, y) = \frac{\sin(n\pi x) \cosh(\sqrt{n^2 \pi^2 - k^2} y)}{n^2}. \quad (1.3)$$

We observe that $\max_{x \in [0,1]} |f_0(x)| \rightarrow 0$ as $n \rightarrow \infty$, while for any fixed $y > 0$ we have that $\max_{x \in [0,1]} |u(x, y)| \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, the third condition above does not hold, and hence the problem is ill-posed. For more detailed information concerning ill-posed problems one can see for instance, [13, 17].

In order to restore the stability of the solution, with respect to given data, regularization methods has to be applied. For this reason, different regularization methods have been considered by various authors for solving the problem (1.1), for instance meshless fading regularization method [10], Fourier regularization method and wavelet regularization method [25], alternating iterative algorithm based on Conjugate gradient method inconjunction with BEM [26, 27], method of fundamental solution [30], Fourier truncation method [31], and the truncation method [32].

1.1 Regularization

The material presented here is quite standard and can be found in [13, 18].

Let $T : X \rightarrow Y$ be a bounded linear operator mapping the Hilbert space X into Y . Consider the equation

$$Tx = y, \quad (1.4)$$

where $y \in Y$ is given and $x \in X$ must be found.

We discuss the conditions 1–3 from the previous section for the well-posedness of equation (1.4). The first condition holds if y belongs to the range $\mathcal{R}(T)$. The second condition holds if $\mathcal{N}(T) = \{0\}$. The third condition is satisfied if $\mathcal{R}(T) = Y$ and $\mathcal{N}(T) = \{0\}$. In this case T^{-1} exists and is continuous. The first two conditions can be repaired sometimes but the last one is typical reason for ill-posedness.

The most common way of restoring stability of ill-posed problem is to approximate T^{-1} by a family of continuous operators. This process is called regularization. A more precise definition is as follows which can be found in [18].

Definition 1.1.1 *A family of linear and bounded operators*

$$R_N : Y \rightarrow X, \quad N = 1, 2, \dots, \quad (1.5)$$

such that

$$\lim_{N \rightarrow 0} R_N(Tx) = x, \quad \forall x \in X, \quad (1.6)$$

is referred to as regularization strategy. Within the strategy $N > 0$ is referred to as the regularization parameter.

The definition above means that the regularized solution $R_N y$ converges to the exact solution $x = T^{-1}y$. For the case of inexact data y^δ and $y \in \mathcal{R}(T)$, satisfying $\|y^\delta - y\| \leq \delta$, where δ is the noise level, the approximate solution x of $y = Tx$ is given by

$$x_N^\delta = R_N y^\delta.$$

The total error in the obtained solution is split into two parts using the triangle inequality and is given by

$$\|x_N^\delta - x\| \leq \|R_N y^\delta - R_N y\| + \|R_N y - x\| \leq \|R_N\| \|y^\delta - y\| + \|R_N(Tx) - x\|,$$

and we can obtain

$$\|x_N^\delta - x\| \leq \delta \|R_N\| + \|R_N(Tx) - x\|.$$

We can see that the first term in the above total error refers to the error in the measured data multiplied by the norm $\|R_N\|$ while the second term is the estimation error. For a uniformly bounded operator $\|R_N\|$ one can see that $\delta \|R_N\| \rightarrow 0$ as $\delta \rightarrow 0$. Furthermore, from Definition 1.1.1 we observe that the estimation error $\|R_N Tx - x\| \rightarrow 0$ for exact data y as $N \rightarrow \infty$.

However, if the norm $\|R_N\|$ is not uniformly bounded, i.e. there exists a sequence $\{N_j\}$ such that $\|R_{N_j}\| \rightarrow \infty$ for $j \rightarrow \infty$. Then the term $\delta \|R_{N_j}\| \rightarrow \infty$ as $N_j \rightarrow \infty$, see [18, Theorem 2.2]. To obtain a small total error, we have to choose an appropriate $N = N(\delta)$ which depend on δ , see [13].

Let us now give the definition of a regularization method which can be found in [13].

Definition 1.1.2 *Let $T : X \rightarrow Y$ be a bounded linear operator between Hilbert space X and Y , $N_0 \in (0, +\infty]$. For every $N \in (0, +N_0)$, let*

$$R_N : Y \rightarrow X \tag{1.7}$$

be a continuous (not necessary linear) operator. The family $\{R_N\}$ is called a regularization or a regularization operator (for T^{-1}), if, for all $y \in \mathcal{D}(T^{-1})$, there exists a parameter choice rule $N = N(\delta, y^\delta)$ such that

$$\limsup_{\delta \rightarrow 0} \{\|R_{N(\delta, y^\delta)} y^\delta - T^{-1}y\| \mid y^\delta \in Y, \|y^\delta - y\| \leq \delta\} = 0 \tag{1.8}$$

holds. Here, $N : \mathbb{R}^+ \times Y \rightarrow (0, N_0)$ is such that

$$\limsup_{\delta \rightarrow 0} \{N(\delta, y^\delta) \mid y^\delta \in Y, \|y^\delta - y\| \leq \delta\} = 0. \tag{1.9}$$

For a specific $y \in \mathcal{D}(T^{-1})$. The pair (N, R_N) is called a (convergent) regularization method for solving $y = Tx$ if (1.8) and (1.9) hold.

For this definition to be successful the method for choosing the regularization parameter has to be considered for the case when inexact data y^δ is given. This method is referred to as parameter selection method. They are characterized into two distinct classes. That is, one based on the noise level δ of the data and another is based on both the noise level δ and availability of inexact data y^δ . For the case when the noise level is known the Discrepancy principle due to Morozov can be applied. We define the regularization parameter using the Discrepancy principle as follows

$$N(\delta, y^\delta) = \sup\{N > 0 \mid \|Tx_N^\delta - y^\delta\| \leq \delta\}. \tag{1.10}$$

When the knowledge of the noise level δ is not available. The method called L-curve method can be used. This method was first introduced by Lawson and Hanson [22]. The L-curve method is based on the compromise between the residual norm $\|Tx_N^\delta - y^\delta\|$ and the norm of the approximate solution $\|x_N^\delta\|$. Using this method a good regularization parameter is achieved at the corner of the curve which is a plot of the residual norm $\|Tx_N^\delta - y^\delta\|$ against the norm of the approximate solution $\|x_N^\delta\|$, by using log-log scale, see [13].

Let us now give an example of a regularization method that can be use to obtain approximate solution to problem (1.1).

Example 1.1.3 The solution (1.2) obtained by solving the Cauchy problem (1.1) for specified Cauchy data f_0 and g_0 is ill-posed. Stability can be restored by truncating the series of solution $u(x, y)$ to obtain a regularized solution as follows

$$u_N(x, y) = \sum_{n=1}^N \sin(n\pi x)(C_n \cosh \mu_n y + \mu_n^{-1} D_n \sinh \mu_n y), \quad (1.11)$$

where $N > 0$ is the regularization parameter, that is chosen appropriately, and C_n and D_n are Fourier sine coefficients. Here, $\cosh \mu_n y$ and $\sinh \mu_n y$ are bounded and given a small perturbation of the measured data f_0 and g_0 the error in the solution $u_N(x, y)$ becomes small. Then we can finally say that the solution depends continuously on the data given.

In this thesis we consider the Robin-Dirichlet alternating iterative algorithm suggested in [9]. The Dirichlet-Neumann alternating iterative algorithm was first proposed by Kozlov and Maz'ya in [19] for solving Cauchy problem for general strongly elliptic, and formally symmetric operators. It was proven in [20] that the Dirichlet-Neumann algorithm converges and appropriate regularizing properties was established.

The Dirichlet-Neumann alternating algorithm was extended by various authors in different directions, see for instance [4, 11, 16, 23, 24, 26]. The same algorithm have been used by Marin, et al. [26] to solve the Cauchy problem for a modified Helmholtz equation, where the subproblems were solved using the boundary element method. In their work a pure imaginary k was considered. Further, Johansson and Kozlov [16] modified the algorithm but this modification is quite difficult to implement.

It has been shown in [7] that the Dirichlet-Neumann algorithm diverges for large values of k^2 in the Helmholtz equation. Similarly, in [9] it was verified that by replacing the Dirichlet-Neumann boundary conditions with the Dirichlet-Robin conditions the algorithm converges for some values of k^2 .

The rate of convergence were quite slow. Therefore, acceleration methods were considered [6, 8]. Furthermore, it was also shown that the Dirichlet-Robin algorithm can be represented as a Landweber iteration for certain operator equation defined on functions on the boundary of the domain. Finally, the operator equation that was defined motivated the use of various accelerated regularization methods.

Let us now state the Cauchy problem for the Helmholtz equation considered

in a bounded domain Ω with the boundary Γ

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f_0 & \text{on } \Gamma_0, \\ \partial_\nu u = g_0 & \text{on } \Gamma_0, \end{cases} \quad (1.12)$$

where f_0 and g_0 are specified Cauchy data on a part Γ_0 of the boundary Γ with a certain noise level, k is a positive real number, called the wave number, ν is the outward unit normal to the boundary Γ and ∂_ν denotes the outward normal derivative, see [7].

1.1.1 Dirichlet-Robin algorithm

We now discuss the Dirichlet-Robin algorithm for solving the Cauchy problem (1.12). The starting point here is to find constants μ_0 and μ_1 such that

$$\int_{\Omega} (|\nabla u|^2 - k^2 u^2) dx + \mu_0 \int_{\Gamma_0} u^2 dS + \mu_1 \int_{\Gamma_1} u^2 dS > 0, \quad (1.13)$$

for all non-zero $u \in H^1(\Omega)$. Let η^0 be an arbitrary initial approximation of Robin condition on Γ_1 . Then to find the first approximation u_0 we solve the well-posed problem

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0 & \Omega, \\ u_0 = f_0 & \Gamma_0, \\ \partial_\nu u_0 + \mu_1 u_0 = \eta^{(0)} & \Gamma_1. \end{cases} \quad (1.14)$$

Having obtained u_{2n} , we solve the following well-posed problem to determine u_{2n+1} with $\phi = u_{2n}$ on Γ_1 :

$$\begin{cases} \Delta u_{2n+1} + k^2 u_{2n+1} = 0 & \text{in } \Omega, \\ \partial_\nu u_{2n+1} + \mu_0 u_{2n+1} = g & \text{on } \Gamma_0, \\ u_{2n+1} = u_{2n} & \text{on } \Gamma_1. \end{cases} \quad (1.15)$$

Finally, we find u_{2n+2} by solving the following boundary value problem with $\eta = \partial_\nu u_{2n+1} + \mu_1 u_{2n+1}$ on Γ_1 :

$$\begin{cases} \Delta u_{2n+2} + k^2 u_{2n+2} = 0 & \text{in } \Omega, \\ u_{2n+2} = u_{2n+1} & \text{on } \Gamma_0, \\ \partial_\nu u_{2n+2} + \mu_1 u_{2n+2} = \partial_\nu u_{2n+1} + \mu_1 u_{2n+1} & \text{on } \Gamma_1. \end{cases} \quad (1.16)$$

It is proved in [1] that those iterations converges to the exact solution of (1.12) (for the data without noise).

1.1.2 Operator equations

In this section we reformulate the Cauchy problem (1.12) as an operator equation. We consider the following well-posed problem

$$\begin{cases} \Delta u + k^2 u = 0, & \Omega, \\ u = 0, & \Gamma_0, \\ \partial_\nu u + \mu_1 u = \eta, & \Gamma_1, \end{cases} \quad (1.17)$$

where $\eta \in H^{-1/2}(\Gamma_1)$ is the Robin boundary condition on Γ_1 . Note that here we assume $f_0 = 0$ on Γ_0 . We define an operator

$$K : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_0),$$

by

$$K\eta = (\partial_\nu u + \mu_0 u)|_{\Gamma_0},$$

where u is the weak solution of the problem (1.17). We can use the following operator equation for solving problem (1.12):

$$K\eta = g_0, \quad (1.18)$$

where $g_0 \in H^{-1/2}(\Gamma_0)$ and $\eta \in H^{-1/2}(\Gamma_1)$.

The operator K is not self adjoint or positive definite. It is also not compact, see [9].

In the next section, we give definitions of iterative methods that has been used extensively in this thesis for solving equation (1.18).

1.2 Iterative methods

Let X be a Hilbert space and let

$$F : X \mapsto X, \quad (1.19)$$

be a linear operator and consider the operator equation

$$Fx = y, \quad (1.20)$$

where $x \in X$ and $y \in X$. Most modern techniques for solving linear systems of equations (1.20) use Krylov subspaces [28]. The Krylov space of dimension n is given by

$$\mathcal{K}_n(F, x^{(0)}) = \text{span}\{x^{(0)}, Fx^{(0)}, F^2x^{(0)}, \dots, F^{n-1}x^{(0)}\}, \quad (1.21)$$

where $x^{(0)} \in X$ is an arbitrary initial guess, see [28].

Note that in this section the operator maps X to X instead of X to Y as previously shown. Otherwise, the above definition of the Krylov subspaces do not make sense.

The iterative methods we will investigate are projection methods [28]. For such a method we let $\mathcal{K}$ and $\mathcal{L}$ be two n -dimensional subspaces of X and we define an approximate solution $x^{(n)}$ by

$$x^{(n)} \in x^{(0)} + \mathcal{K}, \text{ such that } y - Fx^{(n)} \perp \mathcal{L}. \quad (1.22)$$

Generally such a method will always converge since the residuals $r^{(n)} = y - Fx^{(n)}$ are orthogonal to subspace $\mathcal{L}$ of increasing dimension as $n \rightarrow \infty$.

Different projection methods are obtained by making different choices for $\mathcal{K}$ and $\mathcal{L}$. Popular choices for $\mathcal{K}$ and $\mathcal{L}$ are based on the Krylov subspace.

In the next section, we present iterative methods for solving (1.20) that have been used in the thesis.

1.2.1 The Landweber method

The Landweber method is one of the iterative methods for solving linear equations. It was initially investigated by Landweber [21]. In this method, we consider $x^{(0)} \in X$ to be an initial approximation of x . The iterative sequence is computed by

$$x^{(n)} = x^{(n-1)} + \gamma F^*(y - Fx^{(n-1)}), \quad N = 1, 2, \dots, n, \quad (1.23)$$

where $0 < \gamma < \|F\|^{-2}$, here γ is the relaxation parameter and F^* is the adjoint operator.

We now state a theorem for the convergence of this iterations which can be found in [13, Theorem 6.1 pp. 155 – 156].

Theorem 1.2.1 *If y belongs to the domain of $\mathcal{D}(F^{-1})$, then $x^{(n)} \rightarrow F^{-1}y$ as $n \rightarrow \infty$.*

Recall from Section 1.1.2 that K maps $X = H^{-1/2}(\Gamma_1)$ onto a different space $Y = H^{-1/2}(\Gamma_0)$. Here, the formula (1.23) still work out. For our case we consider the kernel of K to have only a zero element. Then K^{-1} exists and the operator K is bounded. The adjoint operator is required to implement the above iteration.

1.2.2 The Conjugate gradient method

Let us now consider another iterative method for solving linear equations called the Conjugate gradient method, see [15]. This method can be applied to positive definite and self adjoint operators. In our case F is not assumed to be positive, or self adjoint, and we apply the method to the normal equation

$$F^*Fx = F^*y, \quad (1.24)$$

where F^* is an adjoint operator. The algorithm can be described as follows, see [28, 13]: First, pick an initial guess $x^{(0)} \in X$, and compute the initial residual

$$r^{(0)} = y - Fx^{(0)} \quad \text{and} \quad z^{(0)} = F^*r^{(0)}, \quad v^{(0)} = z^{(0)}.$$

Compute also

$$w^{(n)} = Fv^{(n)}, \quad \alpha_n = \frac{\|z^{(n)}\|_X^2}{\|w^{(n)}\|_X^2}.$$

Next update the approximate solution

$$x^{(n+1)} = x^{(n)} + \alpha_n v^{(n)},$$

and the residual

$$z^{(n+1)} = z^{(n)} - \alpha_n F^* w^{(n)}.$$

Finally, compute the new search direction

$$v^{(n+1)} = z^{(n+1)} + \beta_n v^{(n)},$$

where

$$\beta_n = \frac{\|z^{(n+1)}\|_X^2}{\|z^{(n)}\|_X^2}.$$

The iterates obtained using this method converges more rapidly for the case of exact data. A proof of convergence for the method can be found in [13, Theorem 7.6 pp. 184 – 186].

Similarly, recall from Section 1.1.2 that $K : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_0)$, where $X = H^{-1/2}(\Gamma_1)$ and $Y = H^{-1/2}(\Gamma_0)$, here we consider K^*K , which maps from $H^{-1/2}(\Gamma_1)$ to $H^{-1/2}(\Gamma_1)$, and thus the normal equation can be solved using the Krylov subspace method. The adjoint operator is required here and we show in [1] that under certain positivity assumption we have an expression for K^* .

1.2.3 The Generalized Minimal Residual method

Let us now consider another iterative technique based on Krylov subspace called the Generalized minimal residual method. In this method, the operators need not to be self-adjoint or positive definite. The algorithm is as follows, see [28]: Choose $x^{(0)}$ to be an initial guess, and compute the initial residual

$$r^{(0)} = y - Fx^{(0)},$$

then, compute $\beta = \|r^{(0)}\|_X$ and $v^{(1)} = r^{(0)}/\beta$. Having $v^{(n)}$ the algorithm proceeds by the following steps: Evaluate

$$w^{(n)} = Fv^{(n)},$$

For all $i = 1, \dots, n$, compute

$$h_{in} = (w^{(n)}, v^{(i)})_X,$$

and replace

$$w^{(n)} := w^{(n)} - h_{in} v^{(i)}.$$

Then, compute $h(n+1, n) = \|w^{(n)}\|_X$, and $v^{(n+1)} = w^{(n)}/h_{(n+1, n)}$. Furthermore, we define the Hessenberg matrix $H_n = \{h_{ij}\}_{1 \leq i \leq n+1, 1 \leq j \leq n}$ and compute

$$\xi = \arg \min_{\xi \in \mathcal{R}^n} \|\beta e_1 - H_n \xi\|_X, \quad (1.25)$$

where e_1 is the first Euclidean basis vector. Finally, we compute the approximate solution

$$x^{(n)} = x^{(0)} + \sum_{i=1}^n v^{(i)} \xi_i. \quad (1.26)$$

We also recall from Section 1.1.2 that $K : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_0)$ here $X = H^{-1/2}(\Gamma_1)$ and $Y = H^{-1/2}(\Gamma_0)$ are of different spaces. The method cannot be applied directly. The operator K has to be modified. We also don't need the adjoint operator and we can pick any scalar product, for instance the L^2 scalar product, see [5].

2 – Summary of papers

Let us now give a summary of the work we have done and the results obtained in this research.

2.0.1 Paper 1: Analysis of Dirichlet-Robin iterations for solving the Cauchy problem for Elliptic equations.

In this paper we implement the alternating iterative algorithm for solving the Cauchy problem for elliptic equations of second order. Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary Γ divided into two disjoint parts Γ_0 and Γ_1 such that they have a common Lipschitz boundary in Γ and $\bar{\Gamma}_0 \cup \bar{\Gamma}_1 = \Gamma$. The Cauchy problem for an elliptic equation is given as follows:

$$\begin{cases} \mathcal{L}u = D_j a^{ij}(x) D_i u + a(x)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \mathcal{N}u = g & \text{on } \Gamma_1, \end{cases} \quad (2.1)$$

where we defined the conormal operator N as

$$\mathcal{N}u = \nu_j a^{ij}(x) D_i u,$$

and $\nu = (\nu_1, \dots, \nu_j)$ is the outward unit normal to the boundary Γ , $D_j = \frac{\partial}{\partial x_j}$, a^{ij} and a are measurable real valued functions such that a is bounded, $a^{ij} = a^{ji}$, and the functions f and g are specified Cauchy data on Γ_0 . Our concern here is to find the real-valued solution to problem (2.1). We consider the Dirichlet-Robin algorithm for solving (2.1). By making the assumption that the elliptic operator with the Dirichlet boundary condition is positive we show that the Dirichlet-Robin algorithm is convergent, provided that the parameters in the Robin conditions are chosen appropriately.

We consider two real-valued measurable bounded functions μ_0 and μ_1 defined on Γ_0 and Γ_1 so that the positivity assumption of the bilinear form as given is satisfied

$$\int_{\Omega} (a^{ij} D_i u D_j u - a u^2) dx + \int_{\Gamma_0} \mu_0 u^2 dS + \int_{\Gamma_1} \mu_1 u^2 dS > 0, \quad (2.2)$$

for all $u \in H^1(\Omega) \setminus \{0\}$.

Given these two bounded real valued measurable functions μ_0 and μ_1 in place, we consider the two auxiliary boundary value problems

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f_0 & \text{on } \Gamma_0, \\ \mathcal{N}u + \mu_1 u = \eta & \text{on } \Gamma_1, \end{cases} \quad (2.3)$$

and

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ Nu + \mu_0 u = g_0 & \text{on } \Gamma_0, \\ u = \phi & \text{in } \Gamma_1. \end{cases} \quad (2.4)$$

We describe the Dirichlet-Robin algorithm for solving (2.1) as follows: Take $f_0 = f$ and $g_0 = g + \mu_0 f$, where f and g are the Cauchy data given in (2.1).

- (1) The first approximation u_0 is obtained by solving (2.3), where η is an arbitrary initial guess for the Robin condition on Γ_1 .
- (2) Having constructed u_{2n} , we find u_{2n+1} by solving (2.4) with $\phi = u_{2n}$ on Γ_1 .
- (3) We then obtain u_{2n+2} by solving (2.3) with $\eta = Nu_{2n+1} + \mu_1 u_{2n+1}$ on Γ_1 .

This algorithm involves solving the well-posed boundary value problem for the original equation.

We define the weak solutions to the two well-posed boundary value problems (2.3) and (2.4) and we show that the problems are well-posed. The main objective of this paper is to prove the convergence of the algorithm denoted by $(u_n(f_0, g_0, \eta))_{n=0}^\infty$ as stated below. The iterations largely depend on f_0, g_0 and η .

Theorem 2.0.1 *Let $f_0 \in H^{1/2}(\Gamma_0)$ and $g_0 \in H^{-1/2}(\Gamma_0)$, and let $u \in H^1(\Omega)$ be the solution to the problem (2.1). Then for $\eta \in H^{-1/2}(\Gamma_1)$, the sequence $(u_n)_{n=0}^\infty$, obtained using the algorithm described above, converges to u in $H^1(\Omega)$.*

We implement numerical experiments by considering the Cauchy problem for the Helmholtz equation, as shown in (1.1). We also implement a finite difference method to solve the two well-posed boundary value problems (2.3) and (2.4). We investigate how the choice of the Robin parameters μ_0 and μ_1 influence the convergence of the iterations. We have seen that if we set $\mu = \mu_0 = \mu_1$ to be positive constant, then for small values of μ , the Dirichlet-Robin algorithm diverges but for sufficiently large values of μ we obtain convergence. However, for very large values of μ , the convergence is slow.

Similarly, we further investigate the dependence of μ on k^2 for the convergence of the algorithm, i.e the smallest value for μ needed to obtain convergence as a function of k^2 . We note that a larger value of k^2 also means a larger value for μ is needed to obtain convergence.

2.0.2 Paper 2: Accelerated Dirichlet-Robin alternating algorithms for solving the Cauchy problem for the Helmholtz equation.

In this paper, we consider the Cauchy problem (2.1) where $\mathcal{L}$ is an Helmholtz operator defined as $\Delta + k^2$ and $\mathcal{N}$ is the normal derivative $\mathcal{N} = \partial_\nu$. Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary Γ divided into two disjoint

parts Γ_0 and Γ_1 such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is also Lipschitz. The Cauchy problem for the Helmholtz equation is given as follows:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f_0 & \text{on } \Gamma_0, \\ \partial_\nu u = g_0 & \text{on } \Gamma_0, \end{cases} \quad (2.5)$$

where f_0 and g_0 are specified Cauchy data on Γ_0 with a certain noise level, k is a positive real number called the wave number, ν is the outward unit normal to the boundary Γ and ∂_ν denotes the outward normal derivative. We are interested in finding real-valued solutions to problem (2.5).

Given two real valued constants μ_0 and μ_1 in place, we consider two auxiliary problems

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u + \mu_1 u = \eta & \text{on } \Gamma_1, \end{cases} \quad (2.6)$$

and

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u + \mu_0 u = g & \text{on } \Gamma_0, \\ u = \phi & \text{on } \Gamma_1. \end{cases} \quad (2.7)$$

We propose Dirichlet-Robin algorithm for solving (2.5), where, in each iteration, one solves sequences of the two well-posed boundary value problem (2.6) and (2.7). We take $f = f_0$ and $g = g_0 + \mu_0 f_0$, where f_0 and g_0 are the specified Cauchy data in (2.5). The specifics of the iterations are

- (1) The first approximation u_0 is obtained by solving (2.6), where η is an arbitrary initial approximation of the Robin condition on Γ_1 .
- (2) Having constructed u_{2n} , we find u_{2n+1} by solving (2.7) with $\phi = u_{2n}$ on Γ_1 .
- (3) We then obtain u_{2n+2} by solving (2.6) with $\eta = \partial_\nu u_{2n+1} + \mu_1 u_{2n+1}$ on Γ_1 .

It is proved in [6] that the convergence is quite slow. To accelerate this convergence we introduce two operator equations for finding solutions to the Cauchy problem (2.5). First, we define the operator

$$N : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_0),$$

by

$$N\eta = (\partial_\nu + \mu_0)u_0(0, \eta)|_{\Gamma_0},$$

where $u_0(f, \eta)$ is the weak solution of the problem (2.6). We can then use the above operator equation to solve (2.5) as follows:

$$N\eta = g_0 + \mu_0 f_0 - (\partial_\nu + \mu_0)u_0(f_0, 0)|_{\Gamma_0}. \quad (2.8)$$

In order to obtain the normal equation corresponding to (2.8) we introduce the operator

$$M : H^{-1/2}(\Gamma_0) \rightarrow H^{-1/2}(\Gamma_1),$$

by

$$M\xi = (\partial_\nu + \mu_1)u_1(\xi, 0)|_{\Gamma_1},$$

where $u_1(g, \phi) \in H^1(\Omega)$ is the weak solution to (2.7). It appears that the adjoint N^* of the operator N , with respect to certain inner products introduced is equal to M and the normal equation takes the form

$$MN\eta = M(g_0 + \mu_0 f_0 - (\partial_\nu + \mu_0)u_0(f_0, 0)|_{\Gamma_0}). \quad (2.9)$$

We also implement the second operator equation B obtained by the first two iterations involving the auxiliary problems (2.6) and (2.7). The operator

$$B : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_1),$$

is defined by

$$B\eta = (\partial_\nu + \mu_1)u_1(0, u_0(0, \eta))|_{\Gamma_1}. \quad (2.10)$$

In order to solve the Cauchy problem (2.5) one can use also the following operator equation. Let $\eta \in H^{-1/2}(\Gamma_1)$. Then the equation

$$(\partial_\nu + \mu_1)u_1(g_0 + \mu_0 f_0, u_0(f_0, \eta)) = \eta \quad \text{on } \Gamma_1, \quad (2.11)$$

can be solved as an operator equation for finding η , and then finding the solution u of (2.5). To write (2.11) as an operator equation we use linearity:

$$u_1(g_0 + \mu_0 f_0, u_0(f_0, \eta)) = u_1(0, u_0(0, \eta)) + u_1(g_0 + \mu_0 f_0, u_0(f_0, 0)).$$

Thus the equation (2.11) can be written as

$$\eta = B\eta + A, \quad (2.12)$$

where

$$A = (\partial_\nu + \mu_1)u_1(g_0 + \mu_0 f_0, u_0(f_0, 0))|_{\Gamma_1}.$$

Operator equations (2.9) and (2.12) are the basis for implementing numerical tests. We present different properties of the operator B and a connections between the two operators B and N .

Finally, we consider three iterative methods for solving the operator equations obtained: Landweber iteration method, the Conjugate gradient method (CG) and the Generalized minimal residual method (GMRES). The first two methods use the Hilbert structure in some appropriate function spaces, while the third one plays an important role for operator equation considered only on a single space such as in (2.9). By reformulating the Cauchy problem as an operator equation we show that the Dirichlet-Robin iterations are equivalent to the classical Landweber iteration, see [21].

In order to evaluate the proposed iterative methods we implement a finite difference method for solving the Helmholtz equation in a simple domain and conduct several numerical experiments. Numerical results show that both CG and GMRES work very well. However, CG is less accurate due to the fact that the original problem is ill-posed and by forming the normal equation we increase the degree of ill-posedness. Hence GMRES is both faster and gives smaller final errors.

References

- [1] P. Achieng, F. Berntsson, J. Chepkorir, and V. A. Kozlov. Analysis of Dirichlet-Robin iterations for solving the Cauchy problem for Elliptic Equations. *Bulletin of the Iranian Mathematical Society*, (Published online), 2020.
- [2] G. Alessandrini. Stable determination of a crack from boundary measurements. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 123(3):497–516, 1993.
- [3] G. Alessandrini, P. L. Del, and L. Rondi. Stable determination of corrosion by a single electrostatic boundary measurement. *Inverse problems*, 19(4):973–984, 2003.
- [4] G. Bastay, T. Johansson, A. V. Kozlov, and D. Lesnic. An alternating method for the stationary Stokes system. *ZAMM, Z. Angew. Math. Mech.*, 86(4):268–280, 2006.
- [5] F. Berntsson, J. Chepkorir, and V. A. Kozlov. Accelerated Dirichlet-Robin alternating algorithm for solving the Cauchy problem for the Helmholtz Equations. *IMA Journal of Applied Mathematics*, (Submitted), 2020.
- [6] F. Berntsson, V. A. Kozlov, L. Mpinganzima, and B. O. Turesson. An accelerated alternating procedure for the Cauchy problem for the Helmholtz equation. *Comput. Math. Appl.*, 68(1-2):44–60, 2014.
- [7] F. Berntsson, V. A. Kozlov, L. Mpinganzima, and B. O. Turesson. An alternating iterative procedure for the Cauchy problem for the Helmholtz equation. *Inverse Probl. Sci. Eng.*, 22(1):45–62, 2014.
- [8] F. Berntsson, V. A. Kozlov, L. Mpinganzima, and B. O. Turesson. Iterative Tikhonov regularization for the Cauchy problem for the Helmholtz equation. *Comput. Math. Appl.*, 73(1):163–172, 2017.
- [9] F. Berntsson, V. A. Kozlov, L. Mpinganzima, and B. O. Turesson. Robin-Dirichlet algorithms for the Cauchy problem for the Helmholtz equation. *Inverse Probl. Sci. Eng.*, 26(7):1062–1078, 2018.
- [10] L. Caillé, L. Marin, and F. Delvare. A meshless fading regularization algorithm for solving the Cauchy problem for the three-dimensional Helmholtz equation. *Numerical Algorithms*, 82(3):869–894, 2019.
- [11] R. Chapko and B. T. Johansson. An alternating potential-based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut. *Comput. Methods Appl. Math.*, 8(4):315–335, 2008.
- [12] T. DeLillo, V. Isakov, N. Valdivia, and L. Wang. The detection of surface vibrations from interior acoustical pressure. *Inverse Problems*, 19(3):507–524, 2003.

- [13] H. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*, volume 175. Springer Science & Business Media, 1996.
- [14] J. Hadamard. *Lectures on Cauchy's problem in Linear Partial Differential Equations*. Dover Publications, New York, 1953.
- [15] M. R. Hestenes and E. Stiefel. Methods of conjugate gradients for solving linear systems. *Journal of research of the National Bureau of Standards*, 49(6):409–436, 1952.
- [16] B. T. Johansson and V. A. Kozlov. An alternating method for Cauchy problems for Helmholtz-type operators in non-homogeneous medium. *IMA J. Appl. Math.*, 74(1):62–73, 2009.
- [17] S. I. Kabanikhin. Definitions and examples of inverse and ill-posed problems. *Journal of Inverse and Ill-Posed Problems*, 16(4):317–357, 2008.
- [18] A. Kirsch. *An Introduction to the Mathematical Theory of Inverse Problems*, volume 120. Springer Science & Business Media, 2011.
- [19] V. A. Kozlov and V. G. Maz'ya. Iterative procedures for solving ill-posed boundary value problems that preserve the differential equations. *Algebra i Analiz*, 1(5):1207–1228, 1989.
- [20] V. A. Kozlov, V. G. Maz'ya, and A. V. Fomin. An iterative method for solving the Cauchy problem for elliptic equations. *Comput. Math. and Math. Phys.*, 31(1):62–73, 1991.
- [21] L. Landweber. An iteration formula for Fredholm integral equations of the first kind. *American journal of mathematics*, 73(3):615–624, 1951.
- [22] C. L. Lawson and R. J. Hanson. *Solving least squares problems*, volume 161. SIAM, 1974.
- [23] D. Lesnic, L. Elliott, and D. B. Ingham. An alternating boundary element method for solving Cauchy problems for the biharmonic equation. *Inverse Problem in Engineering*, 5(2):145–168, 1997.
- [24] D. Lesnic, L. Elliott, and D. B. Ingham. An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation. *Eng. Analy. Bound. Elem.*, 20(2):123–133, 1997.
- [25] H. Lotfinia, N. Chegini, and R. Mokhtari. The bi-Helmholtz equation with Cauchy conditions: ill-posedness and regularization methods. *Inverse Problems in Science and Engineering*, pages 1–23, 2020.
- [26] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic, and X. Wen. An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation. *Comput. Methods Appl. Mech. Engrg.*, 192(5-6):709–722, 2003.

- [27] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic, and X. Wen. Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations. *Comput. Mech.*, 31(3-4):367–377, 2003.
- [28] Y. Saad. *Iterative Methods for Sparse Linear Systems*. SIAM, 2003.
- [29] J. D. Shea, P. Kosmas, S. C. Hagness, and B. D. Van Veen. Three-dimensional microwave imaging of realistic numerical breast phantoms via a multiple-frequency inverse scattering technique. *Medical physics*, 37(8):4210–4226, 2010.
- [30] T. Wei, Y. C. Hon, and L. Ling. Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators. *Eng. Anal. Bound. Elem.*, 31(4):373–385, 2007.
- [31] F. Yang, P. Fan, and X. X. Li. Fourier truncation regularization method for a three-dimensional Cauchy problem of the modified Helmholtz equation with perturbed wave number. *Mathematics*, 7(8):705, 2019.
- [32] F. Yang, P. Zhang, and X. X. Li. The truncation method for the Cauchy problem of the inhomogeneous Helmholtz equation. *Applicable Analysis*, 98(5):991–1004, 2019.

3 – Included Papers

Papers

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