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Noncommutative minimal embeddings and morphisms of pseudo-Riemannian calculi

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ABSTRACT

In analogy with classical submanifold theory, we introduce morphisms of real metric calculi together with noncommutative embeddings. We show that basic concepts, such as the second fundamental form and the Weingarten map, translate into the noncommutative setting and, in particular, we prove a noncommutative analogue of Gauss' equations for the curvature of a submanifold. Moreover, the mean curvature of an embedding is readily introduced, giving a natural definition of a noncommutative minimal embedding, and we illustrate the novel concepts by considering the noncommutative torus as a minimal surface in the noncommutative 3-sphere.

1. Introduction

In recent years, a lot of progress has been made in understanding the Riemannian aspects of noncommutative geometry. The Levi-Civita connection of a metric plays a crucial role in classical Riemannian geometry and it is important to understand to what extent a corresponding noncommutative theory exists. Several impressive results exist, which compute the curvature of the noncommutative torus from the heat kernel expansion and consider analogues of the classical Gauss–Bonnet theorem [7–10]. However, starting from a spectral triple, with the metric implicitly given via the Dirac operator, it is far from obvious if there exists a module together with a bilinear form, representing the metric corresponding to the Dirac operator, not to mention the existence of a Levi-Civita connection. In order to better understand what kind of results one can expect, it is interesting to take a more naive approach, where one starts with a module together with a metric, and tries to understand under what conditions one may discuss metric compatibility, as well as torsion and uniqueness, of a general connection.

In [2,3,18], pseudo-Riemannian calculi were introduced as a framework to discuss the existence of a metric and torsion free connection as well as properties of its curvature. In fact, the theory is somewhat similar to that of Lie–Rinehart algebras, where a real calculus (as introduced in [3]) might be considered as a "noncommutative Lie–Rinehart algebra". Lie–Rinehart algebras have been discussed from many points of view (see e.g. [12,16] and [1] for an overview of metric aspects). Although the existence of a Levi-Civita connection is not always guaranteed in the context of pseudo-Riemannian calculi, it was shown that the connection is unique if it exists. The theory has concrete similarities with classical differential geometry, and several ideas, such as Koszul's formula, have direct analogues in the noncommutative setting. Apart from the noncommutative torus, noncommutative spheres were considered, and a Chern–Gauss–Bonnet type theorem was proven for the noncommutative 4-sphere [2]. Note that there are several approaches to metric aspects of noncommutative geometry, and Levi-Civita connections, which are different but similar in spirit (see e.g. [4,5,11,14,15,17]).

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In this paper, we introduce morphisms of real (metric) calculi and define noncommutative (isometric) embeddings. We show that several basic concepts of submanifold theory extends to noncommutative submanifolds and we prove an analogue of Gauss' equations for the curvature of a submanifold. Moreover, the mean curvature of an embedding is defined, immediately giving a natural definition of a (noncommutative) minimal embedding. As an illustration of the above concepts, the noncommutative torus is considered as a minimal submanifold of the noncommutative 3-sphere.

2. Pseudo-Riemannian calculi

Let us briefly recall the basic definitions leading to the concept of a pseudo-Riemannian calculus and the uniqueness of the Levi-Civita connection. For more details, we refer to [3].

Definition 2.1 (*Real Calculus*)**.** Let $\mathcal{A}$ be a unital $*$-algebra, let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a finite-dimensional (real) Lie algebra and let M be a (right) $\mathcal{A}$-module. Moreover, let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map whose image generates M as an $\mathcal{A}$-module. Then $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is called a *real calculus over* $\mathcal{A}$.

The motivation for the above definition comes from the analogous structures in differential geometry, as seen in the following example.

Example 2.2. Let Σ be a smooth manifold. Then Σ can be represented by the real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ with $\mathcal{A} = \mathcal{C}^\infty(\Sigma)$, $\mathfrak{g} = \mathrm{Der}(\mathcal{C}^\infty(\Sigma))$, $M = \mathrm{Vect}(M)$ (the module of vector fields on Σ) and choosing φ to be the natural isomorphism between the set of derivations of $C^\infty(\Sigma)$ and smooth vector fields on Σ.

Next, since we are interested in Riemannian geometry, one introduces a metric structure on the module M.

Definition 2.3. Suppose that $\mathcal{A}$ is a $*$-algebra and let M be a right $\mathcal{A}$-module. A *hermitian form on M* is a map $h : M \times M \to \mathcal{A}$ with the following properties:

$h1$. $h(m_1, m_2 + m_3) = h(m_1, m_2) + h(m_1, m_3)$
$h2$. $h(m_1, m_2 a) = h(m_1, m_2)a$
$h3$. $h(m_1, m_2) = h(m_2, m_1)^*$

for all $m_1, m_2, m_3 \in M$ and $a \in \mathcal{A}$. Moreover, if $h(m_1, m_2) = 0$ for all $m_2 \in M$ implies that $m_1 = 0$ then h is said to be *nondegenerate*, and in this case we say that h is a *metric on* M. The pair (M, h) is called a *(right) hermitian $\mathcal{A}$-module*, and if h is a metric on M we say that (M, h) is a *(right) metric $\mathcal{A}$-module*.

Definition 2.4 (*Real Metric Calculus*)**.** Suppose that $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is a real calculus over $\mathcal{A}$ and that (M, h) is a (right) metric $\mathcal{A}$-module. If

$$h(\varphi(\partial_1), \varphi(\partial_2))^* = h(\varphi(\partial_1), \varphi(\partial_2))$$

for all $\partial_1, \partial_2 \in \mathfrak{g}$ then the pair $(C_{\mathcal{A}}, h)$ is called a *real metric calculus*.

Example 2.5. Let (Σ, g) be a Riemannian manifold and let $C_{\mathcal{A}}$ be the real calculus from Example 2.2 representing Σ. Then $(C_{\mathcal{A}}, g)$ is a real metric calculus.

In what follows, we shall sometimes require the metric to satisfy a stronger condition than nondegeneracy.

Definition 2.6. Let h be a metric on M and let $\hat{h} : M \to M^*$ (the dual of M) be the mapping given by $\hat{h}(m)(n) = h(m, n)$. The metric h is said to be *invertible* if $\hat{h}$ is invertible.

Now, given a real metric calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$, we will discuss connections on M and their compatibility with the metric. Let us start by recalling the definition of an affine connection for a derivation based calculus.

Definition 2.7. Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ be a real calculus over $\mathcal{A}$. An *affine connection* on $(M, \mathfrak{g})$ is a map $\nabla : \mathfrak{g} \times M \to M$ satisfying

(1) $\nabla_\partial(m + n) = \nabla_\partial m + \nabla_\partial n$,
(2) $\nabla_{\lambda\partial+\partial'} m = \lambda\nabla_\partial m + \nabla_{\partial'} m$,
(3) $\nabla_\partial(ma) = (\nabla_\partial m)a + m\partial(a)$

for $m, n \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{R}$.

The fact that we shall require the connection to be "real" is reflected in the following definition.

Definition 2.8. Let $(C_{\mathcal{A}}, h)$ be a real metric calculus and let ∇ denote an affine connection on $(M, \mathfrak{g})$. Then $(C_{\mathcal{A}}, h, \nabla)$ is called a *real connection calculus* if

$$h\big(\nabla_\partial\varphi(\partial_1), \varphi(\partial_2)\big) = h(\nabla_\partial\varphi(\partial_1), \varphi(\partial_2))^*$$

for all $\partial, \partial_1, \partial_2 \in \mathfrak{g}$.

Definition 2.9. Let $(C_{\mathcal{A}}, h, \nabla)$ be a real connection calculus. We say that $(C_{\mathcal{A}}, h, \nabla)$ is *metric* if

$$\partial(h(m, n)) = h(\nabla_\partial m, n) + h(m, \nabla_\partial n)$$

for all $\partial \in \mathfrak{g}$ and $m, n \in M$, and *torsion-free* if

$$\nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) - \varphi([\partial_1, \partial_2]) = 0$$

for all $\partial_1, \partial_2 \in \mathfrak{g}$. A metric and torsion-free real connection calculus is called a *pseudo-Riemannian calculus*.

A connection fulfilling the requirements of a pseudo-Riemannian calculus is called a *Levi-Civita connection*. In the quite general setup of real metric calculi, where there are few assumptions on the structure of the algebra $\mathcal{A}$ and the module M, the existence of a Levi-Civita connection cannot be guaranteed. However, if it exists, it is unique.

Theorem 2.10 ([3])**.** *Let $(C_{\mathcal{A}}, h)$ be a real metric calculus. Then there exists at most one affine connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus.*

The next result provides us a noncommutative analogue of Koszul's formula, which is a useful tool for constructing the Levi-Civita connection in several examples.

Proposition 2.11 ([3])**.** *Let $(C_{\mathcal{A}}, h, \nabla)$ be a pseudo-Riemannian calculus and assume that $\partial_1, \partial_2, \partial_3 \in \mathfrak{g}$. Then*

$$\begin{aligned} 2h(\nabla_1 E_2, E_3) = {} & \partial_1 h(E_2, E_3) + \partial_2 h(E_1, E_3) - \partial_3 h(E_1, E_2) \\ & - h\left(E_1, \varphi([\partial_2, \partial_3])\right) + h\left(E_2, \varphi([\partial_3, \partial_1])\right) + h\left(E_3, \varphi([\partial_1, \partial_2])\right), \end{aligned} \tag{2.1}$$

where $\nabla_i = \nabla_{\partial_i}$ and $E_i = \varphi(\partial_i)$ for $i = 1, 2, 3$.

As in Riemannian geometry, a connection satisfying Koszul's formula is torsion-free and compatible with the metric.

Proposition 2.12 ([3])**.** *Let $(C_{\mathcal{A}}, h)$ be a real metric calculus, and suppose that ∇ is an affine connection on $(M, \mathfrak{g})$ such that Koszul's formula (2.1) holds. Then $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus.*

A particularly simple case, which is also relevant to our applications, is when M is a free module. The following result then gives a way of constructing the Levi-Civita connection from Koszul's formula.

Corollary 2.13 ([3])**.** *Let $(C_{\mathcal{A}}, h)$ be a real metric calculus and let $\{\partial_1, \dots, \partial_n\}$ be a basis of $\mathfrak{g}$ such that $\{E_a = \varphi(\partial_a)\}_{a=1}^n$ is a basis for M. If there exist $m_{ab} \in M$ such that*

$$\begin{aligned} h(m_{ab}, E_c) = {} & \partial_a h(E_b, E_c) + \partial_b h(E_a, E_c) - \partial_c h(E_a, E_b) \\ & - h\left(E_a, \varphi([\partial_b, \partial_c])\right) + h\left(E_b, \varphi([\partial_c, \partial_a])\right) + h\left(E_c, \varphi([\partial_a, \partial_b])\right), \end{aligned} \tag{2.2}$$

for $a, b, c = 1, \dots, n$, then there exists an affine connection ∇, given by $\nabla_{\partial_a} E_b = m_{ab}$, such that $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus.

3. Real calculus homomorphisms

In order to understand the algebraic structure of real calculi, a first step is to consider morphisms. Via a concept of morphism of real calculi, one can understand when two calculi are considered to be equal (isomorphic) and, from a geometric point of view, what one means by a noncommutative embedding. In this section we introduce homomorphisms of real (metric) calculi and prove several results which, in different ways, shed light on the new concept.

Definition 3.1. Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ and $C_{\mathcal{A}'} = (\mathcal{A}', \mathfrak{g}', M', \varphi')$ be real calculi and assume that $\phi : \mathcal{A} \to \mathcal{A}'$ is a $*$-algebra homomorphism. If there is a map $\psi : \mathfrak{g}' \to \mathfrak{g}$ such that

(ψ1) ψ is a Lie algebra homomorphism
(ψ2) $\delta(\phi(a)) = \phi(\psi(\delta)(a))$ for all $\delta \in \mathfrak{g}', a \in \mathcal{A}$,

then ψ is said to be *compatible* with ϕ. If ψ is compatible with ϕ we define Ψ as $\Psi = \varphi \circ \psi$, and M_Ψ is defined to be the submodule of M generated by $\Psi(\mathfrak{g}')$.

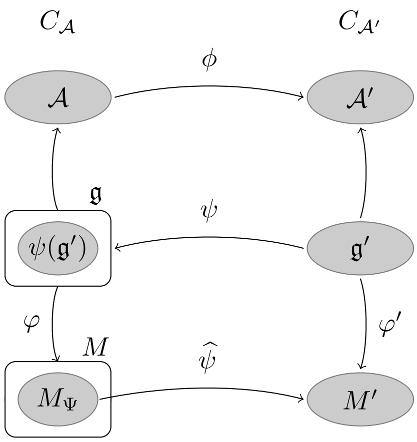

Fig. 1. A real calculus homomorphism $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$.

Furthermore, if there is a map $\widehat{\psi} : M_\Psi \to M'$ such that

($\widehat{\psi}$1) $\widehat{\psi}(m_1 + m_2) = \widehat{\psi}(m_1) + \widehat{\psi}(m_2)$ for all $m_1, m_2 \in M$
($\widehat{\psi}$2) $\widehat{\psi}(ma) = \widehat{\psi}(m)\phi(a)$ for all $m \in M$ and $a \in \mathcal{A}$
($\widehat{\psi}$3) $\widehat{\psi}(\Psi(\delta)) = \varphi'(\delta)$ for all $\delta \in \mathfrak{g}'$,

then $\widehat{\psi}$ is said to be *compatible* with ϕ and ψ, and $(\phi, \psi, \widehat{\psi})$ is called a *real calculus homomorphism* from $C_{\mathcal{A}}$ to $C_{\mathcal{A}'}$ (see Fig. 1 for an illustration of a real calculus homomorphism). If ϕ is a $*$-algebra isomorphism, ψ a Lie algebra isomorphism and $\widehat{\psi}$ is a bijective map then $(\phi, \psi, \widehat{\psi})$ is called a real calculus isomorphism.

Let us try to understand Definition 3.1 in the context of embeddings, where the analogy with classical geometry is rather clear. Thus, let $\phi_0 : \Sigma' \to \Sigma$ be an embedding of Σ' into Σ and let $\phi : C^\infty(\Sigma) \to C^\infty(\Sigma')$ be the corresponding homomorphism of the algebras of smooth functions. In the notation of Definition 3.1 we have

$$\begin{aligned} \mathcal{A} = C^\infty(\Sigma) \quad &\xrightarrow{\phi} \quad \mathcal{A}' = C^\infty(\Sigma') \\ \mathfrak{g} = \mathrm{Der}(\mathcal{A}) \quad &\xleftarrow{\psi} \quad \mathfrak{g}' = \mathrm{Der}(\mathcal{A}') \\ M = \mathrm{Vect}(\Sigma) \supseteq M_\Psi \quad &\xrightarrow{\widehat{\psi}} \quad M' = \mathrm{Vect}(\Sigma'). \end{aligned}$$

First of all, there is no natural map from $\mathrm{Vect}(\Sigma)$ to $\mathrm{Vect}(\Sigma')$ since a vector field $X \in \mathrm{Vect}(\Sigma)$ at a point $p \in \phi_0(\Sigma')$ might not lie in $T_p\Sigma'$ (regarded as a subspace of $T_p\Sigma$). However, vector fields which are tangent to Σ' in this sense may be restricted to Σ'. On the other hand, any vector field $X' \in \mathrm{Vect}(\Sigma')$ (assuming Σ' to be closed) can be extended to a smooth vector field $X \in \mathrm{Vect}(\Sigma)$ such that $X|_{\Sigma'} = X'$. In light of the isomorphism between vector fields and derivations, it is therefore more natural to have a map $\psi : \mathrm{Der}(\mathcal{A}') \to \mathrm{Der}(\mathcal{A})$, corresponding to a choice of extension of vector fields on Σ'. The map $\widehat{\psi}$ then corresponds to the restriction of vector fields on Σ which are tangent to Σ'. Consequently, we consider vector fields in M_Ψ as extensions of vector fields on the embedded manifold.

In noncommutative geometry (in contrast to the classical case) $\mathfrak{g}$ is no longer an $\mathcal{A}$-module, a difference which is captured by the concept of a real calculus. The definition of homomorphism reflects this fact by assuming that every derivation of $\mathcal{A}'$ can be "extended" to a derivation of $\mathcal{A}$ and, furthermore, that every vector field on Σ which is tangent to Σ' (that is, in the image of $\varphi \circ \psi$) can be "restricted" to Σ'.

Next, one can easily check that the composition of two homomorphisms is again a homomorphism.

Proposition 3.2. *Let $C_{\mathcal{A}}$, $C_{\mathcal{A}'}$ and $C_{\mathcal{A}''}$ be real calculi and assume that*

$$(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'} \quad \text{and} \quad (\phi', \psi', \widehat{\psi}') : C_{\mathcal{A}'} \to C_{\mathcal{A}''}$$

are real calculus homomorphisms. Then $(\phi' \circ \phi, \psi \circ \psi', \widehat{\psi}' \circ \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}''}$ is a real calculus homomorphism.

Proof. For convenience, we introduce $\Phi := \phi' \circ \phi$, $\tilde{\psi} := \psi \circ \psi'$ and $\hat{\Psi} := \widehat{\psi}' \circ \widehat{\psi}$. First of all, it is clear that Φ is a $*$-algebra homomorphism and $\tilde{\psi}$ is a Lie algebra homomorphism. For $a \in \mathcal{A}$ and $\delta \in \mathfrak{g}''$ we get that

$$\delta(\Phi(a)) = \phi'(\psi'(\delta)(\phi(a))) = \phi'(\phi(\tilde{\psi}(\delta)(a))) = \Phi(\tilde{\psi}(\delta)(a)),$$

showing that Φ and $\tilde{\psi}$ are compatible, with $M_{\tilde{\psi}}$ being the submodule of M generated by $\tilde{\psi}(\mathfrak{g}'')$. Checking that $\hat{\Psi}(m+n) = \hat{\Psi}(m) + \hat{\Psi}(n)$ and $\hat{\Psi}(ma) = \hat{\Psi}(m)\Phi(a)$ for all $m, n \in M_{\tilde{\psi}}$ and $a \in \mathcal{A}$ is trivial, and for $\delta \in \mathfrak{g}''$ we get

$$\varphi''(\delta) = \widehat{\psi}'(\Psi'(\delta)) = \widehat{\psi}'(\varphi'(\psi'(\delta))) = \widehat{\psi}'(\widehat{\psi}(\Psi(\psi'(\delta)))) = \hat{\Psi}(\varphi \circ \tilde{\psi}(\delta)).$$

Thus $\hat{\Psi}$ is compatible with Φ and $\tilde{\psi}$, and it follows that $(\Phi, \tilde{\psi}, \hat{\Psi})$ is a real calculus homomorphism from $C_{\mathcal{A}}$ to $C_{\mathcal{A}''}$. □

A homomorphism of real calculi $(\phi, \psi, \widehat{\psi})$ consists of three maps, and a natural question is what kind of freedom one has in choosing these maps? Let us start by showing that, given ϕ and ψ, there is at most one $\widehat{\psi}$ such that $(\phi, \psi, \widehat{\psi})$ is a real calculus homomorphism.

Proposition 3.3. *If $(\phi, \psi, \widehat{\psi})$ and $(\phi, \psi, \tilde{\psi})$ are real calculus homomorphisms from $C_{\mathcal{A}}$ to $C_{\mathcal{A}'}$ then $\widehat{\psi} = \tilde{\psi}$.*

Proof. Let $m = \Psi(\delta_i)a^i$ for $\delta_i \in \mathfrak{g}'$ and $a^i \in \mathcal{A}$ be an arbitrary element of M_Ψ. It follows from ($\widehat{\psi}$1)–($\widehat{\psi}$3) that

$$\begin{aligned}\tilde{\psi}(m) &= \tilde{\psi}(\Psi(\delta_i)a^i) = \tilde{\psi}(\Psi(\delta_i))\phi(a^i) = \varphi'(\delta_i)\phi(a^i) = \widehat{\psi}(\Psi(\delta_i))\phi(a^i) \\ &= \widehat{\psi}(\Psi(\delta_i)a^i) = \widehat{\psi}(m). \quad \square\end{aligned}$$

Furthermore, if ϕ is an isomorphism, then the next result shows that ψ is determined uniquely by ϕ. Thus, combined with the previous result we conclude that if $(\phi, \psi, \widehat{\psi})$ is an isomorphism of real calculi, then ψ and $\widehat{\psi}$ are uniquely determined by ϕ.

Proposition 3.4. *If $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is a real calculus homomorphism such that ϕ is an isomorphism, then ψ is a Lie algebra isomorphism with $\psi(\delta) = \phi^{-1} \circ \delta \circ \phi$ for $\delta \in \mathfrak{g}'$.*

Proof. The formula for ψ follows directly from the fact that $\delta(\phi(a)) = \phi(\psi(\delta)(a))$ together with ϕ being an isomorphism. To prove that ψ is an isomorphism, let $\tilde{\psi} : \mathfrak{g} \to \mathfrak{g}'$ be given by $\tilde{\psi}(\partial) = \phi \circ \partial \circ \phi^{-1}$. Then for any $\partial \in \mathfrak{g}$ and $\delta \in \mathfrak{g}'$ it follows that

$$\begin{aligned}&\psi \circ \tilde{\psi}(\partial) = \phi^{-1} \circ \tilde{\psi}(\partial) \circ \phi = \phi^{-1} \circ \phi \circ \partial \circ \phi^{-1} \circ \phi = \partial \\ &\tilde{\psi} \circ \psi(\delta) = \phi \circ \psi(\delta) \circ \phi^{-1} = \phi \circ \phi^{-1} \circ \delta \circ \phi \circ \phi^{-1} = \delta.\end{aligned}$$

Thus ψ is a bijection with inverse $\psi^{-1} = \tilde{\psi}$. Furthermore, ψ^{-1} preserves the Lie bracket:

$$\begin{aligned}\psi^{-1}([\partial_1, \partial_2]) &= \psi^{-1}([\psi \circ \psi^{-1}(\partial_1), \psi \circ \psi^{-1}(\partial_2)]) = \psi^{-1} \circ \psi([\psi^{-1}(\partial_1), \psi^{-1}(\partial_2)]) \\ &= [\psi^{-1}(\partial_1), \psi^{-1}(\partial_2)],\end{aligned}$$

proving that ψ is indeed a Lie algebra isomorphism. □

Given a homomorphism $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$, there is a natural $\mathcal{A}$-module structure on M' given by $m' \cdot a = m'\phi(a)$ for $m' \in M'$ and $a \in \mathcal{A}$. As expected, the right $\mathcal{A}$-modules M and M' are isomorphic when $(\phi, \psi, \widehat{\psi})$ is an isomorphism.

Proposition 3.5. *If $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is a real calculus isomorphism then*

$$M = M_\Psi \simeq M'.$$

Proof. Since ψ is an isomorphism it follows that $\mathfrak{g} = \psi(\mathfrak{g}')$. From this it immediately follows that $M = M_\Psi$, since M_Ψ is defined to be the submodule of M generated by $\mathfrak{g} = \psi(\mathfrak{g}')$. Considering M' as a right $\mathcal{A}$-module, $\widehat{\psi}$ is an $\mathcal{A}$-module homomorphism, and since $\widehat{\psi}$ is assumed to be bijective, we conclude that $M_\Psi \simeq M'$. □

Recalling our previous discussions of real calculus homomorphisms in relation to embeddings, one may consider vector fields in M_Ψ as extensions of vector fields in M. Let us therefore make the following definition.

Definition 3.6. If $m \in M_\Psi$ such that $\widehat{\psi}(m) = m'$ then m is called an *extension of* m'. The set of extensions of m' will be denoted by $\mathrm{Ext}_\Psi(m')$.

3.1. Homomorphisms of real metric calculi

Having introduced the concept of homomorphisms for real calculi, it is natural to proceed to real metric calculi. From the geometric point of view, in the case of embeddings, one would like a homomorphism of real metric calculi to correspond to an isometric embedding. The following definition is straightforward.

Definition 3.7. Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be real metric calculi and assume that $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is a real calculus homomorphism. If

$$h'\big(\varphi'(\delta_1), \varphi'(\delta_2)\big) = \phi\big(h(\Psi(\delta_1), \Psi(\delta_2))\big)$$

for all $\delta_1, \delta_2 \in \mathfrak{g}'$ then $(\phi, \psi, \widehat{\psi})$ is called a *real metric calculus homomorphism*.

Assume that $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to C_{\mathcal{A}}$ is a homomorphism of real calculi. It is natural to ask if there exists a metric h' such that $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}}, h')$ is a homomorphism of real metric calculi, in which case we would call h' the *induced* metric. As it turns out, one cannot guarantee the existence of h', but whenever it exists, it is unique; we state this as follows.

Proposition 3.8. *Let $C_{\mathcal{A}}$ be a real calculus, $(C_{\mathcal{A}}, h)$ a real metric calculus, and let $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to C_{\mathcal{A}'}$ be a real calculus homomorphism. Then there exists at most one hermitian form h' on M' satisfying*

$$h'(\varphi'(\delta_1), \varphi'(\delta_2)) = \phi\big(h(\Psi(\delta_1), \Psi(\delta_2))\big), \quad \delta_1, \delta_2 \in \mathfrak{g}'.$$

Proof. Suppose that h'_1 and h'_2 both fulfill the given conditions for h'. By definition of real calculus homomorphism it is immediately obvious that h'_1 and h'_2 agree on $\varphi'(\mathfrak{g}')$. If we take two arbitrary elements $m, n \in M'$ it follows from the fact that $C_{\mathcal{A}'}$ is a real calculus over $\mathcal{A}'$ that m and n can be written as

$$\begin{aligned} m' &= \varphi'(\delta_i)a^i, \quad \delta_i \in \mathfrak{g}', a^i \in \mathcal{A}', \\ n' &= \varphi'(\delta_j)b^j, \quad \delta_j \in \mathfrak{g}', b^j \in \mathcal{A}'. \end{aligned}$$

Furthermore, one obtains

$$\begin{aligned} h'_1(m', n') &= h'_1\left(\varphi'(\delta_i)a^i, \varphi'(\delta_j)b^j\right) = h'_1\left(\varphi'(\delta_i)a^i, \varphi'(\delta_j)\right) b^j \\ &= (a^i)^* h'_1\left(\varphi'(\delta_i), \varphi'(\delta_j)\right) b^j = (a^i)^* h'_2\left(\varphi'(\delta_i), \varphi'(\delta_j)\right) b^j \\ &= h'_2\left(\varphi'(\delta_i)a^i, \varphi'(\delta_j)\right) b^j = h'_2\left(\varphi'(\delta_i)a^i, \varphi'(\delta_j)b^j\right) = h'_2(m', n'), \end{aligned}$$

since h'_1 and h'_2 are hermitian forms on M' and $h'_1(\varphi'(\delta_i), \varphi'(\delta_j)) = h'_2(\varphi'(\delta_i), \varphi'(\delta_j))$ for $\delta_1, \delta_2 \in \mathfrak{g}'$. Since m' and n' are arbitrary, it follows that $h'_1 = h'_2$. □

Note that if $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ is a homomorphism of real metric calculi, then $\phi\big(h(m, n)\big) = h'(\widehat{\psi}(m), \widehat{\psi}(n))$ for all $m, n \in M_{\Psi}$. In other words

$$\phi\big(h(m, n)\big) = h'(m', n')$$

if $m \in \mathrm{Ext}_{\Psi}(m')$ and $n \in \mathrm{Ext}_{\Psi}(n')$. This is to be compared with the geometrical situation where the inner product of vector fields restricted to the isometrically embedded manifolds equals the inner product of the restricted vector fields.

4. Embeddings of real calculi

In the previous section, we highlighted the analogy with embedded manifolds in order to motivate and understand the different concepts introduced for noncommutative algebras. However, we did not make the distinction between general homomorphisms and embeddings precise. In this section we shall define noncommutative embeddings and introduce a theory of submanifolds, much in analogy with the classical situation. It turns out that one can readily introduce the second fundamental form, and find a noncommutative analogue of Gauss' equation, giving the curvature of the submanifold.

A necessary condition for a map $\phi_0 : \Sigma' \to \Sigma$ to be an embedding, is that ϕ_0 is injective; dually, this corresponds to $\phi : C^\infty(\Sigma) \to C^\infty(\Sigma')$ being surjective. To formulate the next definition, we recall the orthogonal complement of a module. Namely, let $(C_{\mathcal{A}}, h)$ be a real metric calculus. Given any subset $N \subseteq M$, we define $N^\perp = \{m \in M : h(m, n) = 0\}$ and note that $N^\perp$ is a $\mathcal{A}$-module.

Definition 4.1. A homomorphism of real calculi $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is called an *embedding* if ϕ is surjective and there exists a submodule $\tilde{M} \subseteq M$ such that $M = M_{\Psi} \oplus \tilde{M}$. A homomorphism of real metric calculi $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ is called an *isometric embedding* if $(\phi, \psi, \widehat{\psi})$ is an embedding and $M = M_{\Psi} \oplus M_{\Psi}^\perp$.

The surjectivity of ϕ has immediate implications for the maps ψ and $\widehat{\psi}$.

Proposition 4.2. *Assume that $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is a real calculus homomorphism such that ϕ is surjective. Then ψ is injective and $\widehat{\psi}$ is surjective.*

Proof. For the first statement, suppose $\delta \in \ker(\psi)$. Then for any $a \in \mathcal{A}$ it follows that $\psi(\delta)(a) = 0$. Thus, by (ψ2) it follows that

$$\delta(\phi(a)) = \phi(\psi(\delta)(a)) = \phi(0) = 0$$

for any $a \in \mathcal{A}$, and since ϕ is surjective it follows that $\delta(a') = 0$ for every $a' \in \mathcal{A}'$.

For the second statement, let $m' \in M'$. Then m' can be written on the form $m' = \varphi'(\delta_i)b^i$ for some $\delta_i \in \mathfrak{g}'$ and $b^i \in \mathcal{A}'$, and since ϕ is surjective there are $a^i \in \mathcal{A}$ such that $\phi(a^i) = b^i$. It follows that

$$m' = \varphi'(\delta_i)b^i = \widehat{\psi}(\Psi(\delta_i))\phi(a^i) = \widehat{\psi}\left(\Psi(\delta_i)a^i\right),$$

completing the proof. □

Note that Proposition 4.2 gives further motivation for Definition 4.1 since it shows that ψ is injective, in analogy with the injectivity of the tangent map of an embedding. Moreover, it follows from Proposition 4.2 that if $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is an embedding, then every element $m' \in M'$ has at least one extension corresponding to the geometric situation where a vector field on the embedded manifold can be extended to a vector field in the ambient space.

Furthermore, given an embedding $(\phi, \psi, \widehat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$, we define the $\mathcal{A}$-linear projection $P : M \to M_\Psi$ as

$$P(m_\Psi \oplus \tilde{m}) = m_\Psi$$

with respect to the decomposition $M = M_\Psi \oplus \tilde{M}$. The complementary projection will be denoted by $\Pi = \mathbb{1} - P$. (Note that for an embedding of real metric calculi, the projections P and Π are orthogonal with respect to the metric on M.)

In analogy with classical Riemannian submanifold theory (see e.g. [13]), one decomposes the Levi-Civita connection in its tangential and normal parts. Let $(C_{\mathcal{A}}, h, \nabla)$ and $(C_{\mathcal{A}'}, h', \nabla')$ be pseudo-Riemannian calculi and assume that $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ is an isometric embedding and write

$$\nabla_{\psi(\delta)}m = L(\delta, m) + \alpha(\delta, m) \tag{4.1}$$

$$\nabla_{\psi(\delta)}\xi = -A_\xi(\delta) + D_\delta\xi \tag{4.2}$$

for $\delta \in \mathfrak{g}'$, $m \in M_\Psi$ and $\xi \in M_\Psi^\perp$, with

$$L(\delta, m) = P(\nabla_{\psi(\delta)}m) \qquad \alpha(\delta, m) = \Pi(\nabla_{\psi(\delta)}m)$$
$$A_\xi(\delta) = -P(\nabla_{\psi(\delta)}\xi) \qquad D_\delta\xi = \Pi(\nabla_{\psi(\delta)}\xi).$$

In differential geometry, (4.1) is called *Gauss' formula* and (4.2) is called *Weingarten's formula*. Furthermore, $\alpha : \mathfrak{g}' \times M_\Psi \to M_\Psi^\perp$ is called the *second fundamental form* and $A : \mathfrak{g}' \times M_\Psi^\perp \to M_\Psi$ is called the *Weingarten map*. Let us start by showing that the tangential part $L(\delta, m)$ is an extension of the Levi-Civita connection on $(C_{\mathcal{A}'}, h', \nabla')$.

Proposition 4.3. *If $\delta \in \mathfrak{g}'$ and $m \in \mathrm{Ext}_\Psi(m')$ then $L(\delta, m) \in \mathrm{Ext}_\Psi(\nabla'_\delta m')$*

Proof. For the sake of readability, let us first establish some notation. Let $\delta_i \in \mathfrak{g}'$ and let $\partial_i = \psi(\delta_i), E_i = \Psi(\delta_i)$ and $E'_i = \varphi'(\delta_i)$. Moreover, let $h_{ij} = h(E_i, E_j)$ and let $h_{i,[j,k]} = h(E_i, \Psi([\delta_j, \delta_k]))$; likewise, let $h'_{ij} = h(E'_i, E'_j)$ and $h'_{i,[j,k]} = h(E'_i, \varphi'([\delta_j, \delta_k]))$.

With this notation in place, Koszul's formula yields

$$2h(\nabla_i E_j, E_k) = \partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij} - h_{i,[j,k]} + h_{j,[k,i]} + h_{k,[i,j]}$$
$$2h'(\nabla'_i E'_j, E'_k) = \delta_i h'_{jk} + \delta_j h'_{ik} - \delta_k h'_{ij} - h'_{i,[j,k]} + h'_{j,[k,i]} + h'_{k,[i,j]}$$

for all $\delta_i, \delta_j, \delta_k \in \mathfrak{g}'$, and since h' is induced from h it follows that

$$h'_{jk} = \phi(h_{jk})$$
$$h'_{i,[j,k]} = \phi(h_{i,[j,k]})$$
$$\delta_i h'_{jk} = \delta_i\phi(h_{jk}) = \phi(\partial_i(h_{jk}));$$

from this it becomes clear that $h'(\nabla'_i E'_j, E'_k) = \phi(h(\nabla_i E_j, E_k))$. Let $m = E_i a^i \in M_\Psi$ and $n = E_k b^k \in M_\Psi$ be arbitrary elements in M_Ψ, where $a^i, b^k \in \mathcal{A}$. By definition of affine connections it follows that

$$h(\nabla_j m, n) = h(\nabla_j(E_i a^i), E_k b^k) = h((\nabla_j E_i)a^i, E_k b^k) + h(E_i\partial_j(a^i), E_k b^k) = (a^i)^* h(\nabla_j E_i, E_k)b^k + \partial_j(a^i)^* h_{ik} b^k,$$

and we get

$$\begin{aligned}
\phi(h(\nabla_j m, n)) &= \phi(a^i)^* h'(\nabla'_j E'_i, E'_k)\phi(b^k) + \phi(\partial_j(a^i)^*)h'_{ik}\phi(b^k)\\
&= \phi(a^i)^* h'(\nabla'_j E'_i, E'_k)\phi(b^k) + \delta_j(\phi(a^i)^*)h'_{ik}\phi(b^k)\\
&= h'((\nabla'_j E'_i)\phi(a^i), E'_k\phi(b^k)) + h'(E'_i\delta_j(\phi(a^i)), E'_k\phi(b^k))\\
&= h'(\nabla'_j(E'_i\phi(a^i)), E'_k\phi(b^k)) = h'(\nabla'_j(\widehat{\psi}(m)), \widehat{\psi}(n)).
\end{aligned}$$

It now follows that

$$h'(\nabla'_j(\widehat{\psi}(m)), \widehat{\psi}(n)) = \phi(h(\nabla_j m, n)) = \phi(h(P(\nabla_j m), n)) = \phi(h(L(\delta_j, m), n)),$$

which equals $h'(\widehat{\psi}(L(\delta_j, m)), \widehat{\psi}(n))$. Thus,

$$h'(\nabla'_j(\widehat{\psi}(m)), \widehat{\psi}(n)) = h'(\widehat{\psi}(L(\delta_j, m)), \widehat{\psi}(n)),$$

and since h' is nondegenerate and $\widehat{\psi}$ is surjective, it follows that $\widehat{\psi}(L(\delta_j, m)) = \nabla'_j\widehat{\psi}(m)$ which is equivalent to $L(\delta_j, m) \in \mathrm{Ext}_\Psi(\nabla'_j\widehat{\psi}(m))$, and it immediately follows that if $m \in \mathrm{Ext}_\Psi(m')$ then $L(\delta, m) \in \mathrm{Ext}_\Psi(\nabla_\delta m')$ for any $\delta \in \mathfrak{g}'$ and $m' \in M'$. □

In view of the above result, we introduce the notation $L(\delta, m) = \hat{\nabla}'_\delta m$ and conclude that

$$\nabla'_\delta m' = \widehat{\psi}\big(\hat{\nabla}'_\delta m\big) = \widehat{\psi}\big(P(\nabla_{\psi(\delta)} m)\big)$$

if $m \in \mathrm{Ext}_\Psi(m')$, giving a convenient way of retrieving the Levi-Civita connection ∇' from ∇. Next, let us show that the second fundamental form shares the properties of its classical counterpart.

Proposition 4.4. *If $\delta_1, \delta_2 \in \mathfrak{g}'$, $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ then*

$$\alpha\big(\delta_1, \Psi(\delta_2)\big) = \alpha\big(\delta_2, \Psi(\delta_1)\big)$$
$$\alpha\big(\lambda_1\delta_1 + \lambda_2\delta_2, m_1\big) = \lambda_1\alpha(\delta_1, m_1) + \lambda_2\alpha(\delta_2, m_1)$$
$$\alpha(\delta_1, m_1a_1 + m_2a_2) = \alpha(\delta_1, m_1)a_1 + \alpha(\delta_1, m_2)a_2$$

for $m_1, m_2 \in M_\Psi$.

Proof. For the first statement, let $\Delta(\delta_1, \delta_2) = \alpha(\delta_1, \Psi(\delta_2)) - \alpha(\delta_2, \Psi(\delta_1))$. With this notation in place one may use the fact that ∇ is torsion-free to get:

$$\begin{aligned}
0 &= \nabla_{\psi(\delta_1)}\Psi(\delta_2) - \nabla_{\psi(\delta_2)}\Psi(\delta_1) - \varphi([\psi(\delta_1), \psi(\delta_2)])\\
&= \nabla_{\psi(\delta_1)}\Psi(\delta_2) - \nabla_{\psi(\delta_2)}\Psi(\delta_1) - \Psi([(\delta_1), (\delta_2)])\\
&= P(\nabla_{\psi(\delta_1)}\Psi(\delta_2)) - P(\nabla_{\psi(\delta_2)}\Psi(\delta_1)) - \Psi([\delta_1, \delta_2]) + \Delta(\delta_1, \delta_2),
\end{aligned}$$

and since the projection P is linear, together with the fact that $P(\Psi([\delta_1, \delta_2])) = \Psi([\delta_1, \delta_2]) \in M_\Psi$, it follows that

$$\begin{aligned}
0 &= P(\nabla_{\psi(\delta_1)}\Psi(\delta_2) - \nabla_{\psi(\delta_2)}\Psi(\delta_1) - \Psi([\delta_1, \delta_2])) + \Delta(\delta_1, \delta_2)\\
&= P(0) + \Delta(\delta_1, \delta_2) = 0 + \Delta(\delta_1, \delta_2) = \Delta(\delta_1, \delta_2).
\end{aligned}$$

For the second and third statements we use the linearity of the connection:

$$\begin{aligned}
\alpha\big(\lambda_1\delta_1 + \lambda_2\delta_2, m_1\big) &= (\mathbb{1} - P)\big(\nabla_{\psi(\lambda_1\delta_1+\lambda_2\delta_2)} m_1\big)\\
&= (\mathbb{1} - P)\big(\lambda_1\nabla_{\psi(\delta_1)} m_1 + \lambda_2\nabla_{\psi(\delta_2)} m_1\big)\\
&= \lambda_1\alpha\big(\delta_1, m_1\big) + \lambda_2\alpha\big(\delta_2, m_1\big)
\end{aligned}$$

and

$$\begin{aligned}
\alpha(\delta_1, m_1a_1 + m_2a_2) &= (\mathbb{1} - P)\left(\nabla_{\delta_1}(m_1a_1 + m_2a_2)\right)\\
&= (\mathbb{1} - P)\left(\nabla_{\delta_1}(m_1a_1) + \nabla_{\delta_1}(m_2a_2)\right)\\
&= \alpha(\delta_1, m_1a_1) + \alpha(\delta_1, m_2a_2).
\end{aligned}$$

Noting that

$$\begin{aligned}
\alpha(\delta_1, m_1a_1) &= \nabla_{\psi(\delta_1)} m_1a_1 - P(\nabla_{\psi(\delta_1)} m_1a_1)\\
&= (\nabla_{\psi(\delta_1)} m_1)a_1 + m_1\psi(\delta_1)(a_1) - P\big((\nabla_{\psi(\delta_1)} m_1)a_1 + m_1\psi(\delta_1)(a_1)\big)\\
&= (\nabla_{\psi(\delta_1)} m_1)a_1 + m_1\psi(\delta_1)(a_1) - P(\nabla_{\psi(\delta_1)} m_1)a_1 - m_1\psi(\delta_1)(a_1)\\
&= (\nabla_{\psi(\delta_1)} m_1 - P(\nabla_{\psi(\delta)} m_1))a_1 = \alpha(\delta_1, m_1)a_1
\end{aligned}$$

and (similarly) that $\alpha(\delta_1, m_2a_2) = \alpha(\delta_1, m_2)a_2$ the proposition now follows. □

Proposition 4.5. *If $\delta \in \mathfrak{g}'$, $m \in M_\Psi$ and $\xi \in M_\Psi^\perp$ then*

$$h\big(A_\xi(\delta), m\big) = h\big(\xi, \alpha(\delta, m)\big).$$

Proof. Since $h(m, \xi) = 0$ one can use that $(C_{\mathcal{A}}, h, \nabla)$ is metric to see that $0 = \psi(\delta)(h(m, \xi)) = h(\nabla_{\psi(\delta)}\xi, m) + h(\xi, \nabla_{\psi(\delta)} m)$. Using that P is an orthogonal projection, it follows that

$$\begin{aligned}
h(A_\xi(\delta), m) &= -h(P(\nabla_{\psi(\delta)}\xi), m)\\
&= -h(\nabla_{\psi(\delta)}\xi, m) = h(\xi, \nabla_{\psi(\delta)} m) = h\big(\xi, \alpha(\delta, m)\big)
\end{aligned}$$

as desired. □

Having considered properties of L, α and A_ξ, let us now show that D_X has the properties of an affine connection; in differential geometry, D_X is usually identified with a connection on the normal bundle of the submanifold.

Proposition 4.6. *If $\delta_1, \delta_2 \in \mathfrak{g}'$, $\xi_1, \xi_2 \in M_\Psi^\perp$, $\lambda \in \mathbb{R}$ and $a \in \mathcal{A}$ then*

(1) $D_{\delta_1}(\xi_1+\xi_2) = D_{\delta_1}\xi_1 + D_{\delta_1}\xi_1$,
(2) $D_{\lambda\delta_1+\delta_2}\xi_1 = \lambda D_{\delta_1}\xi_1 + D_{\delta_2}\xi_1$,
(3) $D_{\delta_1}(\xi_1 a) = (D_{\delta_1}\xi_1)a + \xi_1\psi(\delta_1)(a)$.

Proof. Note that (1) and (2) follows immediately from the linearity of ∇. To prove (3), one computes the left-hand side directly:

$$\begin{aligned} D_{\delta_1}(\xi_1 a) &= \Pi(\nabla_{\psi(\delta_1)}\xi_1 a) = \Pi((\nabla_{\psi(\delta_1)}\xi_1)a + \xi_1\psi(\delta_1)(a)) \\ &= \Pi((\nabla_{\psi(\delta_1)}\xi_1)a) + \Pi(\xi_1\psi(\delta_1)(a)) = (D_{\delta_1}\xi_1)a + \xi_1\psi(\delta_1)(a), \end{aligned}$$

giving the desired result. □

A classical formula in Riemannian geometry is Gauss' equation, which relates the curvature of the ambient space to the curvature of the submanifold. The next result provides a noncommutative analogue.

Proposition 4.7 (*Gauss' Equation*)**.** *Let $\delta_i \in \mathfrak{g}'$, $\partial_i = \psi(\delta_i) \in \mathfrak{g}$, $E_i = \Psi(\delta_i) \in M_\Psi$ and $E_i' = \varphi'(\delta_i) \in M'$ for $i = 1, 2, 3, 4$ (i.e. E_i is an extension of E_i'). Then*

$$\phi\big(h(E_1, R(\partial_3, \partial_4)E_2)\big) = h'\big(E_1', R'(\delta_3, \delta_4)E_2'\big) + \phi\left(h\big(\alpha(\delta_4, E_1), \alpha(\delta_3, E_2)\big)\right) - \phi\left(h\big(\alpha(\delta_3, E_1), \alpha(\delta_4, E_2)\big)\right). \tag{4.3}$$

Proof. Using the result from Proposition 4.3 one gets that

$$\begin{aligned} R'(\delta_3, \delta_4)E_2' &= \nabla_3'\nabla_4'E_2' - \nabla_4'\nabla_3'E_2' - \nabla'_{[\delta_3,\delta_4]}E_2' \\ &= \nabla_3'\widehat{\psi}(\hat{\nabla}_4'E_2) - \nabla_4'\widehat{\psi}(\hat{\nabla}_3'E_2) - \widehat{\psi}(\hat{\nabla}'_{[\delta_3,\delta_4]}E_2) \\ &= \widehat{\psi}\left(\hat{\nabla}_3'\hat{\nabla}_4'E_2 - \hat{\nabla}_4'\hat{\nabla}_3'E_2 - \hat{\nabla}'_{[\delta_3,\delta_4]}E_2\right). \end{aligned}$$

Setting $\hat{R}(\partial_3, \partial_4)E_2 := \hat{\nabla}_3'\hat{\nabla}_4'E_2 - \hat{\nabla}_4'\hat{\nabla}_3'E_2 - \hat{\nabla}'_{[\delta_3,\delta_4]}E_2$ one obtains

$$\begin{aligned} h'\big(E_1', R'(\delta_3, \delta_4)E_2'\big) &= h'\big(\widehat{\psi}(E_1), \widehat{\psi}(\hat{R}(\partial_3, \partial_4)E_2)\big) = \phi(h\big(E_1, \hat{R}(\partial_3, \partial_4)E_2\big)) \\ &= \phi\left(h\big(E_1, \hat{\nabla}_3'\hat{\nabla}_4'E_2 - \hat{\nabla}_4'\hat{\nabla}_3'E_2 - \hat{\nabla}'_{[\delta_3,\delta_4]}E_2\big)\right) \\ &= \phi\left(h\big(E_1, \nabla_3\hat{\nabla}_4'E_2 - \nabla_4\hat{\nabla}_3'E_2 - \nabla_{[\partial_3,\partial_4]}E_2\big)\right), \end{aligned}$$

since $E_1 \in M_\Psi$. Using the fact that $\nabla_i\hat{\nabla}_j'E_k = \nabla_i(\nabla_j E_k - \alpha(\delta_j, E_k))$ one may write

$$h'\big(E_1', R'(\delta_3, \delta_4)E_2'\big) = \phi\left(h\big(E_1, R(\partial_3, \partial_4)E_2 - \nabla_3\alpha(\delta_4, E_2) + \nabla_4\alpha(\delta_3, E_2)\big)\right),$$

and from this it follows immediately that

$$\phi\big(h(E_1, R(\partial_3, \partial_4)E_2)\big) = h'\big(E_1', R'(\delta_3, \delta_4)E_2'\big) + \phi\left(h\big(E_1, \nabla_3\alpha(\delta_4, E_2)\big)\right) - \phi\left(h\big(E_1, \nabla_4\alpha(\delta_3, E_2)\big)\right).$$

Since $(C_\mathcal{A}, h, \nabla)$ is metric it follows that

$$h(E_1, \nabla_{\psi(\delta)}\xi) = -h(\nabla_{\psi(\delta)}E_1, \xi)$$

for $\xi \in M_\Psi^\perp$, implying that

$$\phi\big(h(E_1, R(\partial_3, \partial_4)E_2)\big) = h'\big(E_1', R'(\delta_3, \delta_4)E_2'\big) + \phi\left(h\big(\nabla_4 E_1, \alpha(\delta_3, E_2)\big)\right) - \phi\left(h\big(\nabla_3 E_1, \alpha(\delta_4, E_2)\big)\right),$$

which completes the proof, since

$$h\big(\nabla_4 E_1, \alpha(\delta_3, E_2)\big) = h\big(\alpha(\delta_4, E_1), \alpha(\delta_3, E_2)\big) \quad \text{and} \quad h\big(\nabla_3 E_1, \alpha(\delta_4, E_2)\big) = h\big(\alpha(\delta_3, E_1), \alpha(\delta_4, E_2)\big). \quad \square$$

5. Free real calculi and noncommutative mean curvature

In the examples we shall consider (the noncommutative torus and the noncommutative 3-sphere), M will be a free module with a basis given by the image of a basis of the Lie algebra $\mathfrak{g}$. Needless to say, the fact that M is a free module implies several simplifications. Although it happens for the torus and the 3-sphere that their modules of vector fields are free (i.e. they are parallelizable manifolds), one expects a projective module in general. However, as originally shown in the case of the noncommutative 4-sphere [2], real calculi can provide a way of performing local computations, in which case the (localized) module of vector fields is free.

Definition 5.1. A real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is called *free* if there exists a basis $\partial_1, \dots, \partial_m$ of $\mathfrak{g}$ such that $\varphi(\partial_1), \dots, \varphi(\partial_m)$ is a basis of M as a (right) $\mathcal{A}$-module.

Note that if there exists a basis $\partial_1, \dots, \partial_m$ of $\mathfrak{g}$ such that $\varphi(\partial_1), \dots, \varphi(\partial_m)$ is a basis of M, then $\varphi(\partial'_1), \dots, \varphi(\partial'_m)$ is a basis of M for any basis $\partial'_1, \dots, \partial'_m$ of $\mathfrak{g}$.

Definition 5.2. A real metric calculus $(C_{\mathcal{A}}, h)$ is called *free* if $C_{\mathcal{A}}$ is free and h is invertible.

An immediate consequence of having an invertible metric, is the existence of a Levi-Civita connection.

Proposition 5.3. *Let $(C_{\mathcal{A}}, h)$ be a free real metric calculus. Then there exists a unique affine connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is a pseudo-Riemannian calculus.*

Proof. Let $\{\partial_i\}$ be a basis of $\mathfrak{g}$. Since $C_{\mathcal{A}}$ is free it follows that $E_i = \varphi(\partial_i)$ provide a basis of M. In this basis one gets the components $h_{ij} = h(E_i, E_j)$ of the metric h, and for notational convenience we set $h_{i,[j,k]} := h(E_i, \varphi[\partial_j, \partial_k])$ and define $K_{ijk} \in \mathcal{A}$ as

$$K_{ijk} := \frac{1}{2}\left(\partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij} - h_{i,[j,k]} + h_{j,[k,i]} + h_{k,[i,j]}\right).$$

Now, define the linear functional $\hat{K}_{ij} \in M^*$ by

$$\hat{K}_{ij}(E_k b^k) := K_{ijk} b^k.$$

Since the metric h is invertible, $m_{ij} = \hat{h}^{-1}(\hat{K}_{ij}) \in M$ is well-defined, and

$$\begin{aligned} 2h(m_{ij}, E_k) &= 2\hat{h}(m_{ij})(E_k) = 2\hat{K}_{ij}(E_k) = 2K_{ijk} \\ &= \partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij} - h_{i,[j,k]} + h_{j,[k,i]} + h_{k,[i,j]}. \end{aligned}$$

From Corollary 2.13 it now follows that there exists a connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian, and from Theorem 2.10 it follows that ∇ is unique. □

Given a free real metric calculus $(C_{\mathcal{A}}, h)$ and a basis $\partial_1, \dots, \partial_m$ of $\mathfrak{g}$, we write

$$E_a = \varphi(\partial_a) \qquad h_{ab} = h(E_a, E_b) \qquad [\partial_a, \partial_b] = f_{ab}^c \partial_c$$

with $f_{pq}^r \in \mathbb{R}$, giving $h(E_a, \varphi([\partial_b, \partial_c])) = h_{ar} f_{bc}^r$. The fact that $\hat{h}$ is invertible and $\{E_a\}_{a=1}^m$ is a basis of M, implies that there exists $h^{ab} \in \mathcal{A}$ such that

$$\hat{h}^{-1}(\hat{E}^a) = E_b h^{ba} \quad \Rightarrow \quad h^{ab} = \hat{E}^a\big(\hat{h}^{-1}(\hat{E}^b)\big) = h\big(\hat{h}^{-1}(\hat{E}^a), \hat{h}^{-1}(\hat{E}^b)\big)$$

where $\{\hat{E}^a\}_{a=1}^m$ is the basis of M^* dual to $\{E_a\}_{a=1}^m$. It follows that $(h^{ab})^* = h^{ba}$ and

$$h^{ab} h_{bc} = h_{cb} h^{ba} = \delta^a{}_c \mathbb{1}.$$

For a free real metric calculus,we introduce the *Christoffel symbols* $\Gamma_{bc}^a \in \mathcal{A}$ as the (unique) coefficients $\nabla_b E_c = E_a \Gamma_{bc}^a$. Let us now derive an explicit formula for the Christoffel symbols in terms of the components of the metric. Indeed, by Koszul's formula it follows that

$$h(E_a \Gamma_{bc}^a, E_d) = h(\nabla_b E_c, E_d) = \frac{1}{2}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} - h_{br} f_{cd}^r + h_{cr} f_{db}^r + h_{dr} f_{bc}^r\right),$$

and since the right hand side is hermitian, one obtains

$$h_{da} \Gamma_{bc}^a = \frac{1}{2}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} - h_{br} f_{cd}^r + h_{cr} f_{db}^r + h_{dr} f_{bc}^r\right).$$

Multiplying from the left by h^{pd} gives

$$\Gamma_{bc}^p = \frac{1}{2} h^{pd}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} - h_{br} f_{cd}^r + h_{cr} f_{db}^r\right) + f_{bc}^p \mathbb{1} \tag{5.1}$$

and, in particular, if $[\partial_a, \partial_b] = 0$ for all $a, b = 1, \dots, m$ then

$$\Gamma_{bc}^a = \frac{1}{2} h^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right), \tag{5.2}$$

in correspondence with the classical formula.

Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be free real metric calculi and let $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ be an isometric embedding. Since ψ is injective, it is easy to see that if $\{\delta_i\}_{i=1}^{m'}$ is a basis of $\mathfrak{g}'$, then $\{\Psi(\delta_i)\}_{i=1}^{m'}$ is a basis of M_Ψ, implying that M_Ψ is a free module of rank m'. Let us now proceed to the define mean curvature, as well as minimality, of an embedding of free real metric calculi. Since we are working with extensions of vector fields on the embedded manifold Σ', rather than tangent vectors at points on Σ', it is more natural to consider the restriction (to Σ') of the inner product of the mean curvature vector with an arbitrary vector, rather than the mean curvature vector itself.

Definition 5.4. Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be free real metric calculi and let $(\phi, \psi, \widehat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$ be an isometric embedding. Given a basis $\{\delta_i\}_{i=1}^{m'}$ of $\mathfrak{g}'$, the *mean curvature* $H_{\mathcal{A}'} : M \to \mathcal{A}'$ of the embedding is defined as

$$H_{\mathcal{A}'}(m) = \phi\left(h\big(m, \alpha(\delta_i, \Psi(\delta_j))\big)\right) h'^{ij}, \tag{5.3}$$

giving trivially $H_{\mathcal{A}'}(m) = 0$ for $m \in M_\Psi$. An embedding is called *minimal* if $H_{\mathcal{A}'}(\xi) = 0$ for all $\xi \in M_\Psi^\perp$.

Remark 5.5. Note that the ordering in (5.3) is natural in the following sense. Considering the restriction of the metric h to M_Ψ, given by $h_{ij} = h(\Psi(\delta_i), \Psi(\delta_j))$ and its inverse h^{ij}, the fact that M is a right module gives a natural definition of the mean curvature as

$$H_{\mathcal{A}'}(m) = \phi\left(h\big(m, \alpha(\delta_i, \Psi(\delta_j))h^{ij}\big)\right) = \phi\left(h\big(m, \alpha(\delta_i, \Psi(\delta_j))\big)\right)\phi(h^{ij}) = \phi\left(h\big(m, \alpha(\delta_i, \Psi(\delta_j))\big)\right) h'^{ij},$$

reproducing the formula in Definition 5.4.

Although defined with respect to a basis of $\mathfrak{g}'$, the mean curvature is independent of the choice of basis. Indeed, if we let h'_{ij} and $\tilde{h}'_{ij}$ denote the components of the metric h' with respect to different bases $\{\delta_i\}$ and $\{\tilde{\delta}_i\}$ of $\mathfrak{g}'$, then there exists a (real) invertible matrix A such that $\tilde{h}' = Ah'A^T$, or equivalently $\tilde{h}'_{ij} = A^k{}_i h'_{kl} A^l{}_j$. Consequently,

$$(\tilde{h}')^{ij} = (A^{-1})^i{}_k h'^{kl} (A^{-1})^j{}_l$$

and it follows that the mean curvature calculated using the basis $\{\tilde{\delta}_i\}$ is

$$\begin{aligned}
H_{\mathcal{A}'}(m) &= \phi\left(h\big(m, \alpha(\tilde{\delta}_i, \Psi(\tilde{\delta}_j))\big)\right)(\tilde{h}')^{ij} \\
&= \phi\left(h\big(m, \alpha(A^k{}_i\delta_k, \Psi(A^l{}_j\delta_l))\big)\right)(A^{-1})^i{}_m h'^{mn}(A^{-1})^j{}_n \\
&= A^k{}_i(A^{-1})^i{}_m A^l{}_j (A^{-1})^j{}_n \phi\left(h\big(m, \alpha(\delta_k, \Psi(\delta_l))\big)\right) h'^{mn} \\
&= \phi\left(h\big(m, \alpha(\delta_k, \Psi(\delta_l))\big)\right)(h')^{kl}
\end{aligned}$$

showing that the definition of $H_{\mathcal{A}'}$ is indeed basis independent.

Let us end this section by noting that it is straight-forward to define the gradient, divergence and Laplace operator for free real metric calculi.

Definition 5.6. Let $(C_{\mathcal{A}}, h)$ be a free real metric calculus and let ∇ denote the Levi-Civita connection. Moreover, let $\{\partial_a\}_{a=1}^m$ be a basis of $\mathfrak{g}$ and set $E_a = \varphi(\partial_a)$. The *gradient* $\mathrm{grad} : \mathcal{A} \to M$ is defined as

$$\mathrm{grad}(a) = E_a h^{ab} \partial_b a$$

for $a \in \mathcal{A}$. The *divergence* $\mathrm{div} : M \to \mathcal{A}$ is defined as

$$\mathrm{div}(m) = (\nabla_{\partial_a} m)^a$$

for $m \in M$, where $\nabla_{\partial_a} m = E_b(\nabla_{\partial_a} m)^b$. The Laplace operator $\Delta : \mathcal{A} \to \mathcal{A}$ is defined as $\Delta(a) = \mathrm{div}\big(\mathrm{grad}(a)\big)$ for $a \in \mathcal{A}$.

Note that it is easy to check that the above definitions are independent of the choice of basis of $\mathfrak{g}$.

6. Minimal tori in the 3-sphere

The 3-sphere has a rich flora of minimal surfaces, and the fact that minimal surfaces of arbitrary genus exist in S^3 is a famous result by Lawson [6]. As an illustration of the concepts we have developed, as well as being our motivating example, we shall consider the noncommutative torus minimally embedded in the noncommutative 3-sphere. However, rather than the round metric on S^3, we will consider more general metrics. Therefore, let us start by recalling the classical situation.

The Clifford torus T^2 is embedded in $S^3 \subseteq \mathbb{R}^4$ via

$$\vec{x} = \frac{1}{\sqrt{2}}(x^1, x^2, x^3, x^4) = (\cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2).$$

With $\delta_1 = \partial_{\varphi_1}$ and $\delta_2 = \partial_{\varphi_2}$, the tangent space at a point is spanned by

$$\delta_1\vec{x} = \frac{1}{\sqrt{2}}(-\sin\varphi_1, \cos\varphi_1, 0, 0) = (-x^2, x^1, 0, 0)$$

$$\delta_2\vec{x} = \frac{1}{\sqrt{2}}(0, 0, -\sin\varphi_2, \cos\varphi_2) = (0, 0, -x^4, x^3).$$

The 3-sphere is embedded in $\mathbb{C}^2$ via

$$\begin{aligned} z &= e^{i\xi_1}\sin\eta \\ w &= e^{i\xi_2}\cos\eta, \end{aligned}$$

and with $\partial_1 = \partial_{\xi_1}$ and $\partial_2 = \partial_{\xi_2}$ the tangent space at a point with $0 < \xi_1, \xi_2 < 2\pi$ and $0 < \eta < \pi/2$ is spanned by

$$\begin{aligned} E_1 &= \partial_1(x^1, x^2, x^3, x^4) = (-x^2, x^1, 0, 0) \\ E_2 &= \partial_2(x^1, x^2, x^3, x^4) = (0, 0, -x^4, x^3) \\ E_\eta &= \partial_\eta(x^1, x^2, x^3, x^4) = (\cos\xi_1\cos\eta, \sin\xi_1\cos\eta, -\cos\xi_2\sin\eta, -\sin\xi_2\sin\eta). \end{aligned}$$

The standard metric on S^3 is given by

$$g = \begin{pmatrix} \sin^2\eta & 0 & 0 \\ 0 & \cos^2\eta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and for $H \in C^\infty(S^3)$ such that $H > 0$ we consider the perturbed metric

$$\tilde{g} = H \begin{pmatrix} \sin^2\eta & 0 & 0 \\ 0 & \cos^2\eta & 0 \\ 0 & 0 & 1 \end{pmatrix} H.$$

Let us now proceed to determine the Levi-Civita connection on $(S^3, \tilde{g})$. The Christoffel symbols are computed using

$$\Gamma^i_{jk} = \frac{1}{2}\tilde{g}^{il}\left(\partial_j\tilde{g}_{kl} + \partial_k\tilde{g}_{jl} - \partial_l\tilde{g}_{jk}\right),$$

giving

$$\begin{aligned} \Gamma^1_{jk} &= \begin{pmatrix} \partial_1(\ln H) & \partial_2(\ln H) & \partial_\eta(\ln H) + \cot\eta \\ \partial_2(\ln H) & -\partial_1(\ln H)\cot^2\eta & 0 \\ \partial_\eta(\ln H) + \cot\eta & 0 & -\partial_1(\ln H)\csc^2\eta \end{pmatrix} \\ \Gamma^2_{jk} &= \begin{pmatrix} -\partial_2(\ln H)\tan^2\eta & \partial_1(\ln H) & 0 \\ \partial_1(\ln H) & \partial_2(\ln H) & \partial_\eta(\ln H) - \tan\eta \\ 0 & \partial_\eta(\ln H) - \tan\eta & -\partial_2(\ln H)\sec^2\eta \end{pmatrix} \\ \Gamma^3_{jk} &= \begin{pmatrix} -\partial_\eta(\ln H)\sin^2\eta - \sin\eta\cos\eta & 0 & \partial_1(\ln H) \\ 0 & -\partial_\eta(\ln H)\cos^2\eta + \sin\eta\cos\eta & \partial_2(\ln H) \\ \partial_1(\ln H) & \partial_2(\ln H) & \partial_\eta(\ln H) \end{pmatrix}. \end{aligned}$$

Thus, the Levi-Civita connection is explicitly given as

$$\begin{aligned} \nabla_1\partial_1 &= \partial_1(\ln H)\partial_1 - \partial_2(\ln H)\tan^2\eta\,\partial_2 - (\partial_\eta(\ln H)\sin^2\eta + \sin\eta\cos\eta)\partial_\eta \\ \nabla_1\partial_2 &= \partial_2(\ln H)\partial_1 + \partial_1(\ln H)\partial_2 = \nabla_2\partial_1 \\ \nabla_1\partial_\eta &= (\partial_\eta(\ln H) + \cot\eta)\partial_1 + \partial_1(\ln H)\partial_\eta = \nabla_\eta\partial_1 \\ \nabla_2\partial_2 &= -\partial_1(\ln H)\cot^2\eta\,\partial_1 + \partial_2(\ln H)\partial_2 + (\sin\eta\cos\eta - \partial_\eta(\ln H)\cos^2\eta)\partial_\eta \\ \nabla_2\partial_\eta &= (\partial_\eta(\ln H) - \tan\eta)\partial_2 + \partial_2(\ln H)\partial_\eta = \nabla_\eta\partial_2 \\ \nabla_\eta\partial_\eta &= -\partial_1(\ln H)\csc^2\eta\,\partial_1 - \partial_2(\ln H)\sec^2\eta\,\partial_2 + \partial_\eta(\ln H)\partial_\eta. \end{aligned}$$

6.1. Embedding the torus into the 3-sphere

For fixed $\eta_0 \in (0, \pi/2)$, let $f_{\eta_0} : T^2 \to (S^3, \tilde{g})$ denote the embedding

$$f_{\eta_0} : (\cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2) \mapsto (e^{i\varphi_1}\sin\eta_0, e^{i\varphi_2}\cos\eta_0),$$

The induced metric on the torus is given by

$$g_{T^2} = \tilde{H}\begin{pmatrix} \sin^2\eta_0 & 0 \\ 0 & \cos^2\eta_0 \end{pmatrix}\tilde{H},$$

where $\tilde{H}(\varphi_1, \varphi_2) = H(\varphi_1, \varphi_2, \eta_0)$. The unit normal of T^2 is $N = \tilde{H}^{-1}\partial_\eta$, and one writes the second fundamental form α as:

$$\alpha(\delta_1, \delta_1) = -\tilde{H}(\partial_\eta(\ln H)|_{\eta_0}\sin^2\eta_0 + \sin\eta_0\cos\eta_0)N$$

$$\alpha(\delta_1,\delta_2)=\alpha(\delta_2,\delta_1)=0$$
$$\alpha(\delta_2,\delta_2)=\tilde{H}(\sin\eta_0\cos\eta_0-\partial_\eta(\ln H)|_{\eta_0}\cos^2\eta_0)N.$$

Calculating the mean curvature of T^2 in $(S^3,\tilde{g})$ yields

$$H_{T^2}=\frac{1}{2}\tilde{g}\big(N,\alpha\big(\delta_i,\delta_j\big)\big)g^{ij}_{T^2}=-\tilde{H}^{-1}(\cot 2\eta_0+\partial_\eta(\ln H)|_{\eta_0})$$

and it follows that T^2 is minimally embedded in $(S^3,\tilde{g})$ if $\partial_\eta(\ln H)|_{\eta_0}=-\cot 2\eta_0$; for instance, one might choose

$$H(\xi_1,\xi_2,\eta)=\exp\left(p(\xi_1,\xi_2)-\frac{r(\eta)\cot 2\eta_0}{r'(\eta_0)}\right)$$

for arbitrary functions p and r, with r having a nonzero derivative at $\eta=\eta_0$. In the classical case, when $H=1$, the embedding is minimal if $\cot 2\eta_0=0$, i.e. $\eta_0=\pi/4$.

7. The noncommutative minimal torus

Let us now apply the framework for noncommutative embeddings to the case of the noncommutative torus and the noncommutative 3-sphere. We shall start by recalling their definitions, as well as their corresponding real metric calculi. For more details, we refer to [3] (however, where only the standard metric on the 3-sphere was considered).

7.1. The noncommutative torus

The noncommutative torus T^2_θ is a unital $*$-algebra generated by the unitary elements U,V subject to the relation $VU=qUV$, with $q=e^{2\pi i\theta}$. Introducing the hermitian elements

$$X^1=\frac{1}{2}(U+U^*)\qquad X^2=\frac{1}{2i}(U-U^*)$$
$$X^3=\frac{1}{2}(V+V^*)\qquad X^4=\frac{1}{2i}(V-V^*)$$

gives $\mathbb{1}=UU^*=(X^1)^2+(X^2)^2$ and $\mathbb{1}=VV^*=(X^3)^2+(X^4)^2$. In analogy with the geometrical setting, let M' be the (right) submodule of $(T^2_\theta)^4$ generated by

$$e_1=(-X^2,X^1,0,0)$$
$$e_2=(0,0,-X^4,X^3).$$

We note that M' is a free T^2_θ-module, since e_1 and e_2 form a basis for M':

$$e_1a+e_2b=0\quad\Rightarrow\quad(-X^2a,X^1a,-X^4b,X^3b)=(0,0,0,0)$$
$$\Rightarrow\quad\begin{cases}\big((X^1)^2+(X^2)^2\big)\,a=UU^*a=a=0\\ \big((X^3)^2+(X^4)^2\big)\,b=VV^*b=b=0.\end{cases}$$

Next, we let $\mathfrak{g}'$ be the (real) Lie algebra generated by the two hermitian derivations δ_1,δ_2, given by

$$\delta_1U=iU\qquad\delta_1V=0$$
$$\delta_2U=0\qquad\delta_2V=iV,$$

satisfying $[\delta_1,\delta_2]=0$. Finally, let $\varphi':\mathfrak{g}'\to M'$ with $\varphi'(\delta_j)=e_j$ for $j=1,2$ and extended by $\mathbb{R}$-linearity, which implies that M' is generated by $\varphi'(\mathfrak{g}')$ as a T^2_θ-module. Hence, we have shown that $C_{T^2_\theta}=(T^2_\theta,\mathfrak{g}',M',\varphi')$ is a real calculus over the noncommutative torus.

As a first illustration of a real calculus homomorphism, let us construct a family of automorphisms of T^2_θ as follows. Let $a,b,c,d\in\mathbb{Z}$ be given such that $ad-bc=1$, and let $\alpha:T^2_\theta\to T^2_\theta$ be the automorphism given by

$$\alpha(U)=U^aV^b,$$
$$\alpha(V)=U^cV^d,$$

with inverse

$$\alpha^{-1}(U)=q^{\frac{1}{2}bd(a-c-1)}U^dV^{-b}$$
$$\alpha^{-1}(V)=q^{\frac{1}{2}ac(d-b-1)}U^{-c}V^a.$$

Once the automorphism α is established, it is a simple task to find a real calculus automorphism from $C_{T^2_\theta}$ to itself by using Proposition 3.4 to find the required Lie algebra homomorphism. Indeed, Proposition 3.4 implies that

$$\psi(\delta_1)(U)=\alpha^{-1}\circ\delta_1\circ\alpha(U)=iaU\qquad\qquad\psi(\delta_2)(U)=\alpha^{-1}\circ\delta_2\circ\alpha(U)=ibU$$

$$\psi(\delta_1)(V) = \alpha^{-1} \circ \delta_1 \circ \alpha(V) = icV \qquad \psi(\delta_2)(V) = \alpha^{-1} \circ \delta_2 \circ \alpha(V) = idV,$$

giving

$$\psi(\delta_1) = a\delta_1 + c\delta_2 \quad \text{and} \quad \psi(\delta_2) = b\delta_1 + d\delta_2.$$

From the compatibility conditions $\widehat{\psi}(\Psi(\delta_i)) = \varphi'(\delta_i)$ one obtains

$$\widehat{\psi}\big(e_1 a + e_2 c\big) = e_1 \quad \text{and} \quad \widehat{\psi}\big(e_1 b + e_2 d\big) = e_2,$$

implying that

$$\widehat{\psi}(e_1) = e_1 d - e_2 c \quad \text{and} \quad \widehat{\psi}(e_2) = -e_1 b + e_2 a.$$

This ensures that $(\alpha, \psi, \widehat{\psi})$ as defined above is an automorphism of the real calculus $C_{T^2_\theta}$.

7.2. The noncommutative 3-sphere

The noncommutative 3-sphere S^3_θ is the unital $*$-algebra generated by Z, Z^*, W, W^* satisfying

$$\begin{array}{llll} WZ = qZW & W^*Z = \bar{q}ZW^* & WZ^* = \bar{q}Z^*W & W^*Z^* = qZ^*W^* \\ Z^*Z = ZZ^* & W^*W = WW^* & WW^* = \mathbb{1} - ZZ^*, & \end{array}$$

with $q = e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$.

Similar to the case of T^2_θ, we introduce

$$\begin{aligned} X^1 &= \frac{1}{2}(Z + Z^*) & X^2 &= \frac{1}{2i}(Z - Z^*) \\ X^3 &= \frac{1}{2}(W + W^*) & X^4 &= \frac{1}{2i}(W - W^*) \\ |Z|^2 &= ZZ^* & |W|^2 &= WW^*, \end{aligned}$$

giving $|Z|^2 = (X^1)^2 + (X^2)^2$ and $|W|^2 = (X^3)^2 + (X^4)^2$; recall that $|Z|^2$ and $|W|^2$ are in the center of S^3_θ and, furthermore, neither of them is a zero divisor (cf. [3]). Let us now construct a real metric calculus for S^3_θ, closely related to the Hopf fibration of the 3-sphere.

Recall from Section 6 that S^3 can be given in terms of the coordinates (ξ_1, ξ_2, η), and we noted that the tangent plane at a given point is spanned by the three vectors

$$\begin{aligned} E_1 &= \partial_1(x^1, x^2, x^3, x^4) = (-x^2, x^1, 0, 0) \\ E_2 &= \partial_2(x^1, x^2, x^3, x^4) = (0, 0, -x^4, x^3) \\ E_\eta &= \partial_\eta(x^1, x^2, x^3, x^4) = (\cos\xi_1 \cos\eta, \sin\xi_1 \cos\eta, -\cos\xi_2 \sin\eta, -\sin\xi_2 \sin\eta). \end{aligned}$$

For the noncommutative analogue, it is apparent how to choose E_1 and E_2, but the analogue of E_η is less clear. Therefore, instead of ∂_η, one considers the derivation $\partial_3 = |z||w|\partial_\eta$, giving

$$E_3 = \partial_3(x^1, x^2, x^3, x^4) = (x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2),$$

which can be used together with E_1 and E_2 to span the tangent space.

Returning to the complex embedding coordinates z and w in $\mathbb{C}^2$, one finds

$$\partial_1(z) = iz \qquad \partial_1(w) = 0 \tag{7.1}$$
$$\partial_2(z) = 0 \qquad \partial_2(w) = iw \tag{7.2}$$
$$\partial_3(z) = z|w|^2 \qquad \partial_3(w) = -w|z|^2, \tag{7.3}$$

and with respect to the basis $\{E_1, E_2, E_3\}$ of the tangent space of S^3, the induced standard metric becomes

$$(h_{ab}) = (h(E_a, E_b)) = \begin{pmatrix} |z|^2 & 0 & 0 \\ 0 & |w|^2 & 0 \\ 0 & 0 & |z|^2|w|^2 \end{pmatrix}. \tag{7.4}$$

Motivated by the above considerations, let M the submodule of the free (right) module $(S^3_\theta)^4$ generated by $\{E_1, E_2, E_3\}$, where

$$E_1 = (-X^2, X^1, 0, 0)$$
$$E_2 = (0, 0, -X^4, X^3)$$
$$E_3 = (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2).$$

In [3] it was shown that M is a free module with a basis $\{E_1, E_2, E_3\}$ and that there exist hermitian derivations $\partial_1, \partial_2, \partial_3$ such that

$$\begin{aligned} &\partial_1(Z) = iZ && \partial_1(W) = 0 \\ &\partial_2(Z) = 0 && \partial_2(W) = iW \\ &\partial_3(Z) = Z|W|^2 && \partial_3(W) = -W|Z|^2, \end{aligned}$$

with $[\partial_a, \partial_b] = 0$ for $a, b = 1, 2, 3$. Let $\mathfrak{g}$ be the (real) Lie algebra generated by ∂_1, ∂_2 and ∂_3, and define $\varphi : \mathfrak{g} \to M$ as the linear map (over $\mathbb{R}$) given by $\varphi(\partial_a) = E_a$ for $a = 1, 2, 3$. From the above considerations, it follows that $C_{S^3_\theta} = (S^3_\theta, \mathfrak{g}, M, \varphi)$ is a real calculus over S^3_θ.

Now, let us proceed to construct a real metric calculus over S^3_θ, in which we shall minimally embed the noncommutative torus. In analogy with Section 6, we choose the hermitian form $h : M \times M \to M$

$$h(m, n) = \sum_{a,b=1}^{3} (m^a)^* h_{ab} n^b,$$

where $m = E_a m^a$, $n = E_b n^b$ and

$$(h_{ab}) = H \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix} H^*,$$

where $H \in S^3_\theta$ is chosen such that HH^* is invertible. Since neither $|Z|^2$ nor $|W|^2$ is a zero divisor, the metric is clearly nondegenerate; furthermore, h_{ab} is hermitian for $a, b = 1, 2, 3$. We conclude that $(C_{S^3_\theta}, h)$ is a real metric calculus.

Next, let us construct a metric and torsion-free connection on $(C_{S^3_\theta}, h)$. In order to achieve this, we will localize the algebra at $|Z|^2$ and $|W|^2$. That is, one extends the algebra of the noncommutative 3-sphere by the inverses of $|Z|^2$ and $|W|^2$. (In principle, for a well-behaved noncommutative localization, one has to check the so called Ore conditions, but since $|Z|^2$ and $|W|^2$ are central, these are trivially fulfilled.) The resulting algebra is denoted by $S^3_{\theta,\mathrm{loc}}$. It is straight-forward to extend the real metric calculus $(C_{S^3_\theta}, h)$ to a real metric calculus $(C_{S^3_{\theta,\mathrm{loc}}}, h)$ (cf. [2] where a similar construction was carried out for the 4-sphere).

Proposition 7.1. *There exists a unique affine connection ∇ such that $(C_{S^3_{\theta,loc}}, h, \nabla)$ is a pseudo-Riemannian calculus with*

$$\begin{aligned} \nabla_1 E_1 &= E_1 H_1 - E_2 |Z|^2 |W|^{-2} H_2 - E_3(|W|^{-2} H_3 + \mathbb{1}) \\ \nabla_1 E_2 &= \nabla_2 E_1 = E_1 H_2 + E_2 H_1 \\ \nabla_1 E_3 &= \nabla_3 E_1 = E_1(H_3 + |W|^2) + E_3 H_1 \\ \nabla_2 E_2 &= -E_1 |W|^2 |Z|^{-2} H_1 + E_2 H_2 + E_3(\mathbb{1} - |Z|^{-2} H_3) \\ \nabla_2 E_3 &= \nabla_3 E_2 = E_2(H_3 - |Z|^2) + E_3 H_2 \\ \nabla_3 E_3 &= -E_1 |W|^2 H_1 - E_2 |Z|^2 H_2 + E_3(H_3 + |W|^2 - |Z|^2), \end{aligned}$$

where $H_a = \frac{1}{2}(HH^*)^{-1}\partial_a(HH^*)$ for $a = 1, 2, 3$.

Proof. Since h is invertible, $(C_{S^3_{\theta,\mathrm{loc}}}, h)$ is a free real metric calculus, implying that the Levi-Civita connection ∇ exists. Moreover, $[\partial_a, \partial_b] = 0$ for all $a, b \in \{1, 2, 3\}$, and thus it follows that the Christoffel symbols for ∇ can be calculated directly using (5.2). For instance,

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2} h^{11} \partial_1 h_{11} = \frac{1}{2}(HH^*)^{-1}\partial_1(HH^*) = H_1 \\ \Gamma^2_{11} &= \frac{1}{2} h^{22}(-\partial_2 h_{11}) = -\frac{1}{2}|Z|^2|W|^{-2}(HH^*)^{-1}\partial_2(HH^*) = -|Z|^2|W|^{-2}H_2 \\ \Gamma^3_{11} &= \frac{1}{2} h^{33}(-\partial_3 h_{11}) = -\frac{1}{2}|W|^{-2}(HH^*)^{-1}\partial_3(HH^*) - \mathbb{1} = -|W|^{-2}H_3 - \mathbb{1}, \end{aligned}$$

giving

$$\nabla_1 E_1 = E_1 H_1 - E_2 |Z|^2 |W|^{-2} H_2 - E_3(|W|^{-2} H_3 + \mathbb{1}).$$

The remaining Christoffel symbols are computed in a completely analogous way. $\square$

7.3. An embedding of the noncommutative torus

Finally, we will now construct an embedding $(\phi, \psi, \widehat{\psi}) : C_{S^3_{\theta,\mathrm{loc}}} \to C_{T^2_\theta}$. To this end, we set

$$\phi(Z) = \lambda U$$
$$\phi(W) = \mu V,$$

where λ and μ are complex nonzero constants such that $|\lambda|^2 + |\mu|^2 = 1$. It is easy to verify that with these conditions ϕ is a $*$-algebra homomorphism. Moreover, since λ and μ are chosen to be nonzero it means that ϕ is surjective as well. With this choice of ϕ it follows that a Lie algebra homomorphism $\psi : \mathfrak{g}' \to \mathfrak{g}$ compatible with ϕ is given by

$$\psi(\delta_1) = \partial_1 \quad \text{and} \quad \psi(\delta_2) = \partial_2,$$

and M_Ψ is the submodule of M generated by E_1 and E_2. Furthermore, with

$$\widehat{\psi}(E_1) = e_1 \quad \text{and} \quad \widehat{\psi}(E_2) = e_2$$

$(\phi, \psi, \widehat{\psi})$ is a real calculus homomorphism. This choice of $(\phi, \psi, \widehat{\psi})$ gives an embedding of $C_{T^2_\theta}$ into $C_{S^3_{\theta,\mathrm{loc}}}$, since by choosing $\tilde{M}$ to be the submodule of M generated by E_3 one gets that $M = M_\Psi \oplus \tilde{M}$.

Let us now find the induced metric h' such that $(\phi, \psi, \widehat{\psi}) : (C_{T^2_\theta}, h') \to (C_{S^3_{\theta,\mathrm{loc}}}, h)$ is an embedding of real metric calculi. Since M' has a basis $\{e_1, e_2\}$ it suffices to calculate $h'(e_i, e_j)$ for $i, j = 1, 2$:

$$h'(e_1, e_1) = \phi(h(E_1, E_1)) = \phi((HH^*)|Z|^2) = |\lambda|^2(\tilde{H}\tilde{H}^*)$$
$$h'(e_1, e_2) = h'(e_2, e_1) = \phi(h(E_1, E_2)) = 0$$
$$h'(e_2, e_2) = \phi(h(E_2, E_2)) = \phi((HH^*)|W|^2) = |\mu|^2(\tilde{H}\tilde{H}^*),$$

with $\tilde{H} = \phi(H)$; it is easy to check that h' is an invertible metric on M', implying that $(C_{T^2_\theta}, h')$ is indeed a free real metric calculus. Moreover, it is clear that $\tilde{M} = M_\Psi^\perp$.

Since M and M' are free modules, Proposition 4.3 can be used to quickly determine the Levi-Civita connection ∇' for $(C_{T^2_\theta}, h')$:

$$\nabla'_1 e_1 = \widehat{\psi}(L(\delta_1, \Psi(\delta_1))) = e_1\tilde{H}_1 - e_2\tilde{H}_2|\lambda|^2|\mu|^{-2}$$
$$\nabla'_1 e_2 = \nabla'_2 e_1 = \widehat{\psi}(L(\delta_1, \Psi(\delta_2))) = e_1\tilde{H}_2 + e_2\tilde{H}_1$$
$$\nabla'_2 e_2 = \widehat{\psi}(L(\delta_2, \Psi(\delta_2))) = -e_1\tilde{H}_1|\lambda|^{-2}|\mu|^2 + e_2\tilde{H}_2,$$

where $\tilde{H}_i = \phi(H_i)$ for $i = 1, 2, 3$. Consequently, one obtains the second fundamental form as

$$\alpha(\delta_1, \Psi(\delta_1)) = -E_3(|W|^{-2}H_3 + \mathbb{1})$$
$$\alpha(\delta_1, \Psi(\delta_2)) = \alpha(\delta_2, \Psi(\delta_1)) = 0$$
$$\alpha(\delta_2, \Psi(\delta_2)) = E_3(\mathbb{1} - |Z|^{-2}H_3),$$

giving the mean curvature

$$\begin{aligned} H_{T^2_\theta}(m) =& \phi\left(h\big(m, \alpha(\delta_1, \Psi(\delta_1))\big)\right)(h')^{11} + \phi\left(h\big(m, \alpha(\delta_2, \Psi(\delta_2))\big)\right)(h')^{22} \\ &= \phi\left(h\big(m, -E_3(|W|^{-2}H_3 + \mathbb{1})\big)\right)|\lambda|^{-2}(\tilde{H}\tilde{H}^*)^{-1} + \phi\left(h\big(m, E_3(\mathbb{1} - |Z|^{-2}H_3)\big)\right)|\mu|^{-2}(\tilde{H}\tilde{H}^*)^{-1} \\ &= \phi\left(h\big(m, E_3\big)\right)\left(|\mu|^{-2} - |\lambda|^{-2} - 2|\lambda|^{-2}|\mu|^{-2}\tilde{H}_3\right)(\tilde{H}\tilde{H}^*)^{-1}. \end{aligned}$$

For the embedded torus, $M_\Psi^\perp$ is the submodule of M generated by the basis element E_3. Hence, the mean curvature is zero if

$$\begin{aligned} 0 = H_{T^2_\theta}(E_3) &= (\tilde{H}\tilde{H}^*)|\lambda|^2|\mu|^2\left(|\mu|^{-2} - |\lambda|^{-2} - 2|\lambda|^{-2}|\mu|^{-2}\tilde{H}_3\right)(\tilde{H}\tilde{H}^*)^{-1} \\ &= (\tilde{H}\tilde{H}^*)\left(|\lambda|^2 - |\mu|^2 - 2\tilde{H}_3\right)(\tilde{H}\tilde{H}^*)^{-1} \\ &= \left(|\lambda|^2 - |\mu|^2\right)\mathbb{1} - 2(\tilde{H}\tilde{H}^*)\tilde{H}_3(\tilde{H}\tilde{H}^*)^{-1} \\ &= \left(|\lambda|^2 - |\mu|^2\right)\mathbb{1} - \phi(\partial_3(HH^*))(\tilde{H}\tilde{H}^*)^{-1}, \end{aligned}$$

implying that the embedding of $(C_{T^2_\theta}, h')$ into $(C_{S^3_\theta}, h)$ is minimal if and only if

$$\phi\big(\partial_3(HH^*)\big) = (|\lambda|^2 - |\mu|^2)\phi(HH^*).$$

In the special case where $\phi(\partial_3(HH^*)) = 0$, the embedding is minimal if $|\lambda| = |\mu| = 1/\sqrt{2}$ (in analogy with the classical case). For the same values of $|\lambda|$ and $|\mu|$ one may also choose, e.g., $H = ZW$ giving $HH^* = |Z|^2|W|^2$ and

$$\phi\big(\partial_3 HH^*\big) = 2|\lambda|^2|\mu|^2\big(|\mu|^2 - |\lambda|^2\big) = 0.$$

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