Boundary Value Problems for Nonlinear Elliptic Equations in Divergence Form

Abubakar Mwasa

Linköping Studies in Science and Technology.
Dissertations No. 2128

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Department of Mathematics
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in Divergence Form**

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Abstract

The thesis consists of three papers focussing on the study of nonlinear elliptic partial differential equations in a nonempty open subset Ω of the n -dimensional Euclidean space $\mathbf{R}^n$. We study the existence and uniqueness of the solutions, as well as their behaviour near the boundary of Ω . The behaviour of the solutions at infinity is also discussed when Ω is unbounded.

In Paper A, we consider a mixed boundary value problem for the p -Laplace equation $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ in an open infinite circular half-cylinder with prescribed Dirichlet boundary data on a part of the boundary and zero Neumann boundary data on the rest. By a suitable transformation of the independent variables, this mixed problem is transformed into a Dirichlet problem for a degenerate (weighted) elliptic equation on a bounded set. By analysing the transformed problem in weighted Sobolev spaces, it is possible to obtain the existence of continuous weak solutions to the mixed problem, both for Sobolev and for continuous data on the Dirichlet part of the boundary. A characterisation of the boundary regularity of the point at infinity is obtained in terms of a new variational capacity adapted to the cylinder.

In Paper B, we study Perron solutions to the Dirichlet problem for the degenerate quasilinear elliptic equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in a bounded open subset of $\mathbf{R}^n$. The vector-valued function $\mathcal{A}$ satisfies the standard ellipticity assumptions with a parameter $1 < p < \infty$ and a p -admissible weight w . For general boundary data, the Perron method produces a lower and an upper solution, and if they coincide then the boundary data are called resolute. We show that arbitrary perturbations on sets of weighted p -capacity zero of continuous (and quasicontinuous Sobolev) boundary data f are resolute, and that the Perron solutions for f and such perturbations coincide. As a consequence, it is also proved that the Perron solution with continuous boundary data is the unique bounded continuous weak solution that takes the required boundary data outside a set of weighted p -capacity zero.

Some results in Paper C are a generalisation of those in Paper A, extended to quasilinear elliptic equations of the form $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$. Here, results from Paper B are used to prove the existence and uniqueness of continuous weak solutions to the mixed boundary value problem for continuous Dirichlet data. Regularity of the boundary point at infinity for the equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ is characterised by a Wiener type criterion. We show that sets of Sobolev p -capacity zero are removable for the solutions and also discuss the behaviour of the solutions at ∞ . In particular, a certain trichotomy is proved, similar to the Phragmén–Lindelöf principle.

Populärvetenskaplig sammanfattning

Många problem i fysik och andra naturvetenskaper modelleras av partiella differentialekvationer. Ett av dessa problem är att minimera någon form av energi, givet vissa randvärden. Detta kan modelleras med elliptiska partiella differentialekvationer som t.ex. laplaceekvationen $\Delta u = 0$. Funktionen u beskriver densiteten hos någon fysikalisk kvantitet i jämvikt, t.ex. elektrostatisk potential eller temperaturfördelningen i en kropp. Laplaceekvationen är också användbar för att studera stationära rotationsfria flöden av Newtonska vätskor, och är en prototyp för linjära elliptiska partiella differentialekvationer.

Men många problem i naturvetenskaperna är ickeinjära och modelleras därför bättre med ickeinjära partiella differentialekvationer. Ett vanligt exempel är den ickeinjära p -laplaceekvationen som minimerar p -energiintegralen bland funktioner med givna randvärden. Den ekvationen kan beskriva flöden hos icke-Newtonska vätskor så som färg, glaciärer och smält plast. Detta kan användas vid formgivning av gjutformar för plastprodukter, vid beskrivning av glaciärflöden, och även inom bildbehandling.

De flesta lösningar till sådana ekvationer är inte tillräckligt glatta för att vara lösningar i klassisk mening. De måste snarare förstås som någon form av generaliserade lösningar, så som svaga lösningar definierade med hjälp av så kallade sobolevrum.

Med hjälp av dessa rum får man en klass av generaliserade lösningar utan att behöva hitta explicita klassiska lösningar. Då behöver man också försöka besvara följande frågor. Existens: Finns dessa lösningar, och i så fall under vilka givna standardförutsättningar? Entydighet: Är lösningarna entydiga? Man kan också undersöka hur lösningarna beter sig i olika områden.

I den här avhandlingen studerar vi några av dessa aspekter hos generaliserade lösningar till vissa ickeinjära elliptiska partiella differentialekvationer både i begränsade och obegränsade områden. Speciellt studeras existens, entydighet och beteende hos generaliserade lösningar i en oändlig cirkulär halvcylinder med givna randdata. Vi beskriver också hur små ändringar i randdata påverkar lösningarna. Resultaten bidrar till den allmänna förståelsen av dessa lösningars beteende.

General summary

Many problems in physics and other natural sciences are modelled by partial differential equations. One of the physical problems is to minimise energy of some kind, subject to boundary conditions. This can be modelled by elliptic partial differential equations such as the Laplace equation $\Delta u = 0$. The function u describes the density of some physical quantity in equilibrium, for example the electrostatic potential or the temperature distribution in a body. The Laplace equation is also useful when describing the steady irrotational flow of Newtonian fluids and it is a prototype of linear elliptic partial differential equations.

However, many problems in natural sciences are nonlinear and as such, some are better described by nonlinear partial differential equations. A common example is the nonlinear p -Laplace equation which minimises the p -energy integral among functions with prescribed boundary data. This equation may describe the flow of non-Newtonian fluids such as paint, glaciers and molten plastics. It can be used in the design of molds for plastic products and in the prediction of glacier movement as well as in image analysis.

Most solutions to such equations are not smooth enough to be solutions in the classical sense, but rather are understood in some generalised way as weak solutions defined by means of the so-called Sobolev spaces.

With these spaces, a class of generalised solutions is established without necessarily finding classical solutions explicitly. This involves also answering the following questions. Existence: Do these solutions exist and if so, under what standard conditions? Uniqueness: Are they the only solutions? And possibly one may ask how do such solutions behave in certain regions?

In this thesis, we explore some of these aspects about the generalised solutions for some nonlinear elliptic partial differential equations in both bounded and unbounded regions. We in particular discuss the existence, uniqueness and the behaviour of the generalised solutions in an infinite circular half-cylinder with prescribed data on the boundary. We also show how small changes on the boundary data influence the solutions. The obtained results contribute to the general understanding of the behaviour of these solutions.

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Contents

Abstract	i
Populärvetenskaplig sammanfattning	iii
General summary	v
Acknowledgments	vii
Contents	ix
I Preliminaries and summary of papers	1
1 Introduction	3
1.1 Background	3
1.2 General objectives of the thesis	5
1.3 Papers included	6
2 Theoretical background	7
2.1 Sobolev spaces and weak solutions	7
2.2 Perron solutions	11
2.3 Capacity and boundary regularity	13
3 Summary of the papers	17
3.1 Paper A: Mixed boundary value problem for p -harmonic functions in an infinite cylinder.	17
3.2 Paper B: Resolutivity and invariance for the Perron method for degenerate equations of divergence type	19
3.3 Paper C: Behaviour at infinity for solutions of a mixed nonlinear elliptic boundary value problem via inversion	19
3.4 Future research	21
References	21
II Papers	23
Paper A	25
BJÖRN, J. and MWASA, A., Mixed boundary value problem for p -harmonic functions in an infinite cylinder, <i>Nonlinear Analysis</i> 202 (2021), 112134.	
Paper B	57
BJÖRN, A., BJÖRN, J. and MWASA, A., Resolutivity and invariance for the Perron method for degenerate equations of divergence type, <i>Preprint</i> 2020.	
Paper C	77
BJÖRN, J. and MWASA, A., Behaviour at infinity for solutions of a mixed non-linear elliptic boundary value problem via inversion, <i>Manuscript</i> 2021.	

Preliminaries and summary of papers

1 – Introduction

1.1. Background

Partial differential equations are used to model many problems in natural sciences, especially in physics. For instance, they are often applied in electromagnetism, electrostatics, fluid dynamics, quantum mechanics, image analysis, among others. The urge to find suitable solutions to the problems that arise in such areas has profusely occupied many branches of mathematics such as functional analysis, complex analysis, potential theory, operator theory and calculus of variations.

One of the key physical problems that often arise e.g. in physics is to minimise an appropriate energy of some kind. This can be modelled by *elliptic partial differential equations*. For example, to minimise the energy integral

$$\int |\nabla u|^2 dx$$

of some physical quantity in a homogeneous medium, the *Laplace equation*

$$\Delta u := \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0, \quad (1.1)$$

is solved, where u is a finite-valued function defined in an open set Ω in the n -dimensional Euclidean space $\mathbf{R}^n$. Solutions to this kind of equations describe for example gravitational (and electrostatic) potentials of force fields, temperature distributions in a medium and steady irrotational fluid flows. The Laplace equation (1.1) is also essential in harmonic analysis and in the study of analytic functions. It is a prototype of elliptic partial differential equations which are *linear*, i.e. the sum of two solutions is also a solution.

However, many physical problems, for example the flow of non-Newtonian fluids such as paint, blood, shampoo, glaciers and molten plastics, are highly nonlinear. They can instead be modelled by a quasilinear counterpart to (1.1), i.e. the so-called p -Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad (1.2)$$

which depends on a parameter p , where $1 < p < \infty$. Equation (1.2) is the Euler–Lagrange equation for the problem of minimising the p -energy

$$\int_{\Omega} |\nabla u|^p dx$$

among functions u with prescribed boundary data on the boundary $\partial\Omega$. Similar to (1.1), the p -Laplace equation (1.2) is a prototype of elliptic partial differential

equations which are *nonlinear*, i.e. the sum of two solutions need not be a solution. Solving partial differential equations is often not direct, but rather by means of Sobolev spaces where solutions and boundary data are understood in a weak sense. In Section 2.1, a brief introduction to Sobolev spaces and weak solutions will be given.

Assume that Ω is a nonempty bounded open set in $\mathbf{R}^n$ and consider a function $f : \partial\Omega \rightarrow \mathbf{R}$. Then the *Dirichlet problem* amounts to finding a function which solves the partial differential equation under consideration in Ω and takes the prescribed boundary data f on $\partial\Omega$. For instance, the Dirichlet problem for the Laplace equation is to find a solution u which satisfies (1.1) in Ω and coincides with f on $\partial\Omega$.

If the prescribed data are taken as the directional derivative of the function u in the direction of the outer normal n , i.e. $\partial u / \partial n = g$, then the problem is called a *Neumann problem*. If a part of the boundary carries a Dirichlet condition and the rest a Neumann condition, then we have a mixed boundary value problem, also called a *Zaremba problem*, which was studied for the first time by Zaremba [24] in 1910.

Attempts to solve the Dirichlet problem for the Laplace equation in a nonempty bounded open set $\Omega \subset \mathbf{R}^n$ date back to the 19th century. The classical approach is to find a sufficiently smooth solution u with prescribed boundary data f so that for all $x_0 \in \partial\Omega$,

$$\lim_{\Omega \ni x \rightarrow x_0} u(x) = f(x_0). \quad (1.3)$$

This necessarily requires f to be continuous on $\partial\Omega$. If condition (1.3) holds at x_0 for all continuous boundary data f , then the point x_0 is said to be *regular*. If all boundary points are regular, the solution u attains its continuous boundary data in the classical sense.

However, equality in (1.3) may fail for some continuous f at some $x_0 \in \partial\Omega$. In this case, we say that the point x_0 is *irregular*. The first examples of such a scenario for the Laplace equation (1.1) were the following:

- (i) In 1911, Zaremba [24] showed that the Dirichlet problem was not solvable in the punctured ball $\{x \in \mathbf{R}^2 : 0 < |x| < 1\}$ with the prescribed boundary data 1 at the origin and 0 on the rest of the boundary.
- (ii) In 1912, Lebesgue [13] showed that not only was the Dirichlet problem unsolvable at isolated irregular points but also at other points, in particular when the boundary has a sufficiently sharp ‘thin cusp’ extending into the interior of the region, the so-called Lebesgue spine

$$E = \{(x, t) : x \in \mathbf{R}^2, t > 0 \text{ and } |x| < e^{-1/t}\} \quad \text{and} \quad \Omega = B(0, 1) \setminus \bar{E}.$$

Regularity of a boundary point $x_0 \in \partial\Omega$ for the Laplace equation $\Delta u = 0$ was characterised by the celebrated *Wiener criterion* which was established in 1924 by Wiener [23]. This was done through exhaustions by regular open sets. With Wiener’s criterion, one measures the thickness of the complement of Ω near x_0 in terms of capacities, see (2.7). If the complement is too thin, then the boundary point $x_0 \in \partial\Omega$ is irregular.

Prior to this, various sufficient geometric conditions for boundary regularity were obtained, such as the exterior ball condition and the exterior cone condition. Apart from the Laplace equation, the Wiener criterion has been obtained for various weighted and nonlinear partial differential equations, e.g. in Fabes–Jerison–Kenig [5], Gariepy–Ziemer [6], Kilpeläinen–Malý [11], Littman–Stampacchia–Weinberger [15], Maz’ya [18] and Mikkonen [19].

In the 1920s, various ways of solving the Dirichlet problem for the Laplace equation (1.1) were given by Perron [20] in 1923, Wiener [23] in 1924 and later BreLOT [3] in 1939. This together led to the so-called PWB-method in the linear case, though it is simply referred to as Perron method for the nonlinear case. This method was independently used by Remak [21] in 1923 for the linear equations and Granlund–Lindqvist–Martio [7] were the first to use it for the study of nonlinear equations.

Even for regular domains, it may not always be the case that the boundary data f are continuous and thus the solution u in Ω cannot attain its boundary data as limits, i.e. equality (1.3) fails at some point $x_0 \in \partial\Omega$. Using the Perron method, one can construct solutions for the Dirichlet problem in Ω with arbitrary boundary data $f : \partial\Omega \rightarrow \mathbf{R}$. This method gives an upper and a lower Perron solution and when these two coincide, we get a reasonable solution for the Dirichlet problem and the boundary data f are called *resolutive*.

Later in the 1960s, the above approaches were extended to nonlinear variants of (1.1) which have since then been studied extensively. Examples of such nonlinear equations include (1.2) and its generalised form

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0. \quad (1.4)$$

For instance, the sufficiency condition of the Wiener criterion for elliptic quasilinear equations was obtained by Maz’ya [18] in 1970. The necessity condition was for $p > n - 1$ proved by Lindqvist–Martio [14] in 1985 and for all $p > 1$ by Kilpeläinen–Malý [11] in 1994. Boundary regularity and Perron solutions for (1.4) were studied e.g. in Malý–Ziemer [16] and Heinonen–Kilpeläinen–Martio [8].

1.2. General objectives of the thesis

In this thesis, we study quasilinear elliptic equations in the n -dimensional Euclidean space $\mathbf{R}^n$ and in particular, (1.2) and its generalisation (1.4). The following are general objectives:

- (a) To prove the existence and uniqueness of continuous weak solutions to the mixed boundary value problem for quasilinear elliptic equations with continuous and Sobolev Dirichlet boundary data, and to study the behaviour of their solutions and the boundary regularity of the point at ∞ . (Papers A and C)
- (b) To show that arbitrary perturbations on sets of (p, w) -capacity zero of continuous boundary data f are resolutive and that the Perron solution for f and its perturbations coincide. Also to prove that the Perron solution is

the unique bounded solution of the Dirichlet problem for (1.4) in a bounded open set Ω that takes the correct continuous boundary data outside a set of (p, w) -capacity zero. (Paper B)

1.3. Papers included

The thesis consists of the following papers.

- (A) BJÖRN, J. and MWASA, A., Mixed boundary value problem for p -harmonic functions in an infinite cylinder, *Nonlinear Analysis* **202** (2021), 112134.
`arXiv:2006.03496`
- (B) BJÖRN, A., BJÖRN, J. and MWASA, A., Resolutivity and invariance for the Perron method for degenerate equations of divergence type, *Preprint* 2020.
`arXiv:2008.00883`
- (C) BJÖRN, J. and MWASA, A., Behaviour at infinity for solutions of a mixed nonlinear elliptic boundary value problem via inversion, *Manuscript* 2021.

2 – Theoretical background

2.1. Sobolev spaces and weak solutions

Sobolev spaces are inevitable when solving problems in the field of partial differential equations, harmonic analysis, etc. It is well known that solving partial differential equations in the classical sense is often difficult but by appealing to Sobolev spaces it is possible to find solutions of such equations in a weak sense. In this section we give some basic facts about Sobolev spaces that are useful in this thesis. For more details about Sobolev spaces and their properties, see for instance, Evans [4], Heinonen–Kilpeläinen–Martio [8], Malý–Ziemer [16], Maz'ya [17] and Ziemer [25].

Unless specified, Ω is a nonempty open set in the n -dimensional Euclidean space $\mathbf{R}^n$, $n \geq 2$ and $1 < p < \infty$. The ball in $\mathbf{R}^n$ centred at x with radius $r > 0$ is denoted by $B(x, r)$.

Definition 2.1. Let $u, v \in L^1_{\text{loc}}(\Omega)$. A function v is the *weak partial derivative* $\partial_j u = \partial u / \partial x_j$ of u for $j = 1, 2, \dots, n$, if

$$\int_{\Omega} u \partial_j \varphi \, dx = - \int_{\Omega} v \varphi \, dx \quad \text{for all functions } \varphi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the space of all infinitely many times continuously differentiable functions with compact support in Ω . The functions in $C_0^\infty(\Omega)$ are called test functions.

If the weak partial derivative $\partial_j u$ exists, then it is uniquely defined up to a set of Lebesgue measure zero. It is worth noting that classical derivatives are always weak derivatives, but in general the converse is not true. The following example demonstrates this case.

Example 2.2. Let $\Omega = (-1, 1)$. Let $u : \Omega \rightarrow \mathbf{R}$ be defined by $u(x) = |x|$. Then integration by parts shows that for all $\varphi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} u(x) \varphi'(x) \, dx = - \int_{\Omega} v(x) \varphi(x) \, dx,$$

where

$$v(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Hence, $u(x)$ has a weak derivative on $(-1, 1)$ but it is not differentiable at $x = 0$ in classical sense.

We use the notation $\nabla u \in \mathbf{R}^n$ to mean the vector of the weak partial derivatives $(\partial_1 u, \partial_2 u, \dots, \partial_n u)$.

Definition 2.3. The Sobolev space $W^{1,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ such that their distributional gradients ∇u exist and belong to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_{\Omega} (|u|^p + |\nabla u|^p) dx \right)^{1/p}.$$

Moreover, the Sobolev space $W_0^{1,p}(\Omega)$ with zero boundary values is the completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, while a function u is in $W_{\text{loc}}^{1,p}(\Omega)$ if and only if it belongs to $W^{1,p}(\Omega')$ for every open set $\Omega' \Subset \Omega$. As usual, $\bar{E} \Subset \Omega$ means that the closure of E , written as $\bar{E}$, is a compact subset of Ω .

Sobolev spaces are function spaces and in particular, $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are Banach spaces.

In general, it is often difficult to solve problems in partial differential equation classically, i.e. with sufficiently many derivatives and continuously attained boundary data. But with the theory of weak derivatives, it is possible to solve such problems in the weak sense by means of Sobolev spaces. For example, consider the Dirichlet problem for the p -Laplace equation (1.2) in Ω . Then multiplying (1.2) by test functions $\varphi \in C_0^\infty(\Omega)$ and integrating by parts, we have

$$0 = \int_{\Omega} \operatorname{div}(|\nabla u|^{p-2} \nabla u) \varphi dx = - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

The boundary terms vanish since $\varphi = 0$ on the boundary $\partial\Omega$ of Ω . Motivated by the above integral identity, the following definition of a weak solution for a general quasilinear equation is given.

Definition 2.4. A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a *weak solution* of the quasilinear equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in Ω if for all test functions $\varphi \in C_0^\infty(\Omega)$, the following integral identity holds

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx = 0.$$

Note that since Sobolev functions have only weak derivatives and do not even need to be continuous, weak solutions of partial differential equations can a priori be quite irregular. It is therefore desirable to study their regularity properties, both inside the open set Ω (interior regularity) and on the boundary (boundary regularity). One concrete way to describe these properties, is to study continuity of solutions.

Before we give definitions for the weighted Sobolev space and equations, it is paramount to first discuss briefly some of the basic properties of weights and, in particular, A_p weights which are of interest in this thesis. For more expositions on weights, see e.g. Heinonen–Kilpeläinen–Martio [8], Kilpeläinen [10], Kufner [12] and Turesson [22].

By a *weight* w on $\mathbf{R}^n$ we mean a locally integrable nonnegative function. Each weight gives rise to a measure on Lebesgue measurable subsets of $\mathbf{R}^n$ through integration. Thus, the weight w is associated with the measure μ as

$$\mu(E) = \int_E w(x) dx \quad \text{or} \quad d\mu(x) = w(x) dx,$$

where dx is the n -dimensional Lebesgue measure. In order to have a good theory, there is need to impose some appropriate conditions on w .

Definition 2.5. We say that w is *p-admissible* with $p \geq 1$ if the following two conditions hold:

- (i) The associated measure μ is *doubling*, i.e. there is a constant $C_d > 0$ (often called doubling constant) such that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)).$$

- (ii) The measure μ supports a *p-Poincaré inequality*, i.e. there is a constant $C_p > 0$ such that for all balls $B = B(x, r)$ in $\mathbf{R}^n$ and for all bounded $u \in C^\infty(B)$,

$$\int_B |u - u_B| d\mu \leq C_p r \left(\int_B |\nabla u|^p d\mu \right)^{1/p},$$

where

$$u_B := \int_B u d\mu = \frac{1}{\mu(B)} \int_B u d\mu$$

is the integral average of u over the ball B .

Example 2.6. Admissible weights include the following:

- (i) Weights belonging to the A_p class are known to be p -admissible for the theory of Sobolev spaces and partial differential equations. By a weight w in *Muckenhoupt's A_p class* we mean that there is a constant $C > 0$ depending on w , p and n such that for all balls $B \subset \mathbf{R}^n$,

$$\left(\int_B w(x) dx \right) \left(\int_B w(x)^{1/(1-p)} dx \right)^{p-1} \leq C |B|^p, \quad \text{if } 1 < p < \infty, \quad (2.1)$$

where $|B|$ is the n -dimensional Lebesgue measure of B . The smallest C in (2.1) is the A_p constant. Moreover, $w \in A_1$ if there is a constant $C > 0$ such that

$$\int_B w(x) dx \leq C |B| \operatorname{ess\,inf}_B w.$$

Since $A_1 \subset A_p$, we have that a weight in A_1 class is p -admissible for each $p > 1$, see [8, Corollary 15.22].

- (ii) Let $w(x) = |x|^\alpha$. Then for each $\alpha > -n$ and $p > 1$, $w(x)$ is p -admissible with the constants for the measure μ depending only on n , p and α , see [8, Section 1.6 and Corollary 15.35]. On the other hand $w(x) \in A_p$ if and only if $-n < \alpha < n(p-1)$, see [8, Chapters 1 and 15].

- (iii) Let w be a weight function. If there exist constants $C_1, C_2 > 0$ such that $C_1 \leq w(x) \leq C_2$, then w belongs to A_p for all $p \geq 1$.

Note that by example (i), A_p weights are p -admissible. However, p -admissible weights on $\mathbf{R}^n$, $n \geq 2$, are in general not A_p weights as observed in example (ii) above. On $\mathbf{R}$, A_p weights and 1-admissible weights are equivalent for $p \geq 1$, see Björn–Buckley–Keith [2].

Every p -admissible weight is also q -admissible for all $q > p$ by Hölder inequality. The following result is the open ended property of p -admissible weights.

Theorem 2.7. ([9, Corollary 1.0.2]) *Let $p > 1$ and let w be a p -admissible weight in the Euclidean space $\mathbf{R}^n$, $n \geq 1$. Then there exists $\varepsilon > 0$ such that w is q -admissible for every $q > p - \varepsilon$.*

Definition 2.8. Let w be a p -admissible weight and $\Omega \subset \mathbf{R}^n$ be an open set. The weighted Sobolev space $H^{1,p}(\Omega, w)$ is defined as the completion of all functions $u \in C^\infty(\Omega)$, such that

$$\|u\|_{H^{1,p}(\Omega, w)} := \left(\int_{\Omega} (|u|^p + |\nabla u|^p) w \, dx \right)^{1/p} < \infty,$$

with respect to the norm $\|u\|_{H^{1,p}(\Omega, w)}$. The space $H_0^{1,p}(\Omega, w)$ is the completion of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega, w)$, while a function u belongs to $H_{\text{loc}}^{1,p}(\Omega, w)$ if and only if it belongs to $H^{1,p}(\Omega', w)$ for every open set $\Omega' \Subset \Omega$.

In other words, a function u belongs to $H^{1,p}(\Omega, w)$ if and only if $u \in L^p(\Omega, w)$ and there is a vector-valued function v such that for some sequence of smooth functions $\varphi_k \in C^\infty(\Omega)$ with $\|\varphi_k\|_{H^{1,p}(\Omega, w)} < \infty$, we have

$$\int_{\Omega} |\varphi_k - u|^p w \, dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} |\nabla \varphi_k - v|^p w \, dx \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

The vector-valued function $v = \nabla u$, is the (Sobolev) gradient of u in $H^{1,p}(\Omega, w)$.

Just like unweighted Sobolev spaces, $H_0^{1,p}(\Omega, w)$ and $H^{1,p}(\Omega, w)$ are Banach spaces with respect to the $H^{1,p}(\Omega, w)$ -norm. Weighted Sobolev spaces are useful in studying degenerate elliptic partial differential equations. Unlike the usual Sobolev spaces, a function in a weighted Sobolev space need not be locally integrable with respect to the Lebesgue measure. For instance, the weight function $w(x) = |x|^{p(n+1)}$ is admissible for $p > 1$. However, the function $u(x) = |x|^{-n}$ lies in $H_{\text{loc}}^{1,p}(\mathbf{R}^n, w)$ and its Sobolev gradient is $\nabla u(x) = -nx|x|^{-n-2}$, but neither u nor ∇u is locally integrable in $\mathbf{R}^n$ and so ∇u is not the distributional gradient of u . But if $w \in A_p$ or more generally if $w^{1/(1-p)} \in L_{\text{loc}}^1(\Omega)$, then every function $u \in H_{\text{loc}}^{1,p}(\Omega, w)$ is a distribution and ∇u is the distributional gradient of u , see [8, p. 13].

Definition 2.4 can be extended to functions in $H_{\text{loc}}^{1,p}(\Omega, w)$, under the following standard ellipticity assumptions for the mapping $\mathcal{A} : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ with a parameter $1 < p < \infty$, a p -admissible weight $w(x)$ and some constants $\alpha, \beta > 0$, see Heinonen–Kilpeläinen–Martio [8, (3.3)–(3.7)]:

Assume that $\mathcal{A}(x, q)$ is measurable in x for every $q \in \mathbf{R}^n$, and continuous in q for a.e. $x \in \mathbf{R}^n$. Also, for all $q \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$, assume that the following hold

$$\begin{aligned} \mathcal{A}(x, q) \cdot q &\geq \alpha w(x) |q|^p \quad \text{and} \quad |\mathcal{A}(x, q)| \leq \beta w(x) |q|^{p-1}, \\ (\mathcal{A}(x, q_1) - \mathcal{A}(x, q_2)) \cdot (q_1 - q_2) &> 0 \quad \text{for all } q_1, q_2 \in \mathbf{R}^n, \quad q_1 \neq q_2, \\ \mathcal{A}(x, \lambda q) &= \lambda |\lambda|^{p-2} \mathcal{A}(x, q) \quad \text{for all } \lambda \in \mathbf{R}, \quad \lambda \neq 0. \end{aligned} \quad (2.2)$$

Definition 2.9. A function $u \in H_{\text{loc}}^{1,p}(\Omega, w)$ is said to be a *supersolution* of (1.4) in Ω if for all nonnegative functions $\varphi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0.$$

A function u is a *subsolution* of (1.4) in Ω if $-u$ is a supersolution of (1.4) in Ω .

A function u is a weak solution of (1.4) if and only if it is both a subsolution and a supersolution, see [8, bottom p. 58]. The sum of two (super)solutions is in general not a (super)solution. However, if u and v are two (super)solutions, then $\min\{u, v\}$ is a supersolution, see [8, Theorem 3.23]. If u is a supersolution and $a, b \in \mathbf{R}$, then $au + b$ is a supersolution provided that $a \geq 0$. Every weak solution of (1.4) has a locally Hölder continuous representative by [8, Theorems 3.70 and 6.6]. This then leads us to the following definition.

Definition 2.10. A function $u : \Omega \rightarrow \mathbf{R}$ is $\mathcal{A}$ -harmonic in an open set Ω if u is a continuous weak solution of (1.4) in Ω .

We remark that $\mathcal{A}$ -harmonic functions do not in general form a linear space. However, if u is $\mathcal{A}$ -harmonic and $a, b \in \mathbf{R}$, then $au + b$ is also $\mathcal{A}$ -harmonic. One of the useful properties that come with such functions is the following Harnack inequality, see [8, Chapter 6].

Theorem 2.11. Assume that u is a nonnegative $\mathcal{A}$ -harmonic function in a connected open set Ω . Then there exists a constant $c > 0$ such that

$$\sup_K u \leq c \inf_K u$$

whenever $K \subset \Omega$ is compact, with the constant c depending on K but not on u .

The Dirichlet problem is solvable for partial differential equations with prescribed Sobolev boundary data. More precisely, [8, Theorem 3.17] shows that if $f \in H^{1,p}(\Omega, w)$, then there is a unique weak solution of (1.4) in Ω such that $u - f \in H_0^{1,p}(\Omega, w)$.

2.2. Perron solutions

As earlier mentioned in Section 1.1, the Perron method is one of the most general methods used to solve the Dirichlet problem in a nonempty open set $\Omega \subset \mathbf{R}^n$

with arbitrary boundary data. We devote this section to some of the fundamental properties of Perron solutions for the quasilinear equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$. For simplicity, we assume that Ω is bounded. The interested reader may see Chapters 7 and 9 in Heinonen–Kilpeläinen–Martio [8] for more detailed material. The Perron method is based on the notion of subharmonic and superharmonic functions.

Definition 2.12. A function $u : \Omega \rightarrow (-\infty, \infty]$ is $\mathcal{A}$ -superharmonic in Ω if

- (i) u is lower semicontinuous,
- (ii) u is not identically ∞ in any component of Ω ,
- (iii) for every open $\Omega' \Subset \Omega$ and all functions $v \in C(\overline{\Omega'})$ which are $\mathcal{A}$ -harmonic in Ω' , we have $v \leq u$ in Ω' whenever $v \leq u$ on $\partial\Omega'$.

A function $u : \Omega \rightarrow [-\infty, \infty)$ is $\mathcal{A}$ -subharmonic in Ω if $-u$ is $\mathcal{A}$ -superharmonic in Ω .

Let u and v be $\mathcal{A}$ -superharmonic. Then $au + b$ and $\min\{u, v\}$ are $\mathcal{A}$ -superharmonic whenever $a \geq 0$ and b are real numbers, but in general $u + v$ is not $\mathcal{A}$ -superharmonic, see [8, Lemmas 7.1 and 7.2]. A function u is $\mathcal{A}$ -harmonic if and only if it is both $\mathcal{A}$ -superharmonic and $\mathcal{A}$ -subharmonic in Ω , although the proof is not trivial since their definitions are different, see [8, Lemma 7.8].

Remark 2.13. $\mathcal{A}$ -Superharmonic functions are closely related to supersolutions. For instance by [8, Theorem 7.16], every supersolution has an $\mathcal{A}$ -superharmonic representative given by the *lower semicontinuous regularization*

$$u^*(x) = \operatorname{ess\,lim\,inf}_{\Omega \ni y \rightarrow x} u(y) \quad \text{for every } x \in \Omega.$$

Conversely, if u is an $\mathcal{A}$ -superharmonic function in Ω , then $u^* = u$ in Ω . If moreover, u is locally bounded from above, then $u \in H_{\operatorname{loc}}^{1,p}(\Omega, w)$ and u is a supersolution of (1.4) in Ω , see [8, Corollary 7.20].

We are now ready to give the definition of Perron solutions which follows [8, Chapter 9].

Definition 2.14. Given an arbitrary function $f : \partial\Omega \rightarrow [-\infty, \infty]$, let $\mathcal{U}_f$ be the set of all $\mathcal{A}$ -superharmonic functions u on Ω bounded from below and such that

$$\liminf_{\Omega \ni y \rightarrow x} u(y) \geq f(x) \quad \text{for all } x \in \partial\Omega.$$

The *upper Perron solution* $\bar{P}f$ of f is defined by

$$\bar{P}f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Omega.$$

Analogously, let $\mathcal{L}_f$ be the set of all $\mathcal{A}$ -subharmonic functions v on Ω bounded from above such that

$$\limsup_{\Omega \ni y \rightarrow x} v(y) \leq f(x) \quad \text{for all } x \in \partial\Omega.$$

The *lower Perron solution* $\underline{P}f$ of f is defined by

$$\underline{P}f(x) = \sup_{v \in \mathcal{L}_f} v(x), \quad x \in \Omega.$$

We remark that if $\mathcal{U}_f = \emptyset$, then $\bar{P}f \equiv \infty$ and if $\mathcal{L}_f = \emptyset$, then $\underline{P}f \equiv -\infty$. For every component Ω' of Ω , $\bar{P}f$ (and $\underline{P}f$) is either $\mathcal{A}$ -harmonic or identically $\pm\infty$ in Ω' , see [8, Theorem 9.2]. The comparison principle, [8, p. 133], between $\mathcal{A}$ -subharmonic and $\mathcal{A}$ -superharmonic functions gives that $\underline{P}f \leq \bar{P}f$.

Clearly, $\underline{P}f = -\bar{P}(-f)$ and $\bar{P}f \leq \bar{P}g$ whenever $f \leq g$ on $\partial\Omega$. If $\alpha \in \mathbf{R}$ and $\beta \geq 0$, then

$$\underline{P}(\alpha + \beta f) = \alpha + \beta \underline{P}f \quad \text{and} \quad \bar{P}(\alpha + \beta f) = \alpha + \beta \bar{P}f.$$

If $\bar{P}f = \underline{P}f$ and they are finite valued, then f is said to be *resolutive* with respect to Ω . In this case, we write $Pf := \bar{P}f$. Continuous functions f are resolutive by [8, Theorem 9.25]. In our Paper B, it is shown that perturbations on sets of Sobolev (p, w) -capacity zero of continuous boundary data f are resolutive and that the Perron solutions for f and its perturbations coincide.

Remark 2.15. If u is a bounded $\mathcal{A}$ -harmonic function in Ω , such that

$$f(x) = \lim_{\Omega \ni y \rightarrow x} u(y) \quad \text{for all } x \in \partial\Omega, \quad (2.3)$$

then $u \in \mathcal{U}_f \cap \mathcal{L}_f$ and thus, using also that $\underline{P}f \leq \bar{P}f$, we see that $\bar{P}f = u = \underline{P}f$, see [8, p. 169].

In Paper B, we show that for continuous functions f , it suffices to assume (2.3) outside a set of Sobolev (p, w) -capacity zero. Exceptional sets of zero Sobolev (p, w) -capacity are natural in (2.3) as demonstrated by the following example.

Example 2.16. Consider the punctured disc $\Omega = \{x \in \Omega : 0 < |x| < 1\} \subset \mathbf{R}^2$. Let $f(x) = 0$ when $|x| = 1$ and $f(0) = 1$ be the boundary values. The function $\log(1/|x|)$ is harmonic in Ω and hence

$$0 \leq \underline{P}f \leq \bar{P}f \leq \varepsilon \log(1/|x|) \quad \text{for all } \varepsilon > 0.$$

Letting $\varepsilon \rightarrow 0$, we have that $\bar{P}f = \underline{P}f = 0$. The solution takes 0 as the boundary value at the origin, despite the fact that $f(0) = 1$.

If u is unbounded, then it may not be true that $u = Pf$.

Example 2.17. The Poisson kernel

$$\frac{1 - |z|^2}{|1 - z|^2}$$

with a pole at 1 is a harmonic function in the unit disc $B(0, 1) \subset \mathbb{C} = \mathbf{R}^2$, which is zero on $\partial B(0, 1) \setminus \{1\}$, while the Perron solution of the zero function $f \equiv 0$ is clearly $Pf \equiv 0$.

2.3. Capacity and boundary regularity

Functions in Sobolev spaces are defined pointwise up to sets of Lebesgue measure zero. However, each of the functions in the Sobolev space $H^{1,p}(\Omega, w)$ has a (p, w) -quasicontinuous representative which is unique up to sets of (p, w) -capacity zero,

see [8, Theorem 4.4]. Capacity is natural in Sobolev spaces and very useful in measuring small sets in a more precise way than measure, giving a better way of understanding the local properties of functions in Sobolev spaces. This is useful for the theory of partial differential equations since one requires to find as accurate pointwise behaviour of solutions as possible. Much as computing capacities explicitly is quite difficult, there are some known estimates. We shall give a few of these capacities later in the section.

The material in this section follows mainly from Chapter 2 in Heinonen–Kilpeläinen–Martio [8]. As earlier pointed out in Section 2.1, we assume that w is a p -admissible weight, as such weights are suitable for the theory of Sobolev spaces and partial differential equations.

Definition 2.18. Let $E \subset \mathbf{R}^n$. The *Sobolev (p, w) -capacity* of a set E is

$$C_{p,w}(E) = \inf_u \int_{\mathbf{R}^n} (|u|^p + |\nabla u|^p) w \, dx, \quad (2.4)$$

where the infimum is taken over all functions $u \in H^{1,p}(\mathbf{R}^n, w)$ such that $0 \leq u \leq 1$ and $u = 1$ in an open set containing E . Such functions are said to be admissible.

Suppose that E is compact, then the infimum in (2.4) can equivalently be taken over all $u \in C_0^\infty(\mathbf{R}^n)$ such that $u = 1$ on the set E , see [8, Lemma 2.36].

The Sobolev (p, w) -capacity is a monotone, subadditive set function. It follows directly from the definition that for all $E \subset \mathbf{R}^n$,

$$C_{p,w}(E) = \inf_{\substack{G \supset E \\ G \text{ open}}} C_{p,w}(G). \quad (2.5)$$

In particular, if $C_{p,w}(E) = 0$ then there exist open sets $U_j \supset E$ with $C_{p,w}(U_j) \rightarrow 0$ as $j \rightarrow \infty$.

Definition 2.19. A function $v : \Omega \rightarrow [-\infty, \infty]$ is (p, w) -*quasicontinuous* in Ω if for every $\varepsilon > 0$ there is an open set G such that $C_{p,w}(G) < \varepsilon$ and the restriction of v to $\Omega \setminus G$ is finite-valued and continuous.

It follows from the outer regularity (2.5) of $C_{p,w}$ that if v is quasicontinuous and $\bar{v} = v$ except possibly for a set of Sobolev (p, w) -capacity zero, then $\bar{v}$ is also quasicontinuous.

Definition 2.20. Let K be a compact subset of an open set $\Omega \subset \mathbf{R}^n$. The *variational (p, w) -capacity* of K in Ω , is

$$\text{cap}_{p,w}(K, \Omega) = \inf_v \int_{\Omega} |\nabla v|^p w \, dx, \quad (2.6)$$

where the infimum is taken over all $v \in C_0^\infty(\Omega)$ satisfying $v \geq 1$ on K .

One can as well get the same capacity if the infimum in (2.6) is taken over functions $v \in H_0^{1,p}(\Omega, w) \cap C(\Omega)$ satisfying $0 \leq v \leq 1$, see [8, pp. 27–28]. Just like $C_{p,w}$, the variational (p, w) -capacity is a monotone and subadditive set function.

The capacity $\text{cap}_{p,w}$ can be extended to noncompact sets using a standard procedure as follows. Let $U \subset \Omega$ be an open set, then

$$\text{cap}_{p,w}(U, \Omega) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \text{cap}_{p,w}(K, \Omega),$$

and if $E \subset \Omega$ is arbitrary, then

$$\text{cap}_{p,w}(E, \Omega) = \inf_{\substack{\Omega \supset U \supset E \\ U \text{ open}}} \text{cap}_{p,w}(U, \Omega).$$

The capacity in (2.6) is very fundamental in the characterisation of boundary regularity at a point via the celebrated Wiener criterion, see condition (2.7) below. For instance, in this thesis the capacity $\text{cap}_{p,w}$ is used in obtaining boundary regularity of the point at infinity in Papers A and C.

In essence, the two capacities $C_{p,w}$ and $\text{cap}_{p,w}$ do not differ so much and for practical relevancy, they are equivalent. In particular, both have the same zero sets. To see this, we recall the following result that gives the comparison between the two capacities (2.4) and (2.6).

Theorem 2.21. ([8, Theorem 2.38]) *Let E be a subset of a ball $B(x, r)$ in $\mathbf{R}^n$. Then there is a constant c depending on p, w and n such that*

$$\frac{C_{p,w}(E)}{1 + cr^p} \leq \text{cap}_{p,w}(E, B(x, 2r)) \leq 4^p \left(1 + \frac{1}{r^p}\right) C_{p,w}(E).$$

The Sobolev (p, w) -capacity and the variational (p, w) -capacity are both Choquet capacities. In particular, for all Borel sets $E \subset \Omega$,

$$\text{cap}_{p,w}(E, \Omega) = \sup_{\substack{K \subset E \\ K \text{ compact}}} \text{cap}_{p,w}(K, \Omega)$$

and similarly

$$C_{p,w}(E, \Omega) = \sup_{\substack{K \subset E \\ K \text{ compact}}} C_{p,w}(K, \Omega).$$

Definitions 2.18 and 2.20 give unweighted capacities when $w \equiv 1$. In that case, we use the notation C_p and cap_p , respectively.

As stated before, it is quite difficult to compute capacities. However, there are some known formulas for capacity in unweighted $\mathbf{R}^n$ and other estimates in weighted $\mathbf{R}^n$. By $a \simeq b$, we mean there exists a positive constant C , independent of a and b , such that $a/C \leq b \leq Ca$.

Example 2.22. ([8, Example 2.12]). The spherical condenser $(B(x, r), B(x, R))$ with $0 < r < R < \infty$ has

$$\text{cap}_p(B(x, r), B(x, R)) \simeq \begin{cases} r^{n-p}, & p < n, \\ R^{n-p}, & p > n, \\ (\log R/r)^{1-n}, & p = n. \end{cases}$$

The following is an example of a capacity estimate in the weighted $\mathbf{R}^n$.

Example 2.23. ([8, Theorems 2.18 and 2.19]). If $0 < r < R$ and $w \in A_p$, then

$$\text{cap}_{p,w}(B(x, r), B(x, R)) \simeq \left(\int_{B(x, R) \setminus B(x, r)} |y - x|^{p(1-n)/(p-1)} w(y)^{1/(1-p)} dy \right)^{1-p}$$

where the comparison constants in $\simeq$ depend on n, p and the A_p constant.

Because capacities measure the size of small sets in a more precise manner than measure, it is natural to give the relation between unweighted capacities and Hausdorff dimensions in connection with exceptional (small) sets in $\mathbf{R}^n$. The following is due to [8, Theorems 2.26 and 2.27]. Let $1 < p \leq n$ and $E \subset \mathbf{R}^n$.

- (i) If E is of p -capacity zero, then the Hausdorff dimension of E is at most $n - p$.
- (ii) If the $(n - p)$ -dimensional Hausdorff measure is finite, then E is of p -capacity zero.

We conclude this introduction by illustrating the role of capacity in the boundary regularity for the Dirichlet problem.

Definition 2.24. A point x on the boundary of Ω is *Sobolev regular* for the equation $\text{div } \mathcal{A}(x, \nabla u) = 0$ if, for every $f \in H^{1,p}(\Omega, w) \cap C(\bar{\Omega})$, the $\mathcal{A}$ -harmonic function Hf in Ω with $Hf - f \in H_0^{1,p}(\Omega, w)$ satisfies

$$\lim_{\Omega \ni y \rightarrow x} Hf(y) = f(x).$$

If $x \in \partial\Omega$ is not (Sobolev) regular, then it is (Sobolev) *irregular*.

Theorem 2.25. ([8, Theorems 8.10]) *The set of all irregular boundary points for (1.4) has (p, w) -capacity zero, where the weight w is as in (2.2).*

It is worth noting that regularity of a point can equivalently be defined in terms of Perron solutions, see e.g. [8, Chapter 9] and Paper B.

The need to settle the question of when does the solution attain its boundary data as limits at a boundary point gave rise to the following celebrated Wiener criterion, proved for the Laplace equation $\Delta u = 0$ by Wiener [23] and in its general form for the equation $\text{div } \mathcal{A}(x, \nabla u) = 0$ in Heinonen–Kilpeläinen–Martio [8, Chapter 21].

Theorem 2.26. *A boundary point $x_0 \in \partial\Omega$ is regular for (1.4) if and if*

$$\int_0^1 \left(\frac{\text{cap}_{p,w}(B(x_0, r) \setminus \Omega, B(x_0, 2r))}{\text{cap}_{p,w}(B(x_0, r), B(x_0, 2r))} \right)^{1/(p-1)} \frac{dr}{r} = \infty. \quad (2.7)$$

Condition (2.7) means that the complement of Ω is (p, w) -thick at $x_0 \in \partial\Omega$. This then reveals that the solution u of (1.4) has a limit $f(x)$ at x_0 .

3 – Summary of the papers

This section covers an overview of the papers appended in this thesis.

3.1. Paper A: Mixed boundary value problem for p -harmonic functions in an infinite cylinder.

In this paper, we consider a mixed boundary value problem for the quasilinear p -Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad (3.1)$$

in $G \setminus F$, where

$$G = B' \times (0, \infty)$$

is an open infinite circular half-cylinder with $B' = \{x' \in \mathbf{R}^{n-1} : |x'| < 1\}$ being the unit ball in $\mathbf{R}^{n-1}$ and the points in $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$, $n \geq 2$, are represented as $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$. The set F is assumed to be a closed subset of $\bar{G}$ and contains the base $B' \times \{0\}$ of G . We impose zero Neumann boundary data on $\partial G \setminus F$, and Dirichlet data on the rest of the boundary, i.e. on $F \cap \partial(G \setminus F)$.

The schematic diagram below shows how the half-cylinder G is transformed into a unit half-ball $T(G)$ under that mapping $x \mapsto \xi = T(x)$, defined on $\mathbf{R}^n$ with values in

$$T(\mathbf{R}^n) = \mathbf{R}^n \setminus \{(\xi', \xi_n) \in \mathbf{R}^n : \xi' = 0 \text{ and } \xi_n \leq 0\},$$

where

$$\xi' = \frac{2e^{-\kappa x_n} x'}{1 + |x'|^2} \quad \text{and} \quad \xi_n = \frac{e^{-\kappa x_n} (1 - |x'|^2)}{1 + |x'|^2}. \quad (3.2)$$

The above transformation was adopted from [1]. Similar to x , we use the notation $\xi = (\xi', \xi_n) = (\xi_1, \dots, \xi_{n-1}, \xi_n) \in \mathbf{R}^n$ for points in the transformed space.

It is more convenient to work with the transformed bounded domain and then map back to the open unbounded cylinder. The p -Laplace operator in G corresponds to a weighted quasilinear elliptic operator on $T(G)$, see [Paper A, Section 3].

Figure 3.1. Transformation of G into $T(G)$.

Properties of the new operator and appropriate function spaces are discussed. Removal of the Neumann data is achieved by reflecting the unit half-ball in the hyperplane $\xi_n = 0$, leaving the boundary of the transformed domain with purely Dirichlet data.

We prove the existence of weak solutions to the mixed boundary value problem for (3.1) with Sobolev and continuous boundary data in Theorems 6.3 and 6.4 of Paper A. We also study boundary regularity of the point at infinity for these solutions. More precisely, for every continuous Dirichlet boundary data with a finite limit at infinity, we show that the solution has the correct limit value at infinity if and only if the Dirichlet part of the boundary is sufficiently large in terms of a condition involving a certain capacity. In particular, the following are the main results proved in the paper.

Theorem 3.1. *Let f be a continuous function on $F_0 = F \cap \partial(G \setminus F)$ such that the limit*

$$f(\infty) := \lim_{\substack{x \in F_0 \\ x_n \rightarrow \infty}} f(x) \quad \text{exists and is finite.} \quad (3.3)$$

Then there exists a bounded continuous weak solution $u \in W_{\text{loc}}^{1,p}(G \setminus F)$ to the p -Laplace equation (3.1) in $G \setminus F$ with zero Neumann boundary data on $\partial G \setminus F$ and Dirichlet boundary data f on F_0 attained as the limit

$$\lim_{G \setminus F \ni x \rightarrow x_0} u(x) = f(x_0)$$

for all $x_0 \in F_0$, except possibly for a set of Sobolev C_p -capacity zero.

Theorem 3.2. *The following are equivalent.*

- (a) *The point at infinity is regular for the mixed boundary value problem considered in Theorem 3.1, that is, for every continuous boundary data f satisfying (3.3), the solution u satisfies*

$$\lim_{\substack{x \in \bar{G} \setminus F \\ x_n \rightarrow \infty}} u(x) = f(\infty).$$

- (b) *The condition*

$$\int_1^\infty \text{cap}_{p, G_{t-1}}(\{x \in F : t \leq x_n \leq 2t\})^{1/(p-1)} dt = \infty \quad \text{holds,}$$

where the capacity $\text{cap}_{p, G_{t-1}}$ is for compact sets

$$K \subset G_{t-1} := \{x \in \bar{G} : x_n > t - 1\}$$

defined by

$$\text{cap}_{p, G_{t-1}}(K) = \inf \int_{G_{t-1}} |\nabla v|^p dx,$$

with the infimum taken over all $v \in C_0^\infty(\mathbf{R}^n)$ satisfying $v = 1$ on K and $v = 0$ on $\bar{G} \setminus G_{t-1}$.

It is also shown that cap_{p,G_t} is a Choquet capacity and that it is for certain sets comparable to the standard Sobolev p -capacity in $\mathbf{R}^n$. The paper ends with two concrete examples illustrating when a point at ∞ is regular and irregular for the mixed boundary value problem for (3.1).

3.2. Paper B: Resolutivity and invariance for the Perron method for degenerate equations of divergence type

In this paper, we study the Dirichlet problem for the quasilinear elliptic equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \quad (3.4)$$

in $\Omega \subset \mathbf{R}^n$ with arbitrary boundary data. The vector-valued function $\mathcal{A}$ in the equation (3.4) satisfies the standard ellipticity and monotonicity assumptions (2.2) with a parameter $1 < p < \infty$ and the equation is degenerate with a p -admissible weight w , see Section 2.1.

We show that arbitrary perturbations on sets of (p, w) -capacity zero of continuous boundary data f are resolute and that the Perron solutions for f and its perturbations coincide. We obtain, as a by-product, that Perron solutions of perturbations of Lipschitz boundary data are the same as their Sobolev solutions. This perturbation result, as well as the equality of the Perron and Sobolev solutions is shown also for quasicontinuous representatives of Sobolev functions.

It is also proved that the Perron solution is the unique bounded solution of (3.4) in Ω that takes the prescribed continuous boundary data outside a set of (p, w) -capacity zero:

Theorem 3.3. *Let $f \in C(\partial\Omega)$. Then there exists a unique bounded $\mathcal{A}$ -harmonic function u in Ω such that*

$$\lim_{\Omega \ni y \rightarrow x} u(y) = f(x) \quad \text{for all } x \in \partial\Omega,$$

except on sets of (p, w) -capacity zero. Moreover $u = Pf$.

3.3. Paper C: Behaviour at infinity for solutions of a mixed nonlinear elliptic boundary value problem via inversion

In this paper, we study a mixed boundary value problem for the nonlinear uniformly elliptic equations of the form

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \quad (3.5)$$

in $G \setminus F$, where the infinite half-cylinder $G = B' \times (0, \infty)$, the closed set $F \subset \bar{G}$, the transformation T and other notions are defined as in Section 3.1.

The existence and uniqueness of continuous weak solutions to the mixed boundary value problem for (3.5) in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$ are proved for the prescribed Dirichlet continuous data on $F_0 := F \cap \partial(G \setminus F)$. The Dirichlet boundary data are shown to be attained as a limit on F_0 outside a set of Sobolev C_p -capacity zero. The regularity of the boundary point at infinity is characterised by Wiener criterion for the equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$. We show that sets of Sobolev C_p -capacity zero are removable for the solutions of (3.5) in $G \setminus F$. The behaviour of the solutions at infinity is also studied and a Phragmén–Lindelöf type trichotomy is proved. In particular we prove the following results.

Theorem 3.4. *Assume that F_0 is unbounded and let $f \in C(F_0)$ be such that*

$$f(\infty) := \lim_{F_0 \ni x \rightarrow \infty} f(x) \quad \text{exists and is finite.}$$

Then there exists a unique bounded continuous weak solution $u \in W_{\operatorname{loc}}^{1,p}(\bar{G} \setminus F)$ of the mixed problem for the equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$ and such that

$$\lim_{G \setminus F \ni x \rightarrow x_0} u(x) = f(x_0) \quad \text{for } C_p\text{-quasievery } x_0 \in F_0.$$

The behaviour of solutions at ∞ is addressed in the following theorem.

Theorem 3.5. *Assume that for some $t \geq 0$, the set $E := F \cap G_t$ is such that $C_p(E) = 0$. Let $u \in W_{\operatorname{loc}}^{1,p}(\bar{G} \setminus F)$ be a weak continuous solution of $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Assume that u is bounded in the set $(G_t \setminus G_\tau) \setminus F$ for every $\tau > t$.*

Then there exist constants $\tau_0 > t$, M , M_0 and $A_0 \geq A \geq 0$, such that exactly one of the following holds.

- (i) *The solution u is bounded in G_{τ_0} and the limit*

$$\lim_{G \setminus F \ni x \rightarrow \infty} u(x) =: u(\infty)$$

exists and is finite. Moreover, for some $\alpha \in (0, 1]$ and all $x \in G_t$,

$$|u(x) - u(\infty)| \lesssim e^{-\kappa \alpha x_n}.$$

- (ii) *The solution u tends roughly linearly either to ∞ or to $-\infty$, i.e. either*

$$M + A\tau \leq u(x', \tau) \leq M_0 + A_0\tau \quad \text{for all } x' \in B' \text{ and } \tau \geq \tau_0, \quad (3.6)$$

or (3.6) holds for $-u$ in place of u .

- (iii) *The solution changes sign and approaches both ∞ and $-\infty$. More precisely,*

$$\max_{x_n = \tau} u(x) \geq M + A\tau \quad \text{and} \quad \min_{x_n = \tau} u(x) \leq M_0 - A\tau \quad \text{for all } \tau \geq \tau_0.$$

3.4. Future research

Some of the possible research works related to what is addressed in the thesis are:

- (1) In relation to Papers *A* and *C*, one possibility is to consider a weighted quasilinear elliptic equation $\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ on the cylinder under similar ellipticity conditions as in Paper *B* and apply the same transformation T as in (3.2). One way to achieve this, could be to find suitable weights on the circular cylinder G so that the corresponding weights on the transformed domain are p -admissible. This is needed in order to use the tools available e.g. in [8].
- (2) Another case is to consider a mixed boundary value problem for the quasilinear elliptic equations on other appropriate sets than the circular cylinder, for instance a cone or a paraboloid. Here a challenge is to determine which mapping works so that the transformed domain is bounded and suitable for removing the Neumann data by reflection.

References

1. BJÖRN, J., Regularity at infinity for a mixed problem for degenerate elliptic operators in a half-cylinder, *Math. Scand.* **81** (1997), 101–126.
2. BJÖRN, J., BUCKLEY, S. and KEITH, S., Admissible measures in one dimension, *Proc. Amer. Math. Soc.* **134** (2005), 703–705.
3. BRELOT, M., Familles de Perron et problème de Dirichlet, *Acta Litt. Sci. Szeged* **9** (1939), 133–153.
4. EVANS, L. C., *Partial Differential Equations*, Graduate Studies in Mathematics **19**, Amer. Math. Soc, Providence, RI, 1998.
5. FABES, E. B., JERISON, D. S. and KENIG, C. E., The Wiener test for degenerate elliptic equations., *Ann. Inst. Fourier (Grenoble)* **32:3** (1982), 151–182.
6. GARIEPY, R. and ZIEMER, W. P., A regularity condition at the boundary for solutions of quasilinear elliptic equations, *Arch. Ration. Mech. Anal.* **67** (1977), 25–39.
7. GRANLUND, S., LINDQVIST, P. and MARTIO, O., Note on the PWB-method in the nonlinear case, *Pacific J. Math.* **125** (1986), 381–395.
8. HEINONEN, J., KILPELÄINEN, T. and MARTIO, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.
9. KEITH, S. and ZHONG, X., The Poincaré inequality is an open ended condition, *Ann. of Math.* **2** (2008), 575–599.
10. KILPELÄINEN, T., Weighted Sobolev spaces and capacity, *Ann. Acad. Sci. Fenn. Math.* **19** (1994), 95–113.
11. KILPELÄINEN, T. and MALÝ, J., The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.* **172** (1994), 137–161.
12. KUFNER, A., *Weighted Sobolev Spaces*, Wiley, New York, 1985.
13. LEBESGUE, H., Sur des cas d'impossibilité du problème de Dirichlet ordinaire. In Vie de la société (in the part C. R. Séances Soc. Math. France (1912)), *p.* 17, *Bull. Soc. Math. France* **41** (1913), 1–62 (Supplément spécial).

14. LINDQVIST, P. and MARTIO, O., Two theorems of N. Wiener for solutions of quasilinear elliptic equations, *Acta Math.* **155** (1985), 153–171.
15. LITTMAN, W., STAMPACCHIA, G. and WEINBERGER, H. F., Regular points for elliptic equations with discontinuous coefficients, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **17** (1963), 43–77.
16. MALÝ, J. and ZIEMER, W. P., *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys and Monographs **51**, Amer. Math. Soc., Providence, RI, 1997.
17. MAZ'YA, V., *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **342**, Springer, Heidelberg, 2011.
18. MAZ'YA, V. G., On the continuity at a boundary point of solutions of quasilinear elliptic equations., *Vestnik Leningrad. Univ.* **25**:13 (1970), 42–55. English translation: *Vestnik Leningrad. Univ. Math.* **3** (1976), 225–242.
19. MIKKONEN, P., *On the Wolff Potential and Quasilinear Elliptic Equations Involving Measures*, Ann. Acad. Sci. Fenn. Ser. Math. Dissertationes **104** (1996).
20. PERRON, O., Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$, *Math. Z.* **18** (1923), 42–54.
21. REMAK, R., Über potentialkonvexe Funktionen, *Math. Z.* **20** (1924), 126–130.
22. TURESSON, B. O., *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lecture Notes in Mathematics **1736**, Springer, Berlin, 2000.
23. WIENER, N., Certain notions in potential theory, *J. Math. Phys.* **3** (1924), 24–51. Reprinted in *Norbert Wiener: Collected works*, Vol. **1**, pp. 364–391, MIT Press, Cambridge, MA, 1976.
24. ZAREMBA, S., Sur un Problème mixte relatif à l'équation de Laplace, *Bull. Int. Acad. Polon. Sci. Cracovie. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math.* (1910), 313–344.
25. ZIEMER, W. P., *Weakly Differentiable Functions*, Graduate Texts in Mathematics **120**, Springer, New York, 1989.

Papers

Papers

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