Parameterized Verification of Synchronized Concurrent Programs

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ABSTRACT

There is currently an increasing demand for concurrent programs. Checking the correctness of concurrent programs is a complex task due to the interleavings of processes. Sometimes, violation of the correctness properties in such systems causes human or resource losses; therefore, it is crucial to check the correctness of such systems. Two main approaches to software analysis are testing and formal verification. Testing can help discover many bugs at a low cost. However, it cannot prove the correctness of a program. Formal verification, on the other hand, is the approach for proving program correctness. Model checking is a formal verification technique that is suitable for concurrent programs. It aims to automatically establish the correctness (expressed in terms of temporal properties) of a program through an exhaustive search of the behavior of the system. Model checking was initially introduced for the purpose of verifying finite-state concurrent programs, and extending it to infinite-state systems is an active research area.

In this thesis, we focus on the formal verification of *parameterized systems*. That is, systems in which the number of executing processes is not bounded a priori. We provide fully-automatic and parameterized model checking techniques for establishing the correctness of safety properties for certain classes of concurrent programs. We provide an open-source prototype tool for most of the techniques and present our experimental results on several benchmarks. We also provide undecidability results for certain reachability problems.

First, we address the problem of automatically checking safety properties for bounded as well as parameterized phaser programs. Phaser programs are concurrent programs that make use of the phaser synchronization construct found in modern languages such as Habanero Java. For the bounded case, we establish the decidability of checking violation of program assertions and the undecidability of checking deadlock-freedom. For the parameterized case, we study different formulations of the verification problem and propose a procedure that can be exact and guaranteed to terminate for some reachability problems even in the presence of unbounded phases and arbitrarily many spawned processes. Second, we propose an approach for automatic verification of parameterized concurrent programs in which shared variables are manipulated by atomic transitions to count and synchronize the spawned processes. For this purpose, we introduce counting predicates that relate counters that refer to the number of processes satisfying some given properties to the variables that are directly manipulated by the concurrent processes. We then combine existing works on the counter, predicate, and constrained monotonic abstraction and build a nested counterexample-based refinement scheme to establish correctness. Third, we introduce Lazy Constrained Monotonic Abstraction for more efficient exploration of well-structured abstractions of infinite-state non-monotonic systems. We propose several heuristics and assess the efficiency of the proposed technique by extensive experiments using our open-source prototype tool. Lastly, we propose a sound but (in general) incomplete procedure for automatic verification of safety properties for a class of fault-tolerant distributed protocols described in the Heard-Of (HO for short) model. The HO model is a popular model for describing distributed protocols. We propose a verification procedure that is guaranteed to terminate even for unbounded number of the processes that execute the distributed protocol.

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POPULÄRVETENSKAPLIG SAMMANFATTNING

Samtidiga program består av flera processer som deltar i beräkningar. Sådana processer kan, jämfört med de sekventiella varianterna, resultera i mer effektiva och tillförlitliga program. Sensornätverk som sprids i en skog för att larma om eld, inbyggda system som möjliggör användningen av olika funktioner i en modern bil eller banksystem som delar transaktioner och beräkningsbelastning på olika processorer är några exempel på system som kan köra samtidiga program. Att kontrollera att samtidiga program är korrekta, dvs. att de beter sig som förväntat, är en komplex uppgift på grund av deras invecklade beteende och det, möjligtvis, stora antalet inblandade processer. Buggar och logiska misstag i datorprogram kan orsaka mänskliga och materiella förluster. Verkliga skräckexemplen inkluderar fall där synkroniseringsfel mellan processer som kontrollerar delar av en strålbehandlingsmaskin resulterade i strålningsöverdoser för flera patienter. Att försäkra sig att samtidiga program beter sig som förväntat är därför viktigt. Formell verifiering och testning är två huvudmetoder för analys av datorprogram. Vid testning ges en uppsättning testfall som input till programmet. För varje testfall anges också det förväntade resultatet. Sedan jämförs programmets resultat mot det förväntade resultatet. Testning är väl etablerat och kan upptäcka många buggar till en låg kostnad. Testning kan dock inte bevisa att ett program är korrekt. Formell verifiering är en samling av tekniker som syftar till att bevisa programkorrekthet. Modellkontroll (model checking) är en formell verifieringsteknik som är lämplig för samtidiga program. Korrektheten är uttryckt med hjälp av formler i temporallogik. Själva verifieringen genomförs med hjälp av en uttömmande sökning av systemets beteende. Tekniken introducerades ursprungligen för att verifiera samtidiga program med ändligt antal tillstånd. Att utöka modellkontroll till system med godtyckliga tillståndstorlek är ett aktivt forskningsområde. I denna avhandling fokuserar vi på formell verifiering av parametriserade system. Det vill säga, system där antalet processer är ändligt men inte begränsas på förhand. Vi tillhandahåller helautomatiska och parametriserade modellkontrolltekniker för att etablera eller motbevisa säkerhetsegenskaper (safety properties) för valda klasser av samtidiga program. Vi presenterar våra experimentella resultat genom flera exempel. Först beaktar vi problemet med att automatiskt kontrollera säkerhetsegenskaper för phaser-program där antalet processer kan vara avgränsade eller parametriserade. Dessa samtidiga program använder sig av phasers: en komplex synkroniseringskonstruktion som implementeras i moderna språk som Habanero Java. För det avgränsade fallet fastställer vi avgörbarhet för att kontrollera om programuttryck respekteras och oavgörbarhet av att kontrollera dödläge-frihet. För det parametriserade fallet studerar vi olika formuleringar av verifieringsproblemet och föreslår en exakt procedur som garanteras avslutas för vissa närbarhetsproblem även med obegränsade antal phaser och godtyckligt många uppkomna processer. Vi föreslår också en metod för automatisk verifiering av parametriserade program där variabler delas och manipuleras av flera processer. Vi introducerar predikat som använder speciella variabler som hänvisar till antalet processer som uppfyller vissa givna egenskaper och vanliga variabler som direkt manipuleras av det samtidiga programmet. Detta möjliggör att resonera kring relationen mellan antalet processer med vissa egenskaper och värden i programvariablerna. Dessutom introducerar vi Lazy Constrained Monotonic Abstraction för effektivare utforskning av välstrukturerade abstraktioner av icke-monotona system. Dessa är system där instanser med flera processer inte nödvändigtvis behöver kunna göra mera än instanser med färre processer. Det gör att analysen av parametriserade varianter blir svårare. Vi föreslår flera heuristik och jämför deras effektivitet med hjälp av omfattande experiment med vår öppen-källkod prototyp. Till slut föreslår vi en sund men (i allmänhet) icke-fullständig procedur för automatisk verifiering av säkerhetsegenskaper för en klass av fel-toleranta distribuerade protokoll som beskrivs i Heard-Of-modellen. Vi föreslår en verifieringsheuristik

som garanteras avslutas även för ett obegränsat antal processer som kör det distribuerade protokollet.

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1

Introduction

In this chapter, we present the motivation, objective, and the main contributions of the thesis. We start by discussing the importance and challenges of finding errors in concurrent programs. Then, we present a summary of related work. We also give a brief introduction to the contributions in this thesis.

1.1 Motivation

Although the first modern computers emerged in 1945 [120], until the mid-1970s, the computers were large, expensive, and not connected. Since the mid-1970s, two major advances have changed the picture. Powerful microprocessors and computer networks were developed around this time [122] which contributed to the production of smaller, faster, and connectable computers.

Since the invention of the first multicore architectures in the 2000s, there have been several options available for concurrent execution of programs. There was also an increasing demand for concurrency. One of the reasons was that sequential programs were not expected to speed up significantly at this point. This is because efficiency of sequential programs relied directly on chip speed, which used to double every 18 months, but stopped doing so around this time.

Concurrency means several computations happen at the same time. There are different approaches to achieving concurrency, which are called concurrency models. We focus in this thesis on *message passing* and *shared memory* [116] concurrency models.

In the message passing model, processes communicate with each other by passing messages over a network. *Distributed systems* [123] are an instance of this model. Distributed systems consist of processes running on autonomous and geographically dispersed computers. The processes interact with each other using a protocol to achieve a common goal. A distributed system appears to the user as a single coherent system.

In the shared memory model, the processes communicate with each other using shared variables. Each process may also have local variables that are not accessible to others. The concurrent use of shared resources can be a source of indeterminacy due to arbitrary interleaving of processes. This may lead to issues such as wrong results, deadlocks, or race conditions. In general, a synchronization mechanism is needed to control access to those resources. Barriers, semaphores, and locks are synchronization mechanisms that are widely used for this purpose. We will consider similar synchronization mechanisms in this work. For instance, we verify concurrent programs using Phasers [28, 41], a flexible synchronization construct that can capture both barriers and semaphores and that is implemented in modern languages such as Habanero Java.

Correctness properties for programs are usually divided into *liveness properties* and *safety properties* [117]. Safety properties intuitively mean that something bad never happens. Such properties include, for instance, the validity of assertions, race condition, and deadlock freedom. On the other hand, liveness properties indicate that something good will eventually happen. In this work, we focus on the verification of safety properties of synchronized concurrent programs.

Checking the correctness of concurrent programs is a complex task due to the interleavings of processes. Sometimes, violation of the correctness properties in such systems causes human or resource losses; therefore, it is crucial to check their correctness. For instance, several deficiencies in the development process of Therac-25 software led to race conditions between concurrent processes that resulted in overdoses of radiation, causing the death of several cancer patients between 1985 and 1987 [129]. The first operational launch attempt of a space shuttle in the Aegis air-defense system failed due to a synchronization problem among its flight-control computers [95].

There are different approaches to analyzing the correctness of software. Such approaches differ in terms of guarantees they provide for the correctness of the target system. An analysis technique is *sound* for specific errors if a program is actually error-free, whenever the technique claims that it is. On the other hand, a technique is *complete* if a program is erroneous whenever the technique claims that it is. An analysis technique that is both sound and complete is called *exact*. Designing an algorithm to answer whether or not a program is correct would amount to solving the halting problem of a Turing machine. Therefore, automatic approaches cannot be exact and guaranteed to terminate in general.

Two main approaches to software analysis are testing and formal verification. Testing is a dynamic analysis approach in which the correctness of a program is checked by providing test cases and validating their outputs. Although testing can help to discover many bugs at a low cost, it cannot prove the correctness of a program since it is not sound. As Dijkstra puts it, "Program testing can be used to show the presence of bugs, but never to show their absence" [49]. For example, the Ptolemy project [45] developed at Berkeley involved a case that illustrated the inability of testing for assuring the correctness of concurrent software. This project used Java threads for concurrent computation. After finalizing the implementation in early 2000, the software was reviewed by concurrency experts and tested extensively using regression tests that achieved 100 percent code coverage. Then, the system was widely used without any problem until it deadlocked in April 2004 [94]. However, if formal verification had been successfully used to check for deadlocks, this bug would not have been missed.

1.2 Formal Verification

The approach for *proving* software correctness is formal verification. Formal verification is a term for several techniques, including deductive verification, abstract interpretation, and model checking.

Deductive verification [74, 92, 125]. In *deductive verification*, correctness of a program is established using axioms and proof rules. The primary approach among deductive verification techniques is the use of Hoare triples [73]. In this technique, the correctness of a program is proved by annotating different blocks of it (e.g., loops and function calls) by Hoare triples. A Hoare triple is of the form $\{S\}P\{Q\}$ where P is the annotated program block, and S is the precondition that P must satisfy to ensure that the postcondition Q holds after its execution, given that P terminates. This approach builds correctness proofs incrementally and manually and may benefit partially from theorem provers to check the proofs. Deductive verification is a time-consuming and complicated task that needs to be performed by experienced logicians or mathematicians. Since it is a costly approach, it is usually used only for systems whose correctness is critical, such as security protocols. Deductive verification can be applied to infinite-state systems. However, this method is not fully automatic, and even if the system is correct, no limit can be put on the amount of time or memory required to obtain the correctness proof.

Abstract interpretation [42, 43, 44] is a verification technique in which the program is analyzed using an abstract domain and abstract transformation. In fact, the concrete program is interpreted in the abstract domain using

an abstraction function. Then, the abstract system is checked for correctness. Abstract interpretation techniques can check both liveness and safety properties. However, they often have difficulties with concurrency. Abstract interpretation has been used successfully in analyzing a wide range of programs, including the flight-control software of the Airbus A380, the world’s largest passenger aircraft [118].

Model checking [32, 37, 111] is a technique for formal verification that aims at automatically establishing the correctness of a program through an exhaustive search of the behavior of the system expressed in terms of temporal properties. Model checking is applied to a model of the program and is suitable for concurrent programs. It was initially introduced for verifying finite-state concurrent systems, and extending it to infinite-state systems is an active research area. The state-space of a system can be infinite because of infinite data types, recursion, or concurrency. The latter is the focus of this thesis.

Deductive verification, abstract interpretation, and model checking have different communities but have gradually exchanged ideas. For instance, model checking has used abstract interpretation ideas to abstract the state-space of the system [14, 36, 51] in order to tackle the infinite state-space problem.

1.3 Objective

In this thesis, our focus is on the formal verification of *parameterized systems*. A system can be parameterized in different ways, such as the size of its arrays or the number of processes that execute it. In this work, we use the term parameterized systems to mean those systems in which the number of executing processes is not bounded a priori. In many concurrent algorithms and distributed protocols such as cloud architectures and sensor networks, an arbitrary number of processes run the program. For this reason, it is interesting to prove the correctness of parameterized systems regardless of the number of processes that execute them. This is called *parameterized verification* [2]. Formal verification is challenging per se, and making it parameterized exacerbates the situation even further.

Our goal is to provide fully-automatic and parameterized model checking techniques for establishing the correctness of certain classes of concurrent programs with regard to safety properties. We consider different synchronization paradigms from those implemented using shared program variables, to dynamic synchronization constructs such as Phasers, to static synchronization of distributed protocols.

1.4 Previous Work

It is proved that the verification problem of parameterized programs is undecidable even for those involving simple processes [11, 103]. That is, there is no exact algorithm for checking the correctness of parameterized programs in general. However, parameterized verification has been an active line of research for decades, and extensive studies have been conducted to find decidable subclasses of the general problem. Pnueli and Zuck [130] survey the works on different techniques for parameterized verification. In general, two paths are taken as a remedy to undecidability: looking for restricted decidable subclasses of parameterized systems or developing sound, but necessarily incomplete verification techniques in the hope of finding proofs for the target systems.

1.4.1 Decidable Subclasses of Parameterized Systems

German and Sistla [64], and Emerson and Namjoshi [56] showed that parameterized verification is decidable for certain classes of synchronously communicating processes. Emerson and Kahlon [55] verify a general parameterized system that can contain several different types of processes. However, it requires all of the transition guards be either conjunctive or disjunctive. Abdulla et al. [2] proved that checking the reachability of an upward closed set of states for well-structured transition systems (WSTS) [2, 59], such as Petri nets [112], is decidable. Well-structured transition systems are infinite-state systems with a special partial order on their states, which is compatible with the transitions of the system. More precisely, the transitions of a WSTS are monotonic with respect to the order of its states (more details in Chapter 3).

1.4.2 Beyond Decidable Subclasses

Inductive techniques. The techniques in [33, 34, 84] are based on induction on the number of processes in a network. Such methods are called inductive methods. Inductive methods usually require the user to provide inductive invariants of the system. Then, they check small instances of the system using those invariants and generalize them to be valid for any instance of the system. The method of *invisible invariants* [12, 109], on the other hand, automatically generates such induction assertions for safety and liveness verification of numerous systems that have finite data. In techniques based on induction, the challenge is to generalize the guessed invariant to find an invariant of the system that proves the property.

Deductive techniques. Owicki and Gries [106] introduced a deductive proof system for parallel programs using Hoare triples. This technique was later extended to parameterized programs in [105]. IronFleet [69] uses a mix-

ture of TLA-style [92] and Hoare-logic to verify implementations of distributed protocols. Pandya and Joseph [108] introduced a logic similar to Hoare triples for safety verification of distributed protocols. In deductive techniques, an expert has to manually model the system and come up with the inductive invariants of the system and use a prover to validate them. Finding the right predicates and invariants for the proof is usually challenging.

Cut-off detection techniques. Empirical evidence suggests that parameterized systems often benefit from the *small model property*. Intuitively, the idea of the small model property is that in order to prove the correctness of a parameterized system, it is enough to analyze only a small number of the processes and their interactions. The upper bound on the number of processes to be analyzed is called the *cut-off point*. For many systems, such a point can be found statically before starting the verification process [80, 124]. For others, the cut-off point detection needs to be dynamic, that is, a dynamic approach starts with a small cut-off point and iteratively increments it in the hope of establishing the proof.

This idea gave rise to works in [6, 80, 81, 124]. The methods presented in [6] and [81] have a dynamic approach to cut-off point detection, unlike the other ones which find it statically. The work in [99] has introduced a cut-off detection technique for a syntactically restricted class of distributed protocols.

Static cut-off detection approaches tend to restrict communication in the underlying systems. For instance, the work in [99] restricts the syntax of the guards used in the communication protocols, and the approach of [80] restricts itself to concurrent programs using locks.

Abstraction-based techniques. Such techniques try to reduce the size of the parameterized system by disregarding details that are unnecessary for the correctness proof. For example, *counter abstraction* [75, 97, 110] abstracts the system by counting the number of processes in each local state. The values of those counters, as well as a shared valuation, represent the state of the abstract system. Counter abstraction is not directly applicable to systems where the local states are infinite. Konnov et al. [79] introduced *parametric interval abstraction* as well as a new form of counter abstraction in which counters range over an abstract finite domain. They used this abstraction for parameterized verification of several distributed protocols [79, 86, 87, 88].

Symmetric concurrent programs can be analyzed by the *predicate abstraction* technique introduced in [51], which aims at reducing the state-space of each process to a finite space. This technique is based on [67], which introduced predicate abstraction for sequential programs. *Environment abstraction* [39] uses ideas from counter abstraction to enrich predicate abstraction.

Thread-modular analysis [60, 70] is another abstraction-based technique in which the state of a parameterized program is represented by the state

of one process, as well as an *environment* which abstracts the interference of other processes. This is intended to address the exponential complexity in the control state of concurrent programs.

As opposed to most of the abstraction techniques that aim to reduce the system to a finite system, *monotonic abstraction* [4, 5, 8] over-approximates a system into an infinite Well-Structured Transition System. Constrained monotonic abstraction (CMA) [3] allows for an iterative refinement scheme for monotonic abstraction (more details in Chapter 3).

In abstraction techniques, in general, it is challenging to find the right abstraction which is precise enough to not result in false alarms, but which disregards details that are irrelevant to the correctness property.

1.5 Contributions

We have made the following contributions for the verification of concurrent systems using complex synchronization constructs:

- We addressed the problem of automatically checking safety properties for *Phaser programs*. Phaser programs are concurrent programs that make use of a complex synchronization construct called *Phasers*. These are implemented in languages such as Habanero Java. We established the decidability of checking the violation of program assertions and the undecidability of checking deadlock-freedom.
- We again addressed the problem of automatically checking safety properties for Phaser programs, but, this time, we considered programs that might generate arbitrarily many processes and phasers (i.e., *parameterized* Phaser programs). We studied different formulations of the verification problem. We also proposed a procedure that is exact and guaranteed to terminate for some reachability problems, even for unbounded phases and arbitrarily many spawned processes.

For the verification of concurrent programs using integer shared variables for synchronization, we have made the following contributions:

- We proposed an approach to automatic verification of parameterized concurrent programs in which shared variables are manipulated by atomic transitions for the purpose of counting and synchronizing the spawned processes. For this purpose, we introduced *counting predicates* that mix counters referring to the number of processes that satisfy some given properties and variables that are directly manipulated by the concurrent processes. We then combined existing works on counter, predicate, and constrained monotonic abstraction and built a nested counterexample-based refinement scheme for establishing correctness.

- We introduced Lazy Constrained Monotonic Abstraction for a more efficient exploration of well-structured abstractions of infinite-state non-monotonic systems. We proposed several heuristics and assessed the efficiency of the proposed technique by performing extensive experiments using our open-source prototype tool.

For the verification of distributed protocols, we have made the following contribution:

- We proposed a sound but (in general) incomplete procedure for automatic verification of safety properties for a class of fault-tolerant distributed protocols described in the Heard-Of (HO for short) model. The HO model [23, 30] is a popular model for describing distributed protocols. HO protocols proceed in rounds where, at each round, each process sends a message to other processes, “hears” from a set of them, and updates its state. Fault descriptions are captured by stating constraints on the possible sets of processes and messages each process hears from (e.g., each process hears from at least half of the processes). This model uniformly describes distributed protocols in the presence of transmission-based faults and captures a wide range of such faults being static or dynamic, transient, or permanent. We proposed a verification procedure that is guaranteed to terminate even for unbounded number of processes that execute the distributed protocol.

For most of the proposed techniques, we have implemented a prototype tool, which is publicly available and shows the application of the proposed method on a set of benchmarks. The results are reported in the thesis.

1.6 Thesis Overview

The following is an overview of the remaining chapters of the thesis.

In Chapters 2 and 3, we give some background and preliminaries. In Chapter 4, we present our first approach for verification of Phaser programs, which sets an upper bound on the number of processes in the program.

In Chapter 5, we push the verification of Phaser programs one step further and introduce another approach that conducts parameterized verification of Phaser programs. As opposed to the previous approach, this one does not set an upper bound on the number of processes and phasers.

In Chapter 6, we present a verification technique for parameterized verification of the safety of concurrent programs that implement complex synchronization schemes. This work combines counter, predicate, and monotonic abstractions. In Chapter 7, we investigate several methods for lazily refining monotonic abstraction.

In Chapter 8, we introduce a novel approach for parameterized verification of fault-tolerant distributed protocols expressed in the Heard-Of model in the presence of different fault models. The approach is followed by proof of soundness and termination.

In Chapter 9, we conclude the thesis by summarizing the primary outcomes of our work and elaborating on potential directions for further development.

1.7 Publication Overview

The content of the thesis is based mainly on the following journal article and conference papers:

1. Zeinab Ganjei, Ahmed Rezine, Petru Eles, and Zebo Peng. “Safety verification of Phaser programs”. In: *Proceedings of the 15th International Conference on Formal Methods in Computer Aided Design, FMCAD 2017, Vienna, Austria, October 2017*. IEEE, 2017, pp. 68–75.
2. Zeinab Ganjei, Ahmed Rezine, Ludovic Henrio, Petru Eles, and Zebo Peng. “On reachability in Parameterized Phaser programs”. In: *Proceedings of the 25th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS 2019, Prague, Czech Republic, April 2019*. Springer, 2019, pp. 299–315.
3. Zeinab Ganjei, Ahmed Rezine, Petru Eles, and Zebo Peng. “Abstracting and Counting Synchronizing Processes”. In: *Proceedings of the 16th International Conference on Verification, Model Checking, and Abstract Interpretation, VMCAI 2015, Mumbai, India, January 2015*. Springer, 2015, pp. 227–244.
4. Zeinab Ganjei, Ahmed Rezine, Petru Eles, and Zebo Peng. “Counting dynamically synchronizing processes”. In: *International Journal on Software Tools for Technology Transfer* 18.5 (2016), pp. 517–534.
5. Zeinab Ganjei, Ahmed Rezine, Petru Eles, and Zebo Peng. “Lazy constrained monotonic abstraction”. In: *Proceedings of the 17th International Conference on Verification, Model Checking, and Abstract Interpretation, VMCAI 2016, St. Petersburg, FL, USA, January 2016*. Springer, 2016, pp. 147–165.
6. Zeinab Ganjei, Ahmed Rezine, Petru Eles, and Zebo Peng. “Verifying safety of parameterized Heard-Of algorithms”. In: *Proceedings of the 8th International Conference on NETWORKed SYStems, NETYS 2020, Marrakech, Morocco, June 2020*. Springer, 2020, pp. 209–226 .

The publications listed above correspond to the chapters of the thesis as follows. The work on Phaser programs verification presented in Paper 1 is

covered in Chapter 4. The approach for parameterized verification of Phaser programs in Paper 2 is presented in Chapter 5. Paper 4 is the journal version of Paper 3. Its verification technique is introduced in Chapter 6 which targets general synchronization schemes using shared variables. Paper 5 introduces a method for optimizing constrained monotonic abstraction and is discussed in detail in Chapter 7. The work in Paper 6 on parameterized verification of a class of Heard-Of algorithms is covered in Chapter 8.

Lastly, the following conference paper uses the verification technique of Chapter 7 but is not discussed in the thesis:

1. Parosh Aziz Abdulla, Mohamed Faouzi Atig, Zeinab Ganjei, Ahmed Rezine, and Yunyun Zhu. “Verification of Cache Coherence Protocols wrt. Trace Filters”. In: *Proceedings of the Formal Methods in Computer-Aided Design, FMCAD 2015, Austin, Texas, USA, September 2015*, IEEE, pp. 9-16.

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2

Background

This work focuses on extending abstraction and model checking of infinite-state systems to new classes of concurrent and distributed programs. In this chapter, we recall several synchronization mechanisms that are widely used in concurrent programs, as well as ideas from abstraction and model checking of safety properties.

2.1 Synchronization Mechanisms

In the following, we recall three widely-used synchronization mechanisms in concurrent systems, namely, barriers, semaphores, and locks. We also briefly introduce Phasers, which are a modern and dynamic synchronization paradigm.

A *barrier* for a group of processes means that any process must stop at this point and cannot proceed until all other processes reach the barrier. Barriers can be either *static* or *dynamic*. With static barriers, the number of processes that have to stop at the barrier is fixed and is usually set at the time of the barrier's instantiation. On the other hand, with dynamic barriers, processes can be dynamically added as participants in the barrier or can remove themselves from the synchronization. Dynamic barriers, however, are more challenging to implement than static barriers.

A *semaphore* [116] is a non-negative integer variable on which only two atomic operations - *wait* and *signal* - can be performed. The *wait* operation decrements the value of the semaphore variable by 1 if it is positive.

Otherwise, the process that executes *wait* is blocked. The *signal* operation increments the value of the semaphore variable by 1. The supporting environment must provide the atomicity of the semaphore implementation. A semaphore with a maximum value of 1 is called a *binary semaphore*.

Locks [116] are another synchronization construct that are usually implemented by binary semaphores and have two operations, namely, *acquire* (instead of *wait*) and *release* (instead of *signal*).

Phasers are a synchronization mechanism introduced with the Habanero Java API [28, 41]. They implement a flexible synchronization paradigm, which is general enough to model static and dynamic barriers and semaphores. Intuitively, a phaser associates two phases (hereafter *wait* and *signal* phases) with each registered process. Apart from creating phasers and registering each other to them, processes can individually issue **wait** and **signal** commands to a phaser they are registered with. Intuitively, **signal** commands are used to inform other registered processes that the issuing process is done with its *signal* phase. The **wait** command is instead used to check whether all registered processes are done with the *wait* phase of the issuing process. This command may get blocked by a process that has not yet finished the corresponding phase.

2.2 Model Checking

Model checking [13, 32, 37, 111] is a fully automated method for analyzing systems modeled as Kripke structures or transition systems. Model checking is widely used for verification of hardware and software systems. It conducts an exhaustive search on the behavior of a system model to determine whether or not the desired specifications hold. To perform Model Checking, first, correctness properties or specifications of systems are written in temporal logic. Then, a decision procedure determines whether the model satisfies those properties. If not, the model checker outputs a counterexample that witnesses that the model violates the correctness property. Model checking is based on the algorithmic exploration of state-spaces.

2.2.1 Transition Systems

Transition systems are often used to capture the semantics of software systems. A transition system is a tuple $\mathcal{TS} = (\mathbf{C}, \mathbf{C}_{init}, \mathbf{T}, \rightarrow)$ where $\mathbf{C} = \{c_1, c_2, \dots\}$ is a set of configurations of the system (for a program, configurations typically correspond to the values of all the program variables at a particular time), $\mathbf{C}_{init} \subset \mathbf{C}$ is the set of initial configurations (i.e., configurations from which the program may start execution), and $\rightarrow \subseteq \mathbf{C} \times \mathbf{T} \times \mathbf{C}$ is a set of transitions between configurations. A *run* ρ is an alternating sequence of configurations and transitions starting with a configuration, $c_0; t_0; c_1; t_1; \dots; c_n$, such that c_0 is an initial configuration and for each $i \geq 0$, we have $c_i \xrightarrow{t_i} c_{i+1}$.

We use $\rightarrow^*$ for the reflexive and transitive closure of $\rightarrow$. We write $c \rightarrow^* c'$ to mean that c' is reachable from c . A configuration is said to be reachable if it is reachable from some initial configuration c_{init} in C_{init} .

A transition system $\mathcal{TS}$ can have an infinite set of configurations. We say that such a system is an *infinite* transition system.

Given a transition system $\mathcal{TS} = (C, C_{init}, T, \rightarrow)$, and a target configuration $c_{tgt} \in C$, the *reachability problem* asks whether the target configuration c_{tgt} is reachable in $\mathcal{TS}$. Also, the *coverability problem* asks whether some configuration $c \in C$ is reachable in $\mathcal{TS}$ such that $c_{tgt} \leq c$ where $\leq$ is a partial order on configurations. Extending a transition system with a labeling function $\mathcal{L} : C \rightarrow 2^A$ that labels its configurations with respect to a non-empty set of atomic propositions A results in a Kripke structure [89].

2.2.2 Specification

The specification of a system states the properties the system must satisfy. Such properties are usually expressed in terms of some logical formalism. The underlying logic determines the expressive power of the specification; for instance, propositional logic has less power than temporal logics. Temporal logics are formalisms that describe temporal (as in sequences) behaviors of reactive systems. Two popular temporal logics are computation-tree logic (CTL) and linear-time logic (LTL) [37]. In LTL (also called propositional temporal logic), one can encode formulae about the future of paths; for instance, a formula that specifies some property will *eventually always* hold on *all* paths. In CTL, on the other hand, the model of time is a tree-like structure (the computation tree) in which the future is not determined; there are different paths in the future, any one of which might be an actual path that is realized. A computation tree is obtained by unfolding the Kripke structure. The root of each computation tree is some initial configuration c_0 . CTL can specify, for instance, that there *exists* a path such that from its second state onward, some property holds.

Two classes of properties are often considered for showing the correctness of a system, i.e., *liveness properties* and *safety properties*. In this thesis, we focus on verifying safety properties for concurrent and distributed systems.

Safety Properties

Intuitively, a safety property expresses that “nothing bad happens” during the execution of the system. In many systems, Safety properties can be captured by determining a set of *bad* configurations and asserting that the system does not end up in one of those. A system is considered to be unsafe if some of its bad configurations are reachable. Otherwise, the system is called safe.

2.2.3 Decidability

The model checking problem $M \models S$ is the decision problem, which asks whether the model M (often given as a Kripke structure) satisfies some temporal property S characterizing configurations of the Kripke structure. This problem is *decidable* for a set of models and properties if there is a terminating procedure that can answer it for any model and property in the considered sets. In this work, where we consider safety properties, this boils down to deciding the reachability of sets of configurations.

2.3 Abstraction

With abstraction, the details that are not relevant to the property to be checked are removed (*abstracted away*), which usually reduces the size of the state-space. This approach is used widely in model checking to tackle the state-space explosion problem. To perform abstraction, one has to implicitly build the abstract transition system from a high-level description of the system [37] and refine it if needed, for example, using the Counterexample-Guided Abstraction Refinement technique (introduced in Section 2.3.1).

In this thesis, we use Existential Abstraction [36]. A transition system $\hat{\mathcal{T}}\mathcal{S} = (\hat{\mathcal{C}}, \hat{\mathcal{C}}_{init}, \hat{\mathcal{T}}, \hat{\rightarrow})$ is an existential abstraction of $\mathcal{T}\mathcal{S} = (\mathcal{C}, \mathcal{C}_{init}, \mathcal{T}, \rightarrow)$ where:

- the abstraction of any initial configuration in $\mathcal{C}_{init}$ is an initial configuration of the abstract system; and
- if two configurations are related with $\rightarrow$, their abstractions must be related with $\hat{\rightarrow}$.

A concretization relation indicates a set of concrete configurations for each abstract configuration. An abstract model constructed using existential abstraction is sound for reachability problems. That is, if some abstract set of configurations is not reachable in the abstract system, then its concretization is not reachable in the concrete one either. Such an abstract model is an *over-approximation* of the concrete model. That is, it might allow more behavior than the concrete model. Hence, if the abstract system is safe, so is the concrete one. An example of such abstraction is Predicate Abstraction (see Section 6.5.1).

2.3.1 Counterexample-Guided Abstraction Refinement

The abstract models obtained through existential abstraction usually contain less information than the original ones. Therefore, model checking the abstract model might produce incorrect results, such as spurious traces to error. The framework of CounterExample-Guided Abstraction Refinement

(CEGAR) [35] is used for refining an abstraction to eliminate false positives and hence, make the abstraction more precise.

The CEGAR framework consists of the following steps:

1. Given a model M and a specification S , generate an initial abstract model $\hat{M}$.
2. Model check $\hat{M}$ with respect to the specification S . If $\hat{M}$ satisfies S , then M also satisfies it and conclude. If a counterexample $\hat{\rho}$ is found, check whether the trace $\hat{\rho}$ is also possible in the concrete model. If this is the case, conclude that M does not satisfy S and stop. Otherwise, $\hat{\rho}$ is spurious. Continue to step 3.
3. Refine the abstract model so that the counterexample $\hat{\rho}$ will not be included in the refined abstract model. Return to step 2.

As stated earlier, each abstract configuration usually represents a set of concrete configurations. For refinement, usually, one or more abstract configurations are partitioned.

3

Preliminaries

In this chapter, we introduce some notations, definitions, and theorems used in the subsequent chapters. We present counter machines, well-structured transition systems, a reachability analysis procedure, monotonic abstraction, and its refinement.

3.1 Preorders, Configurations, and Constraints

The binary relation $\leq$ is a preorder or quasi-ordering over a set A , if it is reflexive and transitive. A quasi-ordering $\leq$ is a well-quasi-ordering ($\mathcal{WQO}$) over a set A if there is no infinite sequence $a_0, a_1, \dots$ of elements in A such that $a_i \not\leq a_j$ for all $i < j$. We often capture possibly infinite sets of configurations with a symbolic representation φ and call it a *constraint*. A constraint can, for instance, be the upward closure of some configuration c with respect to a quasi-ordering $\leq$, or a logical formula that characterizes the configurations in the set. Given a configuration c and a constraint φ , we say that c satisfies φ , and write $c \models \varphi$, if c satisfies conditions imposed by φ . We define the set of configurations *denoted* by (or characterized by) φ as $\llbracket \varphi \rrbracket = \{c \mid c \models \varphi\}$.

The image computation function $\text{next}_t(\varphi)$ gives the set of all of the images of φ with respect to the transition t . $\llbracket \text{next}_t(\varphi) \rrbracket \cup \llbracket \varphi \rrbracket$ can be used to over-approximate the set $\{c' \mid c' \xrightarrow{t} c \text{ and } c \in \llbracket \varphi \rrbracket\}$ of predecessors, or the set $\{c' \mid c \xrightarrow{t} c' \text{ and } c \in \llbracket \varphi \rrbracket\}$ of successors of configurations denoted by φ . We abuse notation and use $\text{next}_t(\Phi)$ on a set of constraints as well.

We say a preorder $\sqsubseteq$ on constraints is *sound* if $\varphi \sqsubseteq \varphi'$ implies $\llbracket \varphi' \rrbracket \sqsubseteq \llbracket \varphi \rrbracket$. Also, an image computation function $\mathbf{next}_t(\varphi)$ used to approximate successor (respectively, predecessor) configurations with respect to transition t is said to be *sound* if for any arbitrary configuration $c \in \llbracket \varphi \rrbracket$ where $c \xrightarrow{t} c'$ (respectively, $c' \xrightarrow{t} c$), c' is in $\llbracket \mathbf{next}_t(\varphi) \rrbracket \cup \llbracket \varphi \rrbracket$. On the other hand, $\mathbf{next}_t(\varphi)$ is *complete* if for all configurations c' denoted by $\mathbf{next}_t(\varphi)$, there exists a configuration $c \in \llbracket \varphi \rrbracket$, where $c \xrightarrow{t} c'$ (respectively, $c' \xrightarrow{t} c$). We say that $\mathbf{next}_t$ is *exact* when it is both sound and complete. We always require soundness of $\mathbf{next}_t$ and $\sqsubseteq$.

With respect to a $\mathbf{next}$ function which is clear from the context, we define a trace $\rho = \varphi_0; t_1; \dots; t_n; \varphi_n$ to be an alternating sequence of constraints and transitions that starts with a constraint. Moreover, for $i : 0 \leq i < n$, $\varphi_{i+1} \in \mathbf{next}_{t_{i+1}}(\varphi_i)$.

3.2 Counter Machines

A counter machine is a tuple $M = (\mathcal{Q}, \mathcal{C}, T, \mathcal{Q}_{init}, q_{trgt})$ where $\mathcal{Q}$ is a finite set of states of the machine, $\mathcal{C}$ is a finite set of counters (i.e., variables ranging over the natural numbers $\mathbb{N}$), T is a finite set of transitions, $\mathcal{Q}_{init} \subseteq \mathcal{Q}$ is a set of initial states of the machine, and q_{trgt} is a state in $\mathcal{Q}$. A transition t in T is of the form $(q : op : q')$ where $src(t) = q$ is the source state, $dst(t) = q'$ is the destination state, and the operation op is either the identity operation $\mathbf{nop}$, or a guarded command $\mathbf{grd} \Rightarrow \mathbf{cmd}$. A guard $\mathbf{grd}$ is a predicate on counters. e is an expression over $\mathcal{C}$ and a command $\mathbf{cmd}$ is a multiple assignment $ct_1, \dots, ct_n := e_1, \dots, e_n$ that involves assignments of values of expressions $e_1, \dots, e_n$ to pairwise different counters $ct_1, \dots, ct_n$ in $\mathcal{C}$.

A *machine configuration* c is a pair $(q, \mathbb{c})$ where q is a state in $\mathcal{Q}$ and $\mathbb{c}$ is a mapping $\mathcal{C} \rightarrow \mathbb{N}$. $\mathcal{C}$ is the set of all such configurations. The set of *initial* machine configurations $\mathcal{C}_{init}$ consists of those configurations from which the execution of the system may start. More specifically, $\mathcal{C}_{init}$ contains configurations of the form $(q, \mathbb{c})$ where $q \in \mathcal{Q}_{init}$ and $\mathbb{c}$ maps every counter to zero. The image computation function $\xrightarrow[t_M]{} \subseteq \mathcal{C} \times T \times \mathcal{C}$ is defined as follows:

$(q, \mathbb{c}) \xrightarrow[t_M]{} (q', \mathbb{c}')$ (or $(q, \mathbb{c}) \xrightarrow{t} (q', \mathbb{c}')$ for short in case M is clear from the context) iff $src(t) = q$, $\mathbb{c} \models \mathbf{grd}$, $dst(t) = q'$, and $\mathbb{c}'(ct_i) = e_i[\mathbb{c}]$ for $i : 1 \leq i \leq |\mathcal{C}|$, where $e_i[\mathbb{c}]$ is the result of evaluating the expression e_i . Considering a preorder $\leq$ on the machine configurations, a set of configurations $\mathcal{C}$ is said to be *upward-closed* if for every configuration $c \in \mathcal{C}$, $c \leq c'$ implies $c' \in \mathcal{C}$. An M run ρ_M is a sequence $(q_0, \mathbb{c}_0); t_1; \dots; t_n; (q_n, \mathbb{c}_n)$. This run is *feasible* if $(q_0, \mathbb{c}_0)$ is initial and $(q_i, \mathbb{c}_i) \xrightarrow{t_{i+1}} (q_{i+1}, \mathbb{c}_{i+1})$ for $i : 0 \leq i < n$. We write $\rightarrow$ for $\cup_{t \in T} \xrightarrow[t_M]{}.$

Counter machines are a popular model for parameterized systems in which both the shared and local configurations of each process are finite. Intuitively, each counter can represent a specific process-local configuration. The value of the counters can capture the number of processes in different local configurations. Moreover, the counter machine state can model the shared configuration of the system, e.g., the valuation of the shared variables. Violation of mutual exclusion can be easily captured for such a system by representing the critical section with a counter that has a value of at least two.

3.2.1 Safety Problem of a Counter Machine

The safety property of a counter machine can be formulated in terms of a set $\mathcal{C}_{bad}$ of bad configurations. These are configurations that should not occur during the execution of the system. Thus, checking the safety property can be reduced to checking the reachability of the set of bad configurations.

If the set $\mathcal{C}_{bad}$ is upward-closed, checking the safety property amounts to checking the reachability of an upward-closed set, which is called the *coverability problem*. For instance, the set $\mathcal{C}_{bad}$ that formulates violation of the mutual exclusion property is an upward-closed set and checking it results in a coverability problem.

The *state reachability* or *control reachability* problem of the counter machine for target state $\mathbf{q}_{tgt}$ is to decide whether there is an $\mathbf{M}$ -feasible run $(\mathbf{q}_0, \mathbb{C}_0; \tau_1; \dots; \tau_n; (\mathbf{q}_n, \mathbb{C}_n))$ such that $\mathbf{q}_n = \mathbf{q}_{tgt}$. We say that the run *covers* the state $\mathbf{q}_n$. The *state reachability* of a counter machine is a *coverability problem*.

We often use classical results about 2-counter Minsky machines and vector addition systems with reset arcs in this work. In the following section we introduce these counter machines and the results that we use.

3.2.2 2-counter Minsky Machine

Let $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{init}, \mathbf{q}_{tgt})$ be a counter machine with two counters $\mathcal{C} = \{\mathbf{ct}_1, \mathbf{ct}_2\}$ initially at zero. Let the transitions in $\mathbf{T}$ be either an increment ($\mathbf{q} : \mathbf{true} \Rightarrow \mathbf{ct}_i := \mathbf{ct}_i + 1 : \mathbf{q}'$), a decrement ($\mathbf{q} : \mathbf{ct}_i \geq 1 \Rightarrow \mathbf{ct}_i := \mathbf{ct}_i - 1 : \mathbf{q}'$) or a test for zero ($\mathbf{q} : \mathbf{ct}_i = 0 \Rightarrow \mathbf{nop} : \mathbf{q}'$) for $i \in \{1, 2\}$. This is called a *2-counter Minsky machine*.

Theorem 1. *The state reachability problem of a 2-counter Minsky machine is undecidable [103].*

The idea is to show that for every Turing machine there is a 2-counter Minsky machine that simulates it. The halting problem of a Turing machine can be reduced to the state reachability problem of a 2-counter Minsky machine. One way is to perform the following encoding with three steps. First, a Turing machine can be simulated by a finite-state machine equipped with

two stacks. Then, a stack can be simulated by two counters. Finally, four counters can be simulated by two counters.

We will often use this result to show that fundamental verification problems of a concurrent system are undecidable. We can reduce the state reachability problem of a Minsky machine into its reachability problem. Hence, the reachability problem of Minsky machines is also undecidable.

3.2.3 Vector Addition Systems

A Vector Addition System (VAS) is a counter machine $M = (\mathcal{Q}, \mathcal{C}, T, \mathcal{Q}_{init}, \mathbf{q}_{trgt})$ with only the increment ($\mathbf{q} : \text{true} \Rightarrow \text{ct}_i := \text{ct}_i + 1 : \mathbf{q}'$) and decrement ($\mathbf{q} : \text{ct}_i \geq 1 \Rightarrow \text{ct}_i := \text{ct}_i - 1 : \mathbf{q}'$) operations. VAS with *reset* arc (or reset-VAS) is a VAS extended with the reset operation, ($\mathbf{q} : \text{true} \Rightarrow \text{ct}_i := 0 : \mathbf{q}'$) for $i \in \{1, \dots, n\}$.

Theorem 2. *The coverability problem of VAS with reset arcs is decidable [54].*

The argument for Theorem 2 is presented in Section 3.3, where we describe a reachability procedure that can be shown to be exact and to terminate for coverability of WSTS (Section 2) in general and VAS with reset arcs in particular.

Theorem 3. *The reachability problem of VAS with reset arcs is undecidable [54].*

A 2-counter Minsky machine can be modeled as a reset-VAS. The intuition is that for every counter ct in the Minsky machine, the encoded VAS will contain two counters, ct and ct' , and their values must always be equal. Increments and decrements will update both ct and ct' in the same way. Each test for zero of ct will be modeled by resetting ct' . In this way, the first time ct is tested for zero in the Minsky machine when $\text{ct} > 0$, the values of ct and ct' in the VAS will become different, which can never be recovered. In the Minsky machine, this step is not possible since $\text{ct} > 0$.

We will use this result to show that checking deadlock freedom for certain phaser programs is undecidable in general.

3.3 Working-List Procedure

A consequence of Theorem 1 is that model checking infinite transition systems is undecidable in general. In the procedure “explore”, we adopt a working-list reachability analysis to search the infinite state-space. The intuition is to adopt constraints instead of configurations in the procedure “explore” below.

Input: A set of constraints Φ_{start} , a predicate **isTarget** over constraints, a set of image computation functions $\mathbf{next}_t$ for each $t \in T$, and a preorder $\sqsubseteq$ over constraints.

Output: **unreachable** or a trace $\rho = \varphi_0; t_1; \varphi_1; \dots; t_n; \varphi_n$

```

1 initialize both working and store to  $\{(\varphi, \varphi) \mid \varphi \in \Phi_{start}\}$ ;
2 while exists  $(\varphi, \rho) \in \mathbf{working}$  do
3   remove  $(\varphi, \rho)$  from working;
4   if isTarget $(\varphi)$  then return  $\rho$ ;
5   foreach  $t \in T$  do
6     foreach  $\varphi' \in \mathbf{next}_t(\varphi)$  do
7       if  $\varphi'' \not\sqsubseteq \varphi'$  for all  $(\varphi'', -) \in \mathbf{store}$  then
8         add  $(\varphi', \varphi'; \rho)$  to both working and store;
9 return unreachable;
```

Procedure $\mathbf{explore}(\Phi_{start}, \mathbf{isTarget}, T, \mathbf{next}, \sqsubseteq)$. A working-list procedure to check whether some target configuration can be obtained from one of the constraints in Φ_{start} using the image computation functions $\mathbf{next}_t$ for each $t \in T$, and the preorder $\sqsubseteq$ on the constraints. The predicate **isTarget** determines whether a constraint denotes a target configuration.

The procedure starts from a set of constraints Φ_{start} . Its goal is to check if some constraint for which **isTarget** evaluates to **true** can be obtained from one of the constraints in Φ_{start} using the image computation functions $\mathbf{next}_t$ for each $t \in T$. The procedure uses a set **working** whose elements are pairs (φ, ρ) where each pair consists of a constraint φ and the trace traversed from some $\varphi_{start} \in \Phi_{start}$ to φ . Such constraints can be obtained from a φ_{start} with a succession of $\mathbf{next} = \bigcup_{t \in T} \mathbf{next}_t$ and from them the procedure will continue the search. The set **store**, on the other hand, stores all the encountered (φ, ρ) during the analysis unless the encountered pair has a constraint φ' that is larger (with respect to $\sqsubseteq$) than some constraint that is already in **store**.

The procedure takes a possibly infinite loop (line 2) in which during each iteration, it first removes a pair (φ, ρ) from **working** (line 3). Then, it checks whether **isTarget** evaluates to **true** for φ (line 4). If so, it returns the trace ρ as a possible reachability witness. Otherwise, the procedure computes images of φ with respect to each of the system transitions (line 6). For each newly generated constraint φ' , the procedure checks whether another constraint $\varphi'' \sqsubseteq \varphi'$ had been previously encountered (line 7). In this case, we say that φ' is *subsumed* by φ'' . Otherwise, a new pair constructed with φ' and the trace traversed to it is added to both **working** and **store** sets (line 8). The procedure terminates when all of the newly generated constraints are subsumed by constraints that had been generated in previous iterations, which are stored in **store**. In such a case, **working** will eventually be empty, and **store** would be the set of constraints that denote configurations reachable from some φ_{start} . Under suitable assumptions, one can show that a target configuration is reachable from a φ_{start} iff a constraint in **store** denotes it.

The procedure “explore” can be used for forward or backward explorations. In forward exploration, Φ_{start} is a set of constraints that denote the set of initial configurations $\mathbf{C}_{init}$, $\mathbf{next}(\varphi)$ computes or approximates successors of the denotation of φ , and $\mathbf{isTarget}(\varphi)$ checks whether a bad configuration in $\mathbf{C}_{bad}$ is denoted by φ . In backward exploration, Φ_{start} is a set of constraints that denote a set of bad configurations $\mathbf{C}_{bad}$, $\mathbf{next}(\varphi)$ computes or approximates predecessors of denotations of φ , and $\mathbf{isTarget}(\varphi)$ checks whether a configuration in $\mathbf{C}_{init}$ is denoted by φ .

We say that $\sqsubseteq$, $\mathbf{isTarget}$ and $\mathbf{next}_t$ are *effective*, if the following conditions are satisfied:

- For every pair of constraints φ_1 and φ_2 , it is possible to check whether $\varphi_1 \sqsubseteq \varphi_2$,
- For every constraint φ , $\mathbf{isTarget}(\varphi)$ can be evaluated,
- For every constraint φ and t , the set $\mathbf{next}_t(\varphi)$ can be computed.

If $\sqsubseteq$, $\mathbf{isTarget}$ and $\mathbf{next}_t$ are effective, we say the procedure “explore” is effective.

In the following, we show some results about the procedure “explore”.

Corollary 1. *Let $\mathbf{working}_k$ and $\mathbf{store}_k$ be the sets **working** and **store** obtained at line 2 of the k^{th} iteration of the loop in procedure “explore”. Suppose $\mathbf{constraintOf}(\mathbf{store}_k)$ gives the constraints of the pairs stored in $\mathbf{store}_k$.*

- (a) $\mathbf{store}_k$ contains Φ_{start}
- (b) $\mathbf{working}_k$ is a subset of $\mathbf{store}_k$
- (c) for each element (φ, ρ) of $\mathbf{store}_k$ such that $(\varphi, \rho) \notin \mathbf{working}_k$ and for each transition $t \in T$, every image constraint φ' in $\mathbf{next}_t(\varphi)$ or some entailing constraint $\varphi'' \sqsubseteq \varphi'$ is in $\mathbf{constraintOf}(\mathbf{store}_k)$

Proof. We call a pair (φ, ρ) of constraint and trace an *element* for short.

Base case: $k = 0$. At line 1 of the procedure “explore”, an element containing the starting constraint is the only element added to both $\mathbf{store}_0$ and $\mathbf{working}_0$. As a result, $\mathbf{working}_0 = \mathbf{store}_0$ and the propositions (a), (b), and (c) hold for $k = 0$.

Suppose the propositions hold up to k . We show they hold for $k + 1$.

For proposition (a), note that an element that is added to **store** is never removed from it, and for proposition (b), note that whenever an element is added to $\mathbf{working}_k$, it is also added to $\mathbf{store}_k$ (lines 1 and 8 of “explore”). The intuition of proposition (c) is as follows. The algorithm always picks an element $(\varphi, \rho) \in \mathbf{working}_k$ and computes its images with respect to all transitions. Therefore, for any element $(\varphi', \rho') \notin \mathbf{working}_k$, the images must

be subsumed by elements in store_k . The only places where store_k and working_k are manipulated, are at lines 1, 3, and 8 of “explore”. At line 3, an element $(\varphi, \rho) \in \text{working}_k$ is removed from working_k . At line 6, for each transition $t \in T$, $\text{next}_t(\varphi)$ is computed, then all elements $(\varphi', \rho'; t; \rho)$ are added to working_k and store_k , in case no other element with constraint $\varphi'' \sqsubseteq \varphi'$ already exists in store_k . Thus, the images of any constraint removed from working_k are entailed by elements in store_k . $\square$

We show the following theorems for the backward “explore”. The forward case is similar.

Theorem 4 (Soundness of “explore”). *If next , isTarget , and $\sqsubseteq$ are sound, and the procedure “explore” returns **unreachable**, then no configuration c denoted by Φ_{start} is reachable from any configuration $c_{\text{init}} \in C_{\text{init}}$.*

Proof. By soundness of $\sqsubseteq$, only those constraints are not added to store that denote configurations already denoted by $\text{constraintOf}(\text{store})$. Moreover, by the soundness of next_t , the images of a constraint together with the constraint itself denote all of its predecessor configurations.

Suppose the algorithm returns **unreachable**. Then, at some iteration, there is no element (φ, ρ) in working . Combined with propositions (a), (b) and (c) of Corollary 1, we have that after termination, $\bigcup_{t \in T} \text{next}_t(\text{constraintOf}(\text{store})) \subseteq \text{constraintOf}(\text{store})$. Hence, all configurations that can reach Φ_{start} are denoted by $\text{constraintOf}(\text{store})$ and they do not contain any configuration in C_{init} by soundness of isTarget . $\square$

Definition 1. *We say isTarget is complete if whenever $\text{isTarget}(\varphi)$ evaluates to **true**, φ denotes some target configuration.*

Theorem 5. *Assume “explore” is a backward exploration. If next and isTarget are complete, and the procedure “explore” returns a counterexample, that counterexample is real.*

Proof. Assume that the procedure returns a trace $\rho = \varphi_n; t_n \dots t_1; \varphi_{\text{start}}$, then, the test at line 4 ensures that $\text{isTarget}(\varphi_n) = \text{true}$ for some (φ_n, ρ) . We proceed by contradiction. If the trace is not real, there is some $0 \leq i < n$, where there is some configuration c'' in $\llbracket \varphi_{i+1} \rrbracket$ which is not in the set $\{c' \mid c' \rightarrow c \text{ and } c \in \llbracket \varphi_i \rrbracket\}$. Intuitively, c'' causes the spurious trace. This contradicts completeness of next . $\square$

Theorem 6. *If the preorder $\sqsubseteq$ is a well-quasi-ordering (WQO), then the procedure “explore” terminates if $\sqsubseteq$, next , and isTarget are effective.*

Proof. Assuming $\sqsubseteq$, **next** and **isTarget** are effective, the only source of non-termination can be the while loop in line 2 if the set **working** never becomes empty. Suppose there is an infinite sequence of constraints as $\varphi_0, \varphi_1 \dots$ obtained from **working** in different iterations of the while loop at line 2. We show that for any $0 \leq i < j$, $\varphi_i \not\sqsubseteq \varphi_j$. This is because (1) **working** is a subset of **store** according to (b), (2) an element is never removed from **store**, and (3) an element is added to **store** only if no smaller element with respect to $\sqsubseteq$ is already there (lines 1 and 8). Such a sequence would therefore violate well-quasi-orderedness. $\square$

Definition 2 (Well-Structured Transition Systems). *A well-structured transition system (WSTS) [2, 59] is a transition system equipped with a quasi-ordering $\leq \subseteq \mathcal{C} \times \mathcal{C}$ on configurations such that the following two conditions hold:*

1. $\leq$ is a well-quasi-ordering ($\mathcal{WQO}$), and
2. $\rightarrow$ is monotonic with respect to $\leq$, i.e. for all $c_1 \leq c'_1$ and transition $c_1 \rightarrow c_2$, there exists a sequence $c'_1 \xrightarrow{*} c'_2$ such that $c_2 \leq c'_2$.

Example 1. For a VAS, let the component-wise ordering $\leq$ be the ordering on the configurations. $\leq$ is a $\mathcal{WQO}$ and $\xrightarrow{t}$ is monotonic with respect to $\leq$. Hence, VAS is a WSTS. We regard a configuration c as a constraint whose denotation is the upward closure of c with respect to $\leq$, i.e. $\llbracket c \rrbracket = \text{Up}_{\leq}(c)$. In this case, we say c is the minimal element of $\llbracket c \rrbracket$, that is, $c = \text{Min}_{\leq}(\llbracket c \rrbracket)$. Let the preorder $\sqsubseteq$ on constraints be defined as follows. $c \sqsubseteq c'$ iff $\text{Min}_{\leq}(\llbracket c \rrbracket) \leq \text{Min}_{\leq}(\llbracket c' \rrbracket)$. $\sqsubseteq$ is clearly effective. It is also sound by transitivity of $\leq$, and is $\mathcal{WQO}$ by $\mathcal{WQO}$ of $\leq$ [72].

It can be shown [2] that we can effectively compute, for any given c and t , a finite set $\text{next}_t(c) = \{c_1, \dots, c_n\}$ such that $\{c' \mid c' \xrightarrow{t} \text{Up}_{\leq}(c)\} = \bigcup_i \text{Up}_{\leq}(c_i)$. **next** is then effective, sound and complete with respect to the predecessor computation.

Assume a VAS with the set of initial and upward closed set of bad configurations $\mathcal{C}_{init}$ and $\mathcal{C}_{bad}$. To use the procedure “explore”, we define φ_{start} to be $\text{Up}_{\leq}(\mathcal{C}_{bad})$, and **isTarget**(φ) iff $\mathcal{C}_{init} \cap \llbracket \varphi \rrbracket \neq \emptyset$.

By $\mathcal{WQO}$ of $\sqsubseteq$ we conclude the reachability analysis of procedure “explore” terminates. It is also exact by soundness and completeness of **next** and **isTarget** and by soundness of $\leq$.

3.4 Monotonic Abstraction

There are several classes of systems that do not fit into the framework of well-structured transition systems. The reason is that it is usually difficult to find a preorder $\leq$ that is a well-quasi-ordering for which transitions are monotonic

(hence, ensuring exactness of the predecessor computation). A solution is to first find a $\mathcal{WQO} \preceq$ on the set of configurations and then apply *monotonic abstraction* [5, 4, 8] in order to *force* monotonicity. For a well-quasi-ordering $\preceq$ on configurations in $\mathcal{C}$, monotonic abstraction results in an abstract transition system $(\hat{\mathcal{C}}, \hat{\mathcal{C}}_{init}, \hat{\mathcal{T}}, \hat{\rightarrow})$ that is an over-approximation of $(\mathcal{C}, \mathcal{C}_{init}, \mathcal{T}, \rightarrow)$ and is monotonic with respect to $\preceq$. The intuition behind monotonic abstraction is that configuration c is allowed to use the outgoing transitions of any smaller configuration $c' \preceq c$. In fact, monotonic abstraction provides a new transition relation $\hat{\rightarrow}$ which is formally defined as follows. $\hat{\rightarrow} := \cup_{t \in \mathcal{T}} \hat{t} \rightarrow$ where $(c_1, c_3) \in \hat{t} \rightarrow$ iff $\exists c_2 \preceq c_1, c_2 \xrightarrow{t} c_3$. Clearly, $\rightarrow \subseteq \hat{\rightarrow}$ for each transition, thus, $\hat{\rightarrow}$ is an over-approximation of $\rightarrow$. The reachability analysis is guaranteed to terminate for the abstract transition system according to Theorem 6 because $\preceq$ is $\mathcal{WQO}$.

Monotonic abstraction has been useful for verifying several classes of parameterized systems such as cache coherence and mutual exclusion protocols [4, 5].

3.4.1 Refinement of Monotonic Abstraction

Monotonic abstraction works well when the initial ordering can generate abstract transition systems that are safe with respect to the desired system properties, such as mutual exclusion. In such cases, it is usually enough to keep track of simple constraints on individual variables to verify the system successfully. We call such approaches *plain* monotonic abstraction. However, there are many classes of systems where the initial ordering suggested by plain monotonic abstraction might not be precise enough to avoid generating spurious counterexamples. In fact, plain monotonic abstraction cannot preserve the required invariants for the correctness of such systems. This often results in generating spurious counterexamples by running the reachability analysis procedure represented in Section 3.3 on the monotonically abstracted system.

Constrained Monotonic Abstraction (CMA) [3] has introduced a counterexample-guided abstraction refinement (CEGAR) [35] algorithm for iterative refinement of monotonic abstraction. The idea of CMA is to start with an initial preorder $\preceq_0$, such as those introduced in previous works on monotonic abstraction [4] and to iteratively strengthen it.

Figure 3.1 gives an overview of CMA. The approach is based on the CEGAR framework. The inputs of the algorithm are the target transition system $(\mathcal{C}, \mathcal{C}_{init}, \mathcal{T}, \rightarrow)$ and an initial preorder $\preceq_0$. The algorithm consists of two main modules, the *reachability checker* and the *counterexample analyzer*. In the i -th iteration of the procedure, the *reachability checker* runs backward reachability analysis on monotonically abstracted systems (procedure “explore” in Section 3.3) and tries to verify the given transition system by first abstracting it using monotonic abstraction with respect to the preorder $\preceq_i$. If it turns out that bad configurations are reachable, a *counterexample* that is generated

Figure 3.1: An overview of the CEGAR algorithm for refining the monotonic abstraction in CMA [3]

by the reachability analysis is fed to the *counterexample analyzer* represented in the procedure “simulate”. The *counterexample analyzer* checks feasibility of the trace on the *concrete* transition system. It then reports either “Real Error” or “Spurious Trace”.

A feasibility check is done by simulating the trace forwardly. If the counterexample is spurious, a constraint φ is found in the i -th loop iteration such that none of the configurations in $\llbracket \varphi_{j+1} \rrbracket$ is the successor of a configuration in $\llbracket \varphi_j \rrbracket$ with respect to the transition $\xrightarrow{t_{j+1}}$ (line 3 of the procedure “simulate”). In this case the algorithm computes the constraint φ' that denotes the set of predecessor configurations of φ_{j+1} with respect to the transition $\xrightarrow{t_{j+1}}$ (line 4) and then using an interpolation technique (e.g. Craig interpolation [78, 101]) computes a constraint φ_S (a so called “Safety Zone”) at line 5.

The algorithm then uses the constraint φ_S to strengthen the preorder $\leq_i$ in the following way. A new preorder $\leq_{i+1}$ is introduced such that $\leq_{i+1} \subset \leq_i$ and is defined as $\leq_{i+1} := \{(c, c') \mid c \leq_i c' \text{ and if } c' \in \llbracket \varphi_S \rrbracket \text{ then } c \in \llbracket \varphi_S \rrbracket\}$. The intuition of preorder strengthening in CMA is to forbid configurations inside a “Safety Zone” to use a transition from some smaller (with respect to $\leq_i$) configuration, which is outside the “Safety Zone”. The strengthened preorder helps monotonic abstraction to generate a more precise abstract transition system in the $(i + 1)$ -th iteration.

It can be shown by contradiction that such strengthening of the preorder preserves $\mathcal{WQO}$; hence, the backward reachability analysis in each iteration of CMA is guaranteed to terminate according to Theorem 6. This approach will be used in Chapter 6 and will be revisited and improved in Chapter 7.

Input: An initial constraint φ_{init} , a set of transitions T , an interpolation function **ITP**, the predecessor and successor computation functions **pre** and **post**, and a counterexample $\rho = \varphi_0; \mathbf{t}_1; \varphi_1; \dots; \mathbf{t}_n; \varphi_n$

Output: "Real Error" or "Spurious Error" with a constraint φ_S

```

1  $\varphi = \varphi_0 \wedge \varphi_{init}$ ;
2 foreach  $j = 0$  to  $n - 1$  do
3   if  $[[\mathbf{post}_{\mathbf{t}_{j+1}}(\varphi)]] = \emptyset$  then
4      $\varphi' = \mathbf{pre}_{\mathbf{t}_{j+1}}(\varphi_{j+1})$ ;
5     return "Spurious Error", ITP( $\varphi, \varphi'$ );
6    $\varphi = \mathbf{post}_{\varphi} \mathbf{t}_{j+1} \wedge \varphi_{j+1}$ ;
7 return "Real Error";

```

Procedure $\text{simulate}(\rho)$, The counterexample analyzer [3]

4

Safety Verification of Phaser Programs

In this chapter, we focus on safety verification of programs that use *phasers* for synchronization [28, 29, 115]. Phasers are modern non-trivial synchronization constructs that support dynamic parallelism with runtime registration and deregistration of spawned tasks. Phasers essentially associate phases to individual tasks and use their runtime values to restrict possible concurrent executions. Unbounded phases may result in infinite transition systems even in the case of programs that create finite numbers of tasks and phasers. We define control reachability (e.g., assertion violation) and plain reachability (e.g., deadlock) and introduce an exact gap-order-based procedure that always terminates when checking control reachability for programs that generate bounded numbers of coexisting tasks and phasers. We also show that verifying plain reachability is undecidable even for a sample program that generates four tasks and three phasers. We then explain how to turn our procedure into a sound analysis for checking plain reachability. We report on experiments with our open-source prototype tool.

4.1 Introduction

In this chapter, we address the problem of statically checking control state reachability (as in the possibility of assertion violations, race conditions or runtime errors) and plain reachability (as in deadlock-freedom) of *Phaser programs* (programs that use the Phaser construct for synchronization). This sophisticated and dynamic construct unifies collective and point-to-point syn-

chronizations. For instance, it allows for dynamic registration and deregistration of tasks. This allows for a more balanced usage of the computing resources compared to static barriers or producer-consumer constructs [114]. The construct can be added to any parallel programming language with a shared address space. For instance, it is implemented in the Habanero Java programming language, and a simplified version of it is also implemented in the `java.util.concurrent` library of the Java programming language. Phasers build on the clock construct from the X10 programming language [28, 29]. Phasers can be created dynamically, and the spawned tasks may get registered with or deregistered from phasers at runtime.

Intuitively, each phaser associates two phases (i.e., *wait* and *signal* phases) to each registered task. Apart from creating phasers and registering their spawned tasks with them, tasks can individually issue **wait** and **signal** commands to a phaser they are registered with. Intuitively, **signal** commands are used to inform other registered tasks that the issuing task is done with its *signal* phase. The **signal** command is non-blocking. It increments the *signal* phase associated with the issuing task on the given phaser. The **wait** command is instead used to check whether all registered tasks are done with (i.e., have a *signal* phase that is strictly larger than) the *wait* phase of the issuing task. This command may get blocked by a task that has not yet finished the corresponding phase. Unlike classical barriers, phasers need not force the registered tasks to wait for each other at every single phase. Instead, they allow them to proceed with the following phases (by issuing **signal** commands), or even to exit the construct by deregistering from it.

The possible runtime creation of tasks and phasers and their registration/deregistration enable a dynamic behavior. Moreover, the spawned tasks can work in different phases, adding flexibility to the synchronization pattern. The dynamic behavior allowed by the phaser construct suits applications that need dynamic load balancing (e.g., for solving non-uniform problems with unpredictable load estimates [128]), but comes at the price of making it easy to introduce programming mistakes such as assertion violations, race conditions, runtime registration errors and, in situations where wait and signal commands are decoupled for maximum flexibility, deadlocks. The generality of the construct also makes it interesting from a theoretical perspective, as many language constructs can be expressed using phasers. For example, synchronization barriers of Single Program Multiple Data programs, the Bulk Synchronous Parallel computation model [126], or promises and futures constructs [24] can be expressed using phasers.

We summarize our contributions presented in this chapter:

- We propose an operational model of phaser programs based on [28, 40, 115] (Section 4.4).
- We show the undecidability of checking deadlock-freedom for programs with fixed numbers of tasks and phasers (Section 4.4).

- We describe an exact gap-order-based symbolic verification procedure for checking control state reachability (as in assertion violations, race conditions, or registration errors) and plain reachability (as in checking deadlock-freedom) in Section 4.5.
- We show termination of the procedure for control state reachability when numbers of tasks and phasers are fixed (Section 4.6).
- We describe how to turn the procedure into a sound over-approximation for plain reachability (Section 4.6).
- We report on our experiments with our open-source tool (Section 4.7).

Related work. We are not aware of automatic formal verification works other than ours (presented in Chapters 4 and 5) that focus on a dynamic synchronization construct that allows tasks to register with and deregister from the construct in runtime. Unlike [93] which proposes an approach for verifying correct synchronization of static and dynamic barriers, we focus on fully automatic verification and consider the richer and more challenging phaser construct. The work of [40] considers the dynamic verification of phaser programs and can therefore only reason about particular program inputs and runs. The work in [10] uses Java Path Finder [68] to explore several runs, but still for only one concrete input at a time. The works in [27, 100] target gap-order systems. Although phaser programs share some of their properties (larger gaps can do more), the results in [27, 100] do not apply since gap-order systems crucially forbid exact increments.

Outline. We describe a phaser program and recall some preliminaries in Sections 4.2 and 4.3. This is followed in Section 4.4 by a formal description of phaser programs and the properties we want to check. We also establish the undecidability of checking deadlock-freedom. In Section 4.5, we introduce our gap-order-based symbolic representation. In Section 4.6 we describe our verification procedure and show decidability of checking control state reachability. We then introduce our relaxation procedure for checking plain reachability. Finally, we report on our experiments and conclude the chapter.

4.2 Illustrative Example

The program listed in Figure 4.1 uses Boolean shared variables `a`, `b`, and `done`. The `main` task creates two phasers (lines 6 and 7). When creating a phaser, the task gets automatically registered with it. The `main` task also creates three other task instances (lines 9 to 11). Several tasks can be registered with several phasers. When a task t is registered with a phaser p , a pair of numbers $(wait_p^t, sig_p^t)$, each in $\mathbb{N} \cup \{+\infty\}$, is associated to the couple (t, p) . The pair represents the individual *wait* and *signal* phases of task t on phaser p .

```

1  bool a, b, done;
2  main()
3  {
4      done = false;
5
6      pr = newPhaser(SIG_WAIT);
7      cn = newPhaser(SIG_WAIT);
8
9      asynch(aPrd, pr(SIG), cn(WAIT));
10     asynch(bPrd, pr(SIG), cn(WAIT));
11     asynch(abCns, pr(WAIT), cn(SIG));
12
13     pr.drop();
14     cn.drop();
15 }
16
17 aPrd(pr(SIG), cn(WAIT))
18 {
19     while(!done){
20         a = true;
21         pr.signal();
22         cn.wait();
23     };
24
25     pr.drop();
26     cn.drop();
27 }
28
29 bPrd(pr(SIG), cn(WAIT))
30 {
31     while(!done){
32         b = true;
33         pr.signal();
34         cn.wait();
35     };
36     pr.drop();
37     cn.drop();
38 }
39
40 abCns(pr(WAIT), cn(SIG))
41 {
42     while(!done){
43         pr.wait();
44         assert(a & b);
45         a = false;
46         b = false;
47
48         if(ndet())
49             done = true;
50         cn.signal();
51     };
52     pr.drop();
53     cn.drop();
54 }

```

Figure 4.1: Two producers and one consumer are synchronized using two phasers. In this construction, the consumer requires both producers to be ahead of it (wrt. `pr` phaser) in order for it to consume their respective products. At the same time, the consumer needs to be ahead of both producers (with respect to `cn` phaser) in order for these to be able to produce their pair of products.

Registration of a task to a phaser can occur in one of three modes: `SIG_WAIT`, `WAIT` and `SIG`. In `SIG_WAIT` mode, a task may issue both `signal` and `wait` commands. In `WAIT` and `SIG` modes, a task may only issue `wait` and `signal` commands to the phaser, respectively. Issuing a `signal` command by a task on a phaser results in the task incrementing its signal phase associated with the phaser. This command is non-blocking. On the other hand, issuing a `wait` command by a task on a phaser p will block until **all** tasks registered with p exhibit signal values on p that are strictly larger than the wait value of the issuing task on phaser p . In this case, the wait phase of the issuing task is incremented. Intuitively, a `signal` command allows the issuing task to state that other tasks need not wait for it to complete its signal phase. In retrospect, a `wait` command allows a task to make sure all registered tasks have moved past its wait phase.

Upon creation of a phaser by a task, its `wait` and `signal` phases are initialized to 0 (except in `WAIT` mode, where the `signal` phase is instead initialized to $+\infty$ in order to not block waiters). The only other way a task may get registered with a phaser is if an already-registered task registers it in the same

Figure 4.2: Sample wait and signal phase values for Figure 4.1. The task names are followed by their program counter. For instance, `main : @15` means the task `main` is at line 15 of the program in Figure 4.1. Observe that there is no bound on the values of the different wait and signal phases a priori. In this example, the difference between signal and wait phases is bounded. This is not always the case in general.

mode (or in WAIT or SIG if the registrar is registered in SIG_WAIT). In this case, wait and signal phases of the newly registered task are initialized to those of the registrar. Therefore, tasks are dynamically registered (e.g., lines 9 to 11). They can also dynamically deregister themselves (e.g., line 35).

A sample configuration for the program in Figure 4.1 is illustrated by an example in Figure 4.2. In this example, two producers and one consumer are synchronized using two phasers. The consumer requires the two producers to be ahead of it (with respect to the phaser that the task `main` referenced with `pr`) in order for it to consume their respective products. At the same time, the consumer needs to be ahead of both producers (with respect to the phaser `main` referenced with `cn`) in order for these to produce their pair of products.

We remind the reader that phasers can be used as barriers for synchronizing dynamic subsets of concurrent tasks. Observe that producers need not, in general, proceed in a lockstep fashion. Producers may produce many items before consumers “catch up”.

We are interested in checking: (a) control state reachability as in assertions (e.g., line 44), race conditions (e.g., mutual exclusion of lines 20 and 45) or runtime errors (e.g., signaling a dropped phaser), and (b) plain reachability as in deadlocks (e.g., a producer at line 22 and a consumer at line 43 with equal phases). Intuitively, both problems concern themselves with the reachability of target sets of program configurations. The difference is that control state reachability defines the targets with the states of the tasks (their control locations and whether they are registered with any phasers). Plain

reachability can also use values or relations between values of involved phases. Observe that control state reachability depends on the values of the actual phases, but these values are not used to define the target sets. For example, assertions are expressed as predicates over Boolean variables (e.g., line 44). Establishing such an assertion requires capturing the constraints imposed by the phasers on the program behaviors.

Our work proposes a sound and complete algorithm for checking control state reachability in cases where a bounded number of tasks and phasers are generated. The algorithm can handle arbitrarily large phases, e.g., generated using nested signaling loops. The algorithm starts from a symbolic representation of all bad configurations and successively computes sets of the predecessor configurations. We show termination based on a well-quasi-ordering argument that imposes restrictions on what can be expressed with our symbolic representation. For instance, putting upper bounds on differences between phases is forbidden. Deadlock configurations cannot be faithfully captured with such restricted representations. Intuitively, a deadlocked configuration will have a cycle, where each involved task is waiting for the task to its right but where the wait phase of each task equals the signal phase of the task it is waiting for. We show the problem of checking deadlock-freedom is undecidable even for programs that only generate a bounded number of tasks and phasers. We explain how to turn our verification algorithm into a sound but incomplete procedure for checking deadlock-freedom. Precision can then be augmented on demand to eliminate false positives.

4.3 Preliminaries

We write $A \uplus B$ to mean the union of disjoint sets A and B . We let $\mathbf{Pfn}(A, B)$ be the set of partial functions from A to B and use $\emptyset_A$ for the empty function over A , i.e., $\emptyset_A(a)$ is undefined (written $\emptyset_A(a) \uparrow$) for all $a \in A$. Given function $g \in \mathbf{Pfn}(A, B)$ we write $g(a) \downarrow$ to mean that $g(a)$ is defined and write $g \setminus \{a\}$ to mean the restriction of g to the domain $A \setminus \{a\}$. We write $g[a \leftarrow b]$ for the function that coincides with g on A except for a that is associated to b . We abuse notation and let $g[\{a_i \leftarrow b_i \mid i \in I\}]$, for a set $\{a_i \mid i \in I\}$ of pairwise different elements, mean the function that coincides with g on A except for each a_i that is associated to the corresponding b_i . We sometimes write a function g as a set $\{a \mapsto g(a) \mid a \in A\}$. It is then implicitly undefined outside of A .

4.4 Phaser Programs

We define the syntax and semantics of a core phaser program language. We make sure the simplified language presented here represents the general-purpose languages that use phasers so that our results have a practical impact.

```

Pr    ::=  bool  $v_1^b, \dots, v_{|B|}^b$ ;
          task1(ph1, ..., phk1) {stmt1}
          ...
          taskn(ph1n, ..., phkn) {stmtn}

stmt  ::=  ph = newPhaser() | asynch(task, ph1, ..., phk)
          | ph.drop() | ph.signal() | ph.wait() | exit
          | stmt; stmt |  $v^b = \text{cond}$  | assert(cond)
          | while(cond) {stmt} | if(cond) {stmt}

cond  ::=  ndet() | true | false |  $v^b$  | cond  $\vee$  cond
          | cond  $\wedge$  cond |  $\neg$ cond

```

A phaser program $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$ consists of a set of tasks Ts including a unique “main” task $\text{main}()\{\text{stmt}\}$. All task instances share a set B of Boolean variables. A task is declared with $\text{task}(\text{ph}_1, \dots, \text{ph}_k)\{\text{stmt}\}$, where $\text{ph}_1, \dots, \text{ph}_k$ are phaser variables that are local to the declared task. A task can also create a new phaser with $\text{ph} = \text{newPhaser}()$ and store a reference the phaser in a local variable ph . We let P be the union of all local phaser variables.

We write s to mean some control sequence in S , and $\text{hd}(\text{s})$ and $\text{tl}(\text{s})$ to respectively mean the head and the tail of the sequence s . We inductively define the finite set S of control sequences as follows. S is the smallest set containing: (i) suffixes of each “ s ” appearing in some “ $\text{task}(\text{ph}_{1_i}, \dots, \text{ph}_{k_i})\{\text{s}\}$ ”, (ii) all $\text{s}_1; \text{while}(\text{cond})\{\text{s}_1\}; \text{tl}(\text{s})$ if s is in S and $\text{hd}(\text{s}) = \text{while}(\text{cond})\{\text{s}_1\}$, and (iii) all $\text{s}_1; \text{tl}(\text{s})$ if s is in S and $\text{hd}(\text{s}) = \text{if}(\text{cond})\{\text{s}_1\}$.

Example 2. Assume there is a task $\text{task}(\text{ph})\{\text{s}\}$ in the program for which $\text{s} = \text{while}(\text{cond})\{\text{ph.signal}(); \text{ph.wait}(); \text{ph.drop}()\}$. The set of control sequences of the program, S , will then contain $\text{while}(\text{cond})\{\text{ph.signal}(); \text{ph.wait}(); \text{ph.drop}()\}$, $\text{ph.drop}()$, and $\text{ph.signal}(); \text{ph.wait}(); \text{while}(\text{cond})\{\text{ph.signal}(); \text{ph.wait}(); \text{ph.drop}()\}$.

Initially, a unique task instance starts executing the $\text{main}()\{\text{stmt}\}$ task. A phaser can recall a pair of values (i.e., wait and signal) for each task that is registered with it. A task can create a new phaser with $\text{ph} = \text{newPhaser}()$, get registered with it (i.e., gets zero as wait and signal values associated with the new phaser) and refer to the phaser with its local variable ph . We simplify the presentation by assuming all registrations to be in SIG_WAIT mode. Including the other modes is a matter of depriving WAIT -registered tasks of a signal value (to ensure they do not block other registered tasks) and of ensuring that issued commands respect registration modes.

A task $\text{task}(\text{ph}_1, \dots, \text{ph}_k)\{\text{stmt}\}$ in Ts takes the phaser variables $\text{ph}_1, \dots, \text{ph}_k$ as parameters (write $\text{paramOf}(\text{task})$ to mean these parameters). A task instance can spawn another task instance with $\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_n)$. The issuing task instance registers the spawned task on the phasers referenced by $\text{ph}_1, \dots, \text{ph}_n$, with its wait and signal values. Spawner and Spawnee execute concurrently. A task can deregister itself from a phaser referenced by a variable ph with $\text{ph.drop}()$.

A task can issue signal or wait commands on a phaser referenced by ph and on which it is registered. A wait command on a phaser blocks until the wait value of the task instance executing the wait on the phaser is strictly smaller than the signal value of all task instances registered with the phaser. In other words, $\text{ph.wait}()$ blocks if ph references a phaser such that at least one of the signal values stored by the phaser is equal to the wait value of the task that tries to perform the wait. A signal command does not block. It only increments the signal value of the task instance executing the signal command on the phaser.

Note that assignment of phaser variables is excluded from the syntax; additionally, we restrict task creation $\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_n)$ and require that parameter variables ph_i are all different. This prevents two variables from reference the same phaser and avoids the need to deal with aliasing: we can reason on the single variable in a process that references a phaser. Extending our work to deal with aliasing is easy but would require heavier notations.

We will need the notions of configurations, partial configurations, and inclusion in order to define the reachability problems we consider in this chapter. We introduce them in the following and assume a phaser program $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$.

Configurations. Configurations of a phaser program describe valuations of its variables, control sequences of its tasks, and registration details of the tasks with the phasers.

A configuration c of $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$ is a tuple $(\mathcal{T}, \mathcal{P}, \ell v, pc, pv, \phi)$, where:

- $\mathcal{T}$ is the current finite set of task identifiers. Let t, u range over $\mathcal{T}$,
- $\mathcal{P}$ is the current finite set of phaser identifiers. Let p, q range over $\mathcal{P}$,
- $\ell v : \text{B} \rightarrow \{\text{true}, \text{false}\}$ is a total mapping that associates a value to each Boolean variable v^b in the set B .
- $pc : \mathcal{T} \rightarrow \text{S}$ is a total mapping that associates tasks to their remaining sequences (i.e., control location),
- $pv : \mathcal{T} \rightarrow \text{Pfn}(\text{P}, \mathcal{P})$ is a total mapping that associates, to each task identifier in $\mathcal{T}$, a partial mapping from the local phaser variables P to phaser identifiers $\mathcal{P}$. It captures the values of the phaser variables P of each task,

- $\phi : \mathcal{P} \rightarrow \text{Pfn}(\mathcal{T}, \mathbb{N}^2)$ is a total mapping that associates to each phaser $p \in \mathcal{P}$ a partial mapping $\phi(p)$ that is defined exactly on the identifiers of the tasks registered with p . For such a task t , $\phi(p)(t)$ is the pair $(\text{wait}_p^t, \text{sig}_p^t)$ representing wait and signal values of t on p .

4.4.1 Semantics of Phaser Programs

Given a program $\text{Pr} = (\mathcal{B}, \mathcal{P}, \mathcal{T}_s)$, the main task $\text{main}()\{\text{stmt}\}$ starts executing its **stmt** from an initial configuration $\mathbf{c}_{\text{init}} = (\{t_0\}, \{\}, \text{bv}_{\text{false}}, \{t_0 \mapsto \text{stmt}\}, \emptyset, \emptyset)$, where a “main” task with identifier t_0 and code **stmt** is the unique initial task. No phasers are present in the initial configuration, and all shared Boolean variables are mapped to **false**. ϕ is the empty function with an empty domain $\emptyset_\emptyset$. After a task t executes a **ph** := **newPhaser**() statement (rule **newPhaser** in Figure 4.3), a new phaser p is associated to the variable **ph** using $p\mathbf{v}$ and $\phi(p)$ becomes the partial function $\{t \mapsto (0, 0)\}$.

Given two configurations $\mathbf{c}$ and $\mathbf{c}'$ with $\mathbf{c} = (\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{pv}, \phi)$, we write $\mathbf{c} \xrightarrow{t} \mathbf{c}'$ if there is a task $t \in \mathcal{T}$ such that one of the rules in Figure 4.3 holds.

We use $\xrightarrow{*}$ for the reflexive transitive closure of $\rightarrow$ and write $\mathbf{c} \xrightarrow{*} \mathbf{c}'$ to mean that $\mathbf{c}'$ is reachable from $\mathbf{c}$. A configuration is said to be reachable if it is reachable from the initial configuration $\mathbf{c}_{\text{init}}$. We explain in the following the operational semantic rules of the phaser programs listed in Figure 4.3.

- Rule **asynch**. This rule is one of the two rules that alter the set of tasks $\mathcal{T}$. Using **asynch**(**task**, **ph**₁, ..., **ph**_{*n*}) statement, a task t spawns a new task u which is not already in the set of tasks $\mathcal{T}$. t is said to be the *parent* of u . According to the definition of the **task**, k distinct phasers in $\mathcal{P}$ need to be passed to u . The wait and signal phases of t will be assigned as phases of u for all of the phasers passed to it. More formally, for each passed phaser p_i , $\phi(p_i)(t)$ will be set to $\phi(p_i)(u)$. Moreover, u will start execution from **s**₁ which is the first statement in the body of **task**.
- Rule **exit**. This rule is the other rule that alters the set of tasks $\mathcal{T}$. A task t can run the **exit** statement to remove itself from the program. Essentially, this boils down to removing the task t from $\mathcal{T}$, removing its remaining control sequence from pc , and removing its signal and wait phases from ϕ .
- Rule **newPhaser**. The set of phasers $\mathcal{P}$ is updated upon the creation of new phasers. When a new phaser p is created by task t , p is added to the set of phasers $\mathcal{P}$, t is registered with p with zero phases as its wait and signal phases.

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{ph} := \text{newPhaser}() \wedge p \notin \mathcal{P} \wedge \\
 pv(t)(\text{ph}) \uparrow \wedge \mathcal{P}' = \mathcal{P} \cup \{p\} \wedge \\
 pv' = pv[t \leftarrow pv(t)[\text{ph} \leftarrow p]] \wedge \phi' = \phi[p \leftarrow \{t \mapsto (0, 0)\}] \\
 pc' = pc[t \leftarrow \text{tl}(pc(t))] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{ph} := \text{newPhaser}()} (\mathcal{T}, \mathcal{P}', bv, pc', pv', \phi') \quad (\text{newPhaser})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k) \{s_1\} \wedge \text{paramOf}(\text{task}) = (q_1, \dots, q_k) \wedge \\
 \text{for each } i : 1 \leq i \leq k. \text{pv}(t)(\text{ph}_i) = p_i \wedge \phi(p_i) \downarrow \wedge \\
 u \notin \mathcal{T} \wedge pv' = pv[u \leftarrow \{q_i \mapsto p_i \mid 1 \leq i \leq k\}] \wedge pc' = pc[u \leftarrow s_1] \wedge \\
 \phi' = \phi[\{p_i \leftarrow \phi(p_i)[u \leftarrow \phi(p_i)(t)] \mid 1 \leq i \leq k\}] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k) \{s_1\}} (\mathcal{T} \cup \{u\}, \mathcal{P}, bv, pc', [t \leftarrow \text{tl}(pc(t))], pv', \phi') \quad (\text{asynch})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{ph.signal}() \wedge pv(t)(\text{ph}) = p \wedge \phi(p)(t) = (wait_p^t, sig_p^t) \wedge \\
 \phi' = \phi[p \leftarrow \phi(p)[t \leftarrow (wait_p^t, 1 + sig_p^t)]] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{ph.signal}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], pv, \phi') \quad (\text{signal})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{ph.wait}() \wedge pv(t)(\text{ph}) = p \wedge \phi(p)(t) = (wait_p^t, sig_p^t) \wedge \\
 \forall u \in \mathcal{T}. (\phi(p)(u) = (wait_p^u, sig_p^u) \Rightarrow wait_p^t < sig_p^u) \wedge \\
 \phi' = \phi[p \leftarrow \phi(p)[t \leftarrow (1 + wait_p^t, sig_p^t)]] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{ph.wait}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], pv, \phi') \quad (\text{wait})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{ph.drop}() \wedge pv(t)(\text{ph}) = p \wedge \phi(p)(t) \downarrow \wedge \phi' = \phi[p \leftarrow \phi(p)[t \leftarrow \uparrow]] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{ph.drop}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], pv, \phi') \quad (\text{drop})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{exit} \wedge pv' = pv \setminus \{t\} \wedge pc' = pc \setminus \{t\} \wedge \\
 \phi' = \phi[\{p \leftarrow (\phi(p) \setminus \{t\}) \mid p \in \mathcal{P}\}] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{exit}} (\mathcal{T} \setminus \{t\}, \mathcal{P}, bv, pc', pv', \phi') \quad (\text{exit})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{assert}(\text{cond}) \wedge bv(\text{cond}) = \text{true} \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \xrightarrow[t]{\text{assert}(\text{cond})} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], pv, \phi) \quad (\text{assert.ok})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = \text{assert}(\text{cond}) \wedge bv(\text{cond}) = \text{false} \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \in \text{badConfs}_{\text{assert}}^{(|\mathcal{T}|, |\mathcal{P}|)} \quad (\text{assert. fault})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = s \wedge (s = \text{ph.drop}() \vee s = \text{ph.signal}() \\
 \vee s = \text{ph.wait}() \vee s = \text{asynch}(\text{task}, \dots, \text{ph}, \dots)) \wedge \\
 (pv(t)(\text{ph}) \uparrow \vee \phi(pv(t)(\text{ph}))(t) \uparrow) \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \in \text{badConfs}_{\text{register}}^{(|\mathcal{T}|, |\mathcal{P}|)} \quad (\text{registration error})
 \end{array}$$

$$\begin{array}{c}
 \{t_0, \dots, t_{n-1}\} \subseteq \mathcal{T} \wedge \{p_0, \dots, p_{n-1}\} \subseteq \mathcal{P} \wedge \forall i : 0 \leq i \leq (n-1). \text{hd}(pc(t_i)) = \text{ph}_i.\text{wait}() \wedge \\
 pv(t_{(i+1)\%n})(q_i) = p_i \wedge pv(t_i)(\text{ph}_i) = p_i \wedge wait_{p_i}^{t_i} = sig_{p_i}^{t_{(i+1)\%n}} \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \in \text{badConfs}_{\text{deadlock}}^{(|\mathcal{T}|, |\mathcal{P}|)} \quad (\text{deadlock})
 \end{array}$$

$$\begin{array}{c}
 \text{hd}(pc(t)) = v^b := \text{cond} \wedge \text{hd}(pc(u)) = s' \wedge t \neq u \wedge \\
 \left(\begin{array}{l} s' = v^b := \text{cond}' \vee v^b \text{ appears in } \text{cond}' \text{ and} \\ (s' = \text{if}(\text{cond}') \{ \text{stmt} \} \vee \\ s' = \text{while}(\text{cond}') \{ \text{stmt}' \} \vee \\ s' = \text{assert}(\text{cond}') \vee s' = v^{b'} = \text{cond}') \end{array} \right) \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi) \in \text{badConfs}_{\text{race}}^{(|\mathcal{T}|, |\mathcal{P}|)} \quad (\text{race})
 \end{array}$$

Figure 4.3: Operational semantics of phaser statements.

- Rule **signal**. The `ph.signal()` statement increments the signal value of the issuing task t on phaser p , i.e. sig_p^t , by one.
- Rule **wait**. The `ph.wait()` statement similarly increments the wait phase of the issuing task, i.e. $wait_p^t$, given that for all other tasks u registered with p the condition $sig_p^t < sig_p^u$ holds. Otherwise, t blocks until the condition holds.
- Rule **drop**. A task t can deregister itself from a phaser p using the `ph.drop()` statement. This will essentially remove the phaser variable `ph` assigned to p and also its wait and signal values at p , i.e. $\phi(p)(t)$. `asynch` and `drop` statements allow dynamic registration with and deregistration from phasers, hence, a lot of freedom in synchronization patterns in phaser programs.
- Rule **registration error**. The mapping $p\mathcal{v}$ associates values to program phaser variables. Accessing variables with undefined values, or phasers to which t is not currently registered leads to registration errors. This rule is used to characterize the set $\text{badConfs}_{\text{register}}^{(n,m)}$ of configurations that lead to registration errors.
- Rules **assert.fault** and **assert.ok**. An `assert(cond)` statement would result in an error if its condition `cond` does not hold. Rule **assert.fault** formally defines the configurations in the set $\text{badConfs}_{\text{assert}}^{(n,m)}$. On the other hand, if the condition `cond` holds, only the control sequence of the executing task t will be updated according to rule **assert.ok**.
- Rule **deadlock** characterizes the deadlocked configurations in $\text{badConfs}_{\text{deadlock}}^{(n,m)}$ and is explained in Definition 3 and Definition 4 below.

Rule **race** formally defines configurations with race conditions, i.e. those in the set $\text{badConfs}_{\text{race}}^{(n,m)}$. A race condition occurs if distinct tasks t and u are about to read and write to the shared variable v^b at the same time.

Control-state Reachability

Checking the possibility of assertion violations, of registration errors and of race conditions amounts to checking reachability of configurations respectively in $\text{badConfs}_{\text{assert}}^{(n,m)}$, $\text{badConfs}_{\text{register}}^{(n,m)}$ and in $\text{badConfs}_{\text{race}}^{(n,m)}$ for some number of tasks n and number of phasers m . In Chapter 4.5, we introduce a complete procedure for checking reachability of such sets of configurations and show it to be sound for programs with fixed upper bounds on numbers of generated phasers and tasks.

Figure 4.4: A sample deadlock configuration with three tasks and three phasers

Plain Reachability and Deadlocks

Besides control state reachability, we are also interested in checking for the possibility of deadlocks. For this, we need to define the notion of a blocked task. Assume in the following a configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$.

Definition 3 (Blocked). *A task $t \in \mathcal{T}$ is blocked at phaser $p \in \mathcal{P}$ by task $u \in \mathcal{T}$ if $\text{hd}(pc(t)) = \text{ph.wait}()$ with $pv(t)(\text{ph}) = p$ and $\phi(p)(t) = (\text{wait}_p^t, _)$ when $\phi(p)(u) = (_, \text{sig}_p^u)$ and $\text{sig}_p^u \leq \text{wait}_p^t$.*

Intuitively, a task t is blocked by a task u if it cannot finish its **wait** command on some phaser because it is waiting for task u that have not issued enough **signal** commands on the same phaser.

Definition 4 (Deadlock). *$(\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$ is a deadlock configuration if each task of a none-empty subset $\mathcal{U} \subseteq \mathcal{T}$ is blocked by some task in $\mathcal{U}$.*

Example 3. In Figure 4.4, a sample deadlock configuration with three tasks and three phasers is illustrated. For each $i : 0 \leq i \leq 2$, task t_i is blocked on the phaser p_i by task $t_{(i+1)\%(n+1)}$. That is, $\text{sig}_{p_i}^{t_{(i+1)\%(n+1)}} = \text{wait}_{p_i}^{t_i}$. The thick arcs represent the blockage of a task on a phaser.

Theorem 7 (Deadlock-Freedom). *It is undecidable in general, even for programs with only three phasers and four tasks, to check for deadlock-freedom.*

Proof. We encode the reachability problem of any given 3-counter reset-VAS (Section 3.2.3) as the reachability problem of a configuration with a simple cycle involving three tasks. Indeed, reachability of configuration $(q_F, 0, 0, 0)$ (three counters x_1, x_2, x_3 with zero values at some control location q_F) is


```

1 bool s0,s1,...,sF;
2
3 //(#a,#b,#c) is a circular
4 //permutation of (1,2,3)
5 bool check,ok,t#a:inc:x#a,
6   t#a:res:x#a,t#a:res:x#b,
7   t#a:dec:x#c;
8
9 main(){
10
11   p1 = newPhaser();
12   p2 = newPhaser();
13   p3 = newPhaser();
14
15   s0=true;
16   ok=false;
17
18   async(tsk1,p1,p3);
19   async(tsk2,p2,p1);
20   async(tsk3,p3,p2);
21
22   while(true){
23     //reachability
24     if(* && sF){
25       p1.signal();
26       p2.signal();
27       p3.signal();
28       check=true;
29       exit;
30     }
31
29 //qi:inc(x#a):qj
30 if (ndet() && si){
31   t#a:inc:x#a = true;
32   while(!ok){};
33   ok=false;
34   si=false;
35   sj=true;
36 };
37 //qi,dec(x#a),qj
38 if(ndet() && si){
39   p#a.signal();
40   t#b:dec:x#a = true;
41   while(!ok){};
42   p#a.wait();
43   ok=false;
44   si=false;
45   sj=true;
46 };
47 //qi:reset(x#a):qj
48 if(ndet() && si){
49   p#c.signal();
50   t#a:reset:x#a = true;
51   while(!ok){};
52   ok=false;
53   p#c.wait();
54   async(tsk#a,p#a,p#c);
55   si=false;
56   sj=true;
57 };
58 };
59}

```

Figure 4.5: (Part I) Encoding reachability of any 3-counter reset-VAS as the reachability of a deadlock configuration of a phaser program. The state of the counter machine q ranges over the set $\{q_0, \dots, q_F\}$. The Boolean variables are $\{ok, check, t#a:inc:x#a, t#a:res:x#a, t#a:res:x#b, t#a:dec:x#c\}$ where $(\#a, \#b, \#c)$ is a circular permutation of $(1, 2, 3)$ that encodes the messages passed among the tasks.

undecidable for reset-VASs based on Theorem 3. Figures 4.5 and 4.6 describe a phaser program, where the main task t_m spawns three tasks $\{t_1, t_2, t_3\}$ such that t_i runs $tski$ for each $i \in \{1, 2, 3\}$. The main task with id t_m orchestrates the simulation and is registered with three phasers $ph1, ph2$ and $ph3$ respectively with identifiers p_1, p_2 and p_3 . Each task t_i is registered with phasers p_i and $p_{(i+1)\%3+1}$. If a task is registered with a phaser p_i , then, it will reference it with a local variable phi , for $i \in \{1, 2, 3\}$. This is illustrated in Figure 4.7.

Intuitively, we encode the value of counter x_i using the difference $sig_{p_i}^{t_i} - wait_{p_i}^{t(i\%3+1)}$, for $i \in \{1, 2, 3\}$. We show that for $i \in \{1, 2, 3\}$ is the inductive invariants of the encoded system are: $wait_{p_i}^{t_m} = sig_{p_i}^{t_m} = wait_{p_i}^{t(i\%3+1)} = sig_{p_i}^{t(i\%3+1)}$

In the beginning when t_1, t_2 and t_3 are spawned at lines 18 to 20, all phases are zero. Hence, the invariant holds.

<pre> 1 tsk1(p1,p3){ 2 while(true){ 3 if(t1:inc:x1){ 4 p1.signal(); 5 ok=true; 6 }; 7 if(t1:dec:x3){ 8 p3.signal(); 9 p3.wait(); 10 ok=true; 11 }; 12 if(t1:res:x1){ 13 t3:res:x1=true; 14 exit; 15 }; 16 if(t1:res:x2){ 17 p1.signal(); 18 p1.wait(); 19 ok=true; 20 }; 21 if(check){ 22 p3.wait(); 23 exit; 24 }; 25 }; 26} </pre>	<pre> 1 tsk2(p2,p1){ 2 while(true){ 3 if(t2:inc:x2){ 4 p2.signal(); 5 ok=true; 6 }; 7 if(t2:dec:x1){ 8 p1.signal(); 9 p1.wait(); 10 ok=true; 11 }; 12 if(t2:res:x2){ 13 t1:res:x2=true; 14 exit; 15 }; 16 if(t2:res:x3){ 17 p2.signal(); 18 p2.wait(); 19 ok=true; 20 }; 21 if(check){ 22 p1.wait(); 23 exit; 24 }; 25 }; 26} </pre>
<pre> 1 tsk3(p3,p2){ 2 while(true){ 3 if(t3:inc:x3){ 4 p3.signal(); 5 ok=true; 6 }; 7 if(t3:dec:x2){ 8 p2.signal(); 9 p2.wait(); 10 ok=true; 11 }; 12 if(t3:res:x3){ 13 t2:res:x3=true; 14 exit; 15 }; 16 if(t3:res:x1){ 17 p3.signal(); 18 p3.wait(); 19 ok=true; 20 }; 21 if(check){ 22 p2.wait(); 23 exit; 24 }; 25 }; 26} </pre>	

Figure 4.6: (Part II) Encoding reachability of any 3-counter reset-VAS as the reachability of a deadlock configuration of a phaser program. The state of the counter machine q ranges over the set $\{q_0, \dots, q_F\}$. The Boolean variables are $\{ok, check, t\#a:inc:x\#a, t\#a:res:x\#a, t\#a:res:x\#b, t\#a:dec:x\#c\}$ where $(\#a, \#b, \#c)$ is a circular permutation of $(1, 2, 3)$ that encodes the messages passed among the tasks.

Figure 4.7: A deadlock configuration generated by proof of the Theorem 7

We explain in the following how we encode the increment, decrement, and reset operations on counter x_1 in a way that its invariant is preserved. The counters x_2 and x_3 are similar.

Increment. To model incrementing counter x_1 , **main** sets $t1:inc:x1$ to **true** (line 31) and waits for **ok** to be **true** (line 32). Because of the message $t1:inc:x1$, only t_1 can proceed. It issues a **signal** on **ph1** at line 4 of its procedure in Figure 4.6. This operation does not involve any variable that is present in the invariant; hence, the invariant is preserved after the increment.

Decrement. To model decrementing counter x_1 , **main** first issues a **ph1.signal()**, then sets $t2:dec:x1$ to **true** at line 40 of Figure 4.5, and finally, waits for **ok** to hold. Because of the message $t2:dec:x1$, only t_2 can participate in decrementing x_1 . It does so by issuing a **ph1.signal()** and **ph1.wait()** at lines 8 and 9 of its procedure in Figure 4.6. Finally, **main** performs **ph1.wait()** at line 42 of Figure 4.5. Hence, the invariant is preserved. Assume $sig_{p1}^{t1} - wait_{p1}^{t2} = k$ before the decrement. Then, $sig_{p1}^{t1} - (wait_{p1}^{t2} + 1) = k - 1$ after the decrement. We are sure that **ph1.wait()** is enabled since by the semantics of counter machines, x_1 can only be decremented if $x_1 > 0$ and we have shown that for every increment of x_1 , sig_{p1}^{t1} is incremented by one. Moreover, **ph1** is only registered with **main**, **tsk1** and **tsk2** and we have shown that all of them have already issued **ph1.signal()**.

Reset. To reset counter x_1 , **main** first issues a `ph3.signal()` and then sets `t1:res:x1` to `true` (line 50 of Figure 4.5) and waits for `ok` to hold. According to `t1:inc:x1`, t_1 will proceed by setting `t3:res:x1` to `true` at line 13 of procedure `tsk1` in Figure 4.6. It then exits at line 14. Now, `tsk3` will issue `ph3.signal()` and `ph3.wait()` and set `ok` to `true` at lines 17 to 19 of its procedure. Thereafter, **main** performs `ph3.wait()` at line 53 of Figure 4.5. Intuitively, **main** does so to wait for t_1 to exit. Note that t_1 and t_3 are the only tasks other than **main** that are registered with `ph3`. The operations on lines 49 and 53 of **main** together with those of the lines 17 and 18 of `tsk3` ensure that the invariant is preserved for `ph3`.

Finally, **main** spawns a new `tsk1` and passes the phasers `ph1` and `ph3` to it. By the semantics of phaser programs, `tsk1` will now copy the phases of **main** on `ph1` and `ph3` which are respectively equal to the phases of `tsk2` and old values of `tsk1` on those phasers (according to the invariant). Hence, $\text{sig}_{p_1}^{t_1} - \text{wait}_{p_1}^{t_2} = 0$. Moreover, $\text{sig}_{p_3}^{t_3} - \text{wait}_{p_3}^{t_1}$ will remain intact.

Finally, asking each task t_i to perform a `wait` on phaser $p_{(i+1)\%3+1}$ ensures that a simple cycle of size 3 is built exactly when the three counters are 0 according to the invariant. $\square$

We have just demonstrated undecidability for checking deadlock in phaser programs. In the next section, we introduce the symbolic representation that will be used in our verification procedure.

4.5 Symbolic Verification of Phaser Programs

We briefly introduce gap-order constraints and use them to define a symbolic representation (hereafter constraints) that we use in Chapter 4.6 for checking reachability.

4.5.1 Gap-order Constraints and Graphs [113, 91, 27, 100]

Gap-order constraints can be regarded as a particular case of octagons or *unit two variables per inequality* (utvpi) constraints. Assume in this section that x and y are integer variables and that k is an integer constant. We use X and Y to mean finite sets of integer variables. A valuation `val` is a total function $X \rightarrow \mathbb{Z}$. Valuations are implicitly extended to preserve constants (i.e., $\text{val}(k) = k$ for any $k \in \mathbb{Z}$).

A *gap-order clause* δ over X is an inequality of the form $a - b \geq k$, where $a, b \in X \cup \{0\}$. A *gap-order constraint* Δ over X is a finite conjunction of gap-order clauses over the same set X . Observe that $(x = y + 2 \wedge y \leq 5)$ is essentially a gap-order constraint because it can be equivalently rewritten as the conjunction $(x - y \geq 2 \wedge y - x \geq -2 \wedge 0 - y \geq -5)$.

Given a gap-order constraint Δ over X and a valuation $\mathbf{val} : X \rightarrow \mathbb{Z}$, we write $\mathbf{val} \models \Delta$ to mean that $\mathbf{val}(a) - \mathbf{val}(b) \geq k$ holds for each gap-order clause $\delta : a - b \geq k$ appearing in Δ . We let $Sat(\Delta)$ be the set $\{\mathbf{val} : X \rightarrow \mathbb{Z} \mid \mathbf{val} \models \Delta\}$.

A gap-order graph (or graph for short) $\wp$ over X is a graph (V, E) with vertices $V = X \cup \{0\}$, where edges in E are of the form $a \xrightarrow{k} b$ with $a, b \in V$ and weight k in $\mathbb{Z} \cup \{-\infty, +\infty\}$. We let $\mathbf{varsOf}(\wp) = X$.

Given a gap-constraint Δ over X , we can build the graph $\mathbf{graphOf}(\Delta)$ with vertices $X \cup \{0\}$ and where E only contains a representative $a \xrightarrow{k} b$ edge for each clause $a - b \geq k$ appearing in Δ .

A valuation $\mathbf{val} : X \rightarrow \mathbb{Z}$ satisfies a graph $\wp = (V, E)$ (written $\mathbf{val} \models \wp$) iff $\mathbf{val}(a) - \mathbf{val}(b) \geq k$ for each $a \xrightarrow{k} b \in E$. We let $Sat(\wp)$ be the set $\{\mathbf{val} : X \rightarrow \mathbb{Z} \mid \mathbf{val} \models \wp\}$. Clearly, $Sat(\mathbf{graphOf}(\Delta)) = Sat(\Delta)$.

The closure $\mathbf{clo}(\wp)$ of a graph $\wp = (V, E)$ is the unique complete graph with the same vertices V and where $a \xrightarrow{k'} b$ is an edge of $\mathbf{clo}(\wp)$ iff $k' \in \mathbb{Z} \cup \{-\infty, +\infty\}$ is the least upper bound of all weight-sums for any path in $\wp$ from a to b . Closure allows us to deduce $(0 - x \geq -7)$ from $(y - x \geq -2 \wedge 0 - y \geq -5)$. The result of the closure procedure is a unique graph $\wp_{\mathbf{false}}$ denoting the graph without any satisfying valuation each time a weight $k = +\infty$ is generated. The closure of a graph can be computed in polynomial time and we get $Sat(\mathbf{clo}(\wp)) = Sat(\wp)$.

We define the *degree* of a graph $\wp$ (written $\mathbf{degreeOf}(\wp)$) to be 0 if no edge in $\mathbf{clo}(\wp)$ has a negative weight apart from $-\infty$. Otherwise, $\mathbf{degreeOf}(\wp)$ is the largest natural number $k \in \mathbb{N}$ such that there is an edge in $\mathbf{clo}(\wp)$ with weight $-k$. For instance, the degree of the graph resulting from $(x - y \geq 2 \wedge y - x \geq -4)$ is 4. We systematically close all manipulated graphs and write $\mathcal{G}(X)$ for the set of closed graphs over X .

Given a graph $\wp$, we write $\wp[y/x]$ to mean the graph obtained by replacing the vertex x with the vertex y . We abuse notation and write $\wp[\{y_i/x_i \mid i \in I\}]$, for pairwise different x_i elements to mean the simultaneous application of the individual substitutions.

For a set of variables Y , we write $\wp \ominus Y$ to mean the graph obtained by removing the variables in Y from the vertices of $\wp$. Given two closed graphs $\wp$ and $\wp'$ over the same X , we write $\wp \leq_{\mathcal{G}} \wp'$ to mean that each directed edge in $\wp$ is labeled with a larger or equal weight in $\wp'$. As a result, $Sat(\wp') \subseteq Sat(\wp)$. Finally, we write $\wp \otimes \wp'$ to mean the closure of the graph obtained by merging the two sets of vertices and edges. As a result, $Sat(\wp \otimes \wp') = Sat(\wp) \cap Sat(\wp')$.

4.5.2 Symbolic Representation

A constraint φ is a tuple $(\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma)$, where the only difference with the definition of a configuration $(\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$ is the adoption of a gap-order constraint γ instead of ϕ . More specifically, $\gamma : \mathcal{P} \rightarrow \cup_{u \subseteq \mathcal{T}} \mathcal{G}(\cup_{t \in u} \{\omega^t, \sigma^t\})$ is a total mapping that associates a gap-order graph to each phaser $p \in \mathcal{P}$.

Intuitively, we use the variables ω^t and σ^t to constrain in graph $\gamma(p)$ possible values of wait ($wait_p^t$) and signal (sig_p^t) phases of each task t registered with phaser p . As a result, we can check if task t is registered with phaser p according to graph $\wp = \gamma(p)$ by checking if $\{\omega^t, \sigma^t\} \subseteq \text{varsOf}(\wp)$. We will write $\text{Reg}(p, \wp)$ to mean the set of tasks $\{t \mid \{\omega^t, \sigma^t\} \subseteq \text{varsOf}(\wp)\}$. We also write $\text{isReg}(t, p, \wp)$ for the predicate $t \in \text{Reg}(p, \wp)$.

Observe that the language semantics impose that, for each phaser p and for any pair t, u of tasks in $\text{Reg}(p, \wp)$, the predicate $0 \leq wait_p^t \leq sig_p^u$ is an invariant. For this reason, we always safely strengthen, in any obtained $\gamma(p) = \wp$, weights k in $\sigma^t \xrightarrow{k} \omega^u$, $\sigma^t \xrightarrow{k} 0$ and $\omega^t \xrightarrow{k} 0$ with $\max(k, 0)$. The following definition helps us characterize constraints for which our procedure terminates.

Definition 5 (Degree and freeness of constraints). *The degree of a constraint $(\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma)$ is 0 if $\mathcal{P}$ is empty. Otherwise, its degree is the greatest degree among all its graphs $\gamma(p)$ for $p \in \mathcal{P}$. Furthermore, a constraint is said to be “free” if, for any $p \in \mathcal{P}$, the only edges in $\gamma(p)$ with weights different from $-\infty$ are edges of the forms (i) $\sigma^t \xrightarrow{k_{(\sigma^t, \omega^u)}} \omega^u$, (ii) $\sigma^t \xrightarrow{k_{(\sigma^t)}} 0$, or (iii) $\omega^t \xrightarrow{k_{(\omega^t)}} 0$ for some $t, u \in \text{Reg}(p, \gamma(p))$ and $k_{(\sigma^t, \omega^u)}, k_{(\sigma^t)}, k_{(\omega^t)} \in \mathbb{N}$.*

Free constraints are only allowed to impose, for the same phaser, non-negative lower bounds on differences between signals and waits, between signals and 0, and between waits and 0. Like degree-0-constraints, free constraints are not allowed to put a positive upper bound on how much larger a signal is than a wait. Unlike degree-0-constraints, they cannot put bounds on the differences among signal values or wait values. For instance, a free constraint cannot impose $\sigma^t - \sigma^u = 0$ while a degree-0-constraint can. Intuitively, freeness does not oblige our verification procedure to maintain exact differences when firing **signal** or **wait** instructions, jeopardizing termination. This will be stated in Section 4.6.

4.5.3 Denotations of Constraints

Given a configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$ and a constraint $\varphi = (\mathcal{T}', \mathcal{P}', bv', pc', pv', \gamma')$, we say that c satisfies φ , and write $c \models \varphi$, if c satisfies (up to a renaming of the tasks and the phasers) the conditions imposed by φ . More concretely, $c \models \varphi$ if $bv = bv'$ and there are bijections $\kappa : \mathcal{T} \rightarrow \mathcal{T}'$ and $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ such that:

- $pc(t) = pc'(\kappa(t))$ for each $t \in \mathcal{T}$;
- $\pi(pv(t)(\text{ph})) = pv'(\kappa(t))(\text{ph})$ for each $t \in \mathcal{T}$ and $\text{ph} \in \mathcal{P}$;
- the renaming of tasks and phasers in ϕ with respect to κ and π satisfies γ , i.e.,

- for each $t \in \mathcal{T}$ and each $p \in \mathcal{P}$, $\text{isReg}(\kappa(t), \pi(p), \gamma(\pi(p)))$ iff $\phi(p)(t) \downarrow$,
- for each $p' \in \mathcal{P}'$, $\text{val}_{p'} \models \gamma(p')$, where $\text{val}_{p'}$ is the valuation that for each $t' \in \mathcal{T}'$ associates $\omega^{t'}$ to $\text{wait}_{p'}^{t'}$ and $\sigma^{t'}$ to $\text{sig}_{p'}^{t'}$ if $\phi(\pi^{-1}(p'))(\kappa^{-1}(t')) = (\text{wait}_{p'}^{t'}, \text{sig}_{p'}^{t'})$. Note that $\gamma(p')$ is by definition a graph over variables $(\omega^{t'}, \sigma^{t'})$.

Intuitively, $\llbracket (\mathcal{T}', \mathcal{P}', \text{bv}', \text{pc}', \text{pv}', \gamma') \rrbracket$ contains all configurations $\mathbf{c}$ with the same number of tasks and phasers and such that there are renamings of tasks and phasers that preserve in $\mathbf{c}$ the correspondence between pc , pv and γ . We write $\llbracket \Phi \rrbracket$, for a set Φ of constraints, to mean the union $\cup_{\varphi \in \Phi} \llbracket \varphi \rrbracket$.

Given a phaser program $(\mathbf{B}, \mathbf{P}, \mathbf{Ts})$, we can characterize with a finite set of constraints all configurations involving n tasks and m phasers and satisfying the premises of rules (**registration error**), (**assert. fault**), (**race**) and (**deadlock**) listed in Figure 4.3.

Lemma 1 (Characterizing badness). *Given a program $(\mathbf{B}, \mathbf{P}, \mathbf{Ts})$ and natural numbers (n, m) , we can exhibit finite sets of constraints $\text{badCstrs}_{\text{race}}^{(n, m)}$, $\text{badCstrs}_{\text{assert}}^{(n, m)}$, $\text{badCstrs}_{\text{register}}^{(n, m)}$ and $\text{badCstrs}_{\text{deadlock}}^{(n, m)}$ such that:*

$$\begin{aligned} \text{badConfs}_{\text{race}}^{(n, m)} &= \llbracket \text{badCstrs}_{\text{race}}^{(n, m)} \rrbracket \\ \text{badConfs}_{\text{assert}}^{(n, p)} &= \llbracket \text{badCstrs}_{\text{assert}}^{(n, m)} \rrbracket \\ \text{badConfs}_{\text{register}}^{(n, p)} &= \llbracket \text{badCstrs}_{\text{register}}^{(n, m)} \rrbracket \\ \text{badConfs}_{\text{deadlock}}^{(n, p)} &= \llbracket \text{badCstrs}_{\text{deadlock}}^{(n, m)} \rrbracket \end{aligned}$$

In addition, we can choose the constraints in $\text{badCstrs}_{\text{deadlock}}^{(n, m)}$ to be of degree 0 while those in $\text{badCstrs}_{\text{race}}^{(n, m)}$, $\text{badCstrs}_{\text{assert}}^{(n, m)}$ or in $\text{badCstrs}_{\text{register}}^{(n, m)}$ to be free.

Proof. Observe that n and m are given natural numbers. Fix a task set $\mathcal{T} = \{t_1, \dots, t_n\}$ of size n and a phaser set $\mathcal{P} = \{p_1, \dots, p_m\}$ of size m . We can therefore enumerate all tuples $(\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{pv})$, where bv , pc and pv are defined as in the configurations.

Let $\mathbf{C}$ be the set of such tuples. Observe that the tuples in $\mathbf{C}$ are missing information about tasks' registration and wait and signal values. We complete this information in the following. Given a set $\mathcal{U} \subseteq \mathcal{T}$ of tasks and a phaser $p \in \mathcal{P}$, we write $\text{topOf}(\mathcal{U}, p)$ to mean the graph of the conjunction $\bigwedge_{t, u \in \mathcal{U}} (\sigma^t \geq \omega^u \geq 0)$. Observe that an invariant of all phaser programs is that signal and wait phases (of all tasks registered with a given phaser) are always non-negative with the signal phases always greater than or equal to the wait phases. For this reason, $\text{topOf}(\mathcal{U}, p)$ is the weakest possible graph, where the set $\mathcal{U}$ is registered with a phaser p . Observe that $\text{topOf}(\mathcal{U}, p)$ is free. We

now finish the definitions of $\text{badCstrs}_{\text{race}}^{(n,m)}$, $\text{badCstrs}_{\text{assert}}^{(n,m)}$, $\text{badCstrs}_{\text{register}}^{(n,m)}$, $\text{badCstrs}_{\text{deadlock}}^{(n,m)}$:

- Add to $\text{badCstrs}_{\text{race}}^{(n,m)}$ all constraints $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv}, \gamma)$, where:
 - $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv})$ is a tuple of $\mathbb{C}$ with two different tasks t and u in $\mathcal{T}$ executing a read or a write on a Boolean variable with at least one of them writing to it (see $\text{badConfs}_{\text{race}}^{(n,m)}$ in Figure 4.3).
 - The total mapping γ associates $\text{topOf}(\mathcal{U}_p, p)$ to each phaser p , where $\mathcal{U}$ is some subset of $\mathcal{T}$. Intuitively, for each phaser p , we consider all registration possibilities (some subset $\mathcal{U} \subseteq \mathcal{T}$) while imposing the weakest possible constraints on the signal and wait phases of the registered tasks. Observe that γ is free.
- Add to $\text{badCstrs}_{\text{assert}}^{(n,m)}$ all constraints $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv}, \gamma)$, where:
 - $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv})$ is a tuple of $\mathbb{C}$ with some task t in $\mathcal{T}$ executing an assertion on a Boolean condition that evaluates to false with $\mathcal{bv}$ (see $\text{badConfs}_{\text{assert}}^{(n,m)}$ in Figure 4.3).
 - The total mapping γ associates $\text{topOf}(\mathcal{U}_p, p)$ to each phaser p , where $\mathcal{U}$ is some subset of $\mathcal{T}$. Observe that γ is free.
- Add to $\text{badCstrs}_{\text{register}}^{(n,m)}$ all constraints $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv}, \gamma)$, where:
 - $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv})$ is a tuple of $\mathbb{C}$ with some task t in $\mathcal{T}$ executing a statement that involves a phaser variable ph .
 - Again, the total mapping γ associates $\text{topOf}(\mathcal{U}_p, p)$ to each phaser p , where $\mathcal{U}$ is some subset of $\mathcal{T}$. In addition, we require that either $\mathcal{pv}(t)(\text{ph}) \uparrow$ or $p = \mathcal{pv}(t)(\text{ph})$ with $t \notin \text{Reg}(p, \gamma(p))$. (see $\text{badConfs}_{\text{register}}^{(n,m)}$ in Figure 4.3). Observe $\gamma(q)$ is free for all q on which γ is defined. Hence, γ is free.
- Add to $\text{badCstrs}_{\text{deadlock}}^{(n,m)}$ all constraints $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv}, \phi)$, where:
 - $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv})$ is a tuple of $\mathbb{C}$, where a set of tasks $t_0, \dots, t_{l-1}$ in $\mathcal{T}$ are executing wait commands on phaser variables $\text{ph}_0, \dots, \text{ph}_{l-1}$ in $\mathcal{P}$.
 - Again, the total mapping γ associates $\text{topOf}(\mathcal{U}_p, p)$ to each phaser p , where $\mathcal{U}$ is some subset of $\mathcal{T}$. However, we require that: $p_i = \mathcal{pv}(t_i)(\text{ph}_i)$ for each $i : 0 \leq i < l$ and, t_i is waiting for $t_{(i+1)\%l}$ in $\gamma(p_i)$, i.e., $\gamma(p_i)$ imposes the wait phase of t_i on p_i is equal to the signal phase of $t_{(i+1)\%l}$ on p_i . In other words, the edge $\omega^{t_i} \xrightarrow{0} \sigma^{t_{(i+1)\%l}}$ in $\gamma(p_i)$ (see $\text{badConfs}_{\text{deadlock}}^{(n,m)}$ in Figure 4.3). Observe that the graphs in γ are not all free since some of them put an

upper bound on how large some signal values are compared to some wait values. They are however of degree 0 since the only negative weights are $-\infty$.

By construction, we have considered all possible tuples $(\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv})$ and registration combinations. We have considered the weakest possible constraints on the phases for the registered tasks. Besides, any configuration in the denotation of the constraints will belong to the corresponding bad set. $\square$

4.5.4 Entailment

We say that a constraint $\varphi = (\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{pc}, \mathcal{pv}, \gamma)$ is weaker than another constraint $\varphi' = (\mathcal{T}', \mathcal{P}', \mathcal{bv}', \mathcal{pc}', \mathcal{pv}', \gamma')$, written $\varphi \sqsubseteq \varphi'$, to mean the following. The two constraints have the same number of phasers and tasks. They agree on the values of the Boolean variables. Moreover, they agree on which tasks are registered with which phasers (up to a renaming on the values of the phaser variables). Also, the constraints on the wait and signal values are stronger in φ' than in φ . More formally, $\varphi \sqsubseteq \varphi'$ if

- $\mathcal{bv} = \mathcal{bv}'$,
- there are bijections $\kappa : \mathcal{T} \rightarrow \mathcal{T}'$ and $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ such that for each $t \in \mathcal{T}$ and $p \in \mathcal{P}$ the following four conditions hold:
 - $\mathcal{pc}(t) = \mathcal{pc}'(\kappa(t))$;
 - $\pi(\mathcal{pv}(t)(\mathbf{ph})) = \mathcal{pv}'(\kappa(t))(\mathbf{ph})$;
 - $\kappa(\mathbf{Reg}(p, \gamma(p))) = \mathbf{Reg}(\pi(p), \gamma'(\pi(p)))$;
 - $\gamma(p) \leq_{\mathcal{G}} \gamma'(\pi(p)) \left[\left\{ \omega^t / \omega^{\kappa(t)}, \sigma^t / \sigma^{\kappa(t)} \mid t \in \mathbf{Reg}(p, \gamma(p)) \right\} \right]$.

Clearly, $\varphi \sqsubseteq \varphi'$ implies $\llbracket \varphi' \rrbracket \subseteq \llbracket \varphi \rrbracket$. We say that $\sqsubseteq$ is sound. We can show that $\sqsubseteq$ is a well-quasi-ordering over constraints of bounded degrees that involve fixed numbers of tasks and phasers. This is due to the fact that $\leq_{\mathcal{G}}$ is itself a well-quasi-ordering over graphs of bounded degrees over a finite set of variables [113, 100].

Lemma 2 ($\mathcal{WQO}$). *Given $k, n, m \in \mathbb{N}$, the entailment relation $\sqsubseteq$ over the set of constraints of degree k involving at most n tasks and m phasers is a well-quasi-ordering.*

Proof. Assume an infinite sequence $\varphi_1, \varphi_2, \dots$ of constraints, where $\varphi_i = (\mathcal{T}_i, \mathcal{P}_i, \mathcal{bv}_i, \mathcal{pc}_i, \mathcal{gap}_i, \mathcal{egap}_i)$, where $|\mathcal{T}_i| = n$ and $|\mathcal{P}_i| = m$ for all $i \geq 1$ and where all appearing graphs have degree k or less. We can assume that $\mathcal{T}_i = \mathcal{T}_j$ and $\mathcal{P}_i = \mathcal{P}_j$ for any $i, j \geq 1$. We show there are $i < j$ such that $\varphi_i \sqsubseteq \varphi_j$. There is a finite number of different values for $\mathcal{bv}_i, \mathcal{pc}_i, \mathcal{pv}_i$ and for the domains of $\gamma_i(p)$ for each $p \in \mathcal{P}$. We can therefore extract an infinite subsequence, where:

- $bv_i = bv_j$, $pc_i = pc_j$, $pv_i = pv_j$, and
- for each $p \in \mathcal{P}$, the domains of $\gamma_i(p)$ and $\gamma_j(p)$ coincide.

Observe that for each phaser $p \in \mathcal{P}$, the graphs $\gamma_i(p)$ are the same up to possible differences on the weights. Let $\leq$ be the component-wise-ordering on vectors. Each graph $\gamma_i(p)$ is of degree k or less. We can therefore organize its weights as a vector $\mathbf{vectorOf}(\gamma_i(p))$ with elements in $\mathbb{Z} \cup \{-\infty, +\infty\}$ but where the only allowed negative elements are those larger than $-k$. The vectors can be organized in a way that $\mathbf{vectorOf}(\gamma_i(p)) \leq \mathbf{vectorOf}(\gamma_j(p))$ means $\gamma_i(p) \leq_g \gamma_j(p)$. We then repeat the following steps for each $p \in \mathcal{P}$: using Higman's lemma [72] and the degree boundedness of the constraints, we extract an infinite sequence of constraints, where $\mathbf{vectorOf}(\gamma_a(p)) \leq \mathbf{vectorOf}(\gamma_b(p))$ if $a < b$. $\square$

Theorem 8. *The derivation rules of Figure 4.8 are sound.*

Proof. Assume $c = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$ and $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma)$ such that $c \in \llbracket \varphi \rrbracket$. By definition of $c \in \llbracket \varphi \rrbracket$, tasks and phasers in c and φ are related to each other by two bijections. Hence, for brevity and without loss of generality, we assume that the tasks and phasers in c and φ are renamed so that they map to each other by the identity relation.

newPhaser. Assume $c' \xrightarrow[t]{\text{ph}:=\text{newPhaser}()} c$ for some c' and assume p is the new phaser id. Since $c \in \llbracket \varphi \rrbracket$ and t is the only task registered with p in c with $\phi(p)(t) = (0, 0)$, then, $(\omega^t = 0, \sigma^t = 0) \models \gamma(p)$ and $\text{isReg}(u, p, \gamma(p)) \Rightarrow u = t$ for each task $u \in \mathcal{T}$. We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{ph}:=\text{newPhaser}()} \varphi'$. Assuming $c' = (\mathcal{T}, \mathcal{P}', bv, pc', pv', \phi')$, two cases are possible for pv' . Either $pv'(t)(\text{ph})$ is undefined in which case the derivation rule **newPhaser I** captures $\varphi' = (\mathcal{T}, \mathcal{P}', bv, pc', pv', \gamma \setminus \{p\})$ that denotes c' ; or, $pv'(t)(\text{ph})$ associates the pair (t, ph) to some phaser id $q \neq p$ which is captured by the **newPhaser II** derivation rule.

signal. Assume $c' \xrightarrow[t]{\text{ph.signal}()} c$ for some c' , where $pv(t)(\text{ph}) = p$. We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{ph.signal}()} \varphi'$. The definition of $c \in \llbracket \varphi \rrbracket$ and the existence of c' ensure that t is registered with p in $\gamma(p)$. The signal phase of the task t on the phaser p in the configuration c was obtained after incrementing the corresponding signal value in c' . Therefore, the values denoted by σ^t and ω^u in $\gamma(p)$ (for all u registered with p in c) can be restricted so that $\sigma^t > \omega^u$ for each u registered with p in $\gamma(p)$. γ' is obtained from γ by enforcing this constraint and by decrementing the signal phase variable σ^t . Hence, if

$$\begin{array}{c}
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{ph} := \text{newPhaser}(); pc(t)] \wedge \\
 pv(t)(\text{ph}) = p \wedge pv' = pv[t \leftarrow pv(t)[\text{ph} \leftarrow \uparrow]] \wedge \\
 \{\omega^t \mapsto 0, \sigma^t \mapsto 0\} \models \gamma(p) \wedge \gamma' = \gamma \setminus \{p\} \wedge \\
 (\text{isReg}(u, p, \gamma(p)) \implies u = t)
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph} := \text{newPhaser}()} (\mathcal{T}, \mathcal{P} \setminus \{p\}, bv, pc', pv', \gamma')
 } \quad (\text{newPhaser I})
 \\
 \\
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{ph} := \text{newPhaser}(); pc(t)] \wedge \\
 pv(t)(\text{ph}) = p \wedge pv' = pv[t \leftarrow pv(t)[\text{ph} \leftarrow q]] \wedge \\
 \{\omega^t \mapsto 0, \sigma^t \mapsto 0\} \models \gamma(p) \wedge \gamma' = \gamma \setminus \{p\} \wedge \\
 (\text{isReg}(u, p, \gamma(p)) \implies u = t)
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph} := \text{newPhaser}()} (\mathcal{T}, \mathcal{P} \setminus \{p\}, bv, pc', pv', \gamma')
 } \quad (\text{newPhaser II})
 \\
 \\
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{ph.signal}(); pc(t)] \wedge pv(t)(\text{ph}) = p \wedge \text{isReg}(t, p, \gamma(p)) \wedge \\
 \wp = (\gamma(p) \otimes \text{graphOf}(\wedge_{u \in \text{Reg}(p, \gamma(p))} (\sigma^t > \omega^u))) \\
 \wedge \text{isSat}(\wp) \wedge \gamma' = \gamma[p \leftarrow ((\wp[\sigma/\sigma^t] \otimes \text{graphOf}(\sigma^t = \sigma - 1)) \ominus \{\sigma\})]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.signal}()} (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')
 } \quad (\text{signal})
 \\
 \\
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{ph.wait}(); pc(t)] \wedge pv(t)(\text{ph}) = p \wedge \text{isReg}(t, p, \gamma(p)) \wedge \\
 \wp = (\gamma(p) \otimes \text{graphOf}(\wedge_{u \in \text{Reg}(p, \gamma(p))} (\omega^t > 0))) \\
 \wedge \text{isSat}(\wp) \wedge \gamma' = \gamma[p \leftarrow ((\wp[\omega/\omega^t] \otimes \text{graphOf}(\omega^t = \omega - 1)) \ominus \{\omega\})]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.wait}()} (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')
 } \quad (\text{wait})
 \\
 \\
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{ph.drop}(); pc(t)] \wedge pv(t)(\text{ph}) = p \wedge \neg \text{isReg}(t, p, \gamma(p)) \wedge \\
 \gamma' = \gamma[p \leftarrow (\gamma(p) \otimes \text{graphOf}((\sigma^t \geq \omega^t \geq 0) \wedge_{u \in \text{Reg}(p, \gamma(p))} (\sigma^u \geq \omega^t \geq 0) \wedge (\sigma^t \geq \omega^u \geq 0)))]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.drop}()} (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')
 } \quad (\text{drop})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \notin \mathcal{T} \wedge pc' = pc[t \leftarrow \text{exit}] \wedge f \in \text{Pfn}(\mathcal{P}, \mathcal{P}) \wedge pv' = pv \sqcup \{t \mapsto f\} \wedge Q \subseteq \mathcal{P} \wedge \\
 \gamma' = \gamma[\{p \leftarrow \gamma(p) \otimes \text{graphOf}((\sigma^t \geq \omega^t \geq 0) \wedge_{u \in \text{Reg}(p, \gamma(p))} (\sigma^u \geq \omega^t \geq 0) \wedge (\sigma^t \geq \omega^u \geq 0)) \mid p \in Q\}]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{exit}} (\mathcal{T} \cup \{t\}, \mathcal{P}, bv, pc', pv', \gamma')
 } \quad (\text{exit})
 \\
 \\
 \frac{
 \begin{array}{l}
 pc' = pc[t \leftarrow \text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s_1\}; pc(t)] \wedge \text{paramOf}(\text{task}) = (q_1, \dots, q_k) \wedge u \in \mathcal{T} \setminus \{t\} \wedge \\
 pv' = pv \setminus \{u\} \wedge pc(u) = s_1 \wedge \forall i : 1 \leq i \leq k. pv(t)(\text{ph}_i) = pv(u)(q_i) = p_i \wedge \\
 \text{isReg}(t, p_i, \gamma(p_i)) \wedge \text{isReg}(u, p_i, \gamma(p_i)) \wedge \\
 \wp_i = (\gamma(p_i) \otimes \text{graphOf}(\omega^t = \omega^u \wedge \sigma^t = \sigma^u)) \wedge \\
 \text{isSat}(\wp_i) \wedge \gamma_0 = \gamma \wedge \gamma_i = \gamma_{i-1}[p_i \leftarrow (\wp_i \ominus \{\omega^u, \sigma^u\})] \\
 \wedge q \notin \{p_1, \dots, p_k\} \neg \text{isReg}(u, q, \gamma(q))
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s_1\}} (\mathcal{T} \setminus \{u\}, \mathcal{P}, bv, pc' \setminus \{u\}, pv', \gamma_k)
 } \quad (\text{asynch})
 \end{array}$$

Figure 4.8: Derivation rules for computing $\text{pre}_{\text{stmt}}(\varphi)$ as union of all $\{\varphi' \mid \varphi \xrightarrow[t]{\text{stmt}} \varphi'\}$ with $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma)$ and $t \in \mathcal{T}$.

$c' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \phi')$, then, the constraint $\varphi' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$ denotes c' and $\varphi \xrightarrow[t]{\text{ph.signal}()} \varphi'$.

wait. Assume $c' \xrightarrow[t]{\text{ph.wait}()} c$ for some c' where $pv(t)(\text{ph}) = p$. We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{ph.wait}()} \varphi'$. The definition of $c \in \llbracket \varphi \rrbracket$ and the existence of c' ensure that t is registered with p in $\gamma(p)$. The wait phase of the task t on the phaser p in the configuration c was obtained after incrementing the corresponding wait value in c' . The value denoted by ω^t in $\gamma(p)$ can therefore be restricted so that $\omega^u > 0$. γ' is obtained from γ by enforcing this constraint and by decrementing the wait phase variable ω^t . Hence, if $c' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \phi')$, then, the constraint $\varphi' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$ denotes c' and $\varphi \xrightarrow[t]{\text{ph.wait}()} \varphi'$.

drop. Assume $c' \xrightarrow[t]{\text{ph.drop}()} c$ for some c' , where $pv(t)(\text{ph}) = p$. We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{ph.drop}()} \varphi'$. The definition of $c \in \llbracket \varphi \rrbracket$ and the existence of c' ensure that t is not registered with p in $\gamma(p)$. Let γ' capture the most relaxed constraint on wait and signal phases of the tasks registered with p , including t , and enforce the invariants of the phaser programs. Hence, if $c' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \phi')$, then, the constraint $\varphi' = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$ denotes c' and $\varphi \xrightarrow[t]{\text{ph.drop}()} \varphi'$.

exit. Assume $c' \xrightarrow[t]{\text{exit}} c$ for some c' . We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{exit}} \varphi'$. The definition of $c \in \llbracket \varphi \rrbracket$ and the existence of c' ensure that t is not a member of $\mathcal{T}$. If $c' = (\mathcal{T}, \mathcal{P}, bv, pc', pv', \phi')$, then, let γ' be built with respect to a set $Q \subseteq \mathcal{P}$ representing phasers to which t was registered in ϕ' . Let also γ' introduce the most relaxed wait and signal phases of t to each phaser in Q . For other tasks, let γ' be the same as γ . Then, the constraint $\varphi' = (\mathcal{T}, \mathcal{P}, bv, pc', pv', \gamma')$ denotes c' and $\varphi \xrightarrow[t]{\text{exit}} \varphi'$.

asynch. Assume $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s_1\}} c$ for some c' , where $\text{paramOf}(\text{task}) = (q_1, \dots, q_k)$. Task t has just spawned the task u in c . We exhibit φ' such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s_1\}} \varphi'$. The definition of $c \in \llbracket \varphi \rrbracket$ ensures that for each $i : 1 \leq i \leq k$, $pv(t)(\text{ph}_i) = pv(u)(q_i) = p_i$ and that γ can be constrained so that t and u are registered with each p_i in γ and have the same phases. Assume γ' is ob-

tained from γ by imposing those constraints and removing the phase variables of u on every phaser p_i . If $\mathbf{c}' = (\mathcal{T}', \mathcal{P}, bv, pc', pv', \phi')$, the constraint $\varphi' = (\mathcal{T}', \mathcal{P}, bv, pc', pv', \gamma')$ denotes $\mathbf{c}'$. Moreover, by definition of the **asynch** derivation rule, $\varphi \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s_1\}} \varphi'$. $\square$

Theorem 9. *The derivation rules of Figure 4.8 are complete.*

Assume $\varphi' = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma)$ is obtained from φ using a rule in Figure 4.8. We exhibit for any $\mathbf{c}' \in \llbracket \varphi' \rrbracket$ a configuration $\mathbf{c} \in \llbracket \varphi \rrbracket$ such that $\mathbf{c}$ is a successor of $\mathbf{c}'$ according to the semantics of the phaser programs introduced in Figure 4.3.

By definition of $\mathbf{c}' \in \llbracket \varphi' \rrbracket$, tasks and phasers in $\mathbf{c}'$ and φ' are related to each other by two bijections. Hence, for brevity and without loss of generality, we assume that the tasks and phasers in $\mathbf{c}'$ and φ' are renamed so that they map to each other by the identity relation.

newPhaser. Assume $\varphi \xrightarrow[t]{\text{ph}:=\text{newPhaser}()} \varphi'$ for some φ and assume p is the new phaser id. We exhibit $\mathbf{c}$ such that $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph}:=\text{newPhaser}()} \mathbf{c}$. Let ϕ' be obtained from ϕ by associating the pair (p, t) to the phases $(0, 0)$, i.e., $\phi'(p)(t) = (0, 0)$. Therefore, if $\varphi = (\mathcal{T}, \mathcal{P}', bv, pc', pv', \gamma')$, then the configuration $\mathbf{c} = (\mathcal{T}, \mathcal{P}', bv, pc', pv', \phi')$ satisfies both $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph}:=\text{newPhaser}()} \mathbf{c}$.

signal. Assume $\varphi \xrightarrow[t]{\text{ph.signal}()} \varphi'$ for some φ , where $pv(t)(\text{ph}) = p$. We exhibit $\mathbf{c}$ such that $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph.signal}()} \mathbf{c}$. Assuming $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$, the configuration $\mathbf{c} = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \phi')$, where ϕ' captures incrementing the signal phase sig_p^t , satisfies both $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph.signal}()} \mathbf{c}$.

wait. Assume $\varphi \xrightarrow[t]{\text{ph.wait}()} \varphi'$ for some φ , where $pv(t)(\text{ph}) = p$. We exhibit $\mathbf{c}$ such that $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph.wait}()} \mathbf{c}$. Assuming $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$, the configuration $\mathbf{c} = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \phi')$, where ϕ' captures incrementing the wait phase wait_p^t , satisfies both $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph.wait}()} \mathbf{c}$.

drop. Assume $\varphi \xrightarrow[t]{\text{ph.drop}()} \varphi'$ for some φ , where $pv(t)(\text{ph}) = p$. We exhibit $\mathbf{c}$ such that $\mathbf{c} \in \llbracket \varphi \rrbracket$ and $\mathbf{c}' \xrightarrow[t]{\text{ph.drop}()} \mathbf{c}$. Assuming $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc', pv, \gamma')$. Let

pv' and ϕ' respectively be obtained from pv and ϕ by removing t from pv and $\phi(p)$. Then, the configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc', pv', \phi')$ satisfies both $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{ph.drop}()} c$.

exit. Assume $\varphi \xrightarrow[t]{\text{ph.drop}()} \varphi'$ for some φ , where $pv(t)(\text{ph}) = p$. We exhibit c such that $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{ph.drop}()} c$. Assuming $\varphi = (\mathcal{T}', \mathcal{P}, bv, pc', pv', \gamma')$, where $\mathcal{T}' = \mathcal{T} \setminus \{t\}$ and pv' and ϕ' are respectively obtained by restricting pv and ϕ to $\mathcal{T}'$, the configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc', pv', \phi')$ satisfies both $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{ph.drop}()} c$.

asynch. Assume $\varphi \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} \varphi'$ for some φ , where $\text{paramOf}(\text{task}) = (q_1, \dots, q_k)$. We exhibit c such that $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} c$. Assuming $\varphi = (\mathcal{T}', \mathcal{P}, bv, pc', pv', \gamma')$, the configuration $c = (\mathcal{T}', \mathcal{P}, bv, pc', pv', \phi')$, where for each $i : 1 \leq i \leq k$, $pv'(t)(\text{ph}_i) = pv'(u)(q_i) = p_i$, and in ϕ' the tasks t and u have the same phase on every phaser p_i , satisfies both $c \in \llbracket \varphi \rrbracket$ and $c' \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} c$.

4.6 Verification Procedure

Input: A set of constraints Φ_{bad} , the initial configuration c_{init} , a set of image computation functions $\text{pre} = \{\text{pre}_{\text{stmt}} | s \in S \text{ and } \text{stmt} = \text{hd}(s)\}$, a preorder $\sqsubseteq$ over constraints, and the upper bounds $t^\bullet$ and $p^\bullet$ (in $\mathbb{N} \cup \{+\infty\}$) on the number of coexisting tasks and phasers.

Output: **unreachable** or a symbolic *trace* $\rho = \varphi_0; \text{stmt}_1; \varphi_1; \dots; \text{stmt}_n; \varphi_n$

```

1 initialize both working and store to  $\{(\varphi, \varphi) \mid \varphi \in \Phi_{bad}\}$ ;
2 while exists  $(\varphi, \rho) \in \text{working}$  do
3   remove  $(\varphi, \rho)$  from working;
4   let  $(\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) = \varphi$ ;
5   if  $|\mathcal{T}| > t^\bullet$  or  $|\mathcal{P}| > p^\bullet$  then continue;
6   if  $c_{init} \in \llbracket \varphi \rrbracket$  then return  $\rho$ ;
7   foreach  $s \in S$  do
8     foreach  $\varphi' \in \text{pre}_{\text{hd}(s)}(\varphi)$  do
9       if  $\varphi'' \notin \varphi'$  for all  $(\varphi'', -) \in \text{store}$  then
10        add  $(\varphi', \varphi'; \text{hd}(s); \rho)$  to both working and store;
11 return unreachable;
```

Procedure `phaser-explore`($\Phi_{bad}, c_{init}, S, \text{pre}, \sqsubseteq, t^\bullet, p^\bullet$). A working-list procedure to check whether some bad configuration denoted by constraints in Φ_{bad} is reachable from the initial configuration c_{init} . The procedure uses the image computation functions pre_{stmt} from Figure 4.8, and the preorder $\sqsubseteq$ on the constraints. The upper bounds $t^\bullet$ and $p^\bullet$ determine the maximum numbers of tasks and phasers in every constraint.

We adapt the reachability analysis introduced in Section 3.3 in backward to analyse the transition systems induced by phaser programs. The only difference compared to that procedure is that we have added line 5 to discard those constraints that do not respect the upper bounds on the number of tasks or phasers. The procedure starts from a set of constraints Φ_{bad} . Its goal is to check if there is a trace from any of those constraints to some constraint whose denotation intersects some initial configuration in c_{init} using the set of image computation functions $\mathbf{pre}_{\mathbf{stmt}}$ for each $\mathbf{stmt}$ in the phaser program.

Φ_{bad} can, for instance, be any subset of $\mathbf{badCstrs}_{\mathbf{deadlock}}^{(n,p)}$ (degree 0) or of $\mathbf{badCstrs}_{\mathbf{assert}}^{(n,p)}$ (free) in case we want to check the possibility of a deadlock or of an assertion violation.

Lemma 3 (Partial correctness). *If the procedure “phaser-explore”($\Phi_{bad}, c_{init}, \mathbf{S}, \mathbf{pre}, \Xi, t^\bullet, p^\bullet$) in Section 4.6 returns **unreachable**, then $c_{init} \not\stackrel{*}{\rightarrow} \llbracket \Phi_{bad} \rrbracket$. If it returns a trace $\varphi_n, \mathbf{stmt}_n, \dots, \mathbf{stmt}_1, \varphi_1$, then, there are $c_n, \dots, c_1$ with $c_n = c_{init}$, $c_1 \in \llbracket \Phi_{bad} \rrbracket$ and $c_i \xrightarrow{\mathbf{stmt}_i} c_{i-1}$ for $i : 1 < i \leq n$.*

Proof. The procedure “phaser-explore”($\Phi_{bad}, c_{init}, \mathbf{S}, \mathbf{pre}, \Xi, t^\bullet, p^\bullet$) with an unbounded number of tasks and phasers is an instantiation of the procedure “explore” in Section 3.3. By soundness and completeness of $\mathbf{pre}$ (respectively, Theorems 8 and 9), and by soundness and completeness of the procedure “explore” (respectively, Theorems 4 and 5), the procedure “phaser-explore”($\Phi_{bad}, c_{init}, \mathbf{S}, \mathbf{pre}, \Xi$) is sound and complete. However, since Ξ is not $\mathcal{WQO}$ for an unbounded number of tasks and phasers, the procedure “phaser-explore” might not terminate. Hence, it is partially exact. $\square$

Theorem 10 (Free termination). *The procedure “phaser-explore”($\Phi_{bad}, c_{init}, \mathbf{S}, \mathbf{pre}, \Xi, t^\bullet, p^\bullet$) terminates for $t^\bullet, p^\bullet \in \mathbb{N}$ and free Φ_{bad} .*

Proof. Ξ is $\mathcal{WQO}$ over the degree-0 constraints with a bounded number of tasks and phasers. Moreover, freeness is preserved by the $\mathbf{pre}$ computation (Figure 4.8). Hence, by Theorem 6 we conclude the procedure “phaser-explore” terminates. $\square$

In order to check the reachability of arbitrary constraints, we may need to force termination. We do this by soundly bounding the degree of generated constraints using a relaxation ϱ_k . The relaxation $\varrho_l((\mathcal{T}, \mathcal{P}, \mathbf{bv}, \mathbf{pc}, \mathbf{pv}, \gamma))$ replaces, in each graph $\gamma(p)$, each weight k'' such that $k'' < -k$ with $-\infty$.

```

1 foreach  $\varphi'' \in \mathbf{pre}(t, \varphi)$  do
2   | Let  $\varphi' = \varrho_k(\varphi'')$ ;

```

Theorem 11 (Forced termination). *Independently of the degree of Φ_{bad} , procedure “phaser-explore”($\Phi_{bad}, c_{init}, S, pre, \sqsubseteq, t^\bullet, p^\bullet$), where line 8 is replaced by the two lines of Figure 6, is sound and guaranteed to terminate.*

Proof. Soundness is due to the validity of $\varrho_k(\varphi) \sqsubseteq \varphi$ while the termination argument relies, similarly to Theorem 10, on well-quasi orderedness of $\sqsubseteq$ on the set of constraints with bounded degree and fixed numbers of tasks and phasers. $\square$

4.7 Experimental Results

We report on experiments with our open source prototype *PhaserVerify*¹ for the verification of phaser programs. We conducted experiments on 12 different programs (some of which are from [40]). We considered both deadlock and assertion reachability problems. For each property, we considered correct and buggy versions. This resulted in 48 different instances with 2 to 3 phasers and 2 to 4 tasks. Our tool uses global phaser and task variables as in [40]. Experiments were conducted on an Intel Core(TM) i7 processor running at 2.9GHz with 8GB of memory. Our implemented procedure does not eagerly concretize the states of the tasks described in the predecessor computation of Chapter 4.5. Instead, we collect conditions on the phases of the tasks that have not yet take any actions and lazily concretize them. Reported times for checking deadlocks are the sums of the times required to check reachability for each cycle. The prototype is only a proof of concept. For instance, the example (10.Running-3) is a variant of (09-Running-2), where a task instance is spawned twice, leading to two symmetrical tasks (out of four). This required up to three orders of magnitude more time to check than (09-Running-2). We believe partial order reduction techniques would help here. Other relevant heuristics would be to make use of priority queues and to organize the minimal sets. All examples are available on the tool homepage.

4.8 Conclusion

We have proposed a gap-order based reachability analysis for phaser programs. We have shown our analysis to be exact and guaranteed to terminate when checking registration, race, and assertion errors. We have established the undecidability of deadlock verification and explained how to turn our analysis into a sound over-approximation. In the next chapter, we discuss the parameterized verification of phaser programs.

¹<https://gitlab.liu.se/live/phaserverify>

program	property	safe / buggy	times
01. Loopless	deadlock:	ok / trace	1s / 1s
	assertion:	ok / trace	1s / 1s
02. Iterative averaging	deadlock:	ok / trace	1s / 1s
	assertion:	ok / trace	1s / 1s
03. Ordered phasers	deadlock:	ok / trace	1s / 1s
	assertion:	ok / trace	13s / 1s
04. Conditional	deadlock:	ok / trace	2s / 1s
	assertion:	ok / trace	4s / 7s
05. Loop Synch.	deadlock:	ok / trace	178s / 145s
	assertion:	ok / trace	7s / 13s
06. Nested forks	deadlock:	ok / trace	2s / 1s
	assertion:	ok / trace	1s / 1s
07. Conditional membership	deadlock:	ok / trace	1s / 1s
	assertion:	ok / trace	12s / 3s
08. Producer-consumer	deadlock:	ok / trace	37s / 222s
	assertion:	ok / trace	79s / 34s
09. Running-2	deadlock:	ok / trace	5s / 1s
	assertion:	ok / trace	26s / 4s
10. Running-3	deadlock:	ok / trace	4318s / 128s
	assertion:	ok / trace	18631s / 54s

Table 4.1: The results of checking assertion violation and deadlock of Phaser programs with PhaserVerify

5

Safety Verification of Parameterized Phaser Programs

In this chapter, we address the problem of statically checking safety properties (such as assertions or deadlocks) for *parameterized* phaser programs. We consider programs that may generate arbitrarily many tasks and phasers. We study different formulations of the verification problem and propose an exact procedure that is guaranteed to terminate for some reachability problems even in the presence of unbounded phases and arbitrarily many spawned tasks. In addition, we prove undecidability results for several problems. Finally, we report on our experimental results.

5.1 Introduction

We focus on the parameterized verification problem of parallel programs that adopt the phaser construct for synchronization [115]. Parameterized verification of phaser programs is a challenging problem due to the arbitrary numbers of involved tasks and phasers.

We build on the operational model for phaser programs introduced in Chapter 4 and consider an additional statement called **next**. This statement consists of a **signal** followed by a **wait**. The **next** command may be associated with a sequence of statements that are to be executed in isolation by one of the registered tasks participating in the command. We call a program that does not use this feature of the next statement *non-atomic*.

This chapter provides general (un)decidability results that we believe will guide verification of other synchronization constructs. We identify combina-

tions of features (e.g., unbounded differences between signal and wait phases, atomic statements) and properties to be checked (e.g., assertions, deadlocks) for which the parameterized verification problem becomes undecidable. These help identify synchronization constructs with enough expressivity to result in undecidable parameterized verification problems. We also provide a symbolic verification procedure that even terminates on fragments with arbitrary phases and numbers of spawned tasks. We get back to possible implications in the conclusion section. We summarize our contributions as follows:

- We propose an exact symbolic verification procedure for checking reachability of sets of configurations for non-atomic phaser programs when only distinguished tasks such as `main` create phasers. Observe that in such programs arbitrarily many tasks and phasers may be generated.
- We prove undecidability results for several reachability problems.
- We show termination of our procedure when checking assertions for non-atomic programs, even when the programs may spawn arbitrarily many tasks.
- We show termination of our procedure when checking deadlock-freedom and assertions for non-atomic programs in which the difference between *signal* and *wait* phases is bounded, even when the programs may spawn arbitrarily many tasks.

Related work. The closest work to ours is [62], which was introduced in Chapter 4 and is the only work on automatic and static formal verification of phaser programs. However, the work in [62] considers a bounded number of tasks and phasers. The approach introduced in this chapter can decide whether some program assertion has been violated even in the presence of arbitrarily many tasks with arbitrarily large phase gaps. This is well beyond [62] and requires a more complex symbolic representation with a more profound termination argument.

Outline. We describe the phaser construct in Section 5.2. We then formally introduce the construct and show that the associated general reachability problem is undecidable in Section 5.3. In Section 5.4, we describe our symbolic representation and state some of its non-trivial properties. We use the representation in Section 5.5 to define a verification procedure and establish decidability results. We refine our undecidability results in Section 5.6. We summarize our findings in Section 5.7.

```

1  bool a, done;
2  main(){
3    done = false;
4    p= newPhaser(SIG_WAIT);
5    c= newPhaser(SIG_WAIT);
6    while(ndet()){
7      asynch(Prod,p:SIG,c:WAIT);
8      asynch(Cons,p:WAIT,c:SIG);
9    }
10   p.drop();
11   c.drop();
12 }

13 Prod(p:SIG,c:WAIT)
14 {
15   while(!done)
16   {
17     p.signal();
18     c.wait();
19     assert(a);
20     a = false;
21   };
22   p.drop();
23   c.drop();
24 }

25 Cons(p:WAIT,c:SIG)
26 {
27   while(!done){
28     p.wait();
29     if(ndet())
30       done = true;
31     a = true;
32     c.signal();
33   };
34   p.drop();
35   c.drop();
36 }

```

Figure 5.1: An unbounded number of producers and consumers are synchronized using two phasers. In this construction, each consumer requires all producers to be ahead of it (w.r.t. the phaser `p`) so that it is able to consume their respective products. At the same time, each consumer needs to be ahead of all producers (w.r.t. the phaser `c`) for the producers to be able to move to the next phase and produce new items.

5.2 Illustrative Example

The program listed in Figure 5.1 uses Boolean shared variables $B = \{a, done\}$. The `main` task creates two phasers (line 4-5). When creating a phaser, the task gets automatically registered with it. The main task also creates an unbounded number of other task instances (lines 7-8).

Tasks are dynamically registered with phasers (e.g., lines 7-8). They can also dynamically deregister themselves (e.g., lines 10-11).

In this example, an unbounded number of producers and consumers are synchronized using two phasers. The consumers require the producers to be ahead of them (with respect to the phaser they reference with `p`) in order for them to consume their products. At the same time, the consumers need to be ahead of all producers (with respect to the phaser referenced with `c`) in order to produce their items.

We are interested in checking: (a) control reachability, as in assertions (e.g., line 19), race conditions (e.g., mutual exclusion of lines 20 and 31) or registration errors (e.g., signaling a dropped phaser), and (b) plain reachability, as in deadlocks (e.g., a producer at line 18 and a consumer at line 28 with equal phases waiting for each other). Intuitively, both problems concern the reachability of target sets of program configurations. The difference is that control state reachability defines the targets only with the states of the tasks (their control locations and whether they are registered with any phasers).

5.3 Phaser Programs and Reachability

We introduced phaser programs in Section 4.4. In this work, we consider the same syntax extended with two statements, namely, `ph.next()`; and

$\text{ph.next()}\{\text{stmt}\}$. We also let arbitrarily many instances of each task be spawned during program execution. Arbitrarily many phasers might also be generated during program execution.

$$\text{stmt} ::= \dots \mid \text{ph.next()}; \mid \text{ph.next()}\{\text{stmt}\};$$

ph.next() is syntactic sugar for a signal followed by a wait. $\text{ph.next()}\{\text{stmt}\}$ is similar to ph.next() . However, the block of code stmt is executed atomically by precisely one of the tasks participating in the synchronization before all tasks continue the execution that follows the barrier. $\text{ph.next()}\{\text{stmt}\}$, thus, requires all tasks to be synchronized on the same statement and is less flexible. The absence of a $\text{ph.next()}\{\text{stmt}\}$ makes a program *non-atomic*. Note that the assignment of phaser variables is excluded from the syntax; additionally, we restrict task creation $\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_n)$ and require that the parameter variables ph_i are all different. This prevents two variables of a single task from referencing the same phaser and avoids the need to deal with aliasing: we can reason on the single variable in a process that references a phaser. Extending our work to deal with aliasing would be easy but would require heavier notations.

We will need the notions of configurations, partial configurations, and inclusion to define the reachability problems we consider in this work. We introduce them in the following and assume a phaser program $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$.

Configurations. The configurations of a phaser program describe valuations of its variables, control sequences of its tasks, and registration details to the phasers.

Control sequences. The set of control sequences S is defined similar to Section 4.4. But, it also contains (i) all $\text{s}_1; \text{ph.next()}\{\}; \text{tl}(\text{s})$ if s is in S and $\text{hd}(\text{s}) = \text{ph.next()}\{\text{s}_1\}$; and (ii) all $\text{ph.signal()}; \text{ph.wait()}; \text{tl}(\text{s})$ if s is in S and $\text{hd}(\text{s}) = \text{ph.next()}\{\}$.

Partial configurations. Partial configurations allow the characterization of sets of configurations by partially stating some of their common characteristics. We only consider configurations that have only one instance of the task *main*. A *partial configuration* c of $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$ is a tuple $(\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{phase})$ where:

- $\mathcal{T}$ is a finite set of task identifiers. We let t, u range over $\mathcal{T}$.
- $\mathcal{P}$ is a finite set of phaser identifiers. We let p, q range over $\mathcal{P}$.
- $\text{bv} : \text{B} \rightarrow \text{B} \cup \{*\}$ fixes the values of some of the shared variables.
- $\text{pc} : \mathcal{T} \rightarrow \text{S} \cup \{*\}$ fixes the control sequences of some of the tasks.

- $phase : \mathcal{T} \rightarrow \text{Pfn}(\mathcal{P}, \mathcal{P} \cup \{-, *\} \times (\mathbb{N}^2 \cup \{(*, *)\}, \text{nreg}))$ is a mapping that associates to each task t in $\mathcal{T}$ a partial mapping stating which phasers are known by the task and with which registration values.

Intuitively, partial configurations are used to state some facts about the valuations of variables and the control sequences of tasks and their registrations. Partial configurations leave some details unconstrained using partial mappings or the symbol $*$. For instance, if $bv(v^b) = *$ in a partial configuration $(\mathcal{T}, \mathcal{P}, bv, pc, phase)$, then the partial configuration does not constrain the value of the shared variable v^b . Moreover, a partial configuration does not constrain the relation between a task t and a phaser p when $phase(t)(p)$ is undefined. Instead, when the partial mapping $phase(t)$ is defined on phaser p , it associates a pair $phase(t)(p) = (v, val)$ to p . If $v \in \mathcal{P} \cup \{-, *\}$ is a variable $ph \in \mathcal{P}$ then the task t in $\mathcal{T}$ uses its variable ph to refer to the phaser p in $\mathcal{P}^1$. If v is the symbol $-$ then the task t does not refer to ph with any of its variables in $\mathcal{P}$. If v is the symbol $*$, then the task might or might not refer to p . The value val in $phase(t)(p) = (v, val)$ is either the value nreg or a pair $(wait, sig)$. The value nreg means the task t is not registered with phaser p . The pair $(wait, sig)$ belongs to $(\mathbb{N} \times \mathbb{N}) \cup \{(*, *)\}$. In this case, task t is registered with phaser p with a symbolic wait phase $wait$ and a symbolic signal phase sig . The value $*$ means that the wait phase $wait$ (respectively signal phase sig) can be any value in $\mathbb{N}$. For instance, $phase(t)(p) = (ph, \text{nreg})$ means variable ph of the task t refers to phaser p but the task is not registered with p . On the other hand, $phase(t)(p) = (-, (*, *))$ means the task t does not refer to p but is registered with it with an arbitrary wait and signal phases.

Concrete configurations. A concrete configuration (or configuration for short) is a partial configuration $(\mathcal{T}, \mathcal{P}, bv, pc, phase)$ where $phase(t)$ is total for each $t \in \mathcal{T}$ and where the symbol $*$ does not appear in any range. It is a tuple $(\mathcal{T}, \mathcal{P}, bv, pc, phase)$ where $bv : \mathcal{B} \rightarrow \mathbb{B}$, $pc : \mathcal{T} \rightarrow \mathcal{S}$, and $phase : \mathcal{T} \rightarrow \text{Fn}(\mathcal{P}, \mathcal{P} \cup \{-\} \times ((\mathbb{N} \times \mathbb{N}) \cup \{\text{nreg}\}))$.

For a concrete configuration $(\mathcal{T}, \mathcal{P}, bv, pc, phase)$, we write $\text{isReg}(phase, t, p)$ to mean the predicate $phase(t)(p) \notin (\mathcal{P} \cup \{-\} \times \{\text{nreg}\})$. The predicate $\text{isReg}(phase, t, p)$ captures whether the task t is registered with phaser p according to the mapping $phase$.

Note that a concrete configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc, phase)$ in this chapter captures the same information as a configuration $c' = (\mathcal{T}, \mathcal{P}, bv, pc, pv, \phi)$ in Chapter 4 if $phase(t)(p) = (ph, \phi(t)(p))$ and $pv(t)(ph) = p$ for all $t \in \mathcal{T}$ and $p \in \mathcal{P}$. The differences in the configuration representations in the chapters are to simplify the presentation in each chapter. Another difference is that in this chapter we do not allow aliasing for the phaser variables in order to further simplify the presentation.

¹The uniqueness of this variable is due to the absence of aliasing discussed before.

Inclusion of configurations. A configuration $c' = (\mathcal{T}', \mathcal{P}', bv', pc', phase')$ includes a partial configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc, phase)$ if renaming and deleting tasks, phasers, or registrations from c' can give a configuration that “matches” c . More formally, c' includes c if $((bv(v^b) \neq bv'(v^b)) \implies (bv(v^b) = *))$ for each $v^b \in \mathcal{B}$ and there are injections $\kappa : \mathcal{T} \rightarrow \mathcal{T}'$ and $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ such for each $t \in \mathcal{T}$ and $p \in \mathcal{P}$:

1. $((pc(t) \neq pc'(\tau(t))) \implies (pc(t) = *))$,
2. either
 - $phase(t)(p)$ is undefined, or
 - $phase(t)(p) = (ph, val)$ and $phase'(\kappa(t))(\pi(p)) = (ph', val')$ with $((ph \neq ph') \implies (ph = *))$ and either $(val = val' = \text{nreg})$ or $val = (wait, sig)$ and $val' = (wait', sig')$ with $((wait \neq wait') \implies (wait = *))$ and $((sig \neq sig') \implies (sig = *))$.

Semantics and Reachability. Given a program $Pr = (\mathcal{B}, \mathcal{P}, \mathcal{T}s)$, the main task $\text{main}()\{\text{stmt}\}$ starts executing stmt from an initial configuration $c_{init} = (\mathcal{T}_{init}, \mathcal{P}_{init}, bv_{init}, pc_{init}, phase_{init})$ where $\mathcal{T}_{init}$ is a singleton, $\mathcal{P}_{init}$ is empty, bv_{init} sets all shared variables to **false** and pc_{init} associates stmt to the unique task in $\mathcal{T}_{init}$.

We write $c \xrightarrow[\text{stmt}]{t} c'$ to mean a task t in c can fire statement stmt and result in configuration c' . See Figure 5.2 for a description of the operational semantics. We write $c \xrightarrow[\text{stmt}]{} c'$ if $c \xrightarrow[\text{stmt}]{t} c'$ for some task t , and $c \rightarrow c'$ if $c \xrightarrow[\text{stmt}]{} c'$ for some statement stmt . We also write $\xrightarrow{+}$ for the transitive closure of $\rightarrow$ and let $\xrightarrow{*}$ be the reflexive transitive closure of $\rightarrow$. Figure 5.3 identifies erroneous configurations.

We are interested in the reachability of sets of configurations (i.e., checking safety properties). We differentiate between two reachability problems depending on whether the target sets of configurations constrain the registration phases or not. The two problems are defined in the following.

Plain reachability. First, we define *equivalent configurations*. A configuration $c = (\mathcal{T}, \mathcal{P}, bv, pc, phase)$ is equivalent to configuration $c' = (\mathcal{T}', \mathcal{P}', bv', pc', phase')$ if $bv = bv'$ and there are bijections $\kappa : \mathcal{T} \rightarrow \mathcal{T}'$ and $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ such that, for all $t \in \mathcal{T}$, $p \in \mathcal{P}$ and $ph \in \mathcal{P} \cup \{-\}$, $pc(t) = pc'(\kappa(t))$ and there are some integers $(k_p)_{p \in \mathcal{P}}$ such that $phase(t)(p) = (ph, (w, s))$ iff $phase'(\kappa(t))(\pi(p)) = (ph, (w + k_p, s + k_p))$. We write $c \sim c'$ to mean that c and c' are equivalent. Intuitively, equivalent configurations simulate each other. We can establish the following:

Lemma 4 (Equivalence). *Assume two configurations c_1 and c_2 . If $c_1 \rightarrow c_2$ and $c'_1 \sim c_1$ then there is a configuration c'_2 such that $c'_2 \sim c_2$ and $c'_1 \rightarrow c'_2$.*

$$\begin{array}{c}
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad \text{hd}(pc(t)) = \text{ph} := \text{newPhaser}() \quad p \notin \mathcal{P} \\
 \mathcal{P}' = \mathcal{P} \cup \{p\} \quad \text{phase}' = \text{phase}[t \leftarrow \{p \leftarrow (\text{ph}, (0, 0))\}] \\
 \text{phase}'' = \text{phase}'[\{u \leftarrow \text{phase}'(u)[p \leftarrow (-, \text{nreg})] \mid u \in \mathcal{T} \setminus \{t\}\}]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph} := \text{newPhaser}()} (\mathcal{T}, \mathcal{P}', bv, pc[t \leftarrow \text{tl}(pc(t))], \text{phase}'')
 } \quad (\text{newPhaser})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad u \notin \mathcal{T} \quad \mathcal{T}' = \mathcal{T} \cup \{u\} \quad \text{ph}_1, \dots, \text{ph}_k \in \mathcal{P} \\
 \text{hd}(pc(t)) = \text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{\text{stmt}\} \quad \text{paramOf}(\text{task}) = (q_1, \dots, q_k) \\
 \text{for each } i : 1 \leq i \leq k. \text{ phase}(t)(p_i) = (\text{ph}_i, (\text{wait}^i, \text{sig}^i)) \quad \text{for each } i, j : 1 \leq i, j \leq k. i \neq j \Rightarrow \text{ph}_i \neq \text{ph}_j \\
 \text{phase}' = \text{phase}[u \leftarrow \{p_i \leftarrow (q_i, (\text{wait}^i, \text{sig}^i)) \mid 1 \leq i \leq k\} \cup \{p \leftarrow (-, \text{nreg}) \mid p \notin \{p_i \mid 1 \leq i \leq k\}\}]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{\text{stmt}\}} (\mathcal{T}', \mathcal{P}, bv, pc'[\{u \leftarrow \text{stmt}\} \cup \{t \leftarrow \text{tl}(pc(t))\}], \text{phase}')
 } \quad (\text{asynch})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad \text{ph} \in \mathcal{P} \quad \text{hd}(pc(t)) = \text{ph.signal}() \quad \text{phase}(t)(p) = (\text{ph}, (\text{wait}^t, \text{sig}^t)) \\
 \text{phase}' = \text{phase}[t \leftarrow \text{phase}(t)[p \leftarrow (\text{ph}, (\text{wait}^t, 1 + \text{sig}^t)]]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph.signal}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], \text{phase}')
 } \quad (\text{signal})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad \text{ph} \in \mathcal{P} \quad \text{hd}(pc(t)) = \text{ph.wait}() \quad \text{phase}(t)(p) = (\text{ph}, (\text{wait}^t, \text{sig}^t)) \\
 \forall u \in \mathcal{T}, \text{var} \in \mathcal{P} \cup \{-\}. (\text{phase}(u)(p) = (\text{var}, (\text{wait}^u, \text{sig}^u)) \Rightarrow \text{wait}^t < \text{sig}^u) \\
 \text{phase}' = \text{phase}[t \leftarrow \text{phase}(t)[p \leftarrow (\text{ph}, (1 + \text{wait}^t, \text{sig}^t)]]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph.wait}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], \text{phase}')
 } \quad (\text{wait})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad \text{ph} \in \mathcal{P} \quad \text{hd}(pc(t)) = \text{ph.next}() \quad pc' = pc[t \leftarrow \text{ph.signal}(); \text{ph.wait}(); \text{tl}(pc(t))]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph.next}()} (\mathcal{T}, \mathcal{P}, bv, pc', \text{phase}')
 } \quad (\text{next})
 \\
 \\
 \frac{
 \begin{array}{l}
 \text{ph} \in \mathcal{P} \quad \mathcal{U} = \{u \mid \text{isReg}(\text{phase}, u, p)\} \quad \forall u \in \mathcal{U}. \text{hd}(pc(u)) = \text{ph.next}()\{\text{stmt}\} \quad t \in \mathcal{U} \\
 pc' = pc[t \leftarrow \text{stmt}; \text{ph.next}(); \text{tl}(pc(t))][\{u \leftarrow \text{ph.next}(); \text{tl}(pc(u)) \mid u \in \mathcal{U} \setminus \{t\}\}]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph.next}()\{\text{stmt}\}} (\mathcal{T}, \mathcal{P}, bv, pc', \text{phase}')
 } \quad (\text{next}\{\text{stmt}\})
 \\
 \\
 \frac{
 \begin{array}{l}
 t \in \mathcal{T} \quad \text{ph} \in \mathcal{P} \quad \text{hd}(pc(t)) = \text{ph.drop}() \quad \text{phase}(t)(p) = (\text{ph}, (\text{wait}, \text{sig})) \\
 \text{phase}' = \text{phase}[t \leftarrow \text{phase}(t)[p \leftarrow (\text{ph}, \text{nreg})]]
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{ph.drop}()} (\mathcal{T}, \mathcal{P}, bv, pc[t \leftarrow \text{tl}(pc(t))], \text{phase}')
 } \quad (\text{drop})
 \\
 \\
 \frac{
 \begin{array}{l}
 \text{hd}(pc(t)) = \text{exit} \quad pc' = pc \setminus \{t\} \quad \text{phase}' = \text{phase} \setminus \{t\}
 \end{array}
 }{
 (\mathcal{T}, \mathcal{P}, bv, pc, \text{phase}) \xrightarrow[t]{\text{exit}} (\mathcal{T} \setminus \{t\}, \mathcal{P}, bv, pc', \text{phase}')
 } \quad (\text{exit})
 \end{array}$$

Figure 5.2: Operational semantics of phaser statements without errors. Each transition corresponds to a task $t \in \mathcal{T}$ executing a statement from a configuration $(\mathcal{T}, \mathcal{P}, bv, pc, \text{phase})$. For instance, the **drop** transition corresponds to a task t executing **ph.drop()** when registered with phaser $p \in \mathcal{P}$ (with phases $(\text{wait}, \text{sig})$) and referring to it with variable **ph**. The result is the same configuration where task t moves to its next statement without being registered with p (albeit still referring to p with **ph**).

$$\begin{array}{c}
 \frac{t \in \mathcal{T} \quad \text{hd}(pc(t)) = \text{assert}(\text{cond}) \quad \llbracket \text{cond} \rrbracket_{bv} = \text{false}}{(\mathcal{T}, \mathcal{P}, bv, pc, phase) \in \text{AssertErrors}} \quad (\text{assertion : errors}) \\
 \\
 \frac{t, u \in \mathcal{T} \quad t \neq u \quad \text{hd}(pc(t)) = b' := \text{cond}' \quad (b' \text{ coincides with } b \text{ or appears in } \text{cond}) \\
 \text{hd}(pc(u)) \in \{b := \text{cond}, \text{if}(\text{cond})\{\text{stmt}\}, \text{while}(\text{cond})\{\text{stmt}\}, \text{assert}(\text{cond})\}}{(\mathcal{T}, \mathcal{P}, bv, pc, phase) \in \text{RaceErrors}} \quad (\text{race : errors}) \\
 \\
 \frac{t \in \mathcal{T} \quad \text{ph} \in \mathcal{P} \quad p \in \mathcal{P} \quad phase(t)(p) = (\text{ph}, \text{nreg}) \\
 \text{hd}(pc(t)) \in \{\text{asynch}(\text{task}, \dots, \text{ph} \dots)\{\text{stmt}\}, \text{ph.signal}(), \text{ph.wait}(), \text{ph.drop}()\}}{(\mathcal{T}, \mathcal{P}, bv, pc, phase) \in \text{RegisterErrors}} \quad (\text{registration : errors}) \\
 \\
 \frac{t_0, \dots, t_{n-1} \in \mathcal{T} \quad \text{ph}_0, \dots, \text{ph}_{n-1} \in \mathcal{P} \cup \{-\} \quad \text{q}_0, \dots, \text{q}_{n-1} \in \mathcal{P} \cup \{-\} \quad p_0, \dots, p_{n-1} \in \mathcal{P} \\
 \forall i : 0 \leq i \leq (n-1). \left\{ \begin{array}{l} \text{hd}(pc(t_i)) = \text{ph}_i.\text{wait}() \\ phase(t_i)(p_i) = (\text{ph}_i, (\text{wait}_{p_i}^{t_i}, \text{sig}_{p_i}^{t_i})) \\ phase(t_{(i+1)\%n})(p_i) = (\text{q}_i, (\text{wait}_{p_i}^{t_{(i+1)\%n}}, \text{sig}_{p_i}^{t_{(i+1)\%n}})) \\ \text{wait}_{p_i}^{t_i} = \text{sig}_{p_i}^{t_{(i+1)\%n}} \end{array} \right.}{(\mathcal{T}, \mathcal{P}, bv, pc, phase) \in \text{DeadlockErrors}} \quad (\text{deadlock : errors})
 \end{array}$$

Figure 5.3: Definition of error configurations. Starting from $(\mathcal{T}, \mathcal{P}, bv, pc, phase)$, error configurations are obtained when tasks execute the above statements under certain conditions. For instance, a deadlock is obtained if tasks in a subset $\{t_0, \dots, t_{n-1}\} \subseteq \mathcal{T}$ form a cycle where each $t_{(i+1)\%n}$ blocks (with its signal phase $\text{sig}_{p_i}^{t_{(i+1)\%n}}$) the wait being executed by t_i on phaser p_i (with wait phase $\text{wait}_{p_i}^{t_i}$).

Proof. According to the semantics in Figure 5.2, the gap between the wait and signal phases (not the absolute value of the phases) determines whether or not a phaser operation is enabled. This is because the phaser operations update the phases either by removing and adding phasers, or subtracting and adding a constant to them. The successors of equivalent configurations will therefore always be equivalent. $\square$

Observe that if the wait value of a task t on a phaser p is equal to the signal of a task t' on the same phaser p in some configuration c , then this is also the case, up to a renaming of the phasers and tasks, in all equivalent configurations. This is particularly relevant for defining deadlock configurations where several tasks are waiting for each other.

Given a program, a starting configuration, and a target partial configuration, the plain reachability problem asks whether a configuration that includes the target partial configuration is reachable. More formally, given a program $\text{Pr} = (\mathcal{B}, \mathcal{P}, \mathcal{T}s)$, a configuration c_{strt} , and a partial configuration c_{trgt} , then $\text{reach}(\text{Pr}, c_{\text{strt}}, c_{\text{trgt}})$ iff $c_{\text{strt}} \xrightarrow{*} c_1$ for $c_1 \sim c_2$ and c_2 includes c_{trgt} . We abuse notation and write $c_{\text{strt}} \xrightarrow{*} \mathbb{C}$ to mean a configuration $c \in \mathbb{C}$ that includes c_{trgt} is reachable from c_{strt} .

Definition 6 (Plain reachability). *For a program Pr , a configuration $\mathbf{c}_{\text{strt}}$, and a partial configuration $\mathbf{c}_{\text{trgt}}$, decide whether $\text{reach}(\text{Pr}, \mathbf{c}_{\text{strt}}, \mathbf{c}_{\text{trgt}})$ holds.*

Control reachability. A partial configuration $\mathbf{c} = (\mathcal{T}, \mathcal{P}, bv, pc, phase)$ is said to be a *control partial configuration* if for all $t \in \mathcal{T}$ and $p \in \mathcal{P}$, either $phase(t)(p)$ is undefined or $phase(t)(p) \in (\mathbf{P} \cup \{-, *\} \times \{(*, *), \text{nreg}\})$. Intuitively, control partial configurations do not constrain phase values. They are enough to characterize, for example, configurations where an assertion is violated (see Figure 5.3).

Definition 7 (Control reachability). *For a program Pr , a configuration $\mathbf{c}_{\text{strt}}$, and a control partial configuration $\mathbf{c}_{\text{trgt}}$, decide whether $\text{reach}(\text{Pr}, \mathbf{c}_{\text{strt}}, \mathbf{c}_{\text{trgt}})$ holds.*

Observe that plain reachability is at least as hard to answer as control reachability since any control partial configuration is also a partial configuration. It turns out the control reachability problem is undecidable for programs that have arbitrarily many tasks and phasers, as stated by the theorem below.

Theorem 12. *Control reachability is undecidable in general.*

Proof. The proof is by reducing the reachability problem of 2-counter Minsky machines to the state reachability problem and using the Theorem 1 that was introduced in Chapter 3. Recall that the state reachability problem of a Minsky machine can be reduced to its reachability problem. State reachability of an arbitrary 2-counter Minsky machine is encoded as the control reachability problem of a phaser program (captured in Figure 5.5).

Assume a 2-counter Minsky machine $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathcal{T}, \mathbf{q}_{\text{init}}, \mathbf{q}_{\text{trgt}})$ with two counters $\mathcal{C} = \{x, y\}$. The phaser program has three tasks **main**, **xTask** and **yTask**. It uses shared Boolean variables to capture the state $\mathbf{q} \in \mathcal{Q}$ and pass information among different task instances. The phaser program builds two chains, one with **xTask** instances for the counter x , and one with **yTask** instances for the counter y . Each chain alternates a phaser and a task and encodes the values of its counter by the number of tasks in it. We explain the encoding for counter x . A similar encoding is used for counter y .

A chain for counter x is the longest sequence of the form $t_0, p_0, t_1, p_1 \dots p_{m-1}, t_m$ where t_0 is running **main** and for all $k : 1 \leq k \leq m$ the tasks are running **xTask**. t_i, p_j or p_j, t_i appear in the chain iff $\text{Reg}(t_i, p_j)$. m is said to be the *length* of the chain. A sample configuration with two chains is illustrated in Figure 5.4. We show that m models the value of counter x .

The idea is to have the phaser program simulate all counter machine transitions, i.e., increments, decrements, and tests for zero. Answering the state reachability of the counter machine amounts to checking whether from the initial configuration of the phaser program, $\mathbf{c}_{\text{init}}$, a configuration is reachable in which the Boolean variable **sF** that corresponds to the target machine state $\mathbf{q}_F$ is **true**. This can be captured with a control partial configuration.

Figure 5.4: A sample configuration generated in the proof of Theorem 12 containing two chains of alternating tasks and phasers.

Increment. For incrementing counter x , **main** starts by executing **signal** and **wait** on xPh which has id p_0 . If $m = 0$, **main** will not block at line 12. It creates t_1 and registers it with p_0 (line 14).

On the other hand, if $m > 0$, we have the chain $t_0, p_0, t_1, p_1 \dots p_{m-1}, t_m$. In this case, t_k , for all $k : 0 \leq k \leq m - 1$ executes **signal** and **wait** on the phaser p_k . Only t_m is registered with the phaser p_m . Hence, only t_m can pass the **wait** at line 61. As a result, only t_m sets **temp** (line 63), creates a new instance of **xTask** with id t_{m+1} , and registers it with p_m (line 64). It will also issue **signal** on p_m (line 65), hence release its parent who is blocked either at line 61 or line 12. Now, every t_i for $i : 0 < i < m$ in the chain will release its parent by signaling at line 73 and will wait until all tasks other than **main** have done so. This is done by waiting at line 74 for **temp** to become **false**. At the end, **main** will reset **temp** and **xInc** and change the state.

Decrement. If $m = 0$, decrementing x is not enabled according to the semantics of the counter machines. In the encoded program, **main** will block at line 29. Otherwise, when $m > 0$, we have the chain $t_0, p_0, t_1, p_1 \dots p_{m-1}, t_m$. In this case, t_k , for all $k : 0 \leq k \leq m - 1$ executes **signal** and **wait** on the phaser p_k . Only t_m is registered with the phaser p_m . Hence, only t_m can pass the **wait** on line 61. Hence, only t_m will set **temp** (line 69) and exit (line 70). Now, every t_i for $i : 0 < i < m$ in the chain will release its parent by signaling at line 73 and will wait until all tasks other than **main** have done so. This is done by waiting at line 74 for **temp** to become **false**. At the end, **main** will reset **temp** and **xDec** and change the state.

Test for zero is encoded as a **signal** followed by a **wait** on xPh (lines 37 and 38). If some t_m exists for $m > 0$ (that is, $x > 0$), **main** will block at line 38. Otherwise, it will only change the state accordingly.

The above encoding ensures that the lengths of the **xTask** and **yTask** chains capture the values of the counters x and y , respectively. As a result, from the initial configuration c_{init} , the phaser program will reach a configuration in which **sF** = **true** iff the counter machine reaches a configuration with the target state q_F .

□

```

1  bool s1, s2, ..., sF;
2  bool xInc, xDec, yInc, yDec, temp;
3  main(){
4      xPh = newPhaser();
5      yPh = newPhaser();
6      while(true){
7          temp = false;
8          //(q1:inc(x):q3)
9          if(ndet() ∧ si){
10             xInc=true;
11             xPh.signal();
12             xPh.wait();
13             if(!temp){
14                 asynch(xTask, xPh);
15             }
16             else{
17                 temp = false;
18             }
19             xInc = false;
20
21             si=false;
22             sj= true;
23         }
24         //(q1:dec(x):q3)
25         if(ndet() ∧ sj){
26             xDec = true;
27             xPh.signal();
28             xPh.wait();
29             while(!temp){};
30             temp =false;
31             xDec = false;
32             si= false;
33             sj= true;
34         }
35         //(q3:test(x):q3)
36         if(ndet() ∧ sj){
37             xPh.signal();
38             xPh.wait();
39             si= false;
40             sj= true;
41         }
42         //(q1:inc(y):q3)
43         if(ndet() ∧ si){
44             ...
45         }
46         //(q3:dec(y):q3)
47         if(ndet() ∧ si){
48             ...
49         }
50         //(q1:test(y):q3)
51         if(ndet() ∧ si){
52             ...
53         }
54     }
55     //***** xTask *****
56     xTask(parent){
57         child=newPhaser();
58         while(true){
59             child.signal();
60             child.wait();
61             if(xInc){
62                 temp=true;
63                 asynch(xTask, child);
64                 parent.signal();
65                 while(temp){};
66             }
67             if(xDec){
68                 temp=true;
69                 exit();
70             }
71         }
72         if(temp){
73             parent.signal();
74             while(temp){};
75         }
76     }
77 }
78
79 //**** yTask ****
80 yTask(parent){
81     ...
82 }

```

Figure 5.5: For the proof of Theorem 12. Encoding a 2-counter Minsky machine with the counters $\{x, y\}$ using two task-phaser-chains with the lengths of the chains capturing the values of each counter. The messages among the tasks are used to orchestrate the simulation and are transmitted in the Boolean variables `temp`, `xInc`, `xDec`, `yInc`, and `yDec`. For instance, `xInc` stands for incrementing the counter x .

We have demonstrated that the control reachability problem is in general undecidable. In the next section, we introduce our symbolic representation that will be used in the verification procedure.

5.4 A Gap-based Symbolic Representation

The symbolic representation we propose builds on the following intuitions. First, observe that the language semantics impose, for each phaser, the invariant that signal values are always greater than or equal to wait values. We can, therefore, assume this fact in our symbolic representation. Also, our reachability problems from Section 5.3 are defined in terms of reachability of equivalence classes, not of individual configurations. This is because configurations that violate the considered properties (see Figure 5.3) are not defined in terms of concrete phase values but rather in terms of relations among them (as well as the registration status, control sequences, and variable values). Finally, we observe that if a wait is enabled with smaller gaps on a given phaser, then it will be enabled with larger ones. We, therefore, propose to track the gaps of the differences between signal and wait values with respect to an existentially quantified level (per phaser) that lies between wait and signal values of all registered tasks (with the phaser that is being considered).

We formally define our symbolic representation and describe a corresponding entailment relation. We also establish a desirable property (namely that of being a well-quasi ordering) on some classes of representations. This property is crucial for the decidability of specific reachability problems (see. Section 5.5).

Named gaps. A *named gap* is associated to a task-phaser pair. It consists of a tuple $(\mathbf{ph}, \mathbf{val})$ in $\mathbb{G} = \left(\mathbf{P} \cup \{-, *\} \times \left(\left(\mathbb{N}^4 \cup \left(\mathbb{N}^2 \times \{\infty\}^2 \right) \right) \cup \{\mathbf{nreg}\} \right) \right)$. Like for partial configurations in Section 5.3, $\mathbf{ph} \in \mathbf{P} \cup \{-, *\}$ constrains variable values. The $\mathbf{val}$ value describes task registration to the phaser. If not registered, $\mathbf{val}$ is $\mathbf{nreg}$. Otherwise, $\mathbf{val}$ is a 4-tuple $(\mathbf{lw}, \mathbf{ls}, \mathbf{uw}, \mathbf{us})$. This tuple intuitively captures, together with some level lev common to all tasks registered with the considered phaser, all concrete wait and signal values $(wait, sig)$ satisfying $\mathbf{lw} \leq (lev - wait) \leq \mathbf{uw}$ and $\mathbf{ls} \leq (sig - lev) \leq \mathbf{us}$. A named gap $(\mathbf{ph}, (\mathbf{lw}, \mathbf{ls}, \mathbf{uw}, \mathbf{us}))$ is said to be *free* if $\mathbf{uw} = \mathbf{us} = \infty$. It is said to be *B-gap-bounded*, for $B \in \mathbb{N}$, if both $\mathbf{uw} \leq B$ and $\mathbf{us} \leq B$ hold. $(\mathbf{ph}, \mathbf{nreg})$ is *B-gap-bounded* for any $B \in \mathbb{N}$. A set $\mathcal{G} \subseteq \mathbb{G}$ is said to be *free* (respectively *B-gap-bounded*) if all its named gaps are free (respectively *B-gap-bounded*). The set $\mathcal{G}$ is said to be *B-good* if each one of its named gaps is either free or *B-gap-bounded*. Finally, $\mathcal{G}$ is said to be *good* if it is *B-good* for some $B \in \mathbb{N}$. Given a set $\mathcal{G}$ of named gaps, we define the partial order $\sqsubseteq$ on $\mathcal{G}$, and write $(\mathbf{ph}, \mathbf{val}) \sqsubseteq (\mathbf{ph}', \mathbf{val}')$, to mean (i) $(\mathbf{ph} \neq \mathbf{ph}' \Rightarrow \mathbf{ph} = *)$, and (ii) $(\mathbf{val} = \mathbf{nreg}) \iff (\mathbf{val}' = \mathbf{nreg})$, and (iii) if $\mathbf{val} = (\mathbf{lw}, \mathbf{ls}, \mathbf{uw}, \mathbf{us})$ and

$\text{val}' = (\text{lw}', \text{ls}', \text{uw}', \text{us}')$ then $\text{lw} \leq \text{lw}'$, $\text{ls} \leq \text{ls}'$, $\text{uw}' \leq \text{uw}$ and $\text{us}' \leq \text{us}$. Intuitively, named gaps are used in the definition of constraints in order to capture relations (i.e., reference, registration, and possible phases) of tasks and phasers. The partial order $(\text{ph}, \text{val}) \sqsubseteq (\text{ph}', \text{val}')$ ensures the relations allowed by $(\text{ph}', \text{val}')$ are also allowed by (ph, val) .

Constraints. A constraint φ of a phaser program $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$ is a tuple $(\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{gap}, \text{egap})$ that denotes a possibly infinite set of configurations. Intuitively, $\mathcal{T}$ and $\mathcal{P}$ respectively represent a minimal set of tasks and phasers that are required in any configuration denoted by the constraint. We only consider constraints that have at most one instance of the task **main**. In addition:

- $\text{bv} : \text{B} \rightarrow \mathbb{B} \cup \{*\}$ and $\text{pc} : \mathcal{T} \rightarrow \mathbb{S} \cup \{*\}$ respectively represent, like for partial configurations, a valuation of the Boolean variables and a mapping of tasks to their control sequences.
- $\text{gap} : \mathcal{T} \rightarrow \text{Fn}(\mathcal{P}, \mathbb{G})$ constrains relations between $\mathcal{T}$ -tasks and $\mathcal{P}$ -phasers by associating to each task t a mapping $\text{gap}(t)$ that defines for each phaser p a named gap $(\text{ph}, \text{val}) \in \mathbb{G}$ capturing the relation of t and p .
- $\text{egap} : \mathcal{P} \rightarrow \mathbb{N}^2$ associates *lower bounds* (ew, es) on gaps of tasks that are registered with $\mathcal{P}$ -phasers but which are not explicitly captured by $\mathcal{T}$. This is described further in the denotations of a constraint below.

We write $\text{isReg}(t, p, \text{gap})$ to mean a task t is registered with a phaser p , i.e., $\text{gap}(t)(p) \notin (\text{P} \cup \{-, *\} \times \{\text{nreg}\})$. A constraint φ is said to be free (respectively B -gap-bounded or B -good) if the set $\mathcal{G} = \{\text{gap}(t)(p) \mid t \in \mathcal{T}, p \in \mathcal{P}\}$ is free (respectively, B -gap-bounded or B -good). The dimension of a constraint is the number of its phasers (i.e., $|\mathcal{P}|$). A set of constraints is said to be K -dimension-bounded if there is natural number K in $\mathbb{N}$ that is larger than the dimension of any of its elements. A set of constraints Φ is said to be free, B -gap-bounded, or B -good if each of its constraints is.

Denotations. For a constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$ and a configuration $\text{c} = (\mathcal{T}_\text{c}, \mathcal{P}_\text{c}, \text{bv}_\text{c}, \text{pc}_\text{c}, \text{phase}_\text{c})$, we write $\text{c} \models \varphi$ (or $\text{c} \in \llbracket \varphi \rrbracket$) to mean φ denotes c . Intuitively, the configuration c should have at least as many tasks (captured by a surjection κ from a subset $\mathcal{T}_\text{c}^1$ of $\mathcal{T}_\text{c}$ to $\mathcal{T}_\varphi$) and phasers (captured by a bijection π from a subset $\mathcal{P}_\text{c}^1$ of $\mathcal{P}_\text{c}$ to $\mathcal{P}_\varphi$). Constraints on the tasks and phasers in $\mathcal{T}_\text{c}^1$ and $\mathcal{P}_\text{c}^1$ ensure target configurations are reachable. Additional constraints on the tasks in $\mathcal{T}_\text{c}^2 = \mathcal{T}_\text{c} \setminus \mathcal{T}_\text{c}^1$ ensure this reachability is not blocked by tasks not captured by $\mathcal{T}_\varphi$. More formally:

1. for each $\text{v}^b \in \text{B}$, $(\text{bv}_\varphi(\text{v}^b) \neq \text{bv}_\text{c}(\text{v}^b)) \implies (\text{bv}_\varphi(\text{v}^b) = *)$, and
2. $\mathcal{T}_\text{c}$ and $\mathcal{P}_\text{c}$ can be written as $\mathcal{T}_\text{c} = \mathcal{T}_\text{c}^1 \uplus \mathcal{T}_\text{c}^2$ and $\mathcal{P}_\text{c} = \mathcal{P}_\text{c}^1 \uplus \mathcal{P}_\text{c}^2$, with

3. $\kappa : \mathcal{T}_c^1 \rightarrow \mathcal{T}_\varphi$ is a surjection and $\pi : \mathcal{P}_c^1 \rightarrow \mathcal{P}_\varphi$ is a bijection, and
4. for $t_c \in \mathcal{T}_c^1$ with $t_\varphi = \kappa(t_c)$, $(pc_\varphi(t_\varphi) \neq pc_c(t_c)) \implies (pc_\varphi(t_\varphi) = *)$, and
5. for each $p_\varphi = \pi(p_c)$, there is a natural level $lev \geq 0$ such that:
 - a) if $t_c \in \mathcal{T}_c^1$ with $t_\varphi = \kappa(t_c)$, $phase_c(t_c)(p_c) = (\mathbf{ph}_c, \mathbf{val}_c)$ and $gap_\varphi(t_\varphi)(p_\varphi) = (\mathbf{ph}_\varphi, \mathbf{val}_\varphi)$, then it is the case that:
 - i. $(\mathbf{ph}_c \neq \mathbf{ph}_\varphi) \implies (\mathbf{ph}_\varphi = *)$, and
 - ii. $(\mathbf{val}_c = \mathbf{nreg}) \iff (\mathbf{val}_\varphi = \mathbf{nreg})$, and
 - iii. if $(\mathbf{val}_c = (\mathbf{wait}, \mathbf{sig}))$ and $(\mathbf{val}_\varphi = (\mathbf{lw}, \mathbf{ls}, \mathbf{uw}, \mathbf{us}))$ then $\mathbf{lw} \leq lev - \mathbf{wait} \leq \mathbf{uw}$ and $\mathbf{ls} \leq \mathbf{sig} - lev \leq \mathbf{us}$.
 - b) if $t_c \in \mathcal{T}_c^2$, then for each $p_\varphi = \pi(p_c)$ with $phase_c(t_c)(p_c) = (\mathbf{ph}_c, (\mathbf{wait}, \mathbf{sig}))$ and $gap_\varphi(p_\varphi) = (\mathbf{ew}, \mathbf{es})$, we have: $(\mathbf{es} \leq \mathbf{sig} - lev)$ and $(\mathbf{ew} \leq lev - \mathbf{wait})$

We say in this case that κ and π witness the denotation of c by φ . Intuitively, for each phaser, the bounds given by gap constrain the values of the phases belonging to tasks captured by $\mathcal{T}_\varphi$ (i.e., those in $\mathcal{T}_c^1$) and registered with the given phaser. This is done with respect to some non-negative level, one per phaser. The same level is used to constrain phases of tasks registered with the phaser but not captured by $\mathcal{T}_\varphi$ (i.e., those in $\mathcal{T}_c^2$). For these tasks, lower bounds are enough as we only want to ensure that they do not block executions to target sets of configurations. We write $\llbracket \varphi \rrbracket$ for $\{c \mid c \models \varphi\}$.

Entailment. We write $\varphi_a \sqsubseteq \varphi_b$ to mean $\varphi_a = (\mathcal{T}_a, \mathcal{P}_a, bv_a, pc_a, gap_a, egap_a)$ is entailed by $\varphi_b = (\mathcal{T}_b, \mathcal{P}_b, bv_b, pc_b, gap_b, egap_b)$. This will ensure that configurations denoted by φ_b are also denoted by φ_a . Intuitively, φ_b should have at least as many tasks (captured by a surjection κ from a subset $\mathcal{T}_b^1$ of $\mathcal{T}_b$ to $\mathcal{T}_a$) and phasers (captured by a bijection π from a subset $\mathcal{P}_b^1$ of $\mathcal{P}_b$ to $\mathcal{P}_a$). Conditions on tasks and phasers in $\mathcal{T}_b^1$ and $\mathcal{P}_b^1$ ensure the conditions in φ_a are met. Additional conditions on the tasks in $\mathcal{T}_b^2 = \mathcal{T}_b \setminus \mathcal{T}_b^1$ ensure at least the $egap_a$ conditions in φ_a are met. More formally:

1. $(bv_a(v^b) \neq bv_b(v^b)) \implies (bv_a(v^b) = *)$, for each $v^b \in B$ and
2. $\mathcal{T}_b$ and $\mathcal{P}_b$ can be written as $\mathcal{T}_b = \mathcal{T}_b^1 \uplus \mathcal{T}_b^2$ and $\mathcal{P}_b = \mathcal{P}_b^1 \uplus \mathcal{P}_b^2$ with
3. $\kappa : \mathcal{T}_b^1 \rightarrow \mathcal{T}_a$ is a surjection and $\pi : \mathcal{P}_b^1 \rightarrow \mathcal{P}_a$ is a bijection, and
4. $(pc_b(t_b) \neq pc_a(t_a)) \implies (pc_b(t_b) = *)$ for each $t_b \in \mathcal{T}_b^1$ with $t_a = \kappa(t_b)$, and
5. for each phaser $p_a = \pi(p_b)$ in $\mathcal{P}_a$:
 - a) if $egap_a(p_a) = (\mathbf{ew}_a, \mathbf{es}_a)$ and $egap_b(p_b) = (\mathbf{ew}_b, \mathbf{es}_b)$ then $\mathbf{ew}_a \leq \mathbf{ew}_b$ and $\mathbf{es}_a \leq \mathbf{es}_b$

- b) for each $t_b \in \mathcal{T}_b^1$ with $t_a = \kappa(t_b)$ and $gap_a(t_a)(p_a) = (\mathbf{ph}_a, \mathbf{val}_a)$, and $gap_b(t_b)(p_b) = (\mathbf{ph}_b, \mathbf{val}_b)$, it is the case that:
- i. $(\mathbf{ph}_b \neq \mathbf{ph}_a) \implies (\mathbf{ph}_a = *)$, and
 - ii. $(\mathbf{val}_b = \mathbf{nreg}) \iff (\mathbf{val}_a = \mathbf{nreg})$, and
 - iii. if $\mathbf{val}_a = (\mathbf{lw}_a, \mathbf{ls}_a, \mathbf{uw}_a, \mathbf{us}_a)$ and $\mathbf{val}_b = (\mathbf{lw}_b, \mathbf{ls}_b, \mathbf{uw}_b, \mathbf{us}_b)$, then $(\mathbf{lw}_a \leq \mathbf{lw}_b)$, $(\mathbf{ls}_a \leq \mathbf{ls}_b)$, $(\mathbf{uw}_b \leq \mathbf{uw}_a)$ and $(\mathbf{us}_b \leq \mathbf{us}_a)$.
- c) for each $t_b \in \mathcal{T}_b^2$ with $gap_b(t_b)(p_b) = (\mathbf{ph}, (\mathbf{lw}_a, \mathbf{ls}_a, \mathbf{uw}_a, \mathbf{us}_a))$, with $egap_a(p_a) = (\mathbf{ew}_a, \mathbf{es}_a)$, both $(\mathbf{ew}_a \leq \mathbf{lw}_b)$ and $(\mathbf{es}_a \leq \mathbf{ls}_b)$ hold.

We say in this case that κ and π witness the entailment of φ_a by φ_b .

Lemma 5 (Constraint entailment). $\varphi_a \sqsubseteq \varphi_b$ implies $\llbracket \varphi_b \rrbracket \subseteq \llbracket \varphi_a \rrbracket$

Proof. Assume a configuration $\mathbf{c} = (\mathcal{T}_c, \mathcal{P}_c, \mathbf{bv}_c, \mathbf{pc}_c, \mathbf{phase}_c)$ is denoted by $\varphi_b = (\mathcal{T}_b, \mathcal{P}_b, \mathbf{bv}_b, \mathbf{pc}_b, \mathbf{gap}_b, \mathbf{egap}_a)$ with $\varphi_a \sqsubseteq \varphi_b$ and $\varphi_a = (\mathcal{T}_a, \mathcal{P}_a, \mathbf{bv}_a, \mathbf{pc}_a, \mathbf{gap}_a, \mathbf{egap}_a)$. We show $\mathbf{c}$ is also denoted by φ_a .

By assumption, we can write $\mathcal{T}_c$ as a partition $\mathcal{T}_c^m \uplus \mathcal{T}_c^e$ and $\mathcal{P}_c$ as a partition $\mathcal{P}_c^m \uplus \mathcal{P}_c^e$ such that a surjection $\kappa_c : \mathcal{T}_c^m \rightarrow \mathcal{T}_b$ and a bijection $\pi_c : \mathcal{P}_c^m \rightarrow \mathcal{P}_b$ witness the denotation of $\mathbf{c}$ by φ_b . Also, we can write $\mathcal{T}_b$ as a partition $\mathcal{T}_b^m \uplus \mathcal{T}_b^e$ and $\mathcal{P}_b$ as a partition $\mathcal{P}_b^m \uplus \mathcal{P}_b^e$ such that a surjection $\kappa_\varphi : \mathcal{T}_b^m \rightarrow \mathcal{T}_a$ and a bijection $\pi_\varphi : \mathcal{P}_b^m \rightarrow \mathcal{P}_a$ witness the entailment of φ_a by φ_b . Let us write $\mathcal{T}_c^m$ as the partition $\mathcal{T}_c^{m,m} \uplus \mathcal{T}_c^{m,e}$ where $\mathcal{T}_c^{m,m} = \kappa_c^{-1}(\mathcal{T}_b^m)$ and $\mathcal{P}_c^m$ as the partition $\mathcal{P}_c^{m,m} \uplus \mathcal{P}_c^{m,e}$ where $\mathcal{P}_c^{m,m} = \pi_c^{-1}(\mathcal{P}_b^m)$. We define κ to be the restriction of $\kappa_\varphi \circ \kappa_c$ to $\mathcal{T}_c^{m,m}$, i.e., $\kappa : \mathcal{T}_c^{m,m} \rightarrow \mathcal{T}_a$ with $\kappa(t) = \kappa_\varphi(\kappa_c(t))$ for each t in $\mathcal{T}_c^{m,m}$. Observe that κ is a well-defined surjection. In addition, we write π to mean the restriction of $\pi_\varphi \circ \pi_c$ to $\mathcal{P}_c^{m,m}$. Observe that π is a well-defined bijection. We show that κ and π witness the denotation of $\mathbf{c}$ by φ_a .

1. we have that, for each $\mathbf{v}^b \in \mathbf{B}$, both $(\mathbf{bv}_b(\mathbf{v}^b) \neq \mathbf{bv}_c(\mathbf{v}^b)) \implies (\mathbf{bv}_b(\mathbf{v}^b) = *)$ and $(\mathbf{bv}_a(\mathbf{v}^b) \neq \mathbf{bv}_b(\mathbf{v}^b)) \implies (\mathbf{bv}_a(\mathbf{v}^b) = *)$ hold. Hence, $(\mathbf{bv}_a(\mathbf{v}^b) \neq \mathbf{bv}_c(\mathbf{v}^b)) \implies (\mathbf{bv}_a(\mathbf{v}^b) = *)$ also holds.
2. $\mathcal{T}_c = \mathcal{T}_c^{m,m} \uplus (\mathcal{T}_c^{m,e} \uplus \mathcal{T}_c^e)$ and $\mathcal{P}_c = \mathcal{P}_c^{m,m} \uplus (\mathcal{P}_c^{m,e} \uplus \mathcal{P}_c^e)$, with
3. $\kappa : \mathcal{T}_c^{m,m} \rightarrow \mathcal{T}_a$ is a surjection and $\pi : \mathcal{P}_c^{m,m} \rightarrow \mathcal{P}_a$ is a bijection, such that
4. we have that $(\mathbf{pc}_b(t_b) \neq \mathbf{pc}_c(t_c)) \implies (\mathbf{pc}_b(t_b) = *)$ for each $t_c \in \mathcal{T}_c^m$ with $t_b = \kappa_c(t_c)$ in $\mathcal{T}_b$, and $(\mathbf{pc}_a(t_a) \neq \mathbf{pc}_b(t_b)) \implies (\mathbf{pc}_a(t_a) = *)$ for each $t_b \in \mathcal{T}_b^m$ with $t_a = \kappa_\varphi(t_b)$ in $\mathcal{T}_a$. Since $\mathcal{T}_b^{m,m} \subseteq \mathcal{T}_b^m$ and the surjection $\kappa : \mathcal{T}_c^{m,m} \rightarrow \mathcal{T}_b$ is the restriction of $\kappa_\varphi \circ \kappa_c$ to $\mathcal{T}_c^{m,m}$, we deduce: $(\mathbf{pc}_a(t_a) \neq \mathbf{pc}_c(t_c)) \implies (\mathbf{pc}_a(t_a) = *)$ for each $t_c \in \mathcal{T}_c^{m,m}$ with $t_a = \kappa(t_c)$ in $\mathcal{T}_a$.
5. for each $p_a = \pi_\varphi(p_b) = \pi_\varphi(\pi_c(p_c)) = \pi(p_c)$, there is a $\mathbf{lev} : 0 \leq \mathbf{lev}$ s.t.:

- a) if $t_c \in \mathcal{T}_c^{m,m}$ with $t_a = \kappa_\varphi(\kappa_c(t_c))$ with $\text{phase}_c(t_c)(p_c) = (\text{ph}_c, \text{val}_c)$, $\text{gap}_b(t_b)(p_b) = (\text{ph}_b, \text{val}_b)$ and $\text{gap}_a(t_a)(p_a) = (\text{ph}_a, \text{val}_a)$, then:
- i. we have $(\text{ph}_a \neq \text{ph}_b) \implies (\text{ph}_a = *)$ and $(\text{ph}_b \neq \text{ph}_c) \implies (\text{ph}_b = *)$. Hence, $(\text{ph}_a \neq \text{ph}_c) \implies (\text{ph}_a = *)$.
 - ii. $(\text{val}_c = \text{nreg}) \iff (\text{val}_b = \text{nreg}) \iff (\text{val}_a = \text{nreg})$, and
 - iii. if $(\text{val}_c = (\text{wait}, \text{sig}))$ and $(\text{val}_b = (\text{lw}_b, \text{ls}_b, \text{uw}_b, \text{us}_b))$ and $(\text{val}_a = (\text{lw}_a, \text{ls}_a, \text{uw}_a, \text{us}_a))$, then $\text{lw}_a \leq \text{lw}_b \leq \text{lev} - \text{wait} \leq \text{uw}_b \leq \text{uw}_a$ and $\text{ls}_a \leq \text{ls}_b \leq \text{sig} - \text{lev} \leq \text{us}_b \leq \text{us}_a$.
- b) if $t_c \in \mathcal{T}_c^{m,e}$ with $t_b = \kappa_c(t_c)$ with $\text{phase}_c(t_c)(p_c) = (\text{ph}_c, \text{val}_c)$, $\text{gap}_b(t_b)(p_b) = (\text{ph}_b, \text{val}_b)$ and $\text{egap}_a(p_a) = (\text{ew}_a, \text{es}_a)$, then:
- i. if $(\text{val}_c = (\text{wait}, \text{sig}))$ and $(\text{val}_b = (\text{lw}_b, \text{ls}_b, \text{uw}_b, \text{us}_b))$ and $\text{egap}_a(p_a) = (\text{ew}_a, \text{es}_a)$ then $\text{ew}_a \leq \text{lw}_b \leq \text{lev} - \text{wait}$ and $\text{es}_a \leq \text{ls}_b \leq \text{sig} - \text{lev}$.
- c) if $t_c \in \mathcal{T}_c^e$ with $\text{phase}_c(t_c)(p_c) = (\text{ph}_c, (\text{wait}, \text{sig}))$ and $\text{egap}_b(p_b) = (\text{ew}_b, \text{es}_b)$ and $\text{egap}_a(p_a) = (\text{ew}_a, \text{es}_a)$, then: $\text{es}_a \leq \text{es}_b \leq \text{sig} - \text{lev}$ and $\text{ew}_a \leq \text{ew}_b \leq \text{lev} - \text{wait}$

□

The remaining part of this section aims to establish the following theorem:

Theorem 13. (Φ, Ξ) is $\mathcal{WQO}$ if Φ is K -dimension-bounded and B -good for some pre-defined $K, B \in \mathbb{N}$.

The idea is to propose an encoding for each constraint $\varphi = (\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{gap}, \text{egap})$ with respect to some arbitrary total orders $<_{\mathcal{P}}$ and $<_{\mathcal{T}}$. We write $\text{enc}(\varphi, <_{\mathcal{T}}, <_{\mathcal{P}})$ for the encoding of φ . We also define an entailment relation $\leq_e$ on the encodings. Then, we show in Lemma 10 that $\text{enc}(\varphi, <_{\mathcal{T}}, <_{\mathcal{P}}) \leq_e \text{enc}(\varphi', <_{\mathcal{T}'}, <_{\mathcal{P}'})$ implies $\varphi \Xi \varphi'$. Finally, we show in Lemma 11 that $\leq_e$ is $\mathcal{WQO}$ if the encoded K -dimension-bounded constraints are B -good for some pre-defined $K, B \in \mathbb{N}$. We start with the named gaps.

Lemma 6. If $\mathcal{G}$ is a good set of named gaps, then, $(\mathcal{G}, \trianglelefteq)$ is $\mathcal{WQO}$.

Proof. Assume an infinite sequence of named gaps $A = (\text{ph}, (\text{lw}_1, \text{ls}_1, \text{uw}_1, \text{us}_1)), (\text{ph}, (\text{lw}_2, \text{ls}_2, \text{uw}_2, \text{us}_2)), \dots$ with the same phaser variable ph . In both cases where A is free or B -good, from the sequence $C = (\text{lw}_1, \text{ls}_1, \text{uw}_1, \text{us}_1), (\text{lw}_2, \text{ls}_2, \text{uw}_2, \text{us}_2), \dots$ we can extract an infinitely increasing sequence $D = D_1, D_2, \dots$ where $\text{uw}_i = \text{uw}_j$ and $\text{us}_i = \text{us}_j$ for all $i, j \geq 1$. In the sequence $(\text{ph}, D_1), (\text{ph}, D_2), \dots$, for every $1 \leq i < j$ we have $(\text{ph}, D_i) \trianglelefteq (\text{ph}, D_j)$. This proves $(\mathcal{G}, \trianglelefteq)$ is $\mathcal{WQO}$. □

A task state of dimension K is any tuple in $(\mathbb{S} \times \mathcal{G}^K)$ where K is a natural in $\mathbb{N}$ (corresponding to the number of phasers in the constraint to be encoded). We write $(\mathbf{s}, g_1, \dots, g_K) \leq (\mathbf{s}', g'_1, \dots, g'_{K'})$ for two task states to mean that

they have the same dimension (i.e., $K = K'$), that $(\mathbf{s} \neq \mathbf{s}' \Rightarrow \mathbf{s} = *)$, and that $g_k \trianglelefteq g'_k$ for each $k : 1 \leq k \leq K$.

Lemma 7. $((\mathbf{S} \times \mathcal{G}^K), \leq)$ is $\mathcal{WQO}$ in case $\mathcal{G}$ is good.

Proof. By Lemma 6, we have that if $\mathcal{G}$ is a good set of named gaps, then, $(\mathcal{G}, \trianglelefteq)$ is $\mathcal{WQO}$. Using Higman's lemma [72], finite words over a set ordered by a $\mathcal{WQO}$ are also $\mathcal{WQO}$. Moreover, $\mathbf{S}$ is a finite set and does not affect $\mathcal{WQO}$. □

Let $\mathcal{M}(\mathbf{S} \times \mathcal{G}^K)$ be the set of finite multisets over $(\mathbf{S} \times \mathcal{G}^K)$. We write $A \forall \leq \exists B$, for A and B two multisets in $\mathcal{M}(\mathbf{S} \times \mathcal{G}^K)$, if each element $a \in A$ can be mapped to an element $b \in B$ for which $a \leq b$.

Lemma 8. $(\mathcal{M}(\mathbf{S} \times \mathcal{G}^K), \forall \leq \exists)$ is $\mathcal{WQO}$ if $\mathcal{G}$ is good.

Proof. Consider an arbitrary infinite sequence $A = A_1, A_2, \dots$ of elements in $\mathcal{M}(\mathbf{S} \times \mathcal{G}^K)$. We build a sequence $B = B_1, B_2, \dots$ where each B_i is an arbitrary word over the elements of A_i for $i \geq 1$. Lemma 7 together with Higman's lemma [72] ensures there exists $0 \leq i < j$ such that B_i can be obtained from B_j by possibly deleting some elements of B_j . Hence, $A_i \forall \leq \exists A_j$. □

We write $A \exists \leq \forall B$, for A and B finite multisets in $\mathcal{M}(\mathbf{S} \times \mathcal{G}^K)$, to mean that each $b \in B$ can be mapped to some $a \in A$ for which $a \leq b$. Rado's structure [76, 98] shows that $(\mathcal{M}(S), \exists \leq \forall)$ need not be $\mathcal{WQO}$ just because $\leq$ is $\mathcal{WQO}$ over S . Still, we establish the following result:

Lemma 9. $(\mathcal{M}(\mathbf{S} \times \mathcal{G}^K), \exists \leq \forall)$ is $\mathcal{WQO}$ if $\mathcal{G}$ is good.

Proof. We proceed by contradiction. Assume, without loss of generality, an infinite sequence $\langle A_1, A_2, \dots \rangle$ of $\leq$ -minimal multisets in $\mathcal{M}(\mathbf{S} \times \mathcal{G}^K)$ such that $A_j \not\exists \leq \forall A_i$ for all $1 \leq j < i$. Notice that:

1. for each $i : 1 \leq j < i$, we can identify an element $a_i^{-j} \in A_i$ such that $a_j \not\leq a_i^{-j}$ for any $a_j \in A_j$. We sometimes write a_i^{-j} as $(\mathbf{s}_i^{-j}, a_i^{-j}[1], a_i^{-j}[2], \dots, a_i^{-j}[K])$, where $a_i^{-j}[k]$ is the k^{th} component of a_i^{-j} .
2. by the definition of $(\mathbf{S} \times \mathcal{G}^K, \leq)$ and the fact that it is a $\mathcal{WQO}$, we can extract a subsequence $\langle A_{i_1}, A_{i_2}, \dots \rangle$ of $\langle A_1, A_2, \dots \rangle$ associated to a sequence of partial mappings $(b_j : \{1, \dots, K\} \rightarrow \mathbb{N})$ such that for each $j : 1 \leq j$:

- a) the j -sequence $a_{i_1}^{-j} \leq a_{i_2}^{-j} \leq a_{i_3}^{-j} \leq \dots$ is $\leq$ -monotone,
- b) each j -sequence has a constant control, i.e., $\mathbf{s}^{-j} = \mathbf{s}_{i_1}^{-j} = \mathbf{s}_{i_2}^{-j} = \dots$,

- c) for each dimension $k : 1 \leq k \leq K$, the projection of a j -sequence $a_{i_1}^{-j} \leq a_{i_2}^{-j} \leq a_{i_3}^{-j} \leq \dots$ on k is either constant or strictly increasing, i.e.,:
- i. $k \in \text{dom}(b_j)$ implies $a_i^{-j}[k] = b_j(k)$ for all $i : j < i$
 - ii. $k \notin \text{dom}(b_j)$ implies $a_i^{-j}[k] \sqsubseteq a_{i'}^{-j}[k]$ but $a_{i'}^{-j}[k] \not\sqsubseteq a_i^{-j}[k]$ for any $i, i' : 1 \leq j < i < i'$

We rename the sequence $\langle A_{i_1}, A_{i_2}, \dots \rangle$ into $\langle A_1, A_2, \dots \rangle$. We obtain a $\exists \leq \forall$ -bad sequence $\langle A_1, A_2, \dots \rangle$ of $\leq$ -minimal sets satisfying the constraints depicted in the following figure:

$$\begin{array}{cccccccc}
 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & \dots A_i \dots \\
 \neg A_1 & a_2^{-1} \leq & a_3^{-1} \leq & a_4^{-1} \leq & a_5^{-1} \leq & a_6^{-1} \leq & a_7^{-1} \leq & \dots a_i^{-1} \dots \\
 \neg A_2 & & a_3^{-2} \leq & a_4^{-2} \leq & a_5^{-2} \leq & a_6^{-2} \leq & a_7^{-2} \leq & \dots a_i^{-2} \dots \\
 \neg A_3 & & & a_4^{-3} \leq & a_5^{-3} \leq & a_6^{-3} \leq & a_7^{-3} \leq & \dots a_i^{-3} \dots \\
 \neg A_4 & & & & a_5^{-4} \leq & a_6^{-4} \leq & a_7^{-4} \leq & \dots a_i^{-4} \dots \\
 \neg A_5 & & & & & a_6^{-5} \leq & a_7^{-5} \leq & \dots a_i^{-5} \dots \\
 \neg A_6 & & & & & & a_7^{-6} \leq & \dots a_i^{-6} \dots \\
 \neg A_j & & & & & & & \dots a_i^{-j} \dots
 \end{array}$$

Some observations about the a_i^{-j} elements:

1. For any $i, j : 1 \leq j < i$, $a_i^{-j} \in A_i$ and $a_j \not\leq a_i^{-j}$ for any $a_j \in A_j$.
2. For any $i, i', j : 1 \leq j < i < i'$, $a_i^{-j} \leq a_{i'}^{-j}$ with $a_i^{-j}[k] = a_{i'}^{-j}[k]$ iff $k \in \text{dom}(b_j)$
3. For any $i, j, j' : 1 \leq j' < j < i$ and $i' : j \leq i' : a_{i'}^{-j'} \not\leq a_i^{-j}$.

Observations (1) and (2) are obtained by construction. Suppose observation (3) does not hold, i.e., suppose $a_{i'}^{-j'} \leq a_i^{-j}$ for some $i, j, j' : 1 \leq j' < j < i$ and $i' : j \leq i'$. Observation (2) and reflexivity of $\leq$ ensure $a_j^{-j'} \leq a_{i'}^{-j'}$. By transitivity of $\leq$, we get $a_j^{-j'} \leq a_i^{-j}$. This contradicts observation (1). Hence observation (3) also holds.

Now, observe that the domain D_j of b_j , for any $j : 1 \leq j$, is a subset of $\{1, \dots, K\}$. Since this domain is finite, there is an infinite number of indices $j : 1 \leq j$ with the same domain $D \subseteq \{1, \dots, K\}$. In other words, an infinite number of j -sequences are constants on the same dimensions D . We can therefore extract a sequence $\langle j_1, j_2, \dots \rangle$ of j -sequences that is increasing on each dimension in D (i.e., $j_a \leq j_b \implies$ for each $k \in D, b_{j_a}(k) \leq b_{j_b}(k)$). In order for observation (3) to hold (i.e., $a_{i'}^{-j'} \not\leq a_i^{-j}$ for any $i, j, j' : 1 \leq j' < j < i$

and $i' : j \leq i'$), we need to have some dimensions on which the j -sequences $(a_i^{-j})_{j_1, j_2, \dots}$ do not increase, these would be the dimensions in $\{1, \dots, K\} \setminus D$. But for these dimensions, $(a_i^{-j})_{i:1,2,\dots}$ is strictly increasing with i . This again contradicts observation (3). $\square$

Lemma 9 will be used in Lemma 11 to show that an entailment relation on encodings of K -dimension-bounded and B -good constraints corresponds to a stronger relation than $\sqsubseteq$ and is $\mathcal{WQO}$ hence, establishing Theorem 13. First, we introduce encodings of constraints and an entailment relation on them.

Encodings of constraints. Given a finite set Q and an associated total order $<_Q$, we write $Q_{[i]} \in Q$ to mean the element of Q with $<_Q$ -index $i \in \{1, \dots, |Q|\}^2$. For instance, given a finite set of phasers $\mathcal{P}$ and an associated total order $<_{\mathcal{P}}$, we write $\mathcal{P}_{[k]}$ to mean the phaser with $<_{\mathcal{P}}$ -index k in $\mathcal{P}$. The encoding of a constraint $\varphi = (\mathcal{T}, \mathcal{P}, bv, pc, gap, egap)$ with respect to total orders $<_{\mathcal{P}}$ and $<_{\mathcal{T}}$, written $\text{enc}(\varphi, <_{\mathcal{T}}, <_{\mathcal{P}})$, is a tuple (bv, acc, env) where:

1. $bv : \mathbb{B} \rightarrow \mathbb{B} \cup \{*\}$ is the same as in φ ,
2. $acc : \{1, \dots, |\mathcal{T}|\} \rightarrow (\mathbb{S} \times \mathcal{G}^{|\mathcal{P}|})$ associates each task $\mathcal{T}_{[i]}$ to a tuple $acc(i) = (\mathbf{s}, g_1, \dots, g_{|\mathcal{P}|})$ where $pc(\mathcal{T}_{[i]}) = \mathbf{s}$ and $gap(\mathcal{T}_{[i]})(\mathcal{P}_{[j]}) = g_j$ for each phaser $\mathcal{P}_{[j]}$ with index j in $\mathcal{P}$,
3. $env : \{1, \dots, |\mathcal{P}|\} \rightarrow \mathbb{N}^2$ associates to each phaser $\mathcal{P}_{[j]}$ in $\mathcal{P}$ the pair $env(j) = egap(\mathcal{P}_{[j]})$.

Observe that if two constraints result in the same encoding, then they can be obtained from each other by renaming the tasks and the phasers. As a consequence, if a constraint is free (respectively B -gap-bounded or B -good), then all constraints resulting in the same encoding will also be free (respectively, B -gap-bounded or B -good). We define the dimension of an encoding (bv, acc, env) to be the size of the domain of env (i.e., the dimension of an encoded constraint). A (possibly infinite) set of encodings E is said to be free (respectively, B -gap-bounded or B -good) if all constraints encoded by any of its elements are free (respectively, B -gap-bounded or B -good). The set is said to be K -dimension-bounded if there is natural K in $\mathbb{N}$ that is larger than the dimension of any of its elements.

Entailment of Encodings. Assume two encodings (bv, acc, env) and (bv', acc', env') with $acc : \{1, \dots, L\} \rightarrow (\mathbb{S} \times \mathcal{G}^M)$, $env : \{1, \dots, M\} \rightarrow \mathbb{N}^2$, $acc' : \{1, \dots, L'\} \rightarrow (\mathbb{S} \times \mathcal{G}^{M'})$, $env' : \{1, \dots, M'\} \rightarrow \mathbb{N}^2$. Write $(bv, acc, env) \leq_e (bv', acc', env')$ iff:

1. for each $v^b \in \mathbb{B}$, $(bv(v^b) \neq bv'(v^b)) \implies (bv(v^b) = *)$, and

²the $<_Q$ -index of an element q in Q is $1 + |\{q' \mid q' \in Q \text{ and } q' <_Q q\}|$

2. $M' = M$ and there is a surjection $h : \{1, \dots, L'\} \rightarrow \{1, \dots, L\}$ such that:

- a) $\text{acc}(h(i)) \trianglelefteq \text{acc}'(i)$ for each index $i \in \{1, \dots, L'\}$,
- b) $\text{env}(j) \leq_2 \text{env}'(j)$ ³ for each index $j \in \{1, \dots, M'\}$,

Lemma 10. *Let $\varphi = (\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{gap}, \text{egap})$ and $\varphi' = (\mathcal{T}', \mathcal{P}', \text{bv}', \text{pc}', \text{gap}', \text{egap}')$. If $\text{enc}(\varphi, <_{\mathcal{T}}, <_{\mathcal{P}}) = (\text{bv}, \text{acc}, \text{env})$ and $\text{enc}(\varphi', <_{\mathcal{T}'}, <_{\mathcal{P}'}) = (\text{bv}', \text{acc}', \text{env}')$, then $(\text{bv}, \text{acc}, \text{env}) \leq_e (\text{bv}', \text{acc}', \text{env}')$ implies $\varphi \sqsubseteq \varphi'$.*

Proof. From $(\text{bv}, \text{acc}, \text{env}) \leq_e (\text{bv}', \text{acc}', \text{env}')$ we deduce $|\mathcal{P}| = |\mathcal{P}'|$ and the existence of a surjection $h : \{1, \dots, |\mathcal{T}'|\} \rightarrow \{1, \dots, |\mathcal{T}|\}$, such that $\text{acc}(h(i)) \trianglelefteq \text{acc}'(i)$ for each $i \in \{1, \dots, |\mathcal{T}'|\}$, and $\text{env}(j) \leq_2 \text{env}'(j)$ for each $j \in \{1, \dots, |\mathcal{P}'|\}$.

1. $\text{bv} = \text{bv}'$, hence, for each $\mathbf{v}^b \in \mathbf{B}$, $(\text{bv}(\mathbf{v}^b) \neq \text{bv}'(\mathbf{v}^b) \implies \text{bv}(\mathbf{v}^b) = *)$.
2. let $\tau : \mathcal{T}' \rightarrow \mathcal{T}$ with $\tau(\mathcal{T}'_{[i]}) = \mathcal{T}_{[h(i)]}$ for each $\mathcal{T}'_{[i]}$ in $\mathcal{T}'$. Let $\pi : \mathcal{P}' \rightarrow \mathcal{P}$ with $\pi(\mathcal{P}'_{[j]}) = \mathcal{P}_{[j]}$ for each $\mathcal{P}'_{[j]}$ in $\mathcal{P}'$. Observe τ is surjective and π is bijective.
3. for each $i \in \{1, \dots, |\mathcal{T}'|\}$, we have that $\text{acc}(h(i)) \trianglelefteq \text{acc}'(i)$. By definition, $\text{acc}(h(i))$ is the tuple $(\text{pc}(\mathcal{T}_{[h(i)]}), \text{gap}(\mathcal{T}_{[h(i)]})(\mathcal{P}_{[1]}), \dots, \text{gap}(\mathcal{T}_{[h(i)]})(\mathcal{P}_{[|\mathcal{P}|]}))$ and $\text{acc}'(i)$ is the tuple $(\text{pc}'(\mathcal{T}'_{[i]}), \text{gap}'(\mathcal{T}'_{[i]})(\mathcal{P}'_{[1]}), \dots, \text{gap}'(\mathcal{T}'_{[i]})(\mathcal{P}'_{[|\mathcal{P}'|]}))$ where $\mathcal{P} = \{\mathcal{P}_{[1]}, \dots, \mathcal{P}_{[|\mathcal{P}|]}\}$ and $\mathcal{P}' = \{\mathcal{P}'_{[1]}, \dots, \mathcal{P}'_{[|\mathcal{P}'|]}\}$. By definition of $\trianglelefteq$, we get:
 - a) $\text{pc}(\mathcal{T}_{[h(i)]}) \neq \text{pc}'(\mathcal{T}'_{[i]}) \implies \text{pc}(\mathcal{T}_{[h(i)]}) = *$. Since $\tau(\mathcal{T}'_{[i]}) = \mathcal{T}_{[h(i)]}$, we get $\text{pc}(\tau(\mathcal{T}'_{[i]})) \neq \text{pc}'(\mathcal{T}'_{[i]}) \implies \text{pc}(\tau(\mathcal{T}'_{[i]})) = *$ for $\mathcal{T}'_{[i]}$ in $\mathcal{T}'$.
 - b) $\text{gap}(\mathcal{T}_{[h(i)]})(\mathcal{P}_{[j]}) \trianglelefteq \text{gap}'(\mathcal{T}'_{[i]})(\mathcal{P}'_{[j]})$ for each $j \in \{1, \dots, |\mathcal{P}'|\}$. Since $\mathcal{P}'_{[j]} = \mathcal{P}_{[j]}$, and if $\text{gap}(\mathcal{T}_{[h(i)]})(\mathcal{P}_{[j]}) = (\text{ph}, \text{val})$ and $\text{gap}'(\mathcal{T}'_{[i]})(\mathcal{P}'_{[j]}) = (\text{ph}', \text{val}')$, we deduce that:
 - i. $(\text{ph} \neq \text{ph}') \implies (\text{ph} = *)$
 - ii. $(\text{val} = \text{nreg}) \iff (\text{val}' = \text{nreg})$
 - iii. if $\text{val} = (\text{lw}, \text{ls}, \text{uw}, \text{us})$ and $\text{val}' = (\text{lw}', \text{ls}', \text{uw}', \text{us}')$, then $\text{lw} \leq \text{lw}'$ and $\text{ls} \leq \text{ls}'$ and $\text{uw} \leq \text{uw}'$ and $\text{us}' \leq \text{us}$
4. for each $j \in \{1, \dots, |\mathcal{P}'|\}$, we have $\text{env}(j) \leq_2 \text{env}'(j)$. By definition, $\text{env}(j) = \text{egap}(\mathcal{P}_{[j]})$ and $\text{env}'(j) = \text{egap}'(\mathcal{P}'_{[j]})$. Since $\pi(\mathcal{P}'_{[j]}) = \mathcal{P}_{[j]}$, we deduce that $\text{egap}(\mathcal{P}_{[j]}) \leq_2 \text{egap}'(\mathcal{P}'_{[j]})$ for each $j \in \{1, \dots, |\mathcal{P}'|\}$.

□

³ $\leq_k$ is the component-wise ordering on tuples of size k .

Lemma 11. $(E, \leq_e)$ is $\mathcal{WQO}$ if the set E of encodings is K -dimension-bounded and B -good for some pre-defined $K, B \in \mathbb{N}$.

Proof. Assume a K -dimension-bounded set E of B -good encodings and an infinite sequence $S_1 = \langle (bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \dots \rangle$. We show the existence of $i, j : 1 \leq i < j$ for which $(bv_i, acc_i, env_i) \leq_e (bv_j, acc_j, env_j)$. Dimension-boundedness of E ensures there are infinitely many encodings in S_1 with the same dimension, say K . We extract the subsequence S_2 consisting of all encodings with dimension K in S_1 . In addition, observe that the set of possible valuations of the Boolean variables is finite. We can, therefore, extract from S_2 an infinite subsequence S_3 where all elements share the same valuation of the Boolean variables. Let us rewrite S_3 , for simplicity, as the sequence $\langle (bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \dots \rangle$. For each $i : 1 \leq i$, we can represent the mapping env_i as the tuple $(env_i(1), env_i(2), \dots, env_i(K))$ in $(\mathbb{N}^2)^K$. Using Higman's lemma, we can extract from S_3 a subsequence S_4 , also renamed to $\langle (bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \dots \rangle$ for simplicity, where $env_i(k) \leq_2 env_j(k)$ for any $i, j : 1 \leq i < j$ and $k : 1 \leq k \leq K$.

For each mapping $acc_i : \{1, \dots, L_i\} \rightarrow (\mathbb{S} \times \mathcal{G}^K)$ in S_4 , we write m_{acc_i} to mean the multiset over $(\mathbb{S} \times \mathcal{G}^K)$ where the number of occurrences of an element $(s, g_1, \dots, g_K)$ coincides with the number of indices j in $\{1, \dots, L_i\}$ for which $acc_i(j) = (s, g_1, \dots, g_K)$. Consider the sequence $\langle m_{acc_1}, m_{acc_2}, m_{acc_3}, \dots \rangle$ of elements in $\mathcal{M}(\mathbb{S} \times \mathcal{G}^K)$. Using the fact that E is B -good together with Lemma 8 and 9, we deduce the existence of $i, j : i < j$ for which $m_{acc_i} \preceq \exists m_{acc_j}$ and $m_{acc_i} \exists \preceq \forall m_{acc_j}$. We can therefore build a surjection $h : \{1, \dots, L_j\} \rightarrow \{1, \dots, L_i\}$ such that $acc_i(h(l)) \preceq acc_j(l)$ for each l in $\{1, \dots, L_j\}$. $\square$

5.5 A Symbolic Verification Procedure

We use the constraints from Section 5.4 as a symbolic representation in the adaptation of the classical working-list reachability analysis described below. This procedure corresponds to an instantiation of the procedure “explore” in Section 3.3. The procedure takes as arguments a set Φ_{bad} of constraints denoting the set of bad configurations. Such sets can easily be built from the partial configurations described in Figure 5.3.

The predecessor computation rules in Figures 5.8 and 5.9 that are used by the procedure need to first concretize the given constraint to explicitly contain the required task(s), phaser(s), phaser variable(s), and control sequence(s). For this purpose, they make use of the rules `concTasksPhasers` (a wrapper for `concretizeTask` and `concretizePhaser`), `concretizeSeq`, and `concretizeVar`. Each concretization rule returns a (possibly empty) set of concrete constraints. Intuitively, `concretizeTask` $(\varphi, t, \mathcal{U}, \mathcal{F})$ makes sure either t that is going to be used in the computation is already in φ , or adds it as a fresh task. `concretizePhaser` $(\varphi, p, \mathcal{U}, \mathcal{F})$ concretizes the phaser p . `concretizeSeq` $(\varphi, t, \mathbf{s})$ makes sure the sequence of the task t is $\mathbf{s}$, which is a

requirement for the predecessor computation rule. Otherwise, it will return an empty set. Finally, **concretizeVar** $(\varphi, t, p, \text{ph})$ ensures t uses the variable **ph** to reference p , which is again required by the predecessor computation rule. In the following sections we introduce and then prove some of the characteristics of the concretization functions, predecessor computation functions, and the verification procedure.

5.5.1 Concretization

In this section, we introduce the concretization functions and prove some properties about them. Assume a constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$. We say a set $\mathcal{U}$ is in φ if $\mathcal{U} \subseteq \mathcal{T}_\varphi$ and a set $\mathcal{F}$ is *fresh* for φ if $\mathcal{F} \cap \mathcal{T}_\varphi = \emptyset$. We say **concretizePhaser** $(\varphi, p_\varphi, \mathcal{U}, \mathcal{F})$ (Figure 5.6) concretizes the phaser p_φ in φ . Rule **concretize phaser 1** is used when phaser p_φ is already concrete and the input constraint will be returned without any modification. Rule **concretize phaser 2** adds a new phaser p_φ to the constraint φ and non-deterministically registers a set of tasks $\mathcal{T}'_1$ in the concrete constraint that map to $\mathcal{T}_1 \subseteq \mathcal{T}$ with p_φ . Moreover, it non-deterministically chooses a set of tasks $\mathcal{T}'_2$ that map to $\mathcal{T}_2 \subseteq \mathcal{T}$ to not be registered with p_φ . The task mappings κ_1 and κ_2 are used to map the tasks from the concrete constraint to constraint φ .

concretizeTask $(\varphi, t_\varphi, \mathcal{U}, \mathcal{F})$ (Figure 5.6) concretizes the task t_φ in φ . Rule **concretize task 1** is used when task t_φ is already concrete and the input constraint will be returned without any modification. Rules **concretize task 2** and **3** consider the case when a new task t_φ will be added that copies a task $u \in \mathcal{T}$. The difference between **concretize task 2** and **3** is in the fact that **concretize task 3**, unlike **concretize task 2** concretizes u (after being renamed to t') as well as t . Rule **concretize task 4** concretizes a task that is non-deterministically registered with a subset of the phasers in $\mathcal{P}_2 \subseteq \mathcal{P}_\varphi$ and not registered to others. Such task needs to have phase bounds that respect the environment of the phasers in $\mathcal{P}_2$.

concTasksPhasers $(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ (Figure 5.7) concretizes the phasers A_φ and the tasks B_φ in φ using **concretizePhaser** $(\varphi, p_\varphi, \mathcal{U}, \mathcal{F})$ and **concretizeTask** $(\varphi, t_\varphi, \mathcal{U}, \mathcal{F})$.

Rule **concretize seq** concretizes the control sequence of a task and rule **concretize var** concretizes the phaser name a task uses for referencing a phaser. Some facts about the concretizations will follow.

Lemma 12. *For a given constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$, a set $\mathcal{U}$ that is in φ , a set $\mathcal{F}$ that is fresh for φ , and a phaser p that is possibly in $\mathcal{P}_\varphi$, **concretizePhaser** $(\varphi, p, \mathcal{U}, \mathcal{F})$ always terminates. Each tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ in **concretizePhaser** $(\varphi, p, \mathcal{U}, \mathcal{F})$ satisfies the following:*

- $\varphi \sqsubseteq \varphi'$,

$$\begin{array}{c}
 \frac{t \in \mathcal{T} \quad pc(t) = s}{(\mathcal{T}, \mathcal{P}, bv, pc, gap, egap) \in \text{concretizeSeq}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, s)} \left(\begin{array}{c} \text{concretize} \\ \text{seq 1} \end{array} \right) \\
 \\
 \frac{t \in \mathcal{T} \quad pc(t) = * \quad pc' = pc[t \leftarrow s]}{(\mathcal{T}, \mathcal{P}, bv, pc', gap, egap) \in \text{concretizeSeq}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, s)} \left(\begin{array}{c} \text{concretize} \\ \text{seq 2} \end{array} \right) \\
 \\
 \frac{t \in \mathcal{T}}{((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \mathcal{U}, \mathcal{F}) \in \text{concretizeTask}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{task 1} \end{array} \right) \\
 \\
 \frac{\begin{array}{c} t \in \mathcal{F} \quad u \in \mathcal{T} \setminus \mathcal{U} \quad \mathcal{T}' = \mathcal{T} \cup \{t\} \\ \mathcal{F}' = \mathcal{F} \setminus \{t\} \quad \mathcal{U}' = \mathcal{U} \cup \{t\} \quad pc' = pc[t \leftarrow pc(u)] \\ gap' = gap[t \leftarrow gap(u)[p \leftarrow gap(u)(p)] \mid p \in \mathcal{P}] \end{array}}{((\mathcal{T}', \mathcal{P}, bv, pc', gap', egap), \mathcal{U}', \mathcal{F}') \in \text{concretizeTask}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{task 2} \end{array} \right) \\
 \\
 \frac{\begin{array}{c} t \neq t' \quad \{t, t'\} \subseteq \mathcal{F} \quad u \in \mathcal{T} \setminus \mathcal{U} \quad \mathcal{T}' = (\mathcal{T} \setminus \{u\}) \cup \{t, t'\} \\ \mathcal{F}' = \mathcal{F} \setminus \{t, t'\} \quad \mathcal{U}' = \mathcal{U} \cup \{t, t'\} \quad pc' = pc[t_i \leftarrow pc(u) \mid t_i \in \{t, t'\}] \\ gap' = gap[t_i \leftarrow gap(u)[p \leftarrow gap(u)(p) \mid p \in \mathcal{P}] \mid t_i \in \{t, t'\}] \end{array}}{((\mathcal{T}', \mathcal{P}, bv, pc', gap', egap), \mathcal{U}', \mathcal{F}') \in \text{concretizeTask}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{task 3} \end{array} \right) \\
 \\
 \frac{\begin{array}{c} t \in \mathcal{F} \quad t \in \mathcal{F} \quad \mathcal{T}' = \mathcal{T} \cup \{t\} \\ \mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2 \text{ for each } p_2 \in \mathcal{P}_2. \quad egap(p_2) = (ew_{p_2}, es_{p_2}) \\ gap' = gap[t \leftarrow map_1 \cup map_2] \quad map_1 = \{p_1 \leftarrow (*, \text{nreg}) \mid p_1 \in \mathcal{P}_1\} \\ map_2 = \{p_2 \leftarrow (*, (ew_{p_2}, es_{p_2}, \infty, \infty)) \mid p_2 \in \mathcal{P}_2\} \quad pc' = pc[t \leftarrow *] \end{array}}{((\mathcal{T}', \mathcal{P}, bv, pc', gap', egap), \mathcal{U} \cup \{t\}, \mathcal{F} \setminus \{t\}) \in \text{concretizeTask}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{task 4} \end{array} \right) \\
 \\
 \frac{p \in \mathcal{P}}{((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \mathcal{U}, \mathcal{F}) \in \text{concretizePhaser}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), p, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{phaser 1} \end{array} \right) \\
 \\
 \frac{\begin{array}{c} p \notin \mathcal{P} \quad \mathcal{P}' = \mathcal{P} \cup \{p\} \quad egap' = egap[p \leftarrow (0, 0)] \\ \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \text{ with } \mathcal{T}_1 \cap \mathcal{T}_2 \text{ possibly non-empty and } \mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{U} = \emptyset \\ \mathcal{T}' = \mathcal{T}'_1 \cup \mathcal{T}'_2 \quad \mathcal{T}'_1 \cap \mathcal{T}'_2 = \emptyset \\ \kappa_1 : \mathcal{T}_1 \rightarrow \mathcal{T}'_1 \quad \kappa_2 : \mathcal{T}_2 \rightarrow \mathcal{T}'_2 \quad \kappa_1, \kappa_2 \text{ are bijective functions} \\ \forall t \in \mathcal{T}_1 \cap \mathcal{T}_2. \kappa_1(t) = t \quad \forall t \in \mathcal{T}_1 \cap \mathcal{T}_2. \kappa_1(t) \in \{(t, 1)\} \cup \mathcal{F} \\ \forall t \in \mathcal{T}_2 \setminus \mathcal{T}_1. \kappa_2(t) = t \quad \forall t \in \mathcal{T}_1 \cap \mathcal{T}_2. \kappa_2(t) \in \{(t, 2)\} \cup \mathcal{F} \\ \kappa_1(t) = t_1 \quad \kappa_2(t) = t_2 \\ \forall t \in \mathcal{T}_1, \forall q \in \mathcal{P}. gap'(t_1)(q) = gap(t)(q) \\ \forall t \in \mathcal{T}_2, \forall q \in \mathcal{P}. gap'(t_2)(q) = gap(t)(q) \\ \forall t \in \mathcal{T}_1. gap'(t_1)(p) = (*, \text{nreg}) \\ \forall t \in \mathcal{T}_2. gap'(t_2)(p) = (*, (0, 0, \infty, \infty)) \end{array}}{((\mathcal{T}', \mathcal{P}', bv, pc, gap', egap'), \mathcal{U} \cup (\mathcal{F} \setminus \mathcal{F}'), \mathcal{F} \setminus \mathcal{T}') \in \text{concretizePhaser}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), p, \mathcal{U}, \mathcal{F})} \left(\begin{array}{c} \text{concretize} \\ \text{phaser 2} \end{array} \right) \\
 \\
 \frac{t \in \mathcal{T} \quad p \in \mathcal{P} \quad gap(t)(p) = (\text{ph}, \text{val})}{(\mathcal{T}, \mathcal{P}, bv, pc, gap, egap) \in \text{concretizeVar}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, p, \text{ph})} \left(\begin{array}{c} \text{concretize} \\ \text{phaser var 1} \end{array} \right) \\
 \\
 \frac{\begin{array}{c} t \in \mathcal{T} \quad p \in \mathcal{P} \quad gap(t)(p) = (*, \text{val}) \\ \forall q \in \mathcal{P}. (gap(t)(q) = (\text{var}, \text{val}) \Rightarrow \text{var} \neq \text{ph}) \\ gap'(t)(p) = gap[t \leftarrow gap(t)[p \leftarrow (\text{ph}, \text{val})]] \end{array}}{(\mathcal{T}, \mathcal{P}, bv, pc, gap', egap) \in \text{concretizeVar}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), t, p, \text{ph})} \left(\begin{array}{c} \text{concretize} \\ \text{phaser var 2} \end{array} \right)
 \end{array}$$

Figure 5.6: Auxiliary functions used in the predecessor computation of the parameterized Phaser instructions (Part I). The function **concTasksPhasers**($\varphi, A, B, \mathcal{U}, \mathcal{F}$) concretizes the set of phasers A and the set of tasks B in the constraint φ . Those phasers or tasks in A or B that are already in φ will be preserved. $\mathcal{U}$ is a subset of B , which is already concretized to unique tasks. $\mathcal{F}$ is a set of fresh task identifiers that does not intersect B .

$$\begin{array}{c}
 \frac{(\varphi', \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}(\varphi, A, B, \mathcal{U}, \mathcal{F}) \quad (\varphi'', \mathcal{U}'', \mathcal{F}'') \in \text{concretizePhaser}(\varphi', p, \mathcal{U}', \mathcal{F}')}{(\varphi'', \mathcal{U}'', \mathcal{F}'') \in \text{concTasksPhasers}(\varphi, A \uplus \{p\}, B, \mathcal{U}, \mathcal{F})} \quad (1) \\
 \\
 \frac{(\varphi', \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}(\varphi, A, B, \mathcal{U}, \mathcal{F}) \quad (\varphi'', \mathcal{U}'', \mathcal{F}'') \in \text{concretizeTask}(\varphi', t, \mathcal{U}', \mathcal{F}')}{(\varphi'', \mathcal{U}'', \mathcal{F}'') \in \text{concTasksPhasers}(\varphi, A, B \uplus \{t\}, \mathcal{U}, \mathcal{F})} \quad (2) \\
 \\
 \frac{}{(\varphi, \mathcal{U}, \mathcal{F}) \in \text{concTasksPhasers}(\varphi, \emptyset, \emptyset, \mathcal{U}, \mathcal{F})} \quad (3) \\
 \\
 \frac{\varphi = (\mathcal{T}, \mathcal{P}, bv, pc, gap, egap) \quad \text{concTasksPhasers}(\varphi, A, B) \quad B = \mathcal{U} \uplus \mathcal{F} \quad \mathcal{U} \subseteq \mathcal{T} \quad \mathcal{F} \cap \mathcal{T} = \emptyset}{\text{concTasksPhasers}(\varphi, A, B, \mathcal{U}, \mathcal{F})} \quad (4)
 \end{array}$$

Figure 5.7: Auxiliary functions used in the predecessor computation of the parameterized Phaser instructions (Part II). The function $\text{concTasksPhasers}(\varphi, A, B, \mathcal{U}, \mathcal{F})$ concretizes the set of phasers A and the set of tasks B in the constraint φ . Those phasers or tasks in A or B that are already in φ will be preserved.

- $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}' \cap \mathcal{T}_{\varphi'} = \emptyset$,
- $\mathcal{U}' = \mathcal{U} \cup (\mathcal{F} \setminus \mathcal{F}')$ and $\mathcal{U}' \subseteq \mathcal{T}_{\varphi'}$,

Proof. By definition of **concretize phaser** 1 and 2 in Figure 5.6. $\square$

Lemma 13. For a given constraint $\varphi = (\mathcal{T}_{\varphi}, \mathcal{P}_{\varphi}, bv_{\varphi}, pc_{\varphi}, gap_{\varphi}, egap_{\varphi})$, a set $\mathcal{U}$ that is in φ , a set $\mathcal{F}$ that is fresh for φ , and a task $t \in \mathcal{U} \cup \mathcal{F}$, $\text{concretizeTask}(\varphi, t, \mathcal{U}, \mathcal{F})$ always terminates. Each tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ in $\text{concretizeTask}(\varphi, p, \mathcal{U}, \mathcal{F})$ satisfies the following:

- $\varphi \sqsubseteq \varphi'$,
- $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}' \cap \mathcal{T}_{\varphi'} = \emptyset$,
- $\mathcal{U}' = \mathcal{U} \cup (\mathcal{F} \setminus \mathcal{F}')$ and $\mathcal{U}' \subseteq \mathcal{T}_{\varphi'}$,

Proof. By definition of **concretize task** 1, 2, 3, and 4 in Figure 5.6. $\square$

Observe that **concretizeVar** $(\varphi, t, p, \text{ph})$ and **concretizeSeq** (φ, t, s) in Figure 5.6 do not change the tasks, phasers, and phases. One can show that after calling them, the identity relations on $\mathcal{T}$ and $\mathcal{P}$ witness $\varphi \sqsubseteq \varphi'$.

Lemma 14. For a given constraint $\varphi = (\mathcal{T}_{\varphi}, \mathcal{P}_{\varphi}, bv_{\varphi}, pc_{\varphi}, gap_{\varphi}, egap_{\varphi})$, a set $\mathcal{U}$ that is in φ , a set $\mathcal{F}$ that is fresh for φ , a task $t \in \mathcal{T}_{\varphi}$, a phaser variable ph , a phaser $p \in \mathcal{P}_{\varphi}$, and s , **concretizeVar** $(\varphi, t, p, \text{ph})$ and **concretizeSeq** (φ, t, s) always terminate and will respectively generate a singleton $\{\varphi'\}$ such that the constraint φ' satisfies $\varphi \sqsubseteq \varphi'$.

Proof. By definition of $\text{concretizeVar}(\varphi, t, p, \text{ph})$ and $\text{concretizeSeq}(\varphi, t, \text{s})$ in Figure 5.6. $\square$

Lemma 15. Assume $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$, A_φ that is a set of phaser identifiers possibly intersecting $\mathcal{P}_\varphi$, a set $\mathcal{U}$ that is in φ , a set $\mathcal{F}$ that is fresh for φ , and $B_\varphi \subseteq \mathcal{U} \cup \mathcal{F}$. $\text{concTasksPhasers}(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ always terminates and returns a finite set of tuples. Each tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ in $\text{concTasksPhasers}(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ satisfies

- $\varphi \sqsubseteq \varphi'$,
- $\mathcal{F}' \subseteq \mathcal{F} \setminus B_\varphi$ and $\mathcal{F}' \cap \mathcal{T}_{\varphi'} = \emptyset$,
- $\mathcal{U}' = \mathcal{U} \cup (\mathcal{F} \setminus \mathcal{F}')$ and $\mathcal{U}' \subseteq \mathcal{T}_{\varphi'}$,

Proof. By Lemmas 12 and 13 and induction on $|A_\varphi| + |B_\varphi|$. $\square$

Definition 8 (κ -uniquely-mapped). For a given configuration $\mathbf{c} = (\mathcal{T}_\mathbf{c}, \mathcal{P}_\mathbf{c}, \text{bv}_\mathbf{c}, \text{pc}_\mathbf{c}, \text{phase}_\mathbf{c})$ and constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$ such that κ and π witness $\mathbf{c} \in \llbracket \varphi \rrbracket$, a task $t \in \mathcal{T}_\mathbf{c}$ is said to be κ -uniquely-mapped iff $\kappa(u) = \kappa(t) \implies u = t$.

Lemma 16. Assume a configuration $\mathbf{c} = (\mathcal{T}_\mathbf{c}, \mathcal{P}_\mathbf{c}, \text{bv}_\mathbf{c}, \text{pc}_\mathbf{c}, \text{phase}_\mathbf{c})$, a constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$ such that κ and π witness $\mathbf{c} \in \llbracket \varphi \rrbracket$, a set $\mathcal{U}$ that is in φ such that κ uniquely maps $\kappa^{-1}(\mathcal{U})$, and a set $\mathcal{F}$ that is fresh for φ . Let $p_\mathbf{c}$ be an arbitrary phaser in $\mathcal{P}_\mathbf{c}$ and p_φ be $\pi(p_\mathbf{c})$ if $p_\mathbf{c}$ is mapped by π or a fresh phaser, otherwise. A tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ exists among $\text{concretizePhaser}(\varphi, p_\varphi, \mathcal{U}, \mathcal{F})$ such that:

- $\mathbf{c} \in \llbracket \varphi' \rrbracket$ and κ' and π' witness the denotation,
- $\pi' = \pi[p_\mathbf{c} \leftarrow p_\varphi]$,
- κ' uniquely maps the tasks in $\kappa'^{-1}(\mathcal{U}')$,

Proof. The concretization of p_φ will be performed in one of the following ways:

- If $p_\mathbf{c}$ is mapped to p_φ by π , the rule (**concretize phaser 1**) will return $(\varphi, \mathcal{U}, \mathcal{F})$, which is the desired tuple. Observe that the tasks are not altered, hence, the returned tuple satisfies the conditions in the lemma.
- If $p_\mathbf{c}$ is not mapped by π , the rule (**concretize phaser 2**) generates a set of tuples $(\varphi', \mathcal{U}', \mathcal{F}')$ in which the concrete constraints account for all possible registrations of the tasks in $\mathcal{T}_\varphi$ to the new phaser p_φ . A constraint $\varphi' = (\mathcal{T}_{\varphi'}, \mathcal{P}_{\varphi'}, \text{bv}_{\varphi'}, \text{pc}_{\varphi'}, \text{gap}_{\varphi'}, \text{egap}_{\varphi'})$ generated by the rule will capture

- the tasks in $\mathcal{T}_1 \subseteq \mathcal{T}_c$ that were mapped by κ but were not registered with p_c can be mapped to $\mathcal{T}'_1 \subseteq \mathcal{T}_{\varphi'}$ by the task mapping $\kappa_1(\kappa)$.
- the tasks in $\mathcal{T}_2 \subseteq \mathcal{T}_c$ that were mapped by κ and are registered with p_c can be mapped to $\mathcal{T}'_2 \subseteq \mathcal{T}_{\varphi'}$ by the task mapping $\kappa_2(\kappa)$.

The task mapping κ' that maps the tasks in $\mathcal{T}_1$ using $\kappa_1(\kappa)$ and those in $\mathcal{T}_2$ using $\kappa_2(\kappa)$ and the phaser mapping $\pi' = \pi[p_c \leftarrow p_{\varphi}]$ witness $c \models \varphi'$. κ' by construction uniquely maps the tasks in $\kappa'^{-1}(\mathcal{U}')$.

□

Lemma 17. Assume a configuration $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$, a constraint $\varphi = (\mathcal{T}_{\varphi}, \mathcal{P}_{\varphi}, bv_{\varphi}, pc_{\varphi}, gap_{\varphi}, egap_{\varphi})$ such that κ and π witness $c \in \llbracket \varphi \rrbracket$, a set $\mathcal{U}$ that is in φ such that κ uniquely maps $\kappa^{-1}(\mathcal{U})$, a set $\mathcal{F}$ that is fresh for φ , and a set $B_c = (B_{c_1} \uplus B_{c_2}) \subseteq \mathcal{T}_c$ such that κ uniquely maps B_{c_1} and does not uniquely map B_{c_2} . Let t_c be an arbitrary task in B_c and t_{φ} be $\kappa(t_c)$ if t_c is in $\mathcal{U}$ or a task in $\mathcal{F}$, otherwise. `concretizeTask`($\varphi, t_{\varphi}, \mathcal{U}, \mathcal{F}$) will generate a set of tuples among which there is a tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ such that

- $c \in \llbracket \varphi' \rrbracket$ and κ' and π witness the denotation,
- κ' uniquely maps the tasks in $\kappa'^{-1}(\mathcal{U}')$,

Proof. The concretization of t_{φ} will be performed in one of the following ways:

- If t_c is in B_{c_1} , it is already uniquely mapped to t_{φ} , the rule (`concretize task 1`) will return $(\varphi, \mathcal{U}, \mathcal{F})$, which is the desired tuple.
- If t_c is mapped by κ but is in B_{c_2} , it is mapped to $\kappa(t_c)$ but not uniquely. The rule (`concretize task 2`) (respectively, `concretize task 3`) return in this case a tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ for which the mappings $\kappa' = \kappa[t_c \leftarrow t_{\varphi}]$ (respectively, $\kappa' = \kappa[t_c \leftarrow t_{\varphi}][u_c \leftarrow u_{\varphi}]$ given that t_c and u_c are both mapped to $\kappa(t_c)$) and π witnesses $c \in \llbracket \varphi' \rrbracket$. The other claims hold by construction.
- If t_c is not mapped by κ , it is in B_{c_2} . Let $\mathcal{P}_c^m \subseteq \mathcal{P}_c$ be the set of mapped phasers by π and $\mathcal{P}_c^{m,r} \subseteq \mathcal{P}_c^m$ be the set of mapped phasers that t_c is registered with. In this case, the rule (`concretize task 4`) returns a set of tuples among which there is a tuple $(\varphi', \mathcal{U}', \mathcal{F}')$ in which t_{φ} is a fresh task that is registered with $\pi(\mathcal{P}_c^{m,r})$ and not with $\pi(\mathcal{P}_c^m \setminus \mathcal{P}_c^{m,r})$. Hence, the task and phaser mappings $\kappa' = \kappa[t_c \leftarrow t_{\varphi}]$ and π witnesses $c \models \varphi'$. The other claims hold by construction.

□

Definition 9 (Match). Assume a tuple (c, A_c, B_c) with $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$, $A_c \subseteq \mathcal{P}_c$, and $B_c \subseteq \mathcal{T}_c$ and $(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ with $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$, A_φ that is a set of phaser identifiers possibly intersecting $\mathcal{P}_\varphi$, $B_\varphi = \mathcal{U} \cup \mathcal{F}$, a set $\mathcal{U}$ that is in φ , and a set $\mathcal{F}$ that is fresh for φ . We say the tuple (c, A_c, B_c) matches $(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ with respect to κ and π if

- κ and π witness $c \in \llbracket \varphi \rrbracket$,
- $\kappa^{-1}(\mathcal{U})$ is κ -uniquely-mapped,
- $|A_c| = |A_\varphi|$ and $|B_c| = |B_\varphi|$,
- $B_c = B_{c_1} \uplus B_{c_2}$,
- $\kappa(B_{c_1}) \subseteq \mathcal{U}$, i.e., B_{c_1} is κ -uniquely-mapped,
- κ does not uniquely map B_{c_2} ,
- $|B_{c_2}| = |\mathcal{F}|$.

Lemma 18. Given $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$, $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$, and κ and π that witness $c \in \llbracket \varphi \rrbracket$, for any $A_c \subseteq \mathcal{P}_c$ and $B_c \subseteq \mathcal{T}_c$, (c, A_c, B_c) matches $(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ with respect to κ and π if

- $|A_c| = |A_\varphi|$ and $|B_c| = |B_\varphi|$,
- $A_\varphi = \pi(A_{c|_\pi}) \uplus (A_\varphi \setminus \text{dom}(\pi))$,
- $B_\varphi = \mathcal{U} \uplus \mathcal{F}$, $\mathcal{U} \subseteq \mathcal{T}_\varphi$, $\mathcal{F} \cap \mathcal{T}_\varphi = \emptyset$,
- $\kappa^{-1}(\mathcal{U})$ is κ -uniquely-mapped.

Proof. By Definition 9 and construction of $c \in \llbracket \varphi \rrbracket$. □

Example 4. Let $c \in \llbracket \varphi \rrbracket$ such that $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$, $p_c \in \mathcal{P}_c$, $t_c \in \mathcal{T}_c$, $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$, and κ and π witness the denotation. $(c, \{p_c\}, \{t_c\})$ will match some $(\varphi, \{p_\varphi\}, \{t_\varphi\}, \mathcal{U}, \mathcal{F})$ for $\mathcal{U} \cup \mathcal{F} = \{t_\varphi\}$.

Lemma 19. Assume a configuration $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$, a constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$, the mappings κ and π , and the sets A_c , B_c , A_φ , and B_φ such that (c, A_c, B_c) matches $(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ with respect to κ and π . For all the tuples $(\varphi', \mathcal{U}', \mathcal{F}')$ in $\text{concTasksPhasers}(\varphi, A_\varphi, B_\varphi, \mathcal{U}, \mathcal{F})$ we have that $\mathcal{U}' = B_\varphi$ and $\mathcal{F}' = \emptyset$ and there exists one such tuple that:

- $\pi' : \text{dom}(\pi) \cup A_c \rightarrow \mathcal{P}_{\varphi'}$,
- $\kappa' : \text{dom}(\kappa) \cup B_c \rightarrow \mathcal{T}_{\varphi'}$,

- κ' uniquely maps B_c ,
- π' and κ' witness $c \in \llbracket \varphi' \rrbracket$.

Proof. By Lemmas 16 and 17 and induction on $|A_\varphi| + |B_\varphi|$. $\square$

Lemma 20. Assume a configuration $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$ and a constraint $\varphi = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$ such that κ and π witness $c \in \llbracket \varphi \rrbracket$. Assume also that for some task t_c and phaser p_c , $\kappa(t_c) = t_\varphi$, $\pi(p_c) = p_\varphi$, $pc_c(t_c) = s$, and $phase(t_c)(p_c) = (ph, val)$. `concretizeVar`($\varphi, t_\varphi, p_\varphi, ph$) and `concretizeSeq`(φ, t_φ, s) will respectively generate a singleton $\{\varphi'\}$ such that the constraint φ' satisfies $c \in \llbracket \varphi' \rrbracket$.

Proof. By definition of `concretizeVar`(φ, t, p, ph) and `concretizeSeq`(φ, t, s) in Figure 5.6. $\square$

5.5.2 Predecessor Computation

In this section, we formally define the predecessor computation functions in Figures 5.8 and 5.9 and prove their soundness and relative completeness.

Theorem 14. Each predecessor computation rule $\xrightarrow{\text{stmt}}$ in Figures 5.9 and 5.8 is sound with respect to the semantic rules of Figure 5.2.

Proof. Assume a configuration c and a constraint φ such that $c \in \llbracket \varphi \rrbracket$.

newPhaser. Assume $c' \xrightarrow[t_c]{ph := \text{newPhaser}() } c$ for some c' where p_c is the new phaser id in c . We exhibit φ' as well as the task and phaser mappings that witness $c' \in \llbracket \varphi' \rrbracket$ such that $\varphi \xrightarrow[t_\varphi]{ph := \text{newPhaser}() } \varphi'$ for some t_φ . The rule starts by concretizing φ according to c and $A_c = \{t_c\}$ and $B_c = \{p_c\}$. Lemmas 19 and 20 ensure that after concretizations, a concrete constraint φ_{conc} is generated such that $c \in \llbracket \varphi_{\text{conc}} \rrbracket$ with respect to some task and phaser mappings κ and π so that t_c is uniquely mapped to t_φ by κ , and p_c is mapped to some p_φ by π . The task t_c in c' creates p_c , hence, it must be the only task registered with p_c or referencing it. This, combined with the definition of κ and π , guarantees that t_φ is the only task in φ which is registered with p_φ or referencing it. The new mappings κ and $\pi' = \pi \setminus \{p_c\}$ witness $c' \in \llbracket \varphi' \rrbracket$.

signal. Assume $c' \xrightarrow[t_c]{ph.\text{signal}() } c$ for some c' . We exhibit $\varphi' = (\mathcal{T}'_\varphi, \mathcal{P}'_\varphi, bv'_\varphi, pc'_\varphi, gap'_\varphi, egap'_\varphi)$ as well as the task and phaser mappings that witness $c' \in \llbracket \varphi' \rrbracket$ such that $\varphi \xrightarrow[t_\varphi]{ph.\text{signal}() } \varphi'$ for some t_φ . The rule starts by

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p\}, \{t\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeVar}(\varphi_2, t, p, \text{ph}) \\
 \varphi_{\text{conc}} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \\
 s', s \in S \quad s' = \text{ph} := \text{newPhaser}(); s \quad pc'' = pc'[t \leftarrow s'] \\
 (0, 0) \models gap'(t)(p) \\
 (\forall u \in \mathcal{T}'. gap'(u)(p) = (\text{ph}_u, \text{val}_u) \quad \text{ph}_u = \text{ph} \text{ or } \text{val}_u \neq \text{nreg} \Rightarrow u = t) \\
 gap'' = gap'[\{u \leftarrow gap'(u) \setminus \{p\} \mid u \in \mathcal{T}'\}] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph} := \text{newPhaser}()} (\mathcal{T}', \mathcal{P}' \setminus \{p\}, bv, pc'', gap'', egap' \setminus \{p\}) \quad (\text{newPhaser})
 \end{array}$$

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p\}, \{t\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeVar}(\varphi_2, t, p, \text{ph}) \\
 \varphi_{\text{conc}} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \quad gap'(t)(p) = (\text{ph}, \text{nreg}) \\
 s', s \in S \quad s' = \text{ph.drop}(); s \quad pc'' = pc'[t \leftarrow s'] \\
 \mathcal{U} = \{u \mid u \in \mathcal{T}' \text{ and } gap'(u)(p) = (\text{val}_u, (\text{lw}_u, \text{ls}_u, \text{uw}_u, \text{us}_u))\} \\
 \text{uwmin} = \min \{\text{uw}_u \mid u \in \mathcal{U}\} \quad \text{usmin} = \min \{\text{us}_u \mid u \in \mathcal{U}\} \\
 \text{lwmax} = \max \{\text{lw}_u \mid u \in \mathcal{U}\} \quad \text{lsmax} = \max \{\text{ls}_u \mid u \in \mathcal{U}\} \\
 (\text{uwmin} < \infty \wedge \text{usmin} < \infty) \implies -\text{uwmin} \leq \delta \leq \text{usmin} \\
 (\text{uwmin} = \text{usmin} = \infty) \implies -\text{lwmax} \leq \delta \leq \text{lsmax} \\
 \text{map}_\delta = \{u \leftarrow gap'(u)[p \leftarrow (\text{val}_u, ((\text{lw}_u + \delta)^+, (\text{ls}_u - \delta)^+, (\text{uw}_u + \delta), (\text{us}_u - \delta)))] \mid u \in \mathcal{U}\} \\
 \text{map}_t = \{t \leftarrow gap'(t)(p)[p \leftarrow (\text{ph}, (0, 0, \infty, \infty))]\} \quad gap'' = gap'[\text{map}_t \cup \text{map}_\delta] \\
 egap'(p) = (\text{ew}, \text{es}) \quad egap'' = egap'[p \leftarrow ((\text{ew} + \delta)^+, (\text{es} - \delta)^+)] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.drop}()} (\mathcal{T}', \mathcal{P}', bv, pc'', gap'', egap'') \quad (\text{drop})
 \end{array}$$

$$\begin{array}{c}
 t \notin \mathcal{T} \quad \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \\
 ((\mathcal{T}', \mathcal{P}', bv', pc', gap', egap'), \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}(\varphi, \{p\}, \{t\}) \\
 \forall p_i \in \mathcal{P}_1. \text{isReg}(gap', u_i, p_i) \quad pc'' = pc'[t \leftarrow \text{exit}] \\
 \forall p \in \mathcal{P}_1. \mathcal{U}^p = \{u \mid u \in \mathcal{T} \text{ and } gap'(u)(p) = (\text{val}_u^p, (\text{lw}_u^p, \text{ls}_u^p, \text{uw}_u^p, \text{us}_u^p))\} \\
 \text{uwmin}^p = \min \{\text{uw}_u^p \mid u \in \mathcal{U}^p\} \quad \text{usmin}^p = \min \{\text{us}_u^p \mid u \in \mathcal{U}^p\} \\
 \text{lwmax}^p = \max \{\text{lw}_u^p \mid u \in \mathcal{U}^p\} \quad \text{lsmax}^p = \max \{\text{ls}_u^p \mid u \in \mathcal{U}^p\} \\
 \text{map}_t = \{p \leftarrow (*, (0, 0, \infty, \infty)) \mid p \in \mathcal{P}_1\} \cup \{p \leftarrow (*, \text{nreg}) \mid p \in \mathcal{P}_2\} \\
 \forall p \in \mathcal{P}_1. (\text{uwmin}^p < \infty \wedge \text{usmin}^p < \infty) \implies -\text{uwmin}^p \leq \delta^p \leq \text{usmin}^p \\
 \forall p \in \mathcal{P}_1. (\text{uwmin}^p = \text{usmin}^p = \infty) \implies -\text{lwmax}^p \leq \delta^p \leq \text{lsmax}^p \\
 \forall p \in \mathcal{P}_1, u \in \mathcal{U}^p. gap'(u)(p) + \delta^p = (\text{val}_u^p, ((\text{lw}_u^p + \delta^p)^+, (\text{ls}_u^p - \delta^p)^+, (\text{uw}_u^p + \delta^p), (\text{us}_u^p - \delta^p))) \\
 gap'' = gap'[\{t \leftarrow \text{map}_t\} \cup \{u \leftarrow gap'(u)[p \leftarrow gap'(u)(p) + \delta^p] \mid u \in \mathcal{U}^p \text{ and } p \in \mathcal{P}_1\}] \\
 egap'' = egap'[p \leftarrow ((\text{ew} + \delta^p)^+, (\text{es} - \delta^p)^+)] \mid p \in \mathcal{P}_1 \text{ and } egap'(p) = (\text{ew}, \text{es}) \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{exit}} (\mathcal{T}', \mathcal{P}', bv, pc'', gap'', egap'') \quad (\text{exit})
 \end{array}$$

Figure 5.8: Derivation rules for computing predecessors wrt. parameterized Phaser instructions **newPhaser()**, **ph.drop()** and **exit**. For each rule, a task t executes the statement. If t does not belong to $\mathcal{T}$, it will be added by concretizing φ . Otherwise, the concretization will preserve it. In **newPhaser()** and **ph.drop()**, a phaser p is required, which is either added by concretization or is preserved by it.

concretizing t_φ and p_φ in φ . Lemmas 19 and 20 ensure that a concrete constraint $\varphi_{\text{conc}} = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$ is generated such that $c \in \llbracket \varphi_{\text{conc}} \rrbracket$ with respect to some task and phaser mappings κ and π so that t_c is uniquely mapped to some t_φ by κ , and p_c is mapped to some p_φ by π . The task t_c in c' has just incremented the signal phase of the task t_c on the phaser p_c . There is a level $lev \geq 0$ that separates the signal phases of the tasks registered with $p_{\text{concTasksPhasers}}$ from their wait phases. Incrementing the signal sig^{t_c} of

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p_1, \dots, p_k\}, \{t, u\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeSeq}(\varphi_2, u, \text{stmt}) \\
 \text{for each } i : 1 \leq i \leq k \quad \varphi_{3+i} \in \text{concretizeVar}(\varphi_{2+i}, t, p_i, \text{ph}_i) \\
 \text{for each } i : 1 \leq i \leq k \quad \varphi_{3+k+i} \in \text{concretizeVar}(\varphi_{2+k+i}, u, p_i, q_i) \\
 \varphi_{\text{conc}} = \varphi_{3+2k} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \\
 s', s \in S \quad \text{paramOf}(\text{task}) = (q_1, \dots, q_k) \quad s' = \text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k) \{ \text{stmt} \}; s \quad pc'' = pc' [t \leftarrow s'] \\
 \text{for each } i : 1 \leq i \leq k \quad gap'(t)(p_i) = (\text{ph}_i, (lw_i^t, ls_i^t, uw_i^t, us_i^t)) \quad gap'(u)(p_i) = (q_i, (lw_i^u, ls_i^u, uw_i^u, us_i^u)) \\
 gap'' = gap' [t \leftarrow gap'(t)] [\{ p_i \leftarrow (\text{ph}_i, (lw_i^t, ls_i^t, uw_i^t, us_i^t)) \cap (lw_i^u, ls_i^u, uw_i^u, us_i^u) \} \mid i : 1 \leq i \leq k \}] \\
 gap'(u)(p) \notin \{-, *\} \times \{\text{nreg}\} \implies p \in \{p_1, \dots, p_k\} \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)} (\mathcal{T}' \setminus \{u\}, \mathcal{P}', bv, pc'' \setminus \{u\}, gap'' \setminus \{u\}, egap') \quad (\text{asynch})
 \end{array}$$

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p\}, \{t\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeVar}(\varphi_2, t, p, \text{ph}) \\
 \varphi_{\text{conc}} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \\
 s', s \in S \quad s' = \text{ph.signal}(); s \quad pc'' = pc' [t \leftarrow s'] \\
 \mathcal{U} = \{u \mid u \in \mathcal{T}' \text{ and } gap'(u)(p) = (var_u, (lw_u, ls_u, uw_u, us_u))\} \quad \text{for each } u \in \mathcal{U}. uw_u \geq 1 \\
 gap'' = gap' [\{ u \leftarrow gap'(u) [p \leftarrow (var_u, ((lw_u - 1)^+, ls_u + 1, uw_u - 1, us_u + 1))] \mid u \in \mathcal{U} \setminus \{t\} \}] \\
 gap''' = gap'' [t \leftarrow gap'' [p \leftarrow (var_t, ((lw_t - 1)^+, ls_t, uw_t - 1, us_t))]] \\
 egap'' = egap' [p \leftarrow (ew - 1)^+, es + 1] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.signal}()} (\mathcal{T}', \mathcal{P}', bv, pc'', gap'', egap'') \quad (\text{signal I})
 \end{array}$$

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p\}, \{t\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeVar}(\varphi_2, t, p, \text{ph}) \\
 \varphi_{\text{conc}} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \\
 s', s \in S \quad s' = \text{ph.signal}(); s \quad pc'' = pc' [t \leftarrow s'] \\
 gap'(t)(p) = (var_t, (lw_t, ls_t, uw_t, us_t)) \quad us_t \geq 1 \\
 gap'' = gap' [t \leftarrow gap'(t) [p \leftarrow (var_t, (lw_t, (ls_t - 1)^+, uw_t, us_t - 1))]] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.signal}()} (\mathcal{T}', \mathcal{P}', bv, pc'', gap'', egap') \quad (\text{signal II})
 \end{array}$$

$$\begin{array}{c}
 (\varphi_1, \mathcal{U}', \mathcal{F}') \in \text{concTasksPhasers}((\mathcal{T}, \mathcal{P}, bv, pc, gap, egap), \{p\}, \{t\}) \\
 \varphi_2 \in \text{concretizeSeq}(\varphi_1, t, s) \quad \varphi_{\text{conc}} \in \text{concretizeVar}(\varphi_2, t, p, \text{ph}) \\
 \varphi_{\text{conc}} = (\mathcal{T}', \mathcal{P}', bv', pc', gap', egap') \\
 s', s \in S \quad s' = \text{ph.wait}(); s \quad pc'' = pc' [t \leftarrow s'] \\
 gap'(t)(p) = (var_t, (lw_t, ls_t, uw_t, us_t)) \\
 gap'' = gap' [t \leftarrow gap'(t) [p \leftarrow (var_t, (lw_t + 1, ls_t, uw_t + 1, us_t))]] \\
 \hline
 (\mathcal{T}, \mathcal{P}, bv, pc, pv, \gamma) \xrightarrow[t]{\text{ph.wait}()} (\mathcal{T}', \mathcal{P}', bv, pc'', gap'', egap') \quad (\text{wait})
 \end{array}$$

Figure 5.9: Derivation rules for predecessor computation wrt. parameterized Phaser instructions `asynch(task, ph1, ..., phk) {stmt}`, `ph.signal()` and `ph.wait()`. For each rule, a task t executes the statement. If t does not belong to $\mathcal{T}$, it will be added by concretizing φ . Otherwise, the concretization will preserve it. In the given statements, one or more phasers are required, which are either added by concretization or are preserved by that.

task t_c on p_c satisfies the phase bounds in φ_{conc} . The case in which $lev < sig^{t_c}$ can be captured by `signal II` (here, $us > 0$ and $lev' = lev$). The case where $lev = sig^{t_c}$ can be captured with `signal I` (here, $uw > 0$ and $lev' = lev - 1$).

wait. Assume $c' \xrightarrow[t_c]{\text{ph.wait}()} c$ for some c' . We exhibit φ' as well as the task and phaser mappings that witness $c' \in \llbracket \varphi' \rrbracket$ such that $\varphi \xrightarrow[t_\varphi]{\text{ph.wait}()} \varphi'$ for some t_φ . Similar to the **signal** rule, this rule starts by concretizing t_c and p_c in c and uniquely maps them to t_φ and p_φ in φ_{conc} . The task t_c in c' has just incremented the wait phase of the task t_c on the phaser p_c . Let $\text{gap}_\varphi(t_\varphi)(p_\varphi) = (\text{ph}, (\text{lw}, \text{ls}, \text{uw}, \text{us}))$. A level $\text{lev} > 0$ exists that shows the signal and wait phases of the tasks registered with p_c respect the phase bounds in φ_{conc} . The same level can be used to show that phases of c' respect the phase bounds in φ' that is generated by the rule.

drop. Assume $c' \xrightarrow[t_c]{\text{ph.drop}()} c$ for some c' . We exhibit $\varphi' = (\mathcal{T}'_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}'_\varphi, \text{gap}'_\varphi, \text{egap}_\varphi)$ such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t_\varphi]{\text{ph.drop}()} \varphi'$ for some task t_φ . Similar to the **signal** rule, this rule starts by concretizing t_c and p_c in c and uniquely maps them to t_φ and p_φ in φ_{conc} . Since $c \in \llbracket \varphi_{\text{conc}} \rrbracket$, there exists a level lev that separates signal and wait phases of the tasks registered with p_φ . The task t_c in c' has just dropped the phaser p_c . By existence of c' , a lev' exists that takes into account the phases of t_φ as well. If $\delta < -\text{lwmax}$ (respectively, $\delta > \text{lsmax}$), we can show $\text{lev} - \text{lwmax}$ (respectively, $\text{lev} + \text{lsmax}$) also separates the phases in c' . Otherwise, $\text{lev} + \delta$ with $-\text{lwmax} \leq \delta \leq \text{lsmax}$ separates the phases in c' . We adapt the bounds for each such δ in this case. In a similar manner, if $\text{usmin} < \infty$ and $\text{uwmin} < \infty$ for some task, then a level $\text{lev} + \delta$ with $-\text{uwmin} \leq \delta \leq \text{usmin}$ must separate the phases in c' . We adapt the phase bounds to this case.

exit. Assume $c' \xrightarrow[t_c]{\text{exit}} c$ for some c' . Proof of soundness of the rule **exit** has a similar approach to that of the **drop** rule. The difference is that **exit** intuitively has to iterate through all the phasers in $\mathcal{P}_c$ with which t_c is registered and drop them. The mappings $\kappa' = \kappa \cup \{t_c \rightarrow t_\varphi\}$ and π witness the denotation.

asynch. Assume $c' \xrightarrow[t_c]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} c$ for some c' where $\text{paramOf}(\text{task}) = (q_1, \dots, q_k)$. The task t_c has just spawned u_c in c . We exhibit $\varphi' = (\mathcal{T}'_\varphi, \mathcal{P}'_\varphi, \text{bv}'_\varphi, \text{pc}'_\varphi, \text{gap}'_\varphi, \text{egap}'_\varphi)$ such that $c' \in \llbracket \varphi' \rrbracket$ and $\varphi \xrightarrow[t_\varphi]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} \varphi'$. The rule starts by concretizing t_φ , u_φ , and p_{i_φ} for every $i : 1 \leq i \leq k$ in φ . Lemmas 19 and 20 ensure that a concrete constraint $\varphi_{\text{conc}} = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, \text{bv}_\varphi, \text{pc}_\varphi, \text{gap}_\varphi, \text{egap}_\varphi)$ is generated such that $c \in \llbracket \varphi_{\text{conc}} \rrbracket$ with respect to some task and phaser mappings κ and π so that t_c and u_c are uniquely mapped to some t_φ and u_φ by κ , and p_{i_c} is mapped to some

p_{i_φ} by π for each $i : 1 \leq i \leq k$. The task t_c in c' has just spawned the task u_c . t_c and u_c are registered with each p_{i_c} for $i : 1 \leq i \leq k$ and have the same phases in gap_c . $c \in \llbracket \varphi_{\text{conc}} \rrbracket$ ensures that for each $i : 1 \leq i \leq k$, gap_φ can be constrained so that t_φ and u_φ are registered with each p_{i_φ} in gap_φ and have the same phases. Hence, the meet of the signal and wait gaps of t_φ and u_φ in gap_φ is not empty. This meet will actually be the phase of t_φ on phasers p_{i_φ} for $i : 1 \leq i \leq k$. $phase'_c$ is then obtained from $phase_c$ by removing the phases of u_c . gap'_φ is also obtained from gap_φ by removing u_φ . The task and phaser mappings $\kappa' = \kappa \setminus \{u\}$ and π witness that φ' denotes c' . $\square$

We define $c_0 \xrightarrow[\mathcal{T}]{\text{stmt}} c_n$ to mean a sequence $c_0 \xrightarrow[t_1]{\text{stmt}} c_1 \dots c_{n-1} \xrightarrow[t_n]{\text{stmt}} c_n$ where $\mathcal{T} = \{t_1, \dots, t_n\}$ are the tasks in c_0 . Observe that this is in the transitive closure of $\xrightarrow[\text{stmt}]{}$.

Theorem 15. *Each predecessor computation rule $\xrightarrow[\text{stmt}]{}$ in Figures 5.9 and 5.8 except for the rule **newPhaser** is complete with respect to the semantic rules of those in Figure 5.2 and $\xrightarrow[\text{ul}]{\text{stmt}}$ defined above. The rule **newPhaser** is complete only when the task **main** executes it.*

Proof. Assume $\varphi \xrightarrow[t_\varphi]{\text{stmt}} \varphi'$ according to Figures 5.9 and 5.8. For any configuration $c' = (\mathcal{T}_{c'}, \mathcal{P}_{c'}, bv_{c'}, pc_{c'}, phase_{c'})$ where $c' \in \llbracket \varphi' \rrbracket$, we exhibit a configuration $c = (\mathcal{T}_c, \mathcal{P}_c, bv_c, pc_c, phase_c)$ such that $c' \xrightarrow[\kappa^{-1}(t_\varphi)]{\text{stmt}} c$ and $c \in \llbracket \varphi \rrbracket$. Note that any predecessor computation rule starts by concretizing φ to some concrete constraint $\varphi_{\text{conc}} = (\mathcal{T}_\varphi, \mathcal{P}_\varphi, bv_\varphi, pc_\varphi, gap_\varphi, egap_\varphi)$. Then, the predecessor φ' is obtained from the φ_{conc} . Lemmas 14 and 15 ensure $\varphi \sqsubseteq \varphi_{\text{conc}}$ for any concrete constraint φ_{conc} . Hence, for every rule we show $c \in \llbracket \varphi_{\text{conc}} \rrbracket$. This implies $c \in \llbracket \varphi \rrbracket$.

Assume κ and π witness $c' \in \llbracket \varphi' \rrbracket$. We show that using $\kappa^{-1}(t_\varphi)$ for $c' \xrightarrow[\kappa^{-1}(t_\varphi)]{\text{stmt}} c$ generates the desired configuration c . The intuition is that if all tasks in $\mathcal{T}_{c'}$ that are associated to t_φ by κ execute **stmt**, we obtain a configuration c that is denoted by some concrete constraint φ_{conc} , hence, denoted by φ .

newPhaser. This rule is only complete when $\kappa^{-1}(t_\varphi)$ is a singleton. To simplify the presentation, we show completeness when **main** executes **newPhaser**, because then we are sure $\kappa^{-1}(t_\varphi)$ is a singleton. Assume t_φ is the task **main** in some φ and $\varphi \xrightarrow[t_\varphi]{\text{ph:=newPhaser}()} \varphi'$. Assume also p_φ is the new phaser id. Let $p_c \notin \mathcal{P}_{c'}$ and $\mathcal{P}_c = \mathcal{P}_{c'} \cup \{p_c\}$. $phase_c$ is obtained from $phase_{c'}$ by assigning $phase_c(t_c)(p_c) = (\text{ph}, (0, 0))$ where t_c is the task **main** in c' . Since $(0, 0) \models gap_\varphi(t_\varphi)(p_\varphi)$, t_c can again be associated with t_φ in φ_{conc}

and $\text{phase}_c(u_c)(p_c) = (-, \text{nreg})$ for all other tasks $u_c \in \mathcal{T}_c' \setminus \{t_c\}$. Moreover, $\text{phase}_\varphi(u_\varphi)(p_\varphi) = (-, \text{nreg})$ for all tasks $u_\varphi \in \mathcal{T}_\varphi' \setminus \{t_c\}$. Hence, for $c = (\mathcal{T}_c, \mathcal{P}_c, \text{bv}_c, p_c, \text{phase}_c)$, we have $c' \xrightarrow[t_c]{\text{ph}:=\text{newPhaser}()} c$ and the task and phaser mappings κ and $\pi' = \pi \cup \{p_c \rightarrow p_\varphi\}$ witness $c \in \llbracket \varphi_{\text{conc}} \rrbracket$.

signal. Assume $\varphi \xrightarrow[t_\varphi]{\text{ph.signal}()} \varphi'$ for some φ . By definition of $c' \in \llbracket \varphi' \rrbracket$, there exists p_c such that $\pi(p_c) = p_\varphi$ and all tasks in $\kappa^{-1}(t_\varphi)$ are associated with t_φ by κ . c' is denoted by φ' and is obtained via **signal I** or **signal II**. There is therefore a level $\text{lev} \geq 0$ which shows that the signal and wait phases of the tasks registered in c' respect the phase bounds in φ' . We can show that $\text{lev} + 1$ captures that the signal and wait phases of the tasks registered in c respect the phase bounds in φ if φ' is obtained via **signal I**, and lev shows that the signal and wait phases of the tasks registered in c respect the phase bounds in φ if φ' is obtained via **signal II**.

In both cases, the task and phaser mappings κ and π should be used for the denotation.

wait. Assume $\varphi \xrightarrow[t_\varphi]{\text{ph.wait}()} \varphi'$ for some φ . By definition of $c' \in \llbracket \varphi' \rrbracket$, there exists p_c such that $\pi(p_c) = p_\varphi$ and each task $t \in \kappa^{-1}(t_\varphi)$ is associated with t_φ by κ . If lev witnesses $c' \in \llbracket \varphi' \rrbracket$, then the same level witnesses $c \in \llbracket \varphi \rrbracket$ with the task and phaser mappings κ and π .

drop. Assume $\varphi \xrightarrow[t_\varphi]{\text{ph.drop}()} \varphi'$ for some φ . By definition of $c' \in \llbracket \varphi' \rrbracket$, there exists p_c such that $\pi(p_c) = p_\varphi$ and each task $t \in \kappa^{-1}(t_\varphi)$ is associated with t_φ by κ . phase_c is obtained from $\text{phase}_{c'}$ by 1) not modifying the phases of the tasks that are not associated with t_φ , 2) assigning $\text{phase}_c(t)(p_c) = (-, \text{nreg})$ for each $t \in \kappa^{-1}(t_\varphi)$. A level $\text{lev} \geq 0$ witnesses these tasks respect the phase bounds in φ' (which is obtained for a certain δ). We can show $\text{lev} - \delta$ witnesses denotation of c by φ_{conc} . The third group of tasks to consider are those in $\mathcal{T}_c \setminus \kappa^{-1}(t_\varphi)$ which are registered with p_c . The phases for these tasks are not modified. Hence, the task and phaser mappings κ and π witness $c \in \llbracket \varphi_{\text{conc}} \rrbracket$.

exit. This proof is very similar to the proof of completeness of **drop**. The only difference is that the proof of **drop** needs to be extended by iterating through all phasers dropped in c' . The mappings $\kappa' = \kappa \setminus \kappa^{-1}(t_\varphi)$ and π witness $c \in \llbracket \varphi_{\text{conc}} \rrbracket$.

asynch. Assume $\varphi \xrightarrow[t_\varphi]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} \varphi'$ for some φ . Let u_φ be the newly spawned task, and $p_{1_\varphi}, \dots, p_{k_\varphi}$ be the phasers passed to u_φ . By definition of $c' \in \llbracket \varphi' \rrbracket$, there exist $p_{1_c}, \dots, p_{k_c}$ that $\pi(p_{i_c}) = p_{i_\varphi}$ for each $i : 1 \leq i \leq k$

and κ associates each task $t \in \kappa^{-1}(t_\varphi)$ to t_φ , and each task u_t spawned by t to u_φ . We obtain phase_c from $\text{phase}_{c'}$ by copying $\text{phase}_{c'}(t)(p_{i_c})$ to $\text{phase}_c(u_t)(p_{i_c})$ for each $t \in \kappa^{-1}(t_\varphi)$ and $i : 1 \leq i \leq k$. Therefore, each $t \in \kappa^{-1}(t_\varphi)$ and u_t can respectively be associated to t_φ and u_φ in φ . As a result, for $c = (\mathcal{T}_c, \mathcal{P}_{c'}, \text{bv}_{c'}, \text{pc}_c, \text{phase}_c)$, we have $c' \xrightarrow[\kappa^{-1}(t_\varphi)]{\text{asynch}(\text{task}, \text{ph}_1, \dots, \text{ph}_k)\{s\}} c$ and the task and phaser mappings $\kappa' = \kappa \cup \{t \rightarrow t_\varphi \mid t \in \kappa^{-1}(t_\varphi)\} \cup \{u_t \rightarrow u_c \mid t \in \kappa^{-1}(t_\varphi)\}$ and π witness $c \in \llbracket \varphi_{\text{conc}} \rrbracket$. $\square$

5.5.3 Verification Procedure

The procedure “paramPhaserExplore”($\Phi_{\text{bad}}, c_{\text{strt}}, \mathbf{S}, \rightarrow, \sqsubseteq$) makes use of a predecessor computation (line 7) that results, for a constraint φ and a statement **stmt**, in a finite set $\text{pre}_{\text{stmt}} = \left\{ \varphi' \mid \varphi \xrightarrow[t]{\text{stmt}} \varphi' \right\}$ with task t which is either a task in φ or is a fresh task. Observe that renaming tasks and phasers in φ does not change the denotation. Therefore, we keep only one representative. We write **pre** for the union of pre_{stmt} for all **stmt**. These computations are described in Figures 5.8 and 5.9. Intuitively, the program statement for which the predecessors set is being computed can be executed by a task that is either captured by the constraint $\varphi = (\mathcal{T}, \mathcal{P}, \text{bv}, \text{pc}, \text{pv}, \gamma)$ explicitly (i.e., $t \in \mathcal{T}$) or implicitly (i.e., $t \notin \mathcal{T}$ but satisfies the environment gaps). For all but **newPhaser** and atomic statements (i.e., **ph.next()**{**s**}), the set pre_{stmt} is exact in the sense that $\left\{ c' \mid c \in \llbracket \varphi \rrbracket \text{ and } c' \xrightarrow{\text{stmt}} c \right\} \subseteq \bigcup_{\varphi' \in \text{pre}_{\text{stmt}}} \llbracket \varphi' \rrbracket \subseteq \left\{ c' \mid c \in \llbracket \varphi \rrbracket \text{ and } c' \xrightarrow{\text{stmt}}^+ c \right\}$. Intuitively, the atomic **ph.next()**{**stmt**} statement can encode a testAndSet operation. Such an operation can be made to be carried by exactly one task. Our representation allows for more tasks (and larger gaps), but the additional tasks may not be able to carry the atomic operation. In fact, we show in Section 5.6 that allowing atomic instructions results in the undecidability of the problems addressed by Theorems 16 and 17. The proposed **newPhaser** predecessor computation is exact when only one task carries it (e.g., **main**). Otherwise, assume a task t in the constraint creates a new phaser p . If in a configuration each task that is mapped (by a surjection) to t creates a phaser, several phasers are created and it is not possible to map all of them by a bijection to p . Without this restriction the predecessor computation for **newPhaser** is sound but not complete.

Using the soundness of $\sqsubseteq$ (Lemma 5) and the soundness of predecessor computation (Lemma 14) we can show soundness of “paramPhaserExplore”.

Lemma 21 (Soundness). *If “paramPhaserExplore”($\Phi_{\text{bad}}, c_{\text{strt}}, \mathbf{S}, \rightarrow, \sqsubseteq$) returns **unreachable**, then $c_{\text{strt}} \xrightarrow{*} \llbracket \Phi_{\text{bad}} \rrbracket$.*

Input: A set of constraints Φ_{bad} , some configuration c_{strt} , a set of program statements S , a set of image computation functions $\xrightarrow[t]{stmt}$ for each $stmt$ and t , and a preorder $\sqsubseteq$ over constraints.

Output: **unreachable** or a trace $\rho = \varphi_0; stmt_1; \varphi_1; \dots; stmt_n; \varphi_n$

```

1 initialize both working and store to  $\{(\varphi, \varphi) \mid \varphi \in \Phi_{bad}\}$ ;
2 while exists  $(\varphi, \rho) \in \text{working}$  do
3   remove  $(\varphi, \rho)$  from working;
4   let  $(\mathcal{T}, \mathcal{P}, bv, pc, gap, egap) = \varphi$ ;
5   if  $c_{strt} \in \llbracket \varphi \rrbracket$  then return  $\rho$ ;
6   foreach  $s \in S$  do
7     foreach  $\varphi' \in pre_{hd(s)}(\varphi)$  do
8       if  $\varphi'' \not\sqsubseteq \varphi'$  for all  $(\varphi'', -) \in \text{store}$  then
9         add  $(\varphi', \varphi'; hd(s); \rho)$  to both working and store;
10 return unreachable;
```

Procedure paramPhaserExplore($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$). A working-list procedure for checking reachability for phaser programs. The procedure checks whether some configuration c_{strt} is reachable from one of the constraints in Φ_{bad} using the image computation functions $\xrightarrow[t]{stmt}$ for each $stmt$ and t , and the preorder $\sqsubseteq$ on the constraints.

Proof. The procedure “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) is an instantiation of the procedure “explore” in Section 3.3. The function **isTarget** in “explore” is instantiated with the denotation function for configurations and constraints. By soundness of both $\rightarrow$ (Theorem 14) and the procedure “explore” (Theorem 4), the procedure “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) is sound. $\square$

Lemma 22 (Relative completeness). *If the procedure “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) returns a trace $\varphi_0; stmt_0; \dots; \varphi_{n-1}; stmt_{n-1}; \varphi_n$ in which only the task **main** has executed **newPhaser**, there are $c_0, \dots, c_n$ with $c_0 = c_{strt}$, $c_n \in \llbracket \Phi_{bad} \rrbracket$ and $c_{i-1} \xrightarrow{stmt_{i-1}^+} c_i$ for $i : 1 < i \leq n$.*

Proof. The procedure “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) is an instantiation of the procedure “explore” in Section 3.3. The function **isTarget** in “explore” is instantiated with the denotation function for configurations and constraints, which is by definition complete. By relative completeness of $\rightarrow$ (Theorem 15) and completeness of the procedure “explore” (Theorem 5), the procedure “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) is complete relative to the runs in which only the task **main** is allowed to execute **newPhaser**. $\square$

We can also show that the procedure terminates if we only manipulate K -bounded-dimension and B -good constraints.

Lemma 23 (Termination). *“paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) terminates if there are $K, B \in \mathbb{N}$ such that all constraints in **store** are K -dimension-bounded and B -good.*

Proof. If all constraints are K -dimension-bounded and B -good for $K, B \in \mathbb{N}$, then according to Theorem 13 the preorder $\sqsubseteq$ is a $\mathcal{WQO}$. $\rightarrow$, $\sqsubseteq$, and $\models$ are effective by definition. Hence, by Theorem 6 we conclude “paramPhaserExplore”($\Phi_{bad}, c_{strt}, S, \rightarrow, \sqsubseteq$) terminates. $\square$

Theorem 16. *Control reachability for non-atomic phaser programs that generate a finite number of phasers is decidable if only `main` is allowed to create phasers.*

Proof. K -boundedness is ensured by systematically dropping the generated constraints in the backward procedure that violate K -dimension-boundedness (as none of the denoted configurations are reachable). Also, the set of the target constraints is free (since we are checking control reachability) and this is preserved by the `pre` computation in Figure 5.6, 5.8, and 5.9. Finally, Lemmas 21, 22, and 23 ensure soundness, relative completeness, and termination. $\square$

Theorem 17. *Plain reachability is decidable for non-atomic phaser programs that generate at most K phasers with B -bounded gaps for each phaser if only `main` is allowed to create phasers.*

Proof. In the backward procedure, systematically dropping the generated constraints that require more than K phasers or gap-values larger than B for some phaser gaps (as none of the denoted configurations are reachable) ensures K -dimension-boundedness and B -goodness. Finally, Lemmas 21, 22, and 23 ensure soundness, relative completeness, and termination. $\square$

5.6 Limitations of Deciding Reachability

Assume a program $\text{Pr} = (B, P, \text{Ts})$ and a configuration c_{strt} . We show a number of parameterized reachability problems to be undecidable. First, we address checking control reachability when restricting to configurations with at most K task-referenced phasers. We call this K -control-reachability.

Definition 10 (K -control-reachability). *Given a program Pr , a configuration c_{strt} , and a partial control configuration c_{trgt} , we write $\text{reach}_K(\text{Pr}, c_{strt}, c_{trgt})$, and say c_{trgt} is K -control-reachable, to mean there are $n + 1$ configurations $(c_i)_{i:0 \leq i \leq n}$, each with at most K reachable phasers (i.e., phasers referenced by at least a task variable) such that $c_{strt} = c_0$ and $c_i \rightarrow c_{i+1}$ for $i : 0 \leq i < n$ with c_n equivalent to a configuration that includes c_{trgt} .*

Theorem 18. *K -control-reachability is undecidable in general.*

```

1 bool s1, s2, ..., sF;
2 bool xDec, yDec;
3 main(){
4   xPh = newPhaser();
5   yPh = newPhaser();
6   while(true){
7     //(qi:inc(x):qj)
8     if(ndet() ∧ si){
9       asynch(xTask, xPh);
10      si= false;
11      sj= true;
12    }
13    //(qi:inc(y):qj)
14    if(ndet() ∧ si){
15      asynch(yTask, yPh);
16      si= false;
17      sj= true;
18    }
19    //(qi:dec(x):qj)
20    if(ndet() ∧ si){
21      xDec = true;
22      xPh.next();
23      xPh.next();
24      assert(!xDec);
25      si= false;
26      sj= true;
27    }
28    //(qi:dec(y):qj)
29    if(ndet() ∧ si){
30      yDec = true;
31      yPh.next();
32      yPh.next();
33      assert(yDec);
34      si= false;
35      sj= true;
36    }
37    //(qi:test(x):qj)
38    if(ndet() ∧ si){
39      xPh.signal();
40      xPh.wait();
41      si= false;
42      sj= true;
43    }
44    //(qi:test(y):qj)
45    if(ndet() ∧ si){
46      yPh.signal();
47      yPh.wait();
48      si= false;
49      sj= true;
50    }
51  }
52 }//(end of main)
53
54 // ***** xTask *****
55 xTask(xPh){
56   while(true){
57     if(xDec){
58       if(ndet()){
59         xDec = false;
60         p1 = newPhaser();
61         xPh.next();
62         exit;
63       }
64       xPh.next();
65       xPh.next();
66     }
67   }
68 }
69
70 // ***** yTask *****
71 yTask(yPh){
72   while(true){
73     if(yDec){
74       if(ndet()){
75         !yDec = false;
76         p1 = newPhaser();
77         yPh.next();
78         exit;
79       }
80       yPh.next();
81       yPh.next();
82     }
83   }
84 }
85
86 }

```

Figure 5.10: Encoding a Minsky machine with the two counters $\{x, y\}$ for the proof of Theorem 18. The value of counter x is represented by the number of instances of `xTask` tasks registered to phaser `xPh`. The construction ensures that runs trying to decrement a counter by more than one will result in configurations with a larger number of phasers than K .

Figure 5.11: A possible configuration for the phaser program in proof of Theorem 18 that consists of two groups of tasks of types `xTask` and `yTask`. The number of tasks in each cluster represents the value of the corresponding counter, namely, x and y

Proof. We encode the state reachability of an arbitrary Minsky machine with counters x and y using K -control-reachability of a suitable phaser program. We assume $K = 3$. The program depicted in Figure 5.10 has three tasks: `main`, `xTask`, and `yTask`. Machine states $q \in Q$ are captured with shared Boolean variables and counter values with the number of `xTask` registered with phaser `xPh` for counter x (respectively, the number of `yTask` registered with phaser `yPh` for the counter y). A sample configuration of this program is illustrated in Figure 5.11. We reason about the correctness of our encoding for counter x . The reasoning for counter y would be similar.

Incrementing the counter x is encoded by spawning an instance of `xTask` and registering it with `xPh` (line 9 of Figure 5.10).

Test for zero is performed by a signal and a wait on `xPh` (lines 39 and 40).

Decrementing the counter x involves asking an `xTask`, via shared variables, to exit (hence, to deregister from `xPh`). However, more than one task might participate in the decrement operation. For this reason, each participating task creates a new phaser. In this way, if more than one `xTask` participates in the decrement, then the number of reachablephasers of an intermediary configuration will be at least 4, and will therefore be discarded because $4 > K$. A possible such configuration is depicted in Figure 5.12.

First, the task `main` sets `xDec` to `true` (line 21). This will satisfy the if-condition at line 58 of any `xTask`. These tasks non-deterministically choose whether or not to participate in the decrement by the if-condition at line 59. `main` performs two `next` operations on the phaser `xPh`. Those `xTask`s that do not participate will perform two `next` operations at lines 65 and 66 in order to not block the ones that participate. If no such task participates, the assertion at line 24 would be violated.

Figure 5.12: A possible configuration for the phaser program in proof of Theorem 18 consisting of two groups of tasks of types `xTask` and `yTask`. This configuration is the result of two `xTask` tasks trying to decrement the counter x which violates K -control boundedness for $K = 3$.

Any participating `xTask` has to create a phaser `p1` before being able to execute `xPh.next()` at line 62. Thereafter, `xTask` can `exit` which enables `main` to not block at `xPh.next()` (line 23).

The above encoding ensures that the two phasers `xPh` and `yPh` appear in all configurations. Moreover, a decrement operation will add one extra phaser per participating `xTask`. Hence, if more than one such task participates, K will exceed 3, and those configurations will be discarded. As a result, from the initial configuration c_{init} , the phaser program will reach a configuration in which `sF = true` via configurations that have at most three reachable phasers, iff the counter machine reaches a configuration with the target state q_F . $\square$

Theorem 19. *The control reachability of phaser programs that generate a finite number of phasers is undecidable if atomic statements are allowed.*

Proof. We encode the state reachability problem of an arbitrary Minsky machine with counters x and y using a phaser program with atomic statements. The phaser program depicted in Figure 5.13 should have three tasks: `main`, `xTask` and `yTask`. `yTask` is similar to `xTask`. The idea is to associate a phaser `xPh` to counter x (respectively, `yPh` to counter y). We introduce in the following the encoding of the Minsky machine operations on the counter x in the phaser program. Counter y is very similar to that.

Increment. `main` spawns a task `xTask` and registers it with `xPh` (line 10).

Decrement. First, `main` sets `xDec` to `true` and performs `next` with atomic statement on `xPh` (lines 18 and 19). One of the existing `xTask`:s will execute the `next` atomic statement (lines 47 and 48) and reset `xDec` and `exit`. Then,

```

1  bool xDec,yDec,s1, s2, ...,sF;
2
3  main(){
4      xPh = newPhaser();
5      yPh = newPhaser();
6      while(true){
7          //(q1:inc(x):q3)
8          if(ndet() && si)
9          {
10             asynch(xTask,xPh);
11             si=false;
12             sj=true;
13         }
14
15         //(q1:dec(x):q3)
16         if(ndet() && si)
17         {
18             xDec=true;
19             xPh.next(){
20                 xDec=false;
21                 exit;
22             }
23
24             si=false;
25             sj=true;
26         }
27
28         //(q1:test(x):q3)
29         if(ndet() && si)
30         {
31             xPh.next();
32             si=false;
33             sj=true;
34         }
35         //(q1:inc(y):q3)
36         //(q1:dec(y):q3)
37         //(q1:test(y):q3)
38         ...
39     }
40
41     //**** xTask ****
42     xTask(xPh){
43         while(true){
44             if(xDec){
45                 xPh.next(){
46                     xDec=false;
47                     exit;
48                 }
49             }
50         }
51     }
52 }

```

Figure 5.13: Encoding a Minsky machine with the two counters $\{x, y\}$ for proof of the Theorem 19. The value of counter x is represented by the number of instances of `xTask` tasks registered with phaser `xPh`. The construction ensures increments or decrements involve exactly one task.

all remaining `xTask` instances perform the statement `xPh.next()` (line 46) without executing its atomic statements.

A test for zero on the counter x is performed by executing a `next` on `xPh` (line 30).

As a result of the encoding, the phaser program will reach from the initial configuration φ_{init} , a configuration in which the Boolean variable `sF` is set iff the counter machine reaches a configuration with state `qF`.

Finally, even with finite numbers of tasks and phasers, but with arbitrary gap-bounds, we can show [62] the following. $\square$

Theorem 20. *Plain reachability of non-atomic phaser programs that generate a finite number of phasers is undecidable if the generated gaps are not bounded.*

5.7 Conclusion

We have studied parameterized plain (e.g., deadlocks) and control (e.g., assertions) reachability problems for phaser programs. We have proposed a verification procedure that is exact for deciding reachability of non-atomic

	Arbitrary numbers of tasks			
	Finite dimension		K -reachability	Arbitrary dimension
Bounded gaps	ctrl atomic ✗ (Theorem 19)	plain non-atomic ✓ (Theorem 17)	ctrl non-atomic ✗ (Theorem 18)	ctrl non-atomic ✗ (Theorem 12)
Arbitrary gaps	ctrl non-atomic ✓ (Theorem 16)	plain non-atomic ✗ (From [62])		

Table 5.1: Findings summary: *ctrl* stands for control reachability and *plain* for plain reachability; *atomic* stands for allowing the `ph.next(){stmt}` atomic instruction and *non – atomic* for forbidding it (resulting in non-atomic programs). Recall that the dimension of a phaser program is the number of dynamically generated phasers. Our reachability procedure is sound for all problems discussed in the table. Problems that are guaranteed to terminate and are complete relative to the runs in which only `main` creates new phasers are marked with ✓. The undecidable problems are marked with ✗.

programs when only distinguished tasks such as `main` create phasers. The procedure can be used for answering both control and plain reachability problems. We summarize our findings in Table 5.1. The procedure is guaranteed to terminate, even for programs that may generate arbitrarily many tasks but finitely many phasers, when checking control reachability or when checking plain reachability with bounded gaps. These results were obtained using a non-trivial symbolic representation for which termination had required showing an $\exists \leq_V$ preorder on multisets on gaps on natural numbers to be a $\mathcal{WQO}$.

We believe our general decidability results are useful for reasoning about synchronization constructs other than phasers. For instance, one phaser with bounded gaps (in fact, one) can capture a traditional static barrier. Similarly, one phaser with one producer and arbitrarily many consumers can be used to capture futures where a “get” instruction can be modeled with a wait. We believe our negative results can also be used. For instance, atomic instructions can be modeled using `testAndSet` operations and may result in the undecidability of the reachability problem. This suggests that more general applications of the work are to be investigated.

6 Counting Dynamically Synchronizing Processes

In this chapter, we address the problem of automatically establishing correctness for parameterized programs that generate an arbitrary number of concurrent processes and manipulate variables ranging over an infinite domain. The programs we consider can make use of shared variables to count and synchronize the spawned processes. This allows them to implement intricate synchronization mechanisms, such as barriers. For this purpose, we introduce *counting predicates* that mix counters referring to the number of processes satisfying certain properties and variables directly manipulated by the concurrent processes. We then combine existing works on counter, predicate, and constrained monotonic abstraction and build a nested counterexample-based refinement scheme for establishing correctness (expressed as non-reachability of configurations that satisfy counting predicates). We have implemented a tool (PACMAN, for predicated constrained monotonic abstraction) and used it to perform parameterized verification on several programs whose correctness crucially depends on precisely capturing the number of processes synchronizing through the use of shared variables.

6.1 Introduction

We consider programs that spawn arbitrarily many concurrent processes that use shared integer variables for counting the number of processes at different stages of their computations. Such counters might be used to implement a synchronization mechanism among processes. Correctness is stated in terms

of safety properties expressed using *counting predicates*. Counting predicates make it possible to express statements about program variables and counters for the number of processes satisfying certain predicates on program variables. Such statements can capture both individual properties, such as assertion violations, and global properties, such as deadlocks or relations between the numbers of processes in certain states.

Handling programs with barriers of arbitrary sizes (i.e., the number of processes participating in the barrier is fixed but arbitrarily large) is not a trivial task even when all processes only manipulate Boolean variables. To enforce the correct behavior of a barrier, a verification procedure needs to capture relations between the number of processes satisfying certain properties, for instance, that all processes are waiting at the barrier before any of them is allowed to cross it. This amounts to testing that the number of processes at specific locations is zero. Capturing increments and decrements of counters in a given counter machine can be done by having counters that track the number of processes satisfying predicates such as being at some program location, and adding the possibility to have a single process change location. In this case, one can argue that checking violations of program assertions is tantamount to checking state reachability of a counter machine. Therefore, no sound verification technique can be complete for such systems, since the problem is undecidable (Theorem 1).

Our approach to getting around this problem builds on the following observation. In case there are no tests for the number of processes satisfying certain properties (e.g., being in specific program locations), symmetric Boolean concurrent programs can be exactly encoded as monotonic counter machines, i.e., essentially vector addition systems (VASs). For such systems, state reachability can be decided using the backward reachability analysis that was introduced in Section 3.3. This approach is exact because of the monotonicity of the induced transition system (more processes can fire more transitions since there are no tests on the numbers of processes). Termination is guaranteed by the well-quasi-ordering of the component-wise ordering on the natural numbers. The induced transition system is not any more monotonic in the presence of tests on the number of processes. In this case we apply monotonic abstraction [4] as introduced in Section 3.4. The idea in monotonic abstraction is to modify the semantics of the entailed tests (e.g., zero tests for barriers), so that processes that do not satisfy the tests are removed (e.g., zero tests are replaced by resets). This results in a monotonic over-approximation of the original transition system in which spurious traces are possible. Indeed, having more processes while respecting important relations between their numbers and certain variables in the original programs does not necessarily allow to fire more transitions (which is what abstracted programs do in such approaches).

To sum up, our approach consists of combining two nested counterexample-guided abstraction refinement loops. Each loop operates

at a different level of abstraction. We summarize our contributions in the following points.

1. We introduce and propose the use of *counting predicates* to express statements about program variables and the number of processes satisfying given predicates on program variables.
2. We implement an abstraction refinement loop (the outer loop in our approach) by leveraging existing symmetric predicate abstraction techniques. We encode the resulting Boolean programs in terms of counter machines where reachability of configurations satisfying a *counting predicate formula* is captured as a state reachability problem.
3. We explain how to strengthen the counter machine using *counting invariants*, i.e., counting predicates that hold on all runs. We automatically generate such invariants using classical thread-modular analysis techniques.
4. We leverage existing constrained monotonic abstraction techniques to implement the inner loop of our approach and address the counter machine state reachability problem.
5. We implement both loops, together with automatic counting invariants generation, in a prototype tool (PACMAN) that has allowed us to automatically establish or refute counting predicate formulas such as deadlock freedom and assertions. All programs we report on may generate arbitrarily many processes.

Related work. Several works consider parameterized verification for concurrent programs. See [130] for a good survey. We report on some relevant techniques. The works in [81, 6] explore finite instances and automatically check for cutoff conditions. Except for checking larger instances, however, there is no straightforward approach to refine the entailed abstractions. [7] strengthens the approach of [6], but cannot capture global properties that involve relations between the number of processes and program variables, as we do in this work. In [57], the authors target verification of Petri nets, which are inherently monotonic and generate invariants by weakening SMT formulas. Although we also target coverability, we do it for counter machines obtained by strengthening monotonic systems into non-monotonic ones. There is no straightforward approach to apply their method to the non-monotonic systems that we consider. The work in [85] gives a generalization of the IC3 algorithm and tries to build inductive invariants for well-structured transition systems. Adapting it to the non-monotonic systems that we work with is not trivial. Similar to [1], we combine auxiliary invariants obtained on specific variables to strengthen a reachability analysis. In [58], the authors propose an approach that involves synthesizing counters to build correctness proofs from program

traces automatically. The approach repeatedly builds safe counting automata and establishes that their language includes traces of a program given as a control flow net. Such nets can model arbitrarily many processes sharing global variables. We can also handle local variables and automatically discover relevant predicates by nesting constrained monotonic abstraction loops within symmetric predicate abstraction loops. In [19], the authors present a highly optimized coverability checking approach for VASs with broadcasts. We need more than the coverability of monotonic systems. In [83], the authors adopt symbolic representations that can track inter-thread predicates. This yields a non-monotonic system, and the authors force monotonicity as in [4, 3]. However, they do not explain how to refine the obtained decidable monotonic abstraction for an undecidable problem. In [16], the authors prove termination for depth-bounded systems by instrumenting a given over-approximation with counters and sending the numerical abstraction to existing termination provers. We automatically generate the abstractions on which we establish safety properties. In addition, and as stated earlier, over-approximating the concurrent programs we target with (monotonic) well-structured transition systems would result in spurious runs.

The closest works to ours are [3, 51, 66]. Monotonic abstraction in [4, 3] is not combined with predicate abstraction, nor does it explicitly target counting properties or dynamic barrier based synchronization. In [51], the authors propose a predicate abstraction framework for concurrent multi-threaded programs. As explained earlier, such abstractions cannot exclude behaviors forbidden by synchronization mechanisms such as barriers. In our work, we build on [51] in order to handle shared and local integer variables. The approach in [66] introduces an invariant synthesis method that uses horn constraints and automatically infers the counters that are required for program verification. In contrast, we use counters for all program locations. Although their approach outperforms ours because of maintaining fewer counters, it is not made clear how to automatically encode the input programs as horn clauses.

Outline. We start by illustrating our approach using an example in Section 6.2 and introduce some preliminaries in Section 6.3. We then define concurrent programs and describe our counting predicates in Section 6.4. Next, we explain the different phases of our nested loop in Section 6.5 and report on our experimental results in Section 6.6. We finally conclude in Section 6.7.

6.2 Illustrative Example

Consider the concurrent program listed in Figure 6.1. In this program, processes share four integer variables, namely `arrived`, `all`, `read` and `cross`. Variable `all` is initialized to 1, and variables `arrived`, `cross` and `read` are initialized to 0. A single process starts executing the program transitions.


```

int arrived,all:= 0, 1;
int cross,read:=0, 0;

process:
tr0. pc0 → pc0 : spawn
tr1. pc0 → pc1 : assume(cross = 0); all := all + 1
tr2. pc1 → pc2 : read := 1
// do some reading before the barrier
tr3. pc2 → pc3 : read := 0
// barrier
tr4. pc3 → pc4 : arrived := arrived + 1
tr5. pc4 → pc5 : assume(arrived = all); cross := 1
...

```

Figure 6.1: A sample parameterized concurrent program in which processes synchronize using shared variables. No matter how many processes are spawned, no process can be at pc_5 when $read > 0$.

Each transition may consist of several statements. Each statement is either a **spawn**, **join**, multi-assignment, or an **assume()**. Arbitrarily many processes get spawned at location pc_0 by transition tr_0 .

Each process executes tr_1 and increments **all** if **cross** is 0, meaning that no process has crossed the barrier. Otherwise, no process can execute the transition tr_1 . It then sets and resets **read**. Intuitively, **read** is greater than zero when there is at least one process doing some reading before the barrier between transitions tr_2 and tr_3 . Transitions tr_4 and tr_5 essentially implement a barrier in the sense that all processes must have reached pc_4 in order for any of them to move to location pc_5 . The first process that crosses the barrier changes **cross** from 0 to 1. As a result, no other process can take transition tr_1 and start working. After the barrier, no process should be left behind. We capture this by asserting that no process at location pc_5 should witness $read > 0$. We write $@pc_5$ to mean the predicate satisfied by all processes at location pc_5 . A process that satisfies $@pc_5 \wedge (read > 0)$ is at location pc_5 and witnesses $read > 0$. A configuration of the program satisfies the predicate $(@pc_5 \wedge (read > 0))^{\#} \geq 1$ if the number of such processes in the configuration is greater than or equal to one. We call such predicate a *counting predicate* (introduced in Sec 6.4). Counting predicates can be used to capture configurations that violate properties other than assertions, e.g., deadlock freedom.

The assertion $(@pc_5 \wedge (read > 0))^{\#} \geq 1$ is never violated under any run starting from a configuration in which a single process starts executing from location pc_0 . To establish this fact, any verification procedure needs to take into account the barrier at tr_5 as well as the two sources of infiniteness, namely, the infinite domain of the shared and local variables and the number of processes that may participate in the run. Apart from [58], which cannot handle local variables, and [66], the closest works to ours deal with these two sources of infiniteness separately and cannot capture facts that relate them,

counters in order to faithfully capture the behavior of the abstracted program using counter abstraction.

From concurrent Boolean programs to counter machines. Given a concurrent Boolean program and a property to be checked, we generate a counter machine that essentially boils down to a vector addition system with transfers (with additional tests for global properties such as deadlock-freedom). Each counter in the machine counts the number of processes at some location with some specific value combination for the local variables. One state in the counter machine represents reaching a configuration that violates the property we want to check. The other states of the counter machines correspond to the possible combinations of the global variables. Such a machine cannot relate the number of processes in certain locations to the predicates that are valid at certain states (for instance, that `all = arrived`). These are essential for the verification of programs where counters are used to synchronize processes. In order to remedy this issue, we make use of *counting invariants* that relate program variables, `all` and `arrived` in the following invariants, to the number of processes at certain locations.

$$\text{all} = \sum_{i \geq 1} (@\text{pc}_i)^{\#}$$

$$\text{arrived} = \sum_{i \geq 4} (@\text{pc}_i)^{\#}$$

We automatically generate such invariants using a simple thread modular analysis that tracks the number of processes satisfying some property.

Example 5 (Thread-modular analysis). To recover the two counting invariants above, we can perform a classical thread modular analysis [70] where we add a shared instrumentation variable `pci` to track the number of processes at location `pci` for $i : 0 \leq i \leq 6$. We use a suitable abstract numerical domain (in this case, polyhedral domain). For the program of Figure 6.1, we obtained the counting invariants mentioned earlier as well as other invariants such as $0 \leq \text{all}, 0 \leq \text{arrived}$ and $\text{arrived} \leq \text{all}$ that helped for pruning the state space (Section 6.6).

Given such counting invariants, we constrain the counter machine and generate a more precise machine that may not be a vector addition system anymore. We explain in Section 6.5 that the resulting state reachability problem is now undecidable in general.

Constrained monotonic abstraction. We monotonically abstract the resulting counter machine in order to answer the state reachability problem. Spurious traces are now possible. Essentially, monotonic abstraction closes the obtained sets of predecessor configurations upwards. This overapproximation might add larger configurations that did not belong to the set

of predecessor configurations. Intuitively, the effect of monotonic abstraction “in forward” on the example of Figure 6.1 is that it “deletes” processes violating the constraint imposed by the barrier [4]. This example illustrates a situation where such approximations yield false positives. To see this, suppose two processes exist. The first process gets to pc_4 and waits. The second process moves to pc_2 . Deleting the second process is allowed by the monotonic abstraction and opens the barrier for the first process. However, the assertion can now be violated because the deleted process did not have time to reset the variable `read`. Constrained monotonic abstraction eliminates spurious traces by refining the preorder used in monotonic abstraction. For the example of Figure 6.1, if the number of processes at pc_1 is zero, then closing upwards will not alter this fact. By doing so, the process deleted in forward at pc_2 is not allowed to be there to start with, and the assertion is automatically established for any number of processes. The inner loop of our approach can automatically perform more elaborate refinements such as comparing the number of processes at different locations. Exact traces of the counter machine are sent to the next step and unreachability of the control location establishes the safety of the concurrent program.

Trace Simulation. Traces obtained in the counter machine reachability problem are real traces as far as the concurrent Boolean program is concerned. Those traces can be simulated on the original program to find new predicates (e.g., using Craig interpolation) and use them in the next iteration of the outer loop.

6.3 Preliminaries

We write $\mathbb{N}$ and $\mathbb{Z}$ to mean the sets of natural and integer values respectively. Given two natural numbers $i, j \in \mathbb{N}$, we use $[i, j]$ to denote the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$. We let $\mathbb{B} = \{\text{true}, \text{false}\}$ be the set of Boolean values.

We write $\mathbf{V}$ and $\mathbf{B}$ to respectively mean a set of integer and Boolean variables. We write X to mean some set $\mathbf{V}$ or $\mathbf{B}$. Similarly, we write v and v^b to respectively mean an integer or a Boolean variable. We also write x to mean a variable of some type.

We write $\text{exprsOf}(\mathbf{V})$ to mean the set of *arithmetic expressions* over the integer variables $\mathbf{V}$. An arithmetic expression e (or expression for short) in $\text{exprsOf}(\mathbf{V})$ is either an integer constant k , an integer variable v in $\mathbf{V}$, or the sum or difference of two expressions as described below:

$$e ::= k \mid v \mid (e + e) \mid (e - e) \mid k \ e \quad v \in \mathbf{V}$$

We let $\sim$ be a comparator in $\{<, \leq, =, \geq, >\}$. We write $\text{predsOf}_{\mathbf{E}}^{\mathbf{B}}$ to mean the set of predicates (i.e., Boolean expressions) over Boolean variables $\mathbf{B}$ and arithmetic expressions $\mathbf{E}$. A predicate π in $\text{predsOf}_{\mathbf{E}}^{\mathbf{B}}$ is either a Boolean value

$\mathbf{b}$, a variable $\mathbf{v}^b$ in $\mathbf{B}$, a comparison of two expressions in $\mathbf{E}$ or a Boolean combination of predicates. It takes the following form:

$$\pi ::= \mathbf{b} \mid \mathbf{v}^b \mid (\mathbf{e} \sim \mathbf{e}) \mid \neg\pi \mid \pi \wedge \pi \mid \pi \vee \pi \quad \mathbf{v}^b \in \mathbf{B}, \mathbf{e} \in \mathbf{E}$$

We write $\mathbf{e}(\mathbf{V})$ to emphasize that only variables in $\mathbf{V}$ are allowed to appear in $\mathbf{e}$. We write $\mathbf{varsOf}(\mathbf{e})$ to mean all integer variables appearing in an expression $\mathbf{e}$ and $\mathbf{varsOf}(\pi)$ to mean all variables appearing in π , namely both Boolean variables appearing in π and all integer variables in $\mathbf{varsOf}(\mathbf{e})$ for each $\mathbf{e}$ appearing in π . We also write $\mathbf{atomsOf}(\pi)$ to mean all atoms appearing in π . An atom α is either a comparison $(\mathbf{e} \sim \mathbf{e})$ of two arithmetic expressions or a Boolean. We write $\mathbf{A}$ to mean a set of atoms. For the set of integer variables $\mathbf{V}$, we write $\mathbf{atomsOf}(\mathbf{V})$ to mean the set of *atoms* over the variables $\mathbf{V}$.

A mapping $\mathbf{x} : X \rightarrow Y$ associates an element in Y to each element in X . We write $\mathbf{x} : X \rightarrow Y$ to mean a partial mapping from X to Y . We write $\mathbf{dom}(\mathbf{x})$ and $\mathbf{image}(\mathbf{x})$ to respectively mean the domain and the image of $\mathbf{x}$ and use $\epsilon_X : X \rightarrow Y$ for the mapping with an empty domain. We often write a partial mapping $\mathbf{x} : X \rightarrow Y$ as the set $\{x \leftarrow \mathbf{x}(x) \mid x \in \mathbf{dom}(\mathbf{x})\}$ and write $\mathbf{x} \cup \mathbf{x}'$ to mean the union of two mappings $\mathbf{x}$ and $\mathbf{x}'$ with disjoint domains. We assume in the following an arithmetic or Boolean expression $\mathbf{exp}$ or any indexed version of it. We write $\mathbf{exp}[\mathbf{v}]$ to mean the evaluation of expression $\mathbf{exp}$ for a mapping $\mathbf{v}$. We evaluate as follows. First, an expression $\mathbf{exp}_{tmp}$ is deduced by syntactically and simultaneously replacing in $\mathbf{exp}$ each occurrence of a variable $\mathbf{v} \in \mathbf{V}$ by the corresponding $\mathbf{v}(\mathbf{v})$. Then, if $\mathbf{varsOf}(\mathbf{exp}_{tmp}) = \emptyset$, $\mathbf{exp}[\mathbf{v}]$ is the constant obtained by evaluating $\mathbf{exp}_{tmp}$. Otherwise, $\mathbf{exp}[\mathbf{v}]$ is taken to be $\mathbf{exp}_{tmp}$ (see Example 6). Let $\mathbf{v} : \mathbf{V} \rightarrow Y$ and $\mathbf{v}' : \mathbf{V}' \rightarrow Y'$ be two mappings. If $\mathbf{V}$ and $\mathbf{V}'$ are disjoint, we write $\mathbf{exp}[\mathbf{v}, \mathbf{v}']$ to mean the evaluation of expression $\mathbf{exp}$ with respect to $\mathbf{v} \cup \mathbf{v}'$. Larger unions of mappings with pairwise disjoint domains are handled in a similar manner. We abuse notation and write $\mathbf{v}[\mathbf{v}']$ to mean $\{\mathbf{v} \leftarrow \mathbf{v}(\mathbf{v}) \mid \mathbf{v} \in \mathbf{V} \setminus \mathbf{V}'\} \cup \{\mathbf{v} \leftarrow \mathbf{v}'(\mathbf{v}) \mid \mathbf{v} \in \mathbf{V} \cap \mathbf{V}'\}$.

Example 6 (Expressions). Let $\pi = \mathbf{v}^b \wedge (\mathbf{v} = \mathbf{v}' + 1)$. Then, $\mathbf{varsOf}(\pi) = \{\mathbf{v}^b, \mathbf{v}, \mathbf{v}'\}$. If we let $\mathbf{v}^b = \{\mathbf{v}^b \leftarrow \mathbf{true}\}$, $\mathbf{v}_1 = \{\mathbf{v} \leftarrow \mathbf{v}' + 5, \mathbf{v}' \leftarrow 3\}$ and $\mathbf{v}_2 = \{\mathbf{v}' \leftarrow -1\}$, then $\pi[\mathbf{v}^b, \mathbf{v}_1] = \mathbf{true} \wedge (\mathbf{v}' + 5 = 3 + 1)$ and $\pi[\mathbf{v}^b, \mathbf{v}_1][\mathbf{v}_2] = \mathbf{true}$.

6.4 Concurrent Programs and Counting Predicates

To simplify the presentation, we assume a concurrent program (or program for short) to consist of a single non-recursive procedure manipulating integer variables. Arguments and return values are passed using shared variables. Programs where arbitrarily many processes run a finite number of procedures can be encoded by having the processes choose a procedure at the beginning.

$$\begin{array}{ll}
\text{prg} & ::= (\text{sh} := (k \mid \star))^* \\
& \text{process :} \\
& (l := (k \mid \star))^* \\
& (\text{pc} \rightarrow \text{pc} : \text{stmt})^+ \\
\text{stmt} & ::= v_1, \dots, v_n := e_1, \dots, e_n \\
& \mid \text{spawn} \mid \text{join} \mid \text{assume}(\pi) \mid \text{stmt}; \text{stmt}
\end{array}$$

Figure 6.3: Syntax of concurrent programs: sh is a shared variable in Sh , l is a local variable in L , $v_1, \dots, v_n$ are pairwise different variables in $\text{Sh} \cup \text{L}$, $e_1, \dots, e_n$ are arithmetic expressions in $\text{exprsOf}(\text{Sh} \cup \text{L})$ and π is a predicate in $\text{predsOf}_{\text{exprsOf}(\text{Sh} \cup \text{L})}$. $\star$ is a non-deterministic boolean value.

Syntax. The procedure of a program $\text{Pr} = (\text{Sh}, \text{L}, \text{Tr})$ is given in terms of a finite set Tr of transitions, each of the form $(\text{pc} \rightarrow \text{pc}' : \text{stmt})$. Each transition involves two program locations pc and pc' and a statement stmt . Transitions operate on two finite sets of integer variables, namely a set Sh of shared variables and a set L of local variables. We write PC to mean the set of all locations appearing in Tr . We always distinguish two locations, namely an entry location pc_0 and an exit location pc_x . Program syntax is described in Figure 6.3.

Semantics. Initially, a single process starts executing the procedure with both local and shared variables initialized as stated in their respective declarations. Executions might involve an arbitrary number of spawned processes. The execution of any process (whether initial or spawned with a **spawn** statement) starts at the entry location pc_0 with the corresponding local variables initialized as stated in their respective declarations. Any process at an exit point pc_x can be eliminated by a process executing a **join** statement. An **assume**(π) statement blocks if the predicate π over local and shared variables does not evaluate to true. Each transition is executed atomically by a single process without interruption from other processes.

A configuration is given in terms of a *shared state* and a *group configuration*. A *shared state* $s : \text{Sh} \rightarrow \mathbb{Z}$ is a mapping that associates an integer to each variable in Sh . We write $\mathbb{S}$ to mean the set of all shared states. An *initial shared state* is a mapping in $\mathbb{S}$ that respects shared variable declarations. We write $\mathbb{S}_{\text{init}}$ to mean the set of all initial shared states. A *process state* is a pair $(\text{pc}, \mathbb{l})$ where the location pc belongs to PC and the *local state* $\mathbb{l} : \text{L} \rightarrow \mathbb{Z}$ maps each local variable to an integer number. We also write $\mathbb{L}$ and $\mathbb{L}_{\text{init}}$ to respectively mean the sets of all local states and all initial local states. A *group configuration* is a multiset $\mathfrak{m}$ over process states. An *initial group configuration* maps all $(\text{pc}, \mathbb{l})$ to 0 except for a single $(\text{pc}_0, \mathbb{l})$, with $\mathbb{l} \in \mathbb{L}_{\text{init}}$, mapped to 1. We write $\mathbb{M}$ and $\mathbb{M}_{\text{init}}$ to mean the sets of all group configurations and initial group

$$\begin{array}{c}
 \frac{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{stmt}} (\mathfrak{s}', \mathbb{l}', \mathfrak{m}')}{(\mathfrak{s}, (\text{pc}, \mathbb{l}) \boxplus \mathfrak{m}) \xrightarrow[\text{Pr}]{(\text{pc} \rightarrow \text{pc}' : \text{stmt})} (\mathfrak{s}', (\text{pc}', \mathbb{l}') \boxplus \mathfrak{m}')} : \text{transition} \\
 \\
 \frac{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{stmt}} (\mathfrak{s}', \mathbb{l}', \mathfrak{m}') \text{ and } (\mathfrak{s}', \mathbb{l}', \mathfrak{m}') \xrightarrow[\text{Pr}]{\text{stmt}'} (\mathfrak{s}'', \mathbb{l}'', \mathfrak{m}'')}{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{stmt}; \text{stmt}'} (\mathfrak{s}'', \mathbb{l}'', \mathfrak{m}'')} : \text{sequence} \\
 \\
 \frac{\mathbb{v} = \{\mathbb{v}_i \leftarrow \mathbf{e}_i[\mathbb{s}, \mathbb{l}] \mid i : 1 \leq i \leq n\}}{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{v_1, \dots, v_n := \mathbf{e}_1, \dots, \mathbf{e}_n} (\mathfrak{s}[\mathbb{v}], \mathbb{l}[\mathbb{v}], \mathfrak{m})} : \text{assign} \\
 \\
 \frac{\pi[\mathbb{s}, \mathbb{l}]}{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{assume}(\pi)} (\mathfrak{s}, \mathbb{l}, \mathfrak{m})} : \text{assume} \\
 \\
 \frac{\mathfrak{m}' = (\text{pc}_0, \mathbb{l}_{init}) \boxplus \mathfrak{m} \text{ with } \mathbb{l}_{init} \in \mathbb{L}_{init}}{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{spawn}} (\mathfrak{s}, \mathbb{l}, \mathfrak{m}')} : \text{spawn} \\
 \\
 \frac{\mathfrak{m} = ((\text{pc}_x, \mathbb{l}') \boxplus \mathfrak{m}')}{(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{join}} (\mathfrak{s}, \mathbb{l}, \mathfrak{m}')} : \text{join}
 \end{array}$$

Figure 6.4: Semantics of concurrent programs. Executions start from some $(\text{pc}_0, \mathfrak{m}_{init})$ with $\mathfrak{m}_{init} \in \mathbb{M}_{init}$. The operator $\boxplus$ is used to add a process state to $\mathfrak{m}$.

configurations, respectively. Finally, a *configuration* is a pair $(\mathfrak{s}, \mathfrak{m})$ consisting of a shared state $\mathfrak{s}$ and a group configuration $\mathfrak{m}$. We write $(\mathfrak{s}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{tr}} (\mathfrak{s}', \mathfrak{m}')$ to mean that the transition tr of the form $(\text{pc} \rightarrow \text{pc}' : \text{stmt})$ applies atomically to configuration $(\mathfrak{s}, \mathfrak{m})$ and changes it to $(\mathfrak{s}', \mathfrak{m}')$. We introduce a relation $\xrightarrow[\text{Pr}]{\text{stmt}}$ in order to describe the steps involved in the semantics of transitions (Figure 6.4). We write $(\mathfrak{s}, \mathbb{l}, \mathfrak{m}) \xrightarrow[\text{Pr}]{\text{stmt}} (\mathfrak{s}', \mathbb{l}', \mathfrak{m}')$, where $\mathfrak{s}, \mathfrak{s}'$ are shared states, $\mathbb{l}, \mathbb{l}'$ are local states, and $\mathfrak{m}, \mathfrak{m}'$ are multisets of group configurations, in order to mean that a process of the program Pr at local state $\mathbb{l}$ when the shared state is $\mathfrak{s}$ and the other process states are captured by $\mathfrak{m}$, can execute the statement stmt and take the program to a configuration where the process has local state $\mathbb{l}'$, the shared state is $\mathfrak{s}'$ and the states of the other processes are captured by $\mathfrak{m}'$. For instance, a process can always execute a join if there is another process at location pc_x (rule *join*). A process executing a multiple assignment atomically updates values of the shared and local variables according to the

values taken by the expressions of the assignment before the execution (rule *assign*). A **Pr**-run ρ starts with a configuration and is an alternating sequence of transitions and configurations $(s_0, m_0); tr_1 \dots; tr_n; (s_n, m_n)$. The run is **Pr-feasible** if $(s_i, m_i) \xrightarrow[\text{Pr}]{tr_{i+1}} (s_{i+1}, m_{i+1})$ for each $i : 0 \leq i < n$ and s_0 and m_0 are initial, i.e., $s_0 \in S_{init}$ and $m_0 \in M_{init}$. Each configuration (s_i, m_i) for $i : 0 \leq i \leq n$ is then said to be *reachable*.

Example 7 (Feasible run). Consider the concurrent program in Figure 6.1. An initial configuration is (s_0, m_0) where $s_0 = \{\text{all} \leftarrow 1, \text{arrived} \leftarrow 0, \text{cross} \leftarrow 0, \text{read} \leftarrow 0\}$ and the initial multiset m_0 associates 1 to the unique process state with location pc_0 and 0 to all other process states. A feasible run is then represented in the following.

s				m
all	arrived	cross	read	pc
1	0	0	0	pc ₀
$\Downarrow tr_0.pc_0 \rightarrow pc_0: \text{spawn}$				
1	0	0	0	pc ₀ , pc ₀
$\Downarrow tr_0.pc_0 \rightarrow pc_0: \text{spawn}$				
1	0	0	0	pc ₀ , pc ₀ , pc ₀
$\Downarrow tr_1.pc_0 \rightarrow pc_1: \text{cross} = 0; \text{all} := \text{all} + 1$				
2	0	0	0	pc ₀ , pc ₀ , pc ₁
$\Downarrow tr_2.pc_1 \rightarrow pc_2: \text{read} := 1$				
2	0	0	1	pc ₀ , pc ₀ , pc ₂
$\Downarrow tr_3.pc_2 \rightarrow pc_3: \text{read} := 0$				
2	0	0	0	pc ₀ , pc ₀ , pc ₃

Counting Predicates. Recall that PC is the set of program locations. We make use of a set of Boolean variables $@PC = \{@pc \mid pc \in PC\}$. Intuitively, a process evaluates $@pc$ to **true** only when it is at location pc . This way, we can build Boolean expressions in the set $\text{predsOf}_{\text{exprsOf}(Sh \cup L)}^{@PC}$. With these predicates we can state facts about both the location of a process, as well as its local variables and the values of the shared variables. For instance, at the fourth step of the run depicted in Example 7, there is one process for which $(@pc_2 \wedge \text{all} \geq 1)$ holds.

We associate a *counting variable* $(\pi)^\#$ to each predicate π in $\text{predsOf}_{\text{exprsOf}(Sh \cup L)}^{@PC}$. Intuitively, in a given program configuration, the variable $(\pi)^\#$ counts the number of processes for which the predicate π holds. We denote the counting variables $\{(\pi)^\# \mid \pi \in \text{predsOf}_{\text{exprsOf}(Sh \cup L)}^{@PC}\}$ with $\Omega_{PC, Sh, L}$. For example, $(@pc_2 \wedge \text{all} \geq 1)^\#$ is a variable that counts the number of processes for which $(@pc_2 \wedge \text{all} \geq 1)$ holds. Such a counting variable is evaluated with respect to a shared state s and a process state (pc, l) . We abuse notation and write $v[s, (pc, l)]$ to mean that the variable v participating in a counting variable, is evaluated to $s(v)$ if $v \in Sh$, to $l(v)$ if $v \in L$, or to $(pc = pc')$ if v is the Boolean variable $@pc'$.

Any predicate in $\text{predsOf}_{\text{exprsOf}(\text{Sh} \cup \Omega_{\text{pc}, \text{sh}, \text{L}})}$ is a *counting predicate*. We need a shared state s and a group configuration m in order to evaluate a counting variable (Equation 6.1) or a counting predicate (Equation 6.2). We abuse notation and write $\omega[\text{s}, \text{m}]$ to mean the evaluation of the counting predicate ω with respect to a configuration (s, m) . The evaluation is performed as follows. Given a configuration (s, m) , a shared variable $\text{sh} \in \text{Sh}$ is evaluated as usual to $\text{sh}[\text{s}, \text{m}] = \text{s}(\text{sh})$; whereas the counting variable $(\pi)^\#$ is evaluated to the number of processes satisfying π in (s, m) .

$$(\pi)^\#[\text{s}, \text{m}] = \sum_{\{(\text{pc}, \text{l}) \text{ s.t. } \pi[\text{s}, (\text{pc}, \text{l})]\}} \text{m}((\text{pc}, \text{l})) \quad (6.1)$$

$$\omega[\text{s}, \text{m}] = \omega[\text{s}, \{(\pi)^\# \leftarrow (\pi)^\#[\text{s}, \text{m}] | (\pi)^\# \in \text{varsOf}(\omega)\}] \quad (6.2)$$

Our counting predicates are quite expressive. For instance, we can capture assertion violations, deadlocks or rich program invariants with them (see Example 8). For any pc , we can define a counting predicate $\text{isEnabled}(\text{pc})$ that captures whether a process currently at location pc can fire some transition. For instance, in the running example of Figure 6.1, $\text{isEnabled}(\text{pc}_0) = \text{true}$ and $\text{isEnabled}(\text{pc}_4) = (\text{arrived} = \text{all})$. If there would have been only one transition from pc_6 consisting of a join operation, then $\text{isEnabled}(\text{pc}_6) = (@\text{pc}_x \geq 1)^\#$.

Example 8 (Counting predicates). The following counting predicates capture configurations of the program of Figure 6.1: ω_1 captures configurations that violate the assertion, ω_2 captures those in which a deadlock occurs, and ω_3 captures an over-approximation of the reachable configurations (i.e., an invariant).

$$\omega_1 = (@\text{pc}_5 \wedge (\text{read} > 0))^\# \geq 1$$

$$\omega_2 = \bigwedge_{\text{pc} \in \text{PC}} (@\text{pc} \wedge \text{isEnabled}(\text{pc}))^\# = 0$$

$$\omega_3 = (\text{all} = \sum_{i \geq 1} (@\text{pc}_i)^\#) \wedge (\text{arrived} = \sum_{i \geq 4} (@\text{pc}_i)^\#)$$

6.5 Relating abstraction layers

In the following, we formally describe the four steps involved in our predicated constrained monotonic abstraction approach (see Figure 6.2).

6.5.1 Predicate abstraction

Predicate abstraction [67] was introduced as a way of dealing with state explosion. In this approach, the state space of a program is approximated via the

```

bool read_leq_0, arrived_leq_all := true, true

process:
tr0b. pc0 → pc0 : spawn
tr1b. pc0 → pc1 : assume(true); arrived_leq_all := ch(arrived_leq_all, false)
tr2b. pc1 → pc2 : read_leq_0 := false
tr3b. pc2 → pc3 : read_leq_0 := true
tr4b. pc3 → pc4 : arrived_leq_all := ch(false, ¬arrived_leq_all)
tr5b. pc4 → pc5 : assume(arrived_leq_all); assume(true)
...

```

Figure 6.5: Predicate abstraction of the program in Figure 6.1 with respect to the predicates $\Pi = \{\text{read_leq_0}, \text{arrived_leq_all}\}$. $\text{ch}(\pi, \pi')$ operator evaluates to **true** if π evaluates to **true**, to **false** if π' evaluates to **true**, and non-deterministically to either **true** or **false**, otherwise.

values of a finite set of predicates over the program variables while maintaining the control structure of the program. The first algorithm for automatizing predicate abstraction of software programs was introduced in [14] and was a key enabling technology for applying finite-state model checkers to programs written in mainstream languages. The work in [51] extended predicate abstraction to shared-variable concurrent software. The authors have developed a symmetry-aware predicate abstraction strategy. The approach takes into account the replicated structure of C programs that consist of many threads executing the same procedure, and generates a Boolean program template whose multi-threaded execution soundly over-approximates the concurrent C program. State explosion during model checking parallel instantiations of this template can then be absorbed by exploiting symmetry. Predicate abstraction is usually combined with a CEGAR loop to add the required predicates and refine the abstraction.

Given a program $\text{Pr} = (\text{Sh}, \text{L}, \text{Tr})$ and a number of predicates Π on the variables $\text{Sh} \cup \text{L}$, we leverage the technique in [51] in order to generate an abstraction in the form of a Boolean program $\text{abst}_\Pi(\text{Pr}) = (\text{Sh}^b, \text{L}^b, \text{Tr}^b)$ where all shared and local variables are Boolean.

To achieve this, Π is partitioned into three sets Π_{shr} , Π_{loc} and Π_{mix} . Predicates in Π_{shr} only mention variables in Sh and those in Π_{loc} only mention variables in L . Predicates in Π_{mix} mention both shared and local variables of Pr . A bijection associates a predicate $\text{originOf}(v^b)$ in Π_{shr} (respectively, $\Pi_{mix} \cup \Pi_{loc}$) to each Boolean variable v^b in Sh^b (respectively, L^b). The function is a bijection as each predicate will be associated to only one Boolean variable (that tracks the value of that predicate) and vice versa.

Example 9 (Predicate abstraction). Consider the concurrent program in Figure 6.1. We implemented the predicate abstraction of [51] which results, for the predicates $\Pi = \{\text{read_leq_0}, \text{arrived_leq_all}\}$, in the Boolean

program of Figure 6.5. A bijection associates the predicates `arrived` $\leq$ `all` and `read` $\leq$ 0 in Π_{shr} respectively, to the Boolean variables `arrived_leq_all` and `read_leq_0` in Sh^b of the Boolean program.

In addition, there are as many transitions in Tr as in Tr^b . For each $(pc \rightarrow pc' : \text{stmt})$ in Tr there is a corresponding $(pc \rightarrow pc' : \text{abst}_{\Pi}(\text{stmt}))$ with the same source and destination locations pc and pc' , but with an abstracted statement $\text{abst}_{\Pi}(\text{stmt})$ that may operate on the variables in $\text{Sh}^b \cup \text{L}^b$. Moreover, abstracted statements can mention the local variables of passive processes, i.e., processes other than the one executing the transition. For this, we make use of the variables $\text{L}_p^b = \{1_p^b \mid 1^b \text{ in } \text{L}^b\}$ where each 1_p^b denotes local variable 1^b of passive processes. We use passive variables to capture broadcasts where local variables of all passive processes need to be updated. Note that such passive variables and broadcast transitions do not exist in the original concurrent programs to be verified, but are introduced after predicate abstraction of those programs as presented in [51]. They are essential to capture the behavior of the processes that exist in the system other than the process that actually executes a transition (respectively, passive and active processes).

Example 10 (Broadcast transition). Consider a concurrent program with shared and local variables sh and 1 , the assignment transition $\text{tr}_0 :: pc_1 \rightarrow pc_2 : \text{sh} := 1$ and the mixed predicate $\text{mx} :: (\text{sh} = 1)$ to be used for the predicate abstraction. Recall from Section 6.5.1 that such predicates are called *mixed predicates* as they contain both local and shared variables. Similar to local predicates, each process will have its own copy of a mixed predicate. However, unlike local predicates, mixed predicates are updated only by broadcast transitions. Before the assignment in tr_0 is executed, consider two passive processes P_1 and P_2 having mixed predicates $\text{mx}_{p_1} :: (\text{sh} = 1_{p_1})$ evaluating to **true** and $\text{mx}_{p_2} :: (\text{sh} = 1_{p_2})$ evaluating to **false** and an active process P_a having $\text{mx}_{p_a} :: (\text{sh} = 1_a)$ evaluating to **true**. The active process will execute tr_0 , hence, $\text{sh} = 1_a$ will hold after the transition. At this point, mx_{p_1} will hold only if $1_a = 1_{p_1}$. But, mx_{p_2} will *not* hold. In fact, after the transition, all passive processes will be notified to update their corresponding mixed predicate with respect to their own valuation and that of the active process. This corresponds to a broadcast (More details in Example 11).

Syntax and semantics of Boolean programs. We describe the syntax of Boolean programs in Figure 6.6. Variables sh^b and 1^b are some variables respectively, in Sh^b and L^b . Variables $v_1^b, \dots, v_n^b$ are pairwise different and belong to $\text{Sh}^b \cup \text{L}^b \cup \text{L}_p^b$. Predicate π is in $\text{predsOf}^{\text{Sh}^b \cup \text{L}^b}$ (i.e., a Boolean combination of the Boolean variables in $\text{Sh}^b \cup \text{L}^b$) and predicates π_i, π'_i are in $\text{predsOf}^{\text{Sh}^b \cup \text{L}^b \cup \text{L}_p^b}$. By construction, the predicate abstraction of concurrent programs [51] does not involve assignments of passive variables to non-passive ones.

$$\begin{aligned}
\text{prog} &::= (\text{sh}^b := (\text{true} \mid \text{false} \mid *))^* \\
&\quad \text{process :} \\
&\quad (\text{l}^b := (\text{true} \mid \text{false} \mid *))^* \\
&\quad (\text{pc} \rightarrow \text{pc} : \text{stmt})^+ \\
\text{stmt} &::= \text{v}_1^b, \dots, \text{v}_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n) \\
&\quad \text{spawn} \mid \text{join} \mid \text{assume}(\pi) \mid \text{stmt}; \text{stmt}
\end{aligned}$$

Figure 6.6: Syntax of concurrent Boolean programs

We describe the semantics of Boolean programs in Figure 6.7. We add the superscript b to mean the Boolean program variant. For instance, we use $\mathbb{S}^b, \mathbb{L}^b$ and $\mathbb{M}^b$ to respectively mean the sets of shared states, local states and group configurations of Boolean programs. The $\text{ch}(\pi, \pi')$ operator evaluates to **true** if π evaluates to **true**, to **false** if π' evaluates to **true**, and non-deterministically to either **true** or **false**, otherwise. Apart from the fact that all variables are now Boolean and that we make use of the **ch** operator, the main difference between Figure 6.7 and Figure 6.4 is the **assign** statement as it may involve passive variables in order to capture broadcasts.

Example 11 (**ch** operator). Consider once again Example 10. Using **mx**, the assignment transition tr_0 will be abstracted to $\text{tr}_0^b :: \text{pc}_1 \rightarrow \text{pc}_2 : \text{mx}, \text{mx}_p := \text{true}, \text{ch}(\text{mx}_p \wedge \text{mx}, \text{mx}_p \oplus \text{mx})$ ($\oplus$ is exclusive-or). Based on the semantics of the **ch** operator, the variable mx_p will evaluate to **true** if both mx_p and **mx** held before the assignment, to **false** if $\text{mx}_p \oplus \text{mx}$ held before the assignment, and to a nondeterministic Boolean value otherwise.

Let $\text{v}_1^b, \dots, \text{v}_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)$ be an assignment. When describing the semantics in Figure 6.7, we write $(\text{s}^b, \mathbb{l}^b, \mathbb{l}_p^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{v}_1^b, \dots, \text{v}_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)} (\text{s}^{b'}, \mathbb{l}^{b'}, \mathbb{l}_p^{b'})$ in order to mean that a process other than the one performing the assignment (i.e., a passive process) with local state $\mathbb{l}_p^b$ can move to $\mathbb{l}_p^{b'}$ when the active process moves from $\mathbb{l}^b$ to $\mathbb{l}^{b'}$ and the shared state changes from s^b to $\text{s}^{b'}$. The new shared state and the new local state of the active process are obtained from their older versions according to Equations 6.3 and 6.4. The local state $\mathbb{l}_p^{b'}$ of a passive process is obtained as follows. First, we change the domain of $\mathbb{l}_p^b$ from $\mathbb{L}^b$ to $\mathbb{L}_p^b$ and obtain $\mathbb{l}_{p,1}^b$ as described in Equation 6.5. Then we apply the assignment to obtain $\mathbb{l}_{p,2}^b$ according to Equation 6.6. Finally, we obtain $\mathbb{l}_p^{b'}$ by changing the domain of $\mathbb{l}_{p,2}^b$ from $\mathbb{L}_p^b$ back to $\mathbb{L}^b$. Notice that at least one $\mathbb{l}_p^{b'}$ exists for each $\mathbb{l}_p^b$. Intuitively, this step corresponds to updating when during a broadcast an active process changes local state together with possibly many passive processes.

$$\mathbb{s}^{b'} = \mathbb{s}^b [\{v_i^b \leftarrow \text{ch}(\pi_i, \pi'_i) [\mathbb{s}^b, \mathbb{l}^b] | v_i^b \in \text{Sh}^b \wedge i : 1 \leq i \leq n\}] \quad (6.3)$$

$$\mathbb{l}^{b'} = \mathbb{l}^b [\{v_i^b \leftarrow \text{ch}(\pi_i, \pi'_i) [\mathbb{s}^b, \mathbb{l}^b] | v_i^b \in \mathbb{L}^b \wedge i : 1 \leq i \leq n\}] \quad (6.4)$$

$$\mathbb{l}_{p,1}^b = \{1_p^b \leftarrow \mathbb{l}_p^b(1^b) | 1^b \in \mathbb{L}^b\} \quad (6.5)$$

$$\mathbb{l}_{p,2}^b = \mathbb{l}_{p,1}^b [\{v_i^b \leftarrow \text{ch}(\pi_i, \pi'_i) [\mathbb{s}^b, \mathbb{l}^b, \mathbb{l}_{p,1}^b] | v_i^b \in \mathbb{L}_p^b \wedge i : 1 \leq i \leq n\}] \quad (6.6)$$

Example 12 (Updating a passive local state). Consider once again Example 10, Example 11, the active local state $\mathbb{l}^b = \{\text{mx} \leftarrow \text{false}\}$ and a passive local state $\mathbb{l}_p^b = \{\text{mx}_p \leftarrow \text{true}\}$. The assignment in the Boolean transition tr_0^b involves both variables mx and mx_p . We use $\mathbb{l}^b$ and $\mathbb{l}_p^b$ for evaluating mx and mx_p , respectively; but the domain of the passive local state $\mathbb{l}_p^b$ is $\mathbb{L}^b$ based on the syntax of Boolean concurrent programs. We need to change its domain to $\mathbb{L}_p^b$ in order to be able to distinguish between passive and non-passive variables. By doing this, we obtain $\mathbb{l}_{p,1}^b = \{\text{mx}_p \leftarrow \text{true}\}$. Then, we update $\mathbb{l}_{p,1}^b$ by the result of the assignment and will have $\mathbb{l}_{p,2}^b = \{\text{mx}_p \leftarrow \text{false}\}$. Finally, we obtain $\mathbb{l}^{b'}$ by changing the domain of $\mathbb{l}_{p,2}^b$ from $\mathbb{L}_p^b$ back to $\mathbb{L}^b$ as $\mathbb{l}^{b'} = \{\text{mx} \leftarrow \text{false}\}$.

An $\text{abst}_{\Pi}(\text{Pr})$ -run $(\mathbb{s}_0^b, \mathbb{m}_0^b); \text{tr}_1^b; \dots; \text{tr}_n^b; (\mathbb{s}_n^b, \mathbb{m}_n^b)$ is a sequence of alternating transitions and configurations starting with a configuration. This run is considered *feasible* if we have that $(\mathbb{s}_i^b, \mathbb{m}_i^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{tr}_{i+1}^b} (\mathbb{s}_{i+1}^b, \mathbb{m}_{i+1}^b)$ for each $i : 0 \leq i < n$ and $\mathbb{s}_0^b, \mathbb{m}_0^b$ are in $\mathbb{S}_{init}^b$ and $\mathbb{M}_{init}^b$, respectively. Configurations $(\mathbb{s}_i^b, \mathbb{m}_i^b)$, for $i : 0 \leq i \leq n$, are then said to be reachable.

Evaluating (counting) predicates in $\text{abst}_{\Pi}(\text{Pr})$. Given a shared state $\mathbb{s}^b$, we write $\text{originOf}(\mathbb{s}^b)$ to mean the predicate $\bigwedge_{\text{sh}^b \in \text{Sh}^b} (\mathbb{s}^b(\text{sh}^b) \Leftrightarrow \text{originOf}(\text{sh}^b))$. We write $\text{originOf}(\mathbb{l}^b)$ to mean $\bigwedge_{1^b \in \mathbb{L}^b} (\mathbb{l}^b(1^b) \Leftrightarrow \text{originOf}(1^b))$. Observe that $\text{varsOf}(\text{originOf}(\mathbb{s}^b)) \subseteq \text{Sh}$ and the variables $\text{varsOf}(\text{originOf}(\mathbb{l}^b)) \subseteq \text{Sh} \cup \mathbb{L}$. We abuse notation and write $\mathbb{s}^b[\mathbb{s}]$ (respectively, $\mathbb{l}^b[\mathbb{s}, \mathbb{l}]$) to mean that $\text{originOf}(\mathbb{s}^b)[\mathbb{s}]$ (respectively, $\text{originOf}(\mathbb{l}^b)[\mathbb{s}, \mathbb{l}]$) holds.

Let π be a predicate in $\text{predsOf}^{\text{@PC} \cup \Pi}$, where all Boolean variables are either predicates in Π or of the form @pc for some $\text{pc} \in \text{PC}$. We write $\pi[\mathbb{s}^b, (\text{pc}, \mathbb{l}^b)]$ to mean the Boolean value obtained by evaluating the result of replacing each Boolean variable $\text{@pc}'$ with true if $\text{pc} = \text{pc}'$ and with false otherwise, and by replacing each π' in Π with $\mathbb{l}^b(v^b)$ or $\mathbb{s}^b(v^b)$ where $\text{originOf}(v^b) = \pi'$.

We can now evaluate a counting variable $(\pi)^\#$, with $\pi \in \text{predsOf}^{\text{@PC} \cup \Pi}$, with respect to a configuration $(\mathbb{s}^b, \mathbb{m}^b)$. We do this by counting the number of process states $(\text{pc}, \mathbb{l}^b)$ in $\mathbb{m}^b$ for which $\pi[\mathbb{s}^b, (\text{pc}, \mathbb{l}^b)]$ holds (see Equation 6.7). We can also replace each counting variable $(\pi)^\#$ in a counting predicate

$$\begin{array}{c}
 \frac{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{stmt}} (\mathbb{s}^{b'}, \mathbb{l}^{b'}, \mathbb{m}^{b'})}{(\mathbb{s}^b, (\text{pc}, \mathbb{l}^b) \boxplus \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{(\text{pc} \rightarrow \text{pc}' : \text{stmt})} (\mathbb{s}^{b'}, (\text{pc}', \mathbb{l}^{b'}) \boxplus \mathbb{m}^{b'})} : \text{transition} \\
 \\
 \frac{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{stmt}} (\mathbb{s}^{b'}, \mathbb{l}^{b'}, \mathbb{m}^{b'}) \quad (\mathbb{s}^{b'}, \mathbb{l}^{b'}, \mathbb{m}^{b'}) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{stmt}'} (\mathbb{s}^{b''}, \mathbb{l}^{b''}, \mathbb{m}^{b''})}{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{stmt}; \text{stmt}'} (\mathbb{s}^{b''}, \mathbb{l}^{b''}, \mathbb{m}^{b''})} : \text{sequence} \\
 \\
 \frac{\begin{array}{l} \mathbb{m} = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle \quad \mathbb{m}^b = \langle \sigma'_1, \sigma'_2, \dots, \sigma'_n \rangle \\ \text{a bijection } h : [1, n] \rightarrow [1, n] \text{ s.t.} \\ \forall i \in [1, n]. \sigma_i = (\text{pc}_p, \mathbb{l}_p^b) \quad \sigma'_{h(i)} = (\text{pc}_p', \mathbb{l}_p^{b'}) \quad \text{pc}_p = \text{pc}_p' \\ (\mathbb{s}^b, \mathbb{l}^b, \mathbb{l}_p^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{v_1^b, \dots, v_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)} (\mathbb{s}^{b'}, \mathbb{l}^{b'}, \mathbb{l}_p^{b'}) \end{array}}{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{v_1^b, \dots, v_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)} (\mathbb{s}^{b'}, \mathbb{l}^{b'}, \mathbb{m}^{b'})} : \text{assign} \\
 \\
 \frac{\pi[\mathbb{s}^b, \mathbb{l}^b] = \text{true}}{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{assume}(\pi)} (\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b)} : \text{assume} \\
 \\
 \frac{\mathbb{m}^{b'} = ((\text{pc}_0, \mathbb{l}_{init}^b) \boxplus \mathbb{m}^b) \quad \mathbb{l}_{init}^b \in \mathbb{L}_{init}^b}{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{spawn}} (\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^{b'})} : \text{spawn} \\
 \\
 \frac{\mathbb{m}^b = ((\text{pc}_x, \mathbb{l}^{b'}) \boxplus \mathbb{m}^{b'})}{(\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{\text{join}} (\mathbb{s}^b, \mathbb{l}^b, \mathbb{m}^{b'})} : \text{join}
 \end{array}$$

Figure 6.7: Semantics of Boolean concurrent programs. Executions start from some $\mathbb{m}_{init}^b$ in $\mathbb{M}_{init}^b$.

ω with its value in (s^b, m^b) , and each shared predicate in Π with its value in s^b (see Equation 6.8). Observe that the obtained expression might still involve shared variables as these can participate in comparisons with counting variables. Such comparisons do not correspond to any predicate in Π and cannot therefore be mapped.

$$(\pi)^\# [s^b, m^b] = \sum_{\{(\text{pc}, \mathbb{I}^b) \mid \pi[s^b, (\text{pc}, \mathbb{I}^b)]\}} m^b((\text{pc}, \mathbb{I}^b)) \quad (6.7)$$

$$\omega[s^b, m^b] = \omega[\{(\pi)^\# \leftarrow (\pi)^\# [s^b, m^b] \mid (\pi)^\# \in \text{varsOf}(\omega)\}] \quad (6.8)$$

Relation between Pr and $\text{abst}_\Pi(\text{Pr})$. Given the group configurations $m = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle$ to $m^b = \langle \sigma'_1, \sigma'_2, \dots, \sigma'_n \rangle$, we let $m^b[s, m]$ mean that there is a bijection h from $[1, n]$ to $[1, n]$ such that for each i in $[1, n]$, if $\sigma_i = (\text{pc}, \mathbb{I})$ then $\sigma'_{h(i)} = (\text{pc}, \mathbb{I}^b)$ and $\mathbb{I}^b[s, \mathbb{I}]$. The *concretization* of an $\text{abst}_\Pi(\text{Pr})$ configuration (s^b, m^b) is $\gamma((s^b, m^b)) = \{(s, m) \mid s^b[s] \wedge m^b[s, m]\}$. The *abstraction* of (s, m) is the singleton $\alpha((s, m)) = \{(s^b, m^b) \mid s^b[s] \wedge m^b[s, m]\}$. We initialize variables in $\text{abst}_\Pi(\text{Pr})$ such that for each pair $(s_{\text{init}}, m_{\text{init}})$ of Pr , there are $(s_{\text{init}}^b, m_{\text{init}}^b)$ that are initial in $\text{abst}_\Pi(\text{Pr})$ so that $\alpha((s_{\text{init}}, m_{\text{init}})) = \{(s_{\text{init}}^b, m_{\text{init}}^b)\}$. The abstraction $\alpha(\rho)$ of a Pr -run $\rho = (s_0, m_0); \text{tr}_1; \dots; \text{tr}_n; (s_n, m_n)$ is the set of $\text{abst}_\Pi(\text{Pr})$ -runs $\{(s_0^b, m_0^b); \text{tr}_1^b; \dots; \text{tr}_n^b; (s_n^b, m_n^b)\}$ where $\alpha((s_i, m_i)) = \{(s_i^b, m_i^b)\}$ and $\text{tr}_i^b = \text{abst}_\Pi(t_i)$. Concretizations of abstract runs are defined in a straightforward manner.

Example 13. Consider the program in Figure 6.1, its corresponding abstraction with respect to the set of predicates $\Pi = \{\text{read_leq_0}, \text{arrived_leq_all}\}$ in Figure 6.5, and the feasible run in Example 7. The initial shared state s_0 defined as $\{\text{all} \leftarrow 1, \text{arrived} \leftarrow 0, \text{cross} \leftarrow 0, \text{read} \leftarrow 0\}$ in the original program will be encoded as the Boolean shared state $s_0^b = \{\text{read_leq_0} \leftarrow \text{true}, \text{arrived_leq_all} \leftarrow \text{true}\}$ in the Boolean program. We have that $\text{originOf}(s_0^b) = (((\text{read} \leq 0) = \text{true}) \wedge ((\text{arrived} \leq \text{all}) = \text{true}))$. Since there are no local variables in this example, we get that $\alpha((s_0, m_0)) = \{(s_0^b, m_0)\}$. Many other states have the same encoding, e.g. $s_1 = \{\text{all} \leftarrow 10, \text{arrived} \leftarrow 2, \text{cross} \leftarrow 0, \text{read} \leftarrow 0\}$ which satisfies $s_0^b[s_1]$, although it is not initial in Pr . We get that $\gamma((s_0^b, m_0)) = \{(s_0, m_0), (s_1, m_0), \dots\}$.

Definition 11. Let the abstraction of the program $\text{Pr} = (\text{Sh}, \text{L}, \text{Tr})$ with respect to Π be the Boolean program $\text{abst}_\Pi(\text{Pr}) = (\text{Sh}^b, \text{L}^b, \text{Tr}^b)$. The abstraction is said to be *effective and sound* if $\text{abst}_\Pi(\text{Pr})$ can be effectively computed and the abstract run in the singleton $\alpha(\rho)$ of any feasible Pr run ρ is $\text{abst}_\Pi(\text{Pr})$ -feasible.

$$\begin{array}{c}
 \frac{\mathbf{t} = (\mathbf{q} : \mathbf{op} : \mathbf{q}') \quad \mathbb{C} \xrightarrow[\mathbf{M}]{\mathbf{op}} \mathbb{C}'}{(\mathbf{q}, \mathbb{C}) \xrightarrow[\mathbf{M}]{\mathbf{t}} (\mathbf{q}', \mathbb{C}')} \quad \text{transition} \\
 \\
 \frac{}{\mathbb{C} \xrightarrow[\mathbf{M}]{\mathbf{nop}} \mathbb{C}} \quad \text{nop} \qquad \frac{\mathbb{C} \xrightarrow[\mathbf{M}]{\mathbf{op}} \mathbb{C}' \quad \mathbb{C}' \xrightarrow[\mathbf{M}]{\mathbf{op}'} \mathbb{C}''}{\mathbb{C} \xrightarrow[\mathbf{M}]{\mathbf{op}:\mathbf{op}'} \mathbb{C}''} \quad \text{sequence} \\
 \\
 \frac{\exists \mathbf{Ax}. \mathbf{grd}[\mathbb{C}] \wedge \mathbb{C}' = \mathbb{C}[\{\mathbf{ct}_i \leftarrow \mathbf{e}_i[\mathbb{C}] \mid i : 1 \leq i \leq n\}]}{\mathbb{C} \xrightarrow[\mathbf{M}]{\mathbf{grd} \Rightarrow (\mathbf{ct}_1, \dots, \mathbf{ct}_n := \mathbf{e}_1, \dots, \mathbf{e}_n)} \mathbb{C}'} \quad \begin{array}{l} \text{guarded} \\ \text{command} \end{array}
 \end{array}$$

Figure 6.8: Semantics of an extended counter machine. A guard **grd** is a predicate in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathcal{C} \cup \mathbf{Ax})}$ in which **Ax** is a set of auxilliary variables used for a broadcast operation.

6.5.2 Encoding into an Extended Counter Machine

Assume a program $\mathbf{Pr} = (\mathbf{Sh}, \mathbf{L}, \mathbf{Tr})$, a set of predicates $\Pi_0 \subseteq \mathbf{predsOf}_{\mathbf{exprsOf}(\mathbf{Sh} \cup \mathbf{L})}$ and two counting predicates, an invariant ω_{inv} in the set $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathbf{Sh} \cup \Omega_{\mathbf{PC}, \mathbf{Sh}, \mathbf{L}})}$ and a target predicate ω_{tgt} in $\mathbf{predsOf}_{\mathbf{exprsOf}(\Omega_{\mathbf{PC}, \mathbf{Sh}, \mathbf{L}})}$. In the following, we write $\mathbf{abst}_{\Pi}(\mathbf{Pr}) = (\mathbf{Sh}^b, \mathbf{L}^b, \mathbf{Tr}^b)$ to mean the abstraction of $\mathbf{Pr}$ with respect to the predicates Π defined as

$$\Pi = \cup_{(\pi) \# \in \mathbf{varsOf}(\omega_{inv}) \cup \mathbf{varsOf}(\omega_{tgt})} \mathbf{atomsOf}(\pi) \cup \Pi_0$$

Intuitively, this step results in the formulation of a state reachability problem of an extended counter machine $\mathit{enc}(\mathbf{abst}_{\Pi}(\mathbf{Pr}))$ that captures reachability of abstractions of ω_{tgt} configurations with $\mathbf{abst}_{\Pi}(\mathbf{Pr})$ runs that take into account the invariant ω_{inv} . An extended counter machine $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{init}, \mathbb{C}_{init}, \mathbf{q}_{tgt})$ is a tuple $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{init}, \mathbb{C}_{init}, \mathbf{q}_{tgt})$. A transition $\mathbf{t}$ in $\mathbf{T}$ is of the form $(\mathbf{q} : \mathbf{op} : \mathbf{q}')$ where $\mathit{src}(\mathbf{t}) = \mathbf{q}$ is the source state, $\mathit{dst}(\mathbf{t}) = \mathbf{q}'$ is the destination state, and the operation $\mathbf{op}$ is either the identity operation **nop**, a guarded command $\mathbf{grd} \Rightarrow \mathbf{cmd}$, or a sequential composition of operations. We use a set **Ax** of auxiliary variables ranging over $\mathbb{N}$. These are meant to be existentially quantified when firing the transitions as explained in the guarded command rule in Figure 6.8 which explains the semantics of counter machines. A guard **grd** is a predicate in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathcal{C} \cup \mathbf{Ax})}$ and a command **cmd** is a multiple assignment $\mathbf{ct}_1, \dots, \mathbf{ct}_n := \mathbf{e}_1, \dots, \mathbf{e}_n$ that involves $\mathbf{e}_1, \dots, \mathbf{e}_n$ in $\mathbf{exprsOf}(\mathcal{C} \cup \mathbf{Ax})$ and pairwise different $\mathbf{ct}_1, \dots, \mathbf{ct}_n$ in $\mathcal{C}$.

Encoding. We describe in the following an extended counter machine $\mathit{enc}(\mathbf{abst}_{\Pi}(\mathbf{Pr}))$ obtained as an encoding of the Boolean program $\mathbf{abst}_{\Pi}(\mathbf{Pr})$.

$$\begin{array}{c}
 \frac{t = (\text{pc} \rightarrow \text{pc}' : \text{stmt}_\Pi) \quad \left[(\text{s}^b, \mathbb{l}^b) : \text{op} : (\text{s}^{b'}, \mathbb{l}^{b'}) \right]_{\text{stmt}_\Pi}}{(\text{q}_{\text{s}^b} : \text{ct}_{(\text{pc}, \mathbb{l}^b)} \geq 1; \text{ct}_{(\text{pc}, \mathbb{l}^b)} := \text{ct}_{(\text{pc}, \mathbb{l}^b)} - 1; \text{op}; \text{ct}_{(\text{pc}', \mathbb{l}^{b'})} := \text{ct}_{(\text{pc}', \mathbb{l}^{b'})} + 1 : \text{q}_{\text{s}^{b'}}) \in \mathbb{T}_t} : \text{transition} \\
 \\
 \frac{}{(\text{q}_{\text{s}^b} : \omega_{\text{trgt}} [\mathbb{V}_{\text{s}^b, \omega_{\text{trgt}}}] : \text{q}_{\text{trgt}}) \in \mathbb{T}_{\text{trgt}}} : \text{target} \\
 \\
 \frac{\left[(\text{s}^b, \mathbb{l}^b) : \text{op} : (\text{s}^{b'}, \mathbb{l}^{b'}) \right]_{\text{stmt}} \quad \left[(\text{s}^{b'}, \mathbb{l}^{b'}) : \text{op}' : (\text{s}^{b''}, \mathbb{l}^{b''}) \right]_{\text{stmt}'}}{\left[(\text{s}^b, \mathbb{l}^b) : \text{op}; \text{op}' : (\text{s}^{b''}, \mathbb{l}^{b''}) \right]_{\text{stmt}; \text{stmt}'}} : \text{sequence} \\
 \\
 \frac{\pi[\text{s}^b, \mathbb{l}^b]}{\left[(\text{s}^b, \mathbb{l}^b) : \text{nop} : (\text{s}^b, \mathbb{l}^b) \right]_\pi} : \text{assume} \\
 \\
 \frac{}{\left[(\text{s}^b, \mathbb{l}^b) : \text{ct}_{(\text{pc}_0, \mathbb{l}^{b_{\text{init}}})} := \text{ct}_{(\text{pc}_0, \mathbb{l}^{b_{\text{init}}})} + 1 : (\text{s}^b, \mathbb{l}^b) \right]_{\text{spawn}}} : \text{spawn} \\
 \\
 \frac{}{\left[(\text{s}^b, \mathbb{l}^b) : \text{ct}_{(\text{pc}_x, \mathbb{l}^{b'})} \geq 1; \text{ct}_{(\text{pc}_x, \mathbb{l}^{b'})} := \text{ct}_{(\text{pc}_x, \mathbb{l}^{b'})} - 1 : (\text{s}^b, \mathbb{l}^b) \right]_{\text{join}}} : \text{join} \\
 \\
 \frac{\text{tf} = \left\{ (\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \mid (\text{s}^b, \mathbb{l}^b, \mathbb{l}_p^b) \xrightarrow{\text{v}_1^b, \dots, \text{v}_n^b := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)} (\text{s}^{b'}, \mathbb{l}^{b'}, \mathbb{l}_p^{b'}) \right\}}{\text{tf} \neq \emptyset} : \text{assign} \\
 \\
 \frac{}{\text{op} = \left(\begin{array}{l} \bigwedge_{\text{pc} \in \text{PC}, \mathbb{l}_p^b \in \mathbb{L}^b} \left(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)} = \sum_{\{(\text{pc}, \mathbb{l}_p^{b'}) \mid (\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}\}} a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})} \right) \Rightarrow \\ \text{cmdOf} \left(\left\{ \text{ct}_{(\text{pc}, \mathbb{l}_p^{b'})} \leftarrow \sum_{\{(\text{pc}, \mathbb{l}_p^b) \mid (\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}\}} a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})} \mid \text{pc} \in \text{PC}, \mathbb{l}_p^b \in \mathbb{L}^b \right\} \right) \end{array} \right)} : \text{assign} \\
 \\
 \left[(\text{s}^b, \mathbb{l}^b) : \text{op} : (\text{s}^{b'}, \mathbb{l}^{b'}) \right]_{\text{assign}}
 \end{array}$$

Figure 6.9: Encoding of the transitions of a Boolean program $(\text{Sh}^b, \text{L}^b, \text{Tr}^b)$, given a counting target ω_{trgt} , to the transitions $\mathbb{T} = \cup_{\text{tr} \in \text{Tr}^b} \mathbb{T}_{\text{tr}} \cup \mathbb{T}_{\text{trgt}}$ of an extended counter machine. In rule *assign*, we write $\text{cmdOf} \left(\left\{ \text{ct}_{(\text{pc}, \mathbb{l}^{b'})} \leftarrow \text{e}_{(\text{pc}, \mathbb{l}^{b'})} \mid \text{pc} \in \text{PC}, \mathbb{l}^{b'} \in \mathbb{L}^b \right\} \right)$ to mean the multiple assignment that simultaneously assigns each $\text{e}_{(\text{pc}, \mathbb{l}^b)}$ to $\text{ct}_{(\text{pc}, \mathbb{l}^b)}$. A transition $\mathbb{C}_1 \xrightarrow[\text{abst}_\Pi(\text{Pr})]{\text{grd} \Rightarrow \text{cmd}} \mathbb{C}_2$ ensures that there is a mapping $\mathbb{C} : \text{Ax} \rightarrow \mathbb{N}$ from the set of auxiliary variables Ax to the set of natural numbers such that for any $\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}$ we have $\mathbb{C}_1(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) = \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{C}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})})$ and for any $\text{ct}_{(\text{pc}, \mathbb{l}_p^{b'})}$ we have $\mathbb{C}_2(\text{ct}_{(\text{pc}, \mathbb{l}_p^{b'})}) = \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{C}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})})$.

Recall that $\mathbf{abst}_\Pi(\mathbf{Pr})$ results from the predicate abstraction of the concurrent program $\mathbf{Pr}$ with respect to some initial predicates Π_0 as well as all predicates in $\mathbf{atomsOf}(\pi)$ for every counting variable $(\pi)^\#$ in $\mathbf{varsOf}(\omega_{inv})$ and $\mathbf{varsOf}(\omega_{tgt})$. The machine $\mathit{enc}(\mathbf{abst}_\Pi(\mathbf{Pr}))$ is a tuple $(\mathcal{Q}, \mathcal{C}, \mathcal{T}, \mathcal{Q}_{init}, \mathbb{C}_{init}, \mathbf{q}_{tgt})$. A state in $\mathcal{Q}$ is either the target state $\mathbf{q}_{tgt}$ or is associated to a shared configuration $\mathbf{s}^b$ of $\mathbf{abst}_\Pi(\mathbf{Pr})$. We write $\mathbf{q}_{\mathbf{s}^b}$ to make the association explicit. There is a bijection that associates a process configuration $(\mathbf{pc}, \mathbb{l}^b)$ to each counter $\mathbf{ct}_{(\mathbf{pc}, \mathbb{l}^b)}$ in $\mathcal{C}$. The assign rule makes use of auxiliary variables. In case of broadcasts, this rule associates an auxiliary variable $a_{(\mathbf{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})}$ to each possible move from process configuration $(\mathbf{pc}, \mathbb{l}_p^b)$ to $(\mathbf{pc}, \mathbb{l}_p^{b'})$.

We will write $\sum_{\{(\mathbf{pc}, \mathbb{l}^b) \mid \pi[\mathbf{s}^b, (\mathbf{pc}, \mathbb{l}^b)]\}} \mathbf{ct}_{(\mathbf{pc}, \mathbb{l}^b)}$ to mean the sum of all counters $\mathbf{ct}_{(\mathbf{pc}, \mathbb{l}^b)}$ in $\mathcal{C}$ for which $\pi[\mathbf{s}^b, (\mathbf{pc}, \mathbb{l}^b)]$ evaluates to **true**. We can define the mapping $\mathbf{w}_{\mathbf{s}^b, \omega}$ that maps each counting variable appearing in a counting predicate ω to a corresponding sum of counters under $\mathbf{s}^b$. More formally, $\mathbf{w}_{\mathbf{s}^b, \omega}((\pi)^\#) = \sum_{\{(\mathbf{pc}, \mathbb{l}^b) \mid \pi[\mathbf{s}^b, (\mathbf{pc}, \mathbb{l}^b)]\}} \mathbf{ct}_{(\mathbf{pc}, \mathbb{l}^b)}$ for each counting predicate $(\pi)^\#$ in $\mathbf{varsOf}(\omega)$. As a result, $\omega_{tgt}[\mathbf{w}_{\mathbf{s}^b, \omega}]$ is the predicate obtained from ω_{tgt} after replacing all counting variables appearing in it by the corresponding sums of counters. Observe that if the predicate ω is in $\mathbf{predsOf}_{\mathbf{exprsOf}(\Omega_{\mathbf{pc}, \mathbf{Sh}, \mathbf{L}})}$ as it is the case for ω_{tgt} , then $\omega[\mathbf{w}_{\mathbf{s}^b, \omega}]$ does not mention any shared variables in $\mathbf{Sh}$. The target predicate ω_{tgt} does not mention any shared variables, because it is to be evaluated in a configuration of the counter machine where no concrete value for shared variables is available. The predicate $\omega_{tgt}[\mathbf{w}_{\mathbf{s}^b, \omega}]$ in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathcal{C})}$ is used in the target rule of the encoding in Figure 6.9.

The set of transitions $\mathcal{T}$ is exactly the set $\cup_{\mathbf{tr} \in \mathbf{Tr}^b} \mathcal{T}_{\mathbf{tr}} \cup \mathcal{T}_{tgt}$ as described in Figure 6.9. We abuse notation and associate to each statement $\mathbf{stmt}$ appearing in $\mathbf{abst}_\Pi(\mathbf{Pr})$ the set $\mathit{enc}(\mathbf{stmt})$ of tuples $[(\mathbf{s}^b, \mathbb{l}^b) : \mathbf{op} : (\mathbf{s}^{b'}, \mathbb{l}^{b'})]_{\mathbf{stmt}}$ generated in Figure 6.9. Given a group configuration $\mathbf{m}^b$, we write $\mathbb{C}_{\mathbf{m}^b}$ to mean the mapping associating $\mathbf{m}^b((\mathbf{pc}, \mathbb{l}^b))$ to each counter $\mathbf{ct}_{(\mathbf{pc}, \mathbb{l}^b)}$ in $\mathcal{C}$. We let $\mathcal{Q}_{init}$ be the set $\{\mathbf{q}_{\mathbf{s}_0^b} \mid \mathbf{s}_0^b \in \mathbb{S}_{init}^b \text{ of } \mathbf{abst}_\Pi(\mathbf{Pr})\}$, and $\mathbb{C}_{init}$ be the mapping $\{\mathbb{C}_{\mathbf{m}^b} \mid \mathbf{m}^b((\mathbf{pc}_0, \mathbb{l}^b)) = 1 \text{ for a single } \mathbb{l}^b \in \mathbb{L}_{init}^b \text{ in } \mathbf{abst}_\Pi(\mathbf{Pr}) \text{ and } 0 \text{ otherwise}\}$. We associate a program configuration $(\mathbf{s}^b, \mathbf{m}^b)$ to each machine configuration $(\mathbf{q}_{\mathbf{s}^b}, \mathbb{C}_{\mathbf{m}^b})$. The machine $\mathit{enc}(\mathbf{abst}_\Pi(\mathbf{Pr}))$ encodes $\mathbf{abst}_\Pi(\mathbf{Pr})$ as specified in Lemma 26.

We state in Lemma 26 that the reachability problem of the obtained counter machine is equivalent to the reachability in $\mathbf{abst}_\Pi(\mathbf{Pr})$ of Boolean configurations satisfying ω_{tgt} . For this, we make use of Lemma 24 and Lemma 25. Intuitively, these relate executions of the Boolean abstraction to the ones of the counter machine encoding.

Lemma 24 (translation). *For any stmt appearing in the program $\text{abst}_\Pi(\text{Pr})$, $(s^b, \ell^b, m^b) \xrightarrow[\text{abst}_\Pi(\text{Pr})]{\text{stmt}} (s^{b'}, \ell^{b'}, m^{b'})$ iff $\mathbb{C}_{m^b} \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{op}} \mathbb{C}_{m^{b'}}$ for $a \left[(s^b, \ell^b) : \text{op} : (s^{b'}, \ell^{b'}) \right]_{\text{stmt}}$ in $\text{enc}(\text{stmt})$.*

Proof. By induction on the number of atomic statements in stmt . $\square$

Lemma 25 (translation and abstraction). *Any configuration (s^b, m^b) is reachable in $\text{abst}_\Pi(\text{Pr})$ iff $(q_{s^b}, \mathbb{C}_{m^b})$ is reachable in $\text{enc}(\text{abst}_\Pi(\text{Pr}))$.*

Proof. We show that (s^b, m^b) is reachable via a run of length n in $\text{abst}_\Pi(\text{Pr})$ iff $(q_{s^b}, \mathbb{C}_{m^b})$ is reachable via a run of the same length in $\text{enc}(\text{abst}_\Pi(\text{Pr}))$. We proceed by induction on the number of $\text{abst}_\Pi(\text{Pr})$ transitions appearing in the runs. By construction, s_0^b and m_0^b are initial iff $q_{s_0^b}$ and $\mathbb{C}_{m_0^b}$ are also initial. Let (s^b, m^b) be a reachable $\text{abst}_\Pi(\text{Pr})$ configuration and $(q_{s^b}, \mathbb{C}_{m^b})$ be the corresponding $\text{enc}(\text{abst}_\Pi(\text{Pr}))$ configuration. We will not consider runs in $\text{enc}(\text{abst}_\Pi(\text{Pr}))$ that involve T_{tgt} transitions as these lead to error states and not to configurations of the form $(q_{s^b}, \mathbb{C}_{m^b})$. We show $(s^b, m^b) \xrightarrow[\text{abst}_\Pi(\text{Pr})]{(\text{pc} \rightarrow \text{pc}': \text{stmt})} (s^{b'}, m^{b'})$ iff

$$(q_{s^b}, \mathbb{C}_{m^b}) \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{(q_{s^b} : \text{ct}_{(\text{pc}, \ell^b)} \geq 1 ; \text{ct}_{(\text{pc}, \ell^b)} := \text{ct}_{(\text{pc}, \ell^b)} - 1 ; \text{op} ; \text{ct}_{(\text{pc}', \ell^{b'})} := \text{ct}_{(\text{pc}', \ell^{b'})} + 1 ; q_{s^{b'}})} (q_{s^{b'}}, \mathbb{C}_{m^{b'}})$$

for some $\left[(s^b, \ell^b) : \text{op} : (s^{b'}, \ell^{b'}) \right]_{\text{stmt}}$ in $\text{enc}(\text{stmt})$. Semantics of Boolean programs in Figure 6.7 ensure that $(s^b, m^b) \xrightarrow[\text{abst}_\Pi(\text{Pr})]{(\text{pc} \rightarrow \text{pc}': \text{stmt})} (s^{b'}, m^{b'})$ iff $m^b =$

$$(\text{pc}, \ell^b) \oplus m_1^b \text{ and } m^{b'} = (\text{pc}', \ell^{b'}) \oplus m_1^{b'} \text{ and } (s^b, \ell^b, m_1^b) \xrightarrow[\text{abst}_\Pi(\text{Pr})]{\text{stmt}} (s^{b'}, \ell^{b'}, m_1^{b'}).$$

Lemma 24 ensures that this is equivalent to $\mathbb{C}_{m_1^b} \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{op}} \mathbb{C}_{m_1^{b'}}$ for some

$$\left[(s^b, \ell^b) : \text{op} : (s^{b'}, \ell^{b'}) \right]_{\text{stmt}} \text{ in } \text{enc}(\text{stmt}). \text{ Observe that } \mathbb{C}_{m_1^b} \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{op}} \mathbb{C}_{m_1^{b'}}$$

$$\text{is equivalent to } \mathbb{C}_{(\text{pc}, \ell^b) \oplus m_1^b} \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{ct}_{(\text{pc}, \ell^b)} \geq 1 ; \text{ct}_{(\text{pc}, \ell^b)} := \text{ct}_{(\text{pc}, \ell^b)} - 1 ; \text{op} ; \text{ct}_{(\text{pc}', \ell^{b'})} := \text{ct}_{(\text{pc}', \ell^{b'})} + 1} \mathbb{C}_{(\text{pc}', \ell^{b'}) \oplus m_1^{b'}}.$$

$\square$

Lemma 26 (translation reachability). *Target q_{tgt} is $\text{enc}(\text{abst}_\Pi(\text{Pr}))$ reachable iff a configuration (s^b, m^b) is reachable in $\text{abst}_\Pi(\text{Pr})$ such that $\omega_{\text{tgt}}[s^b, m^b]$ holds.*

Proof. Lemma 25 ensures that (s^b, m^b) is $\text{abst}_\Pi(\text{Pr})$ reachable iff $(q_{s^b}, \mathbb{C}_{m^b})$ is $\text{enc}(\text{abst}_\Pi(\text{Pr}))$ reachable. We conclude by observing that $\omega_{\text{tgt}}[s^b, m^b]$ holds iff the evaluation of the target predicate in the counter machine holds, i.e. $\omega_{\text{tgt}}[q_{s^b}, \mathbb{C}_{m^b}]$ holds. $\square$

6.5.3 Encoding precision

We argue in the following that the obtained counter machine results in a monotonic transition system for which the reachability problem is decidable. In fact, predicate abstraction forces monotonicity. For example, in Figure 6.1, transitions tr_4 and tr_5 correspond to a barrier which is non-monotonic. However, the abstracted Boolean program in Figure 6.5 that corresponds to it consists of only monotonic transitions. This happens because the relation between the number of processes in different program locations and the program variables is lost. This corresponds to a loss of precision that makes it impossible to establish correctness of programs such as the one depicted in Figure 6.1. We explain how we retrieve some of that precision by strengthening the abstraction.

Consider the Boolean program $\text{abst}_\Pi(\text{Pr})$ obtained after predicate abstraction. If a configuration $(s^{b'}, m^{b'})$ is obtained from (s^b, m^b) using some transition, then the same transition can obtain a larger configuration (i.e., has the same shared state $s^{b'}$ and more processes at the same process states in $m^{b'}$) than $(s^{b'}, m^{b'})$ from any configuration larger than (s^b, m^b) . Lemma 27 shows that indeed, all transitions in Figure 6.9 (except for the rule *target*) are monotonic with respect to the ordering $\leq$ defined by $(q, c) \leq (q', c')$ iff $q = q'$ and $c \leq c'$ with respect to the component-wise ordering, i.e. $c \leq c'$ iff $c(\text{ct}) \leq c'(\text{ct})$ for each $\text{ct} \in \mathcal{C}$.

Lemma 27 (monotonicity). *Transitions $(q : \text{op} : q')$ generated by all rules in Figure 6.9, except for the target rule are monotonic with respect to $\leq$.*

Proof. Let op be some operation appearing in a generated transition $(q : \text{op} : q')$ of $\text{enc}(\text{abst}_\Pi(\text{Pr}))$. We say that an operation op is monotonic with respect to $\leq$ if for each c_1, c_2, c_3 such that $c_1 \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{op}} c_2$ and $c_1 \leq c_3$ there exists a c_4 such that $c_3 \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{op}} c_4$ and $c_2 \leq c_4$. Observe that $(q : \text{op} : q')$ is monotonic with respect to $\leq$ iff op is monotonic with respect to $\leq$. In addition, observe that if both op and op' are monotonic, then so is $\text{op}; \text{op}'$. It is therefore enough to show monotonicity of nop , $\text{ct} \geq 1$, $\text{ct} := \text{ct} + 1$, $\text{ct} := \text{ct} - 1$ and $\text{grd} \Rightarrow \text{cmd}$. The first four cases are straightforward. We show $\text{grd} \Rightarrow \text{cmd}$ is monotonic. Suppose we are given c_1, c_2, c_3 such that $c_1 \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{grd} \Rightarrow \text{cmd}} c_2$ and $c_1 \leq c_3$. We exhibit a c_4 such that $c_3 \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{grd} \Rightarrow \text{cmd}} c_4$ and $c_2 \leq c_4$. The operation $\text{grd} \Rightarrow \text{cmd}$, resulted from the assign rule in Figure 6.9. This was defined with respect to two pairs (s^b, l^b) and $(s^{b'}, l^{b'})$. We fix these two pairs. By the semantics of counter machines (Figure 6.8) and of the translation of the *assign* statement in Figure 6.9, the fact that $c_1 \xrightarrow[\text{enc}(\text{abst}_\Pi(\text{Pr}))]{\text{grd} \Rightarrow \text{cmd}} c_2$, ensures that there is a mapping $\sigma : A \rightarrow \mathbb{N}$

such that for any $\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}$ we have $\mathbb{C}_1(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) = \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{O}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})})$ and for any $\text{ct}_{(\text{pc}, \mathbb{l}_p^{b'})}$ we have $\mathbb{C}_2(\text{ct}_{(\text{pc}, \mathbb{l}_p^{b'})}) = \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{O}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})})$. Since $\mathbb{C}_1 \leq \mathbb{C}_3$, then for all $\text{ct}_{(\text{pc}, \mathbb{l}_p^b)} \in \mathcal{C}$, we have that $\mathbb{C}_3(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) = \mathbb{C}_1(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) + r_{(\text{pc}, \mathbb{l}_p^b)} = \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{O}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})}) + r_{(\text{pc}, \mathbb{l}_p^b)}$ where $r_{(\text{pc}, \mathbb{l}_p^b)} \geq 0$. The idea is to send these “excedents” along the enabled transfers. Fix such a $(\text{pc}, \mathbb{l}_p^b)$.

By definition of $(s^b, \mathbb{l}_p^b) \xrightarrow[\text{abst}_{\Pi}(\text{Pr})]{v_1, \dots, v_n := \text{ch}(\pi_1, \pi'_1), \dots, \text{ch}(\pi_n, \pi'_n)} (s^{b'}, \mathbb{l}_p^{b'})$ in Section

6.5.1 and of tf in Figure 6.9, we know there is at least one $\mathbb{l}_p^{b'}$ such that $(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}$. We define $\mathbb{C}_4(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) := \sum_{(\mathbb{l}_p^b, \mathbb{l}_p^{b'}) \in \text{tf}} \mathbb{O}'(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})})$ where we have $\mathbb{O}'(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})}) := \mathbb{O}(a_{(\text{pc}, \mathbb{l}_p^b, \mathbb{l}_p^{b'})}) + r_{(\text{pc}, \mathbb{l}_p^b)} \delta_{\mathbb{l}_p^b, \mathbb{l}_p^{b'}}$ with $\delta_{\mathbb{l}_1^b, \mathbb{l}_2^b}$ iff $\mathbb{l}_1^b$ is identical to $\mathbb{l}_2^b$. So, $\mathbb{C}_2(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}) \leq \mathbb{C}_4(\text{ct}_{(\text{pc}, \mathbb{l}_p^b)})$. We repeat the process for each counter $\text{ct}_{(\text{pc}, \mathbb{l}_p^b)}$ in $\mathcal{C}$. This results in a $\mathbb{C}_4$ where $\mathbb{C}_2 \leq \mathbb{C}_4$ and for which the same transition (i.e. assign for the two pairs $(s^b, \mathbb{l}_p^b)$ and $(s^{b'}, \mathbb{l}_p^{b'})$) is possible using the mapping $\mathbb{O}'$. $\square$

In fact, even the rule *target* results in monotonic machine transitions for all counting predicates ω_{trgt} that denote upward closed sets of processes (since the intersection of two upward closed sets is also upward closed). This is, for instance, the case for predicates capturing assertion violations but not for those capturing deadlocks (see Section 6.4). An encoding $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ is said to be monotonic if all its transitions are monotonic. Checking assertion violations on abstractions obtained as in Section 6.5.1 always results in monotonic encodings.

Lemma 28 (decidability). *State reachability of all monotonic encodings is decidable.*

Proof. The ordering $\leq$ is a well-quasi-ordering on the set of configurations of $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ [48]. Monotonicity of the transition rules means the obtained counter machines result in well-structured transition systems [2, 59]. It is known that state reachability is decidable for such systems [2]. $\square$

Strengthening. Monotonic encodings correspond to coarse overapproximations. Intuitively, bad configurations (such as those in which a deadlock occurs, or those obtained in a backward exploration for a program with barriers such as the one in Figure 6.1) are no more guaranteed to be upward closed. This loss of precision makes verification out of the reach of techniques solely based on monotonic encodings. To regain some of the lost precision, we constrain the runs using *counting invariants*. This is done by strengthening the counter machine transitions in order to only allow configurations that are allowed by ω_{inv} .

Example 14. Consider the concurrent program of Figure 6.1 with shared state $\mathfrak{s} = \{\text{all} \leftarrow 4, \text{arrived} \leftarrow 0, \text{cross} \leftarrow 0, \text{read} \leftarrow 0\}$ and process configuration $\mathfrak{m} = \{\text{pc}_3 \leftarrow 1, \text{pc}_1 \leftarrow 3\}$. After transition tr_4 , $\mathfrak{s}' = \{\text{all} \leftarrow 4, \text{arrived} \leftarrow 1, \text{cross} \leftarrow 0, \text{read} \leftarrow 0\}$ and $\mathfrak{m}' = \{\text{pc}_4 \leftarrow 1, \text{pc}_1 \leftarrow 3\}$. Note that executing transition tr_5 (barrier) at $(\mathfrak{s}', \mathfrak{m}')$ is impossible because of the barrier condition. Abstraction of the program with respect to the predicates $\pi_1 = \text{read_leq_0}$, $\pi_2 = \text{arrived_leq_all}$, $\pi_3 = \text{all_leq_arrived}$ and $\pi_4 = \text{cross_leq_0}$ yields abstract transitions tr_4^b and tr_5^b . It turns out that starting from the abstract configuration $(\mathfrak{s}^b, \mathfrak{m}^b)$ the two transitions can be executed as follows:

$\mathfrak{s}^b$				$\mathfrak{m}^b$
π_1	π_2	π_3	π_4	pc
true	true	false	true	pc ₂ , pc ₃
$\Downarrow \text{tr}_4^b. \text{arrived_leq_all} :=$ $\text{ch}(\text{arrived_leq_all} \wedge \neg \text{all_leq_arrived}, \text{all_leq_arrived})$ $\text{all_leq_arrived} := \text{ch}(\text{all_leq_arrived}, \text{false})$				
true	true	true	true	pc ₂ , pc ₄
$\Downarrow \text{tr}_5^b. \text{arrived_leq_all} \wedge \text{all_leq_arrived}; \text{cross_leq_0} := \text{false}$				
true	true	true	false	pc ₂ , pc ₅

The steps above are not feasible in the original program, because tr_4^b changed predicate all_leq_arrived to **true** although $\text{all} \leq \text{arrived}$ does not hold in the original program. This results in a false positive.

Our solution to recover some of the lost precision is to strengthen the transitions of the counter machine using counting predicates (i.e., predicates that relate shared variables to the number of processes) that are valid in all runs (counting invariants), for example:

$$\omega_{inv} : \text{all} = \sum_{i \geq 1} (@\text{pc}_i)^\# \wedge \text{arrived} = \sum_{i \geq 4} (@\text{pc}_i)^\#.$$

Note that the counting invariant ω_{inv} as opposed to the target predicates ω_{tgt} (Section 6.5.2) contains shared variables. This is to regain the information about the shared variables that was lost due to abstraction. Given an abstraction of a shared state $\mathfrak{s}^b$, we first conjunct ω_{inv} with $\text{originOf}(\mathfrak{s}^b)$ and project away the shared variables. We get a predicate that only contains counting variables. Then, we substitute the counting variables with their corresponding counters to obtain a predicate that only mentions counter machine variables. We finally strengthen all counter machine transitions that involve the state $\mathfrak{q}_{\mathfrak{s}^b}$.

Example 15 (Strengthening). Consider the Boolean shared state $\mathfrak{s}^b$ with $\text{originOf}(\mathfrak{s}^b) = ((\text{read} \leq 0) \wedge (\text{arrived} \leq \text{all}) \wedge (\text{all} \leq \text{arrived}) \wedge \neg(\text{cross} \leq 0))$. After the conjunction with the invariant and projection of the shared variables, we obtain: $\exists \text{Sh}. (\text{originOf}(\mathfrak{s}^b) \wedge \omega_{inv}) = ((@\text{pc}_1)^\# + (@\text{pc}_2)^\# + (@\text{pc}_3)^\#) = 0$. After substitution, we get $\text{str}(\mathfrak{s}^b, \omega_{inv}) = (\text{ct}_{(\text{pc}_1, \emptyset)} + \text{ct}_{(\text{pc}_2, \emptyset)} +$

$$\begin{array}{c}
\frac{(\mathbf{q}_{s^b} : \mathbf{op} : \mathbf{q}_{s^{b'}})}{(\mathbf{q}_{s^b} : \mathbf{str}(s^b, \omega_{inv}); \mathbf{op}; \mathbf{str}(s^{b'}, \omega_{inv}) : \mathbf{q}_{s^{b'}})} \\
\\
\frac{(\mathbf{q}_{s^b} : \omega_{trgt}[\mathbb{W}_{s^b, \omega_{trgt}}] : \mathbf{q}_{trgt})}{(\mathbf{q}_{s^b} : \mathbf{str}(s^b, \omega_{inv}); \omega_{trgt}[\mathbb{W}_{s^b, \omega_{trgt}}] : \mathbf{q}_{trgt})}
\end{array}$$

Figure 6.10: Strengthening of an extended counter machine transition using a counting invariant ω_{inv} . For this, let $\mathbf{str}(s^b, \omega_{inv}) = (\exists \text{Sh.}(\mathbf{originOf}(s^b) \wedge \omega_{inv}))[\mathbb{W}_{s^b, \omega_{inv}}]$.

$\mathbf{ct}_{(\mathbf{pc}_3, \emptyset)} = 0$). Each counter $\mathbf{ct}_{(\mathbf{pc}_i, \emptyset)}$ is the counter for the processes at location $\mathbf{pc}_i$ where the processes have no local state. In other words, in the corresponding Boolean program, there exists no local or mixed predicates. We add the condition $(\mathbf{ct}_{(\mathbf{pc}_1, \emptyset)} + \mathbf{ct}_{(\mathbf{pc}_2, \emptyset)} + \mathbf{ct}_{(\mathbf{pc}_3, \emptyset)} = 0)$ as a statement at the beginning (respectively, at the end) of each transition outgoing from (respectively, incoming to) $\mathbf{q}_{s^b}$.

Strengthening is described in Figure 6.10. We need Lemma 29 and Lemma 30 in order to establish soundness of the strengthening step in Lemma 31. Lemma 29 states that including local and mixed predicates in the predicate abstraction step makes it possible to track the number of processes satisfying Boolean combinations of those predicates in the Boolean program abstraction.

Lemma 29 (abstraction and counting variables). *If $(s^b, \mathbb{m}^b) \in \alpha((s, \mathbb{m}))$ then for each predicate π belonging to $\mathbf{predsOf}_{\mathbf{exprsOf}(\text{ShUL})}$ s.t. $\mathbf{atomsOf}(\pi) \subseteq \Pi$, $\sum_{\{(\mathbf{pc}, \mathbb{l}) \mid \pi[s, (\mathbf{pc}, \mathbb{l})]\}} \mathbb{m}((\mathbf{pc}, \mathbb{l})) = \sum_{\{(\mathbf{pc}, \mathbb{l}^b) \mid \pi[s^b, (\mathbf{pc}, \mathbb{l}^b)]\}} \mathbb{m}^b((\mathbf{pc}, \mathbb{l}^b))$.*

Proof. By definition of $\alpha((s, \mathbb{m}))$, $\mathbb{m}^b[s, \mathbb{m}]$ holds. This means that there is a bijection h from $\mathbb{m} = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle$ to $\mathbb{m}^b = \langle \sigma'_1, \sigma'_2, \dots, \sigma'_n \rangle$ s.t. we can associate to each $\sigma_i = (\mathbf{pc}, \mathbb{l})$ in $\mathbb{m}$ a $\sigma_{h(i)} = (\mathbf{pc}', \mathbb{l}^b)$ in $\mathbb{m}^b$ such that $\mathbf{pc} = \mathbf{pc}'$ and $\mathbb{l}^b[s, \mathbb{l}]$. Let π be a Boolean combination of predicates in $\Pi_{mix} \cup \Pi_{loc}$. By construction, we have that $\pi[s^b, (\mathbf{pc}, \mathbb{l}^b)] \Leftrightarrow \pi[s, (\mathbf{pc}, \mathbb{l})]$. In fact, $\pi[s^b, (\mathbf{pc}, \mathbb{l}^b)]$ and $\pi[s, (\mathbf{pc}, \mathbb{l})]$ coincide on each π s.t. $\mathbf{atomsOf}(\pi) \subseteq \Pi$. This implies that for π with $\mathbf{atomsOf}(\pi) \subseteq \Pi$, $\sum_{\{(\mathbf{pc}, \mathbb{l}) \mid \pi[s, (\mathbf{pc}, \mathbb{l})]\}} \mathbb{m}(\mathbf{pc}, \mathbb{l}) = \sum_{\{(\mathbf{pc}, \mathbb{l}^b) \mid \pi[s^b, (\mathbf{pc}, \mathbb{l}^b)]\}} \mathbb{m}^b(\mathbf{pc}, \mathbb{l}^b)$. $\square$

Lemma 30 (abstraction and strengthening). *Assume $(s^b, \mathbb{m}^b) \in \alpha((s, \mathbb{m}))$. If $\omega_{inv}[s, \mathbb{m}]$ holds, then $(\exists \text{Sh.}(\mathbf{originOf}(s^b) \wedge \omega_{inv}))[\mathbb{m}^b]$ also holds.*

Proof. We know that $\omega_{inv}[s, \mathbb{m}]$ holds by assumption and $\mathbf{originOf}(s^b)[s]$ holds since $(s^b, \mathbb{m}^b) \in \alpha((s, \mathbb{m}))$. Then, $(\mathbf{originOf}(s^b) \wedge \omega_{inv})[s, \mathbb{m}]$ also holds, and hence, the predicate $((\mathbf{originOf}(s^b) \wedge \omega_{inv})[s])[\mathbb{m}]$. Since for each $(\pi)^\#$ appearing in $\mathbf{varsOf}(\omega_{inv})$, $\mathbf{atomsOf}(\pi) \subseteq \Pi$, by Lemma 29 we can replace

$\mathfrak{m}$ with $\mathfrak{m}^b$ and obtain the predicate $((\text{originOf}(s^b) \wedge \omega_{inv})[s])[m^b]$. Finally, the existence of s ensures that $(\exists \text{Sh.}(\text{originOf}(s^b) \wedge \omega_{inv}))[m^b]$ holds. $\square$

Lemma 31 (soundness of strengthening). *If a Pr configuration satisfying $\omega_{tgt} \in \text{predsOf}_{\text{exprsOf}(\Omega_{\text{PC}, \text{Sh}, \text{L}})}$ is reachable in Pr , then $\mathbf{q}_{tgt}$ is reachable in $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ after any strengthening with respect to some Pr invariant $\omega_{inv} \in \text{predsOf}_{\text{exprsOf}(\text{Sh} \cup \Omega_{\text{PC}, \text{Sh}, \text{L}})}$.*

Proof. We proceed by contradiction. Assume a ω_{tgt} configuration is reachable in Pr while $\mathbf{q}_{tgt}$ is not reachable in a counter machine M obtained as the strengthening (with respect to a Pr counting invariant ω_{inv}) of an encoding of an abstraction $\text{abst}_{\Pi}(\text{Pr})$ of the program Pr as explained in Section 6.5.1. Let a feasible run ρ_{Pr} without an $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))_{str}$ -feasible run $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))_{str}}$ where $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))_{str}$ is a strengthening of machine $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ with respect to an invariant ω_{inv} .

According to Definition 11, for each run ρ_{Pr} , a non-empty set $\alpha(\rho_{\text{Pr}})$ of $\text{abst}_{\Pi}(\text{Pr})$ -feasible runs exist. Moreover, based on Lemma 26, for each run $\rho_{\text{abst}_{\Pi}(\text{Pr})} \in \alpha(\rho_{\text{Pr}})$ there exists an $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ -feasible run $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))}$ (before strengthening). So, if the run $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))_{str}}$ does not exist, it is because the run $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))}$ was not possible after the strengthening phase.

Let $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))} = (\mathbf{q}_{s_0^b}, \mathbb{C}_{\mathfrak{m}_0^b}; \mathbf{t}_1; \dots; (\mathbf{q}_{s_n^b}, \mathbb{C}_{\mathfrak{m}_n^b})$ and $\rho_{\text{Pr}} = (s_0, \mathfrak{m}_0; \mathbf{t}_1; \dots; (s_n, \mathfrak{m}_n)$ with $(s_i^b, \mathfrak{m}_i^b) \in \alpha((s_i, \mathfrak{m}_i))$ for each $i : 0 \leq i \leq n$. Because $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))}$ is removed after strengthening, then there exists a step $(\mathbf{q}_{s^b}, \mathbb{C}_{\mathfrak{m}^b}) \xrightarrow[\text{enc}(\text{abst}_{\Pi}(\text{Pr}))]{(\mathbf{q}_{s^b} : \text{op} : \mathbf{q}_{s'^b})} (\mathbf{q}_{s'^b}, \mathbb{C}'_{\mathfrak{m}^b})$ in $\rho_{\text{enc}(\text{abst}_{\Pi}(\text{Pr}))}$ such that its corresponding step is impossible. According to the strengthening rule in Figure 6.10, $\text{str}(s^b, \omega_{inv}) = (\exists \text{Sh.} \text{originOf}(s^b) \wedge \omega_{inv})[\mathbb{W}_{s^b, \omega_{inv}}]$. So, the fact that the mentioned step is not possible after strengthening implies that either $(\mathbf{q}_{s^b}, \mathbb{C}_{\mathfrak{m}^b})$ does not satisfy $\text{str}(s^b, \omega_{inv})$, or that $(\mathbf{q}_{s'^b}, \mathbb{C}'_{\mathfrak{m}^b})$ does not satisfy $\text{str}(s'^b, \omega_{inv})$. Both alternatives violate the fact that ω_{inv} is an invariant, that $(s, \mathfrak{m})$ and $(s', \mathfrak{m}')$ are both reachable, and Lemma 30. $\square$

The resulting machine is not monotonic in general and we can encode the state reachability of a 2-counter Minsky machine.

Lemma 32. *State reachability is in general undecidable after strengthening.*

Proof. Let $\text{M} = (\mathcal{Q}, \mathcal{C}, \mathcal{T}, \mathcal{Q}_{init}, \mathcal{C}_{init}, \mathbf{q}_{tgt})$ be an arbitrary 2-counter Minsky machine as introduced in Section 3.2.2. We describe a concurrent program in Figure 6.11 together with predicates Π_0 , a counting invariant ω_{inv} and a counting predicate ω_{tgt} . The obtained strengthened encoding captures the counter machine M in the sense that solving the reachability of the target state of the encoding is equivalent to checking reachability of $\mathbf{q}_{tgt}$ for M , which is undecidable in general.

Each process of the concurrent program manipulates a local variable $\mathbf{tl}$ and the shared variables $\mathbf{ct}_1, \mathbf{ct}_2, \mathbf{st}$, and $\mathbf{ts}$. The local variable $\mathbf{tl}$ ranges over the three distinct constants $\mathbf{k}_{\mathbf{pm}}, \mathbf{k}_{\mathbf{p1}}, \mathbf{k}_{\mathbf{p2}}$. These are used to identify three types of process instances. The shared variable $\mathbf{ts}$ ranges over the constants in $\{\mathbf{k}_{\mathbf{pm}}, \mathbf{k}_{\mathbf{p1}}, \mathbf{k}_{\mathbf{p2}}, \mathbf{dec}_1, \mathbf{dec}_2\}$. Exactly one instance (called the main instance, henceforth) of the processes executes transitions starting with the guard $\mathbf{ts} = \mathbf{k}_{\mathbf{pm}}$. The main instance spawns (and joins with) two kinds of auxiliary processes, those that execute the transition with the assumption that $\mathbf{tl} = \mathbf{ts} = \mathbf{k}_{\mathbf{p1}}$ and those that execute the transition when $\mathbf{tl} = \mathbf{ts} = \mathbf{k}_{\mathbf{p2}}$. Henceforth, these will be referred to as $\mathbf{k}_{\mathbf{p1}}$ and $\mathbf{k}_{\mathbf{p2}}$ instances. The variable $\mathbf{st}$ ranges over $|\mathcal{Q}|$ distinct constants $\{\mathbf{k}_q \mid q \in \mathcal{Q}\}$ and encodes the state $q \in \mathcal{Q}$ of the counter machine $\mathbf{M}$. Finally, the variable $\mathbf{ct}_1$ is used to capture the number of $\mathbf{k}_{\mathbf{p1}}$ instances, and $\mathbf{ct}_2$ the number of $\mathbf{k}_{\mathbf{p2}}$ instances. Figure 6.11 represents the encoding of operations on $\mathbf{ct}_1$. Encoding of $\mathbf{ct}_2$ operations is similar.

Finally, we let $\Pi_0 = \{\mathbf{ct}_1 = 0, \mathbf{ct}_2 = 0\} \cup \{\mathbf{st} = \mathbf{k}_q \mid q \in \mathcal{Q}\}$, $\omega_{\mathbf{trgt}} = (\mathbf{st} = \mathbf{k}_{\mathbf{trgt}})^\# \geq 1$ and:

$$\omega_{\mathbf{inv}} = \left(\begin{array}{l} (@\mathbf{pc}_x)^\# \leq 1 \\ \wedge (\mathbf{ct}_1 = (@\mathbf{pc}_1 \wedge \mathbf{tl} = \mathbf{k}_{\mathbf{p1}})^\#) \\ \wedge (\mathbf{ct}_2 = (@\mathbf{pc}_1 \wedge \mathbf{tl} = \mathbf{k}_{\mathbf{p2}})^\#) \end{array} \right)$$

Predicate abstraction will maintain the predicates $\mathbf{ct}_i = 0$, but will lose their connection with the number of the corresponding instances. Strengthening re-establishes this connection and introduces tests for zero on the counters tracking $\mathbf{k}_{\mathbf{p1}}, \mathbf{k}_{\mathbf{p2}}$ instances, exactly at the encodings of the transitions $\mathbf{tr}_{\mathbf{tst}}$. $\square$

6.5.4 Constrained monotonic abstraction and preorder refinement

This step addresses the state reachability problem for an extended counter machine $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{\mathbf{init}}, \mathcal{C}_{\mathbf{init}}, \mathbf{q}_{\mathbf{trgt}})$. As stated in Lemma 32, this problem is undecidable in general for strengthened encodings. The idea of monotonic abstraction [4] as explained in Section 3.4 is to force monotonicity with respect to a well-quasi-ordering $\leq$ on the set of machine configurations. We adapt the backward reachability analysis procedure “explore” introduced in Section 3.3. An extended counter machine $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{\mathbf{init}}, \mathcal{C}_{\mathbf{init}}, \mathbf{q}_{\mathbf{trgt}})$ induces the infinite-state transition system $(\mathcal{C}, \mathbf{T}, \rightarrow)$ where $\mathcal{C}$ is the set of configurations. A configuration is a tuple $(\mathbf{q}, \mathbf{c})$. $\rightarrow$ is the transition relation defined in figure 6.8. We start with the preorder $\leq_0 \subseteq \mathcal{C} \times \mathcal{C}$ such that for $\mathbf{c} = (\mathbf{q}, \mathbf{c})$ and $\mathbf{c}' = (\mathbf{q}', \mathbf{c}')$, $\mathbf{c} \leq \mathbf{c}'$ iff (1) $\mathbf{q} = \mathbf{q}'$, and (2) $\mathbf{c} \leq \mathbf{c}'$ with respect to the natural component-wise ordering defined as $\bigwedge_{\mathbf{ct} \in \mathcal{C}} \mathbf{c}(\mathbf{ct}) \leq \mathbf{c}'(\mathbf{ct})$. Intuitively, $\mathbf{c} \leq_0 \mathbf{c}'$ holds if $\mathbf{c}'$ can be

```

int ct1 := 0;
int ct2 := 0;
int st := kq0;
int ts := kpm;

process:
int tl := ts
// auxiliary processes:
trp1. pc0 → pc1 : assume((tl = kp1) ∧ (ts = kp1)); ct1 := ct1 + 1; ts := kpm;
trp2. pc1 → pcx : assume((tl = kp1) ∧ (ts = dec1)); ct1 := ct1 - 1; ts := kpm;

// main process:
// (q, inc(x1), q')
trinc. pc0 → pc0 : assume((ts = kpm) ∧ (tl = kpm) ∧ (st = kq)); ts := kp1; st := kq'; spawn
// (q, dec(x1), q')
trdec. pc0 → pc1 : assume((ts = kpm) ∧ (tl = kpm) ∧ (st = kq) ∧ (ct1 ≥ 1)); ts := dec1;
trdec. pc1 → pc0 : assume((ts = kpm) ∧ (tl = kpm)); st := kq'; join
// (q, test(x1), q')
trtest. pc0 → pc0 : assume((ts = kpm) ∧ (tl = kpm) ∧ (st = kq)); assume(ct1 = 0); st := kq';
...

```

Figure 6.11: Encoding a 2-counter Minsky machine for strengthened counter machines obtained as encoding of Boolean concurrent programs.

obtained by “adding more processes to” $\mathcal{C}$. Using this ordering results in a WSTS. The procedure “explore” requires that we can compute membership, upward closure, entailment with respect to the preorder $\leq$, and predecessor computations of an upward closed set. As explained in Example 1 in Section 3.3, the procedure can work with upward closed sets.

If no run is found, then `not_reachable` is returned. We modify the algorithm slightly so that upon finding a run at line 5, the result of the counterexample analyzer procedure introduced in Section 3.4.1, “simulate($M, \leq, \rho$)” is returned. The counterexample analyzer simulates the obtained run on the concrete machine M . If the run is possible, it is sent to the fourth step of our approach (described in Section 6.5.5). Otherwise, the upward closure $\text{Up}_{\leq}((q, \mathcal{C}))$ responsible for the spurious trace is identified and an interpolant I (with $\text{varsOf}(I) \subseteq \mathcal{C}$) is used to refine the preorder as follows: $\leq_{i+1} := \{((q, \mathcal{C}), (q, \mathcal{C}')) \mid (q, \mathcal{C}) \leq_i (q, \mathcal{C}) \wedge (\mathcal{C} \models I \Leftrightarrow \mathcal{C}' \models I)\}$.

Although stronger, the new preorder is again a well-quasi-ordering and the trace is guaranteed to be eliminated in the next round. This is explained in detail in Chapter 7.

Lemma 33 (CMA [3]). *All steps involved in the procedure “explore” are effectively computable and each instantiation of the procedure is sound and terminates given the preorder is a well-quasi-ordering.*

example	Pr	$enc(\text{abst}_{\Pi}(\text{Pr}))$	outer loop		inner loop		time(s)
			num.	preds.	num.	preds.	
cyclic barrier	5:3:12	13:31:79	4	5	5	1	30
		13:4:35	2	2	2	0	15
		13:8:64	4	3	6	2	35
dynamic barrier	5:2:8	8:8:44	3	3	3	0	20
		8:1:14	2	1	2	0	3
		8:1:8	1	0	1	0	1
flag	5:2:9	8:16:123	4	4	6	2	34
		8:8:67	3	2	3	0	5
		8:4:22	3	2	5	2	18
semaphore	4:2:7	7:32:89	5	5	5	0	68
		7:7:24	3	3	3	0	14
		7:16:66	4	4	6	2	17
maximum	5:2:8	18:16:172	5	5	8	3	489
		18:4:51	3	3	3	0	18
		8:8:48	4	3	6	2	21
parent-child	2:3:10	9:16:48	3	4	5	2	76
		9:1:16	2	1	2	0	2
		9:4:35	3	2	3	0	5
as-many	3:2:6	8:4:34	3	2	6	3	68
		8:1:9	2	1	2	0	2
		8:2:8	2	1	2	0	2
locals	6:3:13	14:8:47	3	3	3	0	16
		14:2:24	2	1	2	0	8
		14:16:95	4	4	4	0	58
shareds	7:3:11	13:32:130	4	5	10	6	160
		13:2:21	2	1	2	0	6
		12:2:20	2	1	2	0	17

Table 6.1: The results of checking assertion violation and deadlock with PAC-MAN

6.5.5 Simulation on the original concurrent program

A run of the counter machine $(\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{init}, \mathcal{C}_{init}, \mathbf{q}_{trgt})$ is simulated by this step on the original concurrent program $\text{Pr} = (\text{Sh}, \text{L}, \text{Tr})$. This is possible because each step of the counter machine run corresponds to a unique and concrete transition of Pr . This step is classical in counterexample-guided abstraction refinement approaches. In our case, we need to differentiate the variables belonging to different processes during the simulation. As usual in such frameworks, if the trace turns out to be possible, then we have captured a concrete run of Pr that violates an assertion, and we report it. Otherwise, we deduce predicates that make the run infeasible (Sect. 6.5.1).

Theorem 21 (predicated constrained monotonic abstraction). *Assume an effective and sound predicate abstraction. If the constrained monotonic abstraction step returns `not_reachable`, then no configuration satisfying ω_{trgt} is reachable in Pr . If a Pr run is returned by the simulation step, then it reaches a configuration where ω_{trgt} holds. Every iteration of the outer loop (predicate abstraction refinement) terminates given that the inner loop (monotonic abstraction refinement) terminates. Every iteration of the inner loop terminates.*

Proof. By Lemma 25 and Lemma 29, if the concretization of a run of an encoding is Pr -feasible, then ω_{trgt} is Pr -reachable. Soundness is given by soundness of the predicate abstraction and strengthening (Lemma 31) and soundness of the CMA step (Lemma 33). Termination of each iteration of the inner loop is by well-quasi-ordering (Lemma 33). Termination of each iteration of the outer loop, given that the inner one terminates, is by effectiveness of the predicate abstraction (Definition 11) and construction of the encoding (Figure 6.9) and strengthening (Figure 6.10). $\square$

Notice that there is no general guaranty that we establish or refute the safety property (the problem is undecidable). For instance, it may be the case that one of the loops does not terminate (although each one of their iterations does) or that we need to add predicates that relate the local variables of two different processes (something the predicate abstraction framework we use cannot express).

6.6 Experimental results

We report on experiments with our open-source tool prototype PACMAN (for predicated constrained monotonic abstraction)¹. We have conducted our experiments on an Intel Core(TM) i7 processor running at 2.9GHz with 8GB of memory. The reported examples that require refinements of the natural preorder are challenging for existing techniques, either because the examples require stronger orderings than the usual preorder, or because they involve counting target predicates that are not expressed in terms of violations of program assertions. All predicate abstraction predicates and counting invariants have been derived automatically. For the counting invariants, we implemented a thread-modular analysis operating on the polyhedra numerical domain. We make use of several optimizations as explained in the following.

- Invariants and unsatisfiable combinations. We discard Boolean mappings corresponding to unsatisfiable combinations of predicates, for example $\neg(\text{arrived} \leq \text{all}) \wedge \neg(\text{all} \leq \text{arrived})$ in Figure 6.1. We also use automatically generated invariants (such as $(\text{arrived} \leq \text{all}) \wedge$

¹<https://gitlab.liu.se/live/pacman>

(`arrived` ≥ 0) in the same example) to filter the state space. Such heuristics dramatically help our state space exploration algorithm.

- Auxiliary variables. To make the analysis possible, we try to use as few auxiliary variables as possible. An auxiliary variable $a_{(\text{pc}, 1^b, 1^{b'})}$ captures how many processes in local state $(\text{pc}, 1^b)$ will move to state $(\text{pc}, 1^{b'})$ because of the broadcast transition due to a multiple assignment. Each time this number is constant, which turns out to happen often, we discard the auxiliary variable $a_{(\text{pc}, 1^b, 1^{b'})}$.

For each example, we give the number of transitions and variables both in `Pr` and in the resulting counter machine.

The tuples under the `Pr` column of Table 6.1 respectively refer to the size of the original program, i.e., number of variables, procedures and transitions in the original program. The tuples under the $\text{enc}(\text{abst}_{\Pi}(\text{Pr}))$ column refer to the size of the counter machine, i.e., number of counters, states and transitions in the extended counter machine. We also state the number of refinement steps and predicates automatically obtained in both refinement loops. We mention the required total time for the analysis. The tool and examples are accessible².

We report on experiments checking assertion violation and deadlock freedom in Table 6.1. For each example, we check three versions: the first one is correct and PACMAN establishes that, the second one is modified so that an assertion can be violated, and the third one is modified so that deadlock is possible. for the latter two cases, PACMAN exhibits faulty runs as expected.

6.7 Conclusions

We have presented a technique, predicated constrained monotonic abstraction, for the automated verification of concurrent programs whose correctness depends on synchronization between arbitrarily many processes, for example by means of barriers implemented using integer counters and tests. We have introduced counting predicates and used them in a framework that combines predicate, counter and constrained monotonic abstraction. Our prototype implementation gave encouraging results and managed to automatically establish or refute program assertions and deadlock freedom. Chapter 5 showed that barriers could be modeled using phasers and explained that checking reachability of phaser programs with arbitrarily many tasks is decidable if the phaser program does not have atomic statements. However, in this chapter, we proved that checking reachability of strengthened counter programs that model barriers is undecidable. These two results do not contradict each other. The intuition is that strengthening and atomic transitions in the obtained boolean programs of this chapter allow to relate tests for zero to the

²<https://gitlab.liu.se/live/pacman>

number of processes (barriers) and atomic transitions (by enforcing that only a single process can run a sequence of instructions).

In Chapter 7 we investigate how to improve scalability of the constraint monotonic abstraction approach used in this chapter by lazily refining monotonic abstraction.

7 Lazy Constrained Monotonic Abstraction

In this chapter, we introduce Lazy Constrained Monotonic Abstraction (Lazy CMA for short) for lazily and soundly exploring well-structured abstractions of infinite and non-monotonic transition systems. Monotonic abstraction was used in the verification approach that was introduced in Chapter 6. We consistently improve the monotonic abstraction approach by localizing refinements and by partially preserving the explored state space each time a refinement step is required for the ordering. For this purpose, we adapt ideas from the classical lazy predicate abstraction.

7.1 Introduction

Well-structured transition systems (WSTS:s for short) may be everywhere [59], but not all transition systems are well-structured [3, 61]. Problems such as state reachability (e.g, safety) have been shown to be decidable for WSTS:s [2, 59]. This led to the development of algorithms that could check safety for systems ranging from lossy channels and Petri Nets to concurrent programs and broadcast protocols [63, 82, 96]. Many interesting examples of such systems, including list manipulating programs [26], cache coherence protocols [47] and mutex algorithms [5] are “almost” well-structured in the sense that they would have been well-structured if it was not for a number of transitions that violate the required assumptions. We build on the framework of Constrained Monotonic Abstraction (CMA for short) [3] where we derive well-structured abstractions for infinite-state systems that are “almost” well-structured.

Systems where only some transitions are non-monotonic can be approximated using WSTS:s by adding abstract transitions to restore monotonicity (monotonic abstraction). The resulting abstraction is also infinite-state, and reachability of its upward-closed sets is decidable. However, the obtained abstractions may fail to enforce invariants that are crucial for establishing unreachability of bad configurations in the original system. We explained in Chapter 6 how we automatically account for the number of processes synchronizing with (dynamic) barriers when establishing or refuting local (e.g., assertions) and global (e.g., deadlock freedom) properties of concurrent programs that manipulate arbitrarily many processes. Crucial invariants of such systems enforce an inherently non-monotonic behavior (e.g., a barrier transition that is enabled on a configuration is disabled if more processes are considered in a larger configuration).

Checking safety for such non-monotonic systems is not guaranteed to terminate without abstraction. Plain monotonic abstraction [5, 65] makes use of sets that are upward-closed with respect to natural orderings as a sound symbolic representation. As stated earlier, this ensures termination if the preorder that is used is a $\mathcal{WQO}$ [2]. Of course, this comes at the price of possible false positives. In [3], the authors adapted the existing counterexample-guided abstraction refinement (CEGAR) ideas to refine the ordering in plain monotonic abstraction. The preorder is strengthened by only relating configurations that happen to be in the same equivalence class, as defined by the Craig interpolants obtained from the false positives. The new preorder is also a $\mathcal{WQO}$, and hence, termination is again ensured. As implemented, the predicates are applied on all generated minimal elements to separate upward-closed sets and the exploration has to restart from scratch each time a new refinement predicate is encountered.

We address these inefficiencies by adopting a lazy approach. Like in lazy predicate abstraction [71], we strive to localize the application of the refinement predicates and to reuse the explored state space. However, a major difference with lazy predicate abstraction is that the number of “control locations” (i.e., the locations to which subsets of the refinement predicates are mapped) is a priori unbounded (as opposed to the number of program locations of a non-parameterized system). In this chapter, we propose three heuristics that can be applied both in backward and forward (something plain monotonic abstraction is incapable of). All three heuristics adopt a backtracking mechanism to reuse, as much as possible, the state space that has been explored so far. Schematically, the first heuristic (*point-based*) associates refinement predicates to minimal elements. The second heuristic (*order-based*) associates the refinement predicates to preorder-related minimal elements. The third heuristic (*descendants-based*) uses for a child the preserved predicates of its parent. We describe in detail the different approaches and state the soundness and termination of each refinement step. In addition, we experimentally

compare the heuristics against each other and against the eager approach (CMA) on our open-source tool <https://gitlab.liu.se/live/zaama>.

Related work. Coverability of non-monotonic systems is undecidable in general. Tests for zero are one source of non-monotonicity. The work in [25] introduces a methodology for checking coverability by using an extended Karp-Miller acceleration for the case of Vector Addition Systems (VAS:s for short) with at most one test for zero. Our approach is more general and tackles coverability and reachability for counter machines with arbitrary tests.

Verification methods can be lazy in different ways. For instance, Craig interpolants obtained from program runs can be directly used as abstractions [102], or abstraction predicates can be lazily associated to program locations [71]. Such techniques are now well established [15, 38, 127]. Unlike these approaches, we address lazy exploration for transition systems with “infinite control”. The existing abstract WSTS:s that do not allow to refine the ordering being used [82, 96], cannot model transfers for the local variables [58], or make use of accelerations without termination guarantees [18]. For example, in [82] the authors leverage the combination of an exact forward reachability and of an aggressive backward approximation, while in [96], the explicit construction of a Petri Net is avoided.

The work in [85] gives a generalization of the IC3 algorithm and tries to build inductive invariants for well-structured transition systems. There is no straightforward approach to adapt it to the kind of non-monotonic systems that we work with.

We believe the approach proposed here can be combined with such techniques. To the best of our knowledge, there is no previous work that considered making the preorder refinement of a WSTS abstraction lazy.

Outline. We start in Section 7.2 with some preliminaries. We describe the adopted symbolic representation in Section 7.3 and go through a illustrative example in Section 7.4. This is followed by a description of the eager and lazy procedures in Section 7.5. We finally report on our experiments in Section 7.6 and conclude in Section 7.7.

7.2 Preliminaries

For a set of integer variables V , we write $\text{atomsOf}(V)$ to mean the set of *atoms* over the variables V . An atom α is either a Boolean **true** or **false** or an inequality $e \sim e'$ of two arithmetic expressions; where $\sim \in \{<, \leq, \geq, >\}$. We write A to mean a set of atoms. Observe that the negation of an atom can be expressed as an atom. We often write ψ to mean a conjunction of atoms, or *conjunct* for short, and use Ψ to mean a set of conjuncts. We use $\text{predsOf}_{\text{exprsOf}(V)}$ to mean arbitrary conjunctions and disjunctions of atoms

over the set of integer variables $\mathbf{V}$. We can rewrite any Presburger predicate over $\mathbf{V}$ in negated normal form and replace the negated inequalities with the corresponding atoms to obtain an equivalent predicate π in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathbf{V})}$.

A mapping $\mathbf{x} : U \rightarrow V$ associates an element in V to each element in U . We write $\mathbf{x} : U \rightarrow V$ to mean a partial mapping from U to V . We write $\mathbf{dom}(\mathbf{x})$ and $\mathbf{image}(\mathbf{x})$ to respectively mean the domain and the image of $\mathbf{x}$ and use $\epsilon_U : U \rightarrow V$ for the mapping with an empty domain. We often write a partial mapping $\mathbf{x} : U \rightarrow V$ as the set $\{u \leftarrow \mathbf{x}(u) \mid u \in \mathbf{dom}(\mathbf{x})\}$ and write $\mathbf{x} \cup \mathbf{x}'$ to mean the union of two mappings $\mathbf{x}$ and $\mathbf{x}'$ with disjoint domains. Given a partial mapping $\mathbf{v} : X \rightarrow \mathbf{exprsOf}(X)$, we write $\mathbf{e}[\mathbf{v}]$ to mean the substitution of $\mathbf{V}$ variables in $\mathbf{e}$ by their respective $\mathbf{v}$ images and the natural evaluation of the result. As usual, $\mathbf{e}[\mathbf{v}]$ is a well defined integer value each time $\mathbf{v}$ is a total mapping to $\mathbb{Z}$. This is generalized to (sets of) atoms, conjuncts and predicates.

We let $\mathbb{V}$ (resp. $\mathbb{V}_{\geq 0}$) be the set of all total mappings $\mathbf{V} \rightarrow \mathbb{Z}$ (resp. $\mathbf{V} \rightarrow \mathbb{N}$). We write $\mathbb{0}_{\mathbf{V}}$ for the total mapping $\mathbf{V} \rightarrow \{0\}$. The denotation of a conjunct ψ over $\mathbb{V}$ (resp. $\mathbb{V}_{\geq 0}$), written $\llbracket \psi \rrbracket_{\mathbf{v}}$ (resp. $\llbracket \psi \rrbracket_{\mathbf{v}_{\geq 0}}$), is the set of all total mappings $\mathbf{v}$ in $\mathbb{V}$ (resp. in $\mathbb{V}_{\geq 0}$) s.t. $\psi[\mathbf{v}]$ evaluates to **true**. We generalize to sets of atoms or conjuncts by taking the union of the individual denotations. Let $\preceq$ be the preorder over $\mathbb{V}_{\geq 0}$ defined by $\mathbf{v} \preceq \mathbf{v}'$ iff $\mathbf{v}(\mathbf{v}) \leq \mathbf{v}'(\mathbf{v})$ for each integer variable $\mathbf{v} \in \mathbf{V}$. Given a predicate ψ in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathbf{V})}$, we say that a set $M \subseteq \llbracket \psi \rrbracket_{\mathbf{v}_{\geq 0}}$ is minimal for ψ if: (i) $\mathbf{v} \not\preceq \mathbf{v}'$ for any pair of different $\mathbf{v}, \mathbf{v}' \in M$, and (ii) for any $\mathbf{v}' \in \llbracket \psi \rrbracket_{\mathbf{v}_{\geq 0}}$, there is a $\mathbf{v} \in M$ s.t. $\mathbf{v} \preceq \mathbf{v}'$. We recall the following facts from Linear Programming and [72].

Lemma 34. *For a finite set of natural variables $\mathbf{V}$, the preorder $\preceq$ is a well-quasi-ordering. In addition, we can compute a finite and unique minimal set (written $\mathbf{Min}_{\preceq}(\pi)$) for any predicate π in $\mathbf{predsOf}_{\mathbf{exprsOf}(\mathbf{V})}$.*

7.3 Symbolic representation

Assume an extended counter machine $(\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathcal{Q}_{init}, \mathcal{C}_{init}, \mathbf{q}_{trgt})$ as introduced in Chapter 6. We introduce the symbolic representations used in our reachability procedures in Section 7.5.

Boxes. A *box* $\mathbb{b}$ over a set $\mathbf{A}$ of atoms is a partial mapping from $\mathbf{A}$ to Booleans $\mathbb{B}$. Intuitively, a box corresponds to a bitvector denoting an equivalence class in the classical predicate abstraction. We use it to constrain the upward closure. The predicate $\psi_{\mathbb{b}}$ of a box $\mathbb{b}$ is $\bigwedge_{\alpha \in \mathbf{dom}(\mathbb{b})} ((\mathbb{b}(\alpha) \wedge \alpha) \vee (\neg \mathbb{b}(\alpha) \wedge \neg \alpha))$ (**true** is used for the empty box). Observe that this predicate is indeed a conjunct for any fixed box $\mathbb{b}$ and that $\llbracket \psi_{\mathbb{b}} \rrbracket$ does not need to be finite. We write $\mathbb{b}_{\mathbf{true}}$ for the box of the **true** conjunct. We will say that a box $\mathbb{b}$ is weaker than (or is entailed by) a box $\mathbb{b}'$ if $\psi_{\mathbb{b}'} \Rightarrow \psi_{\mathbb{b}}$ is valid. We abuse notation and write $\mathbb{b} \Leftarrow \mathbb{b}'$. Observe that this is equivalent to $\llbracket \psi_{\mathbb{b}} \rrbracket \subseteq \llbracket \psi_{\mathbb{b}'} \rrbracket$.

Constraints. A *constraint* over a set A of atoms is a triplet $\varphi = (q, c, b)$ where $\text{stateOf}(\varphi) = q \in \mathcal{Q}$ is the state of the constraint, $\text{val}(\varphi) = c$ is its minimal valuation, and $\text{box}(\varphi) = b$ over A is its box. We use Φ to mean a set of constraints. A constraint (q, c, b) is well-formed if $\psi_b[c]$ holds. We only consider well-formed constraints. We write $\text{clo}(c, b)$ to mean the conjunct $(\bigwedge_{ct \in \mathcal{C}} (c(ct) \trianglelefteq ct) \wedge \psi_b)$. Intuitively, $\text{clo}(c, b)$ denotes those valuations that are both “in the box” and in the $\trianglelefteq$ -upward closure of c . We let $\llbracket (q, c, b) \rrbracket$ be the set $\{(q, c') \mid c' \in \llbracket \text{clo}(c, b) \rrbracket\}$. This set contains at least (q, c) by well-formedness. Given two constraints (q, c, b) and (q', c', b') , we write $(q, c, b) \sqsubseteq (q', c', b')$ to mean that: (i) $q = q'$, and (ii) $c \trianglelefteq c'$, and (iii) $b \Leftarrow b'$. Observe that $\varphi \sqsubseteq \varphi'$ implies $\llbracket \varphi' \rrbracket \subseteq \llbracket \varphi \rrbracket$. A subset Φ of a set of constraints Φ' is minimal if: (i) $\varphi_1 \not\sqsubseteq \varphi_2$ for any pair of different constraints $\varphi_1, \varphi_2 \in \Phi$, and (ii) for any $\varphi' \in \Phi'$, there is a $\varphi \in \Phi$ such that $\varphi \sqsubseteq \varphi'$. We can show the following lemma.

Lemma 35. *For a finite set of atoms A over $\mathcal{C}$, the ordering $\sqsubseteq$ is a well-quasi-ordering ($\mathcal{WQO}$) over the set of well-formed constraints over A . In addition, we can compute, for any set Φ of constraints, a finite $\sqsubseteq$ -minimal subset $\text{Min}_{\sqsubseteq}(\Phi)$.*

Image computations. Assume a conjunct ψ over $\mathcal{C}$ and a guarded command $\text{op} = \text{grd} \Rightarrow \text{cmd}$ for some $t \in T$. Recall that Ax is a set of auxiliary variables ranging over $\mathbb{N}$, grd is a predicate in $\text{predsOf}_{\text{exprsOf}(\mathcal{C} \cup Ax)}$, and cmd is of the form $\text{ct}_1, \dots, \text{ct}_n := e_1, \dots, e_n$ where, for each $i : 1 \leq i \leq n$, ct_i is in $\mathcal{C}$ and e_i is in $\text{exprsOf}(\mathcal{C} \cup Ax)$. We let $\mathcal{C}'$ be the set of primed versions of all variables appearing in the left-hand side of cmd . We write $\text{pre}_{\text{op}}(\psi)$ to mean a set of conjuncts whose disjunction is equivalent to $(\exists \mathcal{C}' \cup Ax. (\bigwedge_{1 \leq i \leq n} (\text{ct}'_i = e_i) \wedge \text{grd} \wedge \psi[\text{ct} \leftarrow \text{ct}' \mid \text{ct}' \in \mathcal{C}']))$. We also write $\text{post}_{\text{op}}(\psi)$ to mean a set of conjuncts whose disjunction is equivalent to $(\exists \mathcal{C} \cup Ax. (\bigwedge_{1 \leq i \leq n} (\text{ct}'_i = e_i) \wedge \text{grd} \wedge \psi))[\{\text{ct}' \leftarrow \text{ct} \mid \text{ct} \in \mathcal{C}\}]$. We naturally extend $\text{pre}_{\text{op}}(\psi)$ and $\text{post}_{\text{op}}(\psi)$ to sets of conjuncts.

Lemma 36. *Assume an extended counter machine transition $t \in T$ and conjuncts Ψ . We can compute $\text{pre}_{\text{op}(t)}(\Psi)$ and $\text{post}_{\text{op}(t)}(\Psi)$ such that $\llbracket \text{pre}_{\text{op}(t)}(\Psi) \rrbracket$ equals $\{c \mid (\text{src}(t), c) \xrightarrow{t} (\text{dst}(t), c') \text{ with } c' \in \llbracket \Psi \rrbracket\}$ and $\llbracket \text{post}_{\text{op}(t)}(\Psi) \rrbracket$ equals $\{c' \mid (\text{src}(t), c) \xrightarrow{t} (\text{dst}(t), c') \text{ with } c \in \llbracket \Psi \rrbracket\}$.*

Grounded constraints and symbolic sets. A *grounded constraint* is a pair $\gamma = ((q, c, b), \psi)$ that consists of a constraint $\text{constraintOf}(\gamma) = (q, c, b)$ and a conjunct $\text{groundOf}(\gamma) = \psi$. It is well-formed if: (q, c, b) is well-formed, $\psi \Rightarrow \text{clo}(c, b)$ is valid, and $c \in \llbracket \psi \rrbracket$. We only manipulate well-formed grounded constraints. Intuitively, the ground ψ in

Figure 7.1: The counter machine to the right captures the behaviour of the concurrent program to the left. It makes use of one counter per program location. It involves the following guarded commands: $op_{init} ::= (ct_0, ct_1, ct_2, ct_3, ct_4 := 1, 0, 0, 0, 0)$, $op_0 ::= ct_0 \geq 1 \Rightarrow ct_0 := ct_0 + 1$, $op_1 ::= ct_0 \geq 1 \Rightarrow ct_0, ct_1 := ct_0 - 1, ct_1 + 1$, $op_2 ::= ct_1 \geq 1 \Rightarrow ct_1, ct_2 := ct_1 - 1, ct_2 + 1$, $op_3 ::= (ct_2 \geq 1 \wedge ct_0 + ct_1 = 0) \Rightarrow (ct_2, ct_3 := ct_2 - 1, ct_3 + 1)$, $op_4 ::= ct_3 \geq 1 \Rightarrow ct_3, ct_4 := ct_3 - 1, ct_4 + 1$, and $op_{err} : (ct_3 \geq 1)$. The resulting system is not well-structured because of the test for zero in op_3 .

$((q, c, b), \psi)$ represents the “non-approximated” part of the $\preceq$ -upward closure of c . This information will be needed for refining the preorder during the analysis. We abuse notation and write $\text{constraintOf}(\Gamma)$, resp. $\text{groundOf}(\Gamma)$, to mean the set of constraints, resp. grounds, of a set Γ of grounded constraints. A trace ρ of length n is a sequence starting with a grounded constraint followed by n transitions and grounded constraints. We say that two traces $(\varphi_0, \psi_0); t_1; (\varphi_1, \psi_1); \dots; t_n; (\varphi_n, \psi_n)$ and $(\varphi'_0, \psi'_0); t'_1; (\varphi'_1, \psi'_1); \dots; t'_{n'}; (\varphi'_{n'}, \psi'_{n'})$ are equivalent if: (i) $n = n'$, and (ii) t_i is the same as t'_i for each $i : 1 \leq i \leq n$, and (iii) $\varphi_i \sqsubseteq \varphi'_i$, $\varphi'_i \sqsubseteq \varphi_i$ and $\psi_i \Leftrightarrow \psi'_i$ for each $i : 0 \leq i \leq n$. A *symbolic set* **Sym** is a set of pairs of grounded constraints and traces. Given a symbolic set **Sym**, we also use $\text{constraintOf}(\text{Sym})$ to mean all constraints φ appearing in some $((\varphi, \psi), \rho)$ in **Sym**. Recall that we can compute a set $\text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{Sym}))$ of $\sqsubseteq$ -minimal constraints for $\text{constraintOf}(\text{Sym})$.

7.4 Illustrative Example

We use the example depicted in Figure 7.1 to give an intuition of the lazy heuristics described in this chapter. A more detailed description follows in Section 7.5.

In the concurrent program to the left of Figure 7.1, a single process starts executing the program and it can spawn arbitrarily many concurrent processes using transition tr_0 . Assume all of the processes asynchronously run the same program. Each process can then set the shared flag `read` to `true`, and

perform some reading followed by resetting `read`. All of the processes wait at the barrier. Observe that `read` should be `false` after the barrier since all processes that reached `pc3` must have executed `tr2`. The assertion at `pc3` should therefore hold no matter how many processes are spawned. Capturing the barrier behaviour is crucial to establish that it is not possible to violate the assertion. The barrier behaviour while having the possibility that a single process changes location is inherently non-monotonic (adding more processes does not keep the barrier open). In Chapter 6, we introduced our approach that combines different abstraction techniques to automatically generate non-monotonic counter machines such as the one to the right of Figure 7.1. For this case, the assertion in the concurrent program is violated iff the target state is reachable in the counter machine. We explain briefly in the following how such counter machines are generated.

Our tool, PACMAN (Chapter 6), takes as input concurrent programs similar to the one to left of Figure 7.1. It automatically performs predicate, counter and monotonic abstractions on them and generates counter machines that over-approximate the behaviour of the original program. It then tries to solve the reachability problem for those machines.

Given a concurrent program, PACMAN starts by generating concurrent Boolean programs by performing predicate abstraction and incrementally improving the abstraction in a CEGAR loop [50]. This results in a Boolean concurrent program that has the same control flow graph as the original program, but consists of only Boolean variables. To the obtained Boolean program, PACMAN applies counter abstraction to generate a counter machine. Intuitively, a counter is associated to each local state valuation of a process (which consists of the location and the valuation of the local variables of the process). Each state in the machine is also associated to a valuation of shared variables. An extra state is reserved for the target. The statements of the Boolean program are then translated as transitions of the counter machine.

For instance, in Figure 7.1, counters `cti`, for $i : 0 \leq i \leq 4$, correspond respectively to the number of processes in program location `pci` (the processes have no local variables here). Similarly, each transition `opi` is associated with each `tri`. There are two additional transitions `opinit` and `operr` to model transitions involving initial or target states.

Note that the original concurrent program has non-monotonic invariants. For instance, transitions such as barriers, or any transition that tests variables representing the number of processes satisfying some property, do not necessarily stay enabled if we add more processes to the program. At the same time, the Boolean concurrent programs generated above are inherently monotonic. This corresponds to a loss of precision. Thus, proving correctness of those programs whose correctness depends on respecting the non-monotonic behaviour (e.g., the one enforced by a barrier) can become impossible. As a remedy to this fact, PACMAN automatically strengthens counter machine transitions by enforcing barrier invariants or by deriving new invariants (e.g.,

using an instrumented thread-modular analysis) to regain some of the precision. This helped us to verify several challenging benchmarks. For example, consider the transition tr_3 in the program to the left of Figure 7.1. At the moment a process crosses the barrier and it is the first one to do so, there should be no process before location pc_2 . This fact holds afterwards and forbids any process from setting the flag `read` when some process is checking the assertion. The transition op_3 is its corresponding transition in the strengthened counter machine. To ease the presentation of the example, op_3 is strengthened with the guard $(\text{ct}_0 + \text{ct}_1 = 0)$. (Observe that this is a simplification to ease the presentation; we can more faithfully capture the barrier by combining the test with a global flag.)

Plain monotonic abstraction [4] proceeds backwards while systematically closing upwards with respect to the ordering $\leq$ on $\mathbf{C}$ defined as follows. Recall that for configurations $\mathbf{c} = (\mathbf{q}, \mathbf{c})$ and $\mathbf{c}' = (\mathbf{q}', \mathbf{c}')$, $\mathbf{c} \leq \mathbf{c}'$ iff (1) $\mathbf{q} = \mathbf{q}'$, and (2) $\mathbf{c} \sqsubseteq \mathbf{c}'$. The trace depicted in Figure 7.2 is a generated false positive. In this description, for $i : 0 \leq i \leq 7$, we write $\gamma_i = (\varphi_i, \psi_i)$ to mean the grounded constraint with ψ_i and the constraint $\varphi_i = (\mathbf{q}_i, \mathbf{c}_i, \mathbb{b}_i)$. Intuitively, the grounded constraint represents “exact” valuations while the constraint captures over-approximations that are of the form $(\mathbf{q}_i, \mathbf{c}_i, \mathbb{b}_i)$ where $\mathbf{c}_i \sqsubseteq \mathbf{c}$ and $\mathbf{c}$ satisfies $\psi_{\mathbb{b}_i}$. The computation starts from the grounded constraint $\gamma_7 = ((\text{trgt}, \mathbf{c}_7, \mathbb{b}_{\text{true}}), \psi_7)$ where ψ_7 is $\bigwedge_{\text{cte} \in \mathcal{C}} (\text{ct} \geq 0)$ (always implicit). For γ_7 , the exact and the over-approximated parts coincide.

The trace then computes $\psi_6 = (\text{ct}_3 \geq 1)$ which captures the valuations of the predecessors of $(\text{trgt}, \mathbf{c}_7, \mathbb{b}_{\text{true}})$ with respect to the transition $(\text{rd}, \text{op}_{\text{err}}, \text{trgt})$. This set happens to be upward-closed and there is no need for approximation, hence $\gamma_6 = ((\text{rd}, \mathbf{c}_6, \mathbb{b}_{\text{true}}), \psi_6)$. Valuations of the exact predecessors of $(\text{rd}, \mathbf{c}_6, \mathbb{b}_{\text{true}})$ with respect to $(\text{rd}, \text{op}_3, \text{rd})$ are captured with the conjunct $\psi_5 = (\text{ct}_0 = \text{ct}_1 = 0 \wedge \text{ct}_2 \geq 1)$. According to monotonic abstraction, these are approximated with the conjunct $(\text{ct}_0 \geq 0 \wedge \text{ct}_1 \geq 0 \wedge \text{ct}_2 \geq 1)$. Continuing to compute the predecessors and closing upwards leads to the constraint φ_0 which involves the initial state `init`. The trace is reported as a possible reachability witness. It is well-known [4] that upward-closed sets are not preserved by non-monotonic transitions (such as those transitions involving op_3 in Figure 7.1). At the same time, maintaining an exact analysis makes it impossible to guarantee termination.

Following the trace in the forward direction (from the left of Figure 7.1), it turns out that the upward closure that resulted in γ_5 is the one that made the spurious trace possible. Indeed, it is its approximation that allowed the counter ct_1 to be non-zero. This new value for ct_1 is the one that allowed the machine to execute $(\neg \text{rd}, \text{op}_1, \text{rd})$ backwards from φ_5 , making it possible to reach the initial state. The constraint φ_5 is the *pivot* constraint of the trace. Constrained monotonic abstraction (CMA) proposes to refine the used ordering by strengthening it with a relevant predicate. In this case, $\text{ct}_1 \leq 0$ is used for strengthening, but in general (the atoms of) any predicate in

Figure 7.2: A spurious trace generated by monotonic abstraction. The γ_5 constraint introduces the first over-approximation that makes the spurious trace possible. The configuration $(\mathbf{rd}, \mathbf{c}_5)$ is the pivot configuration of the spurious trace.

$\text{predsOf}^{\text{expresOf}(\mathcal{C})}$ that separates the exact predecessors from the reachable part of the upward closure would do.

Different approaches can be taken regarding how to continue the exploration by using the constraint monotonic abstraction to remove the spurious counterexample. We briefly compare them in the following discussion of the example.

Eager CMA [3]. The exploration is restarted from scratch and $(\mathbf{ct}_1 \leq 0)$ is used to systematically partition *all* exact predecessors. The upward closure is constrained to not alter the refinement predicate. Therefore, all generated valuations are approximated with the stronger ordering. Localizing refinement can make possible both the reuse of a potentially large part of the explored state space and application of the (slower) refinement to a smaller number of sets.

Lazy CMA. When backtracking, we only eliminate those constraints that were obtained as descendants of a constraint that caused the spurious trace and therefore needs to be refined. We refer to this constraint as the *pivot constraint*, and to its minimal configuration as the *pivot configuration*. In fact, we identify three localization heuristics:

- *point-based-lazy.* We associate the refinement predicates with the pivot configurations. Later in the exploration, when we hit a new pivot configuration, we constrain with respect to those predicates that were already associated with it.

- *order-based-lazy*. The point-based approach may be too localized as there is an infinite number of pivot configurations. For instance, a similar trace can continue, after $(\mathbf{rd}, \mathbf{ct}_2 = 1)$, with $\mathbf{op}_1$ and get to the minimal configuration sending $\mathbf{ct}_2$ to 2. This one is different from the mapped pivot configuration, and hence, we need to introduce a new pivot configuration with the same predicate $\mathbf{ct}_0 \leq 0$. This approach considers the predicates of all larger or smaller pivot configurations. The idea is that if the predicate was important for the mapped pivot configuration, then it must have been to separate it from a reachable upward closed part, and hence, it may be relevant.
- *descendants-based-lazy*. In addition to associating refinement predicates with pivot configurations as in the point-based approach, this heuristic leverages the fact that predicates may remain relevant for a sequence of transitions. Here we compare the exact predecessors with the predicates used to constrain the upward closure of the parent. If those predicates still hold for the predecessors, then we maintain them when closing upwards. This heuristic bears similarity to the forward propagation of clauses in IC3 [85]. In the IC3 algorithm the clauses are propagated in the trace from a preceding formula to the succeeding one if they still hold.

7.5 State Reachability Checking

In this section, we describe four different forward CMA variants (*eager*, *point-based-lazy*, *order-based-lazy* and *descendants-based-lazy*). The four procedures can also be applied backwards (as described in the experiments of Section 7.6). The four variants use grounded constraints as symbolic representations for a possibly infinite number of machine configurations. The symbolic representation is refined using atoms that are obtained using a counterexample-guided refinement scheme. The difference between the four variants lies in the way the discovered predicates (in fact, atoms for simplifying the presentation) are associated with the new symbolic representations and in the way backtracking is carried out. We start by introducing the basic “partition” procedure.

Input: A state $\mathbf{q}$, a conjunct ψ and a finite set of atoms $\mathbf{A}$
Output: A well-formed set of grounded constraints

```

1  $\Gamma := \emptyset$ ;
2 foreach ( $\text{total } \mathbb{b} : \mathbf{A} \rightarrow \mathbb{B}$ ) do
3   | foreach ( $\mathbb{c} \in \text{Min}_{\leq}(\psi \wedge \psi_{\mathbb{b}})$ ) do  $\Gamma := \Gamma \cup ((\mathbf{q}, \mathbb{c}, \mathbb{b}), \psi \wedge \mathbf{c1o}(\mathbb{c}, \mathbb{b}))$  ;
4 return  $\Gamma$ ;
```

Procedure $\text{partition}(\mathbf{q}, \psi, \mathbf{A})$, the constraint partitioning procedure common to all variants.

Partition. “ $\text{partition}(\mathbf{q}, \psi, \mathbf{A})$ ” partitions ψ according to all atoms in $\mathbf{A}$. Each obtained conjunct is further decomposed according to its $\leq$ -minimal val-

uations. Conjuncts are then used to build a well-formed grounded constraint $((q, c, \mathbb{b}), \psi')$ where $\mathbb{b}$ is a box over A . Observe that the disjunction of the grounds of the obtained grounded constraints is equivalent to ψ . Soundness is stated in Lemma 37.

Lemma 37. *Assume a finite set A of atoms. For any conjunct ψ , it is the case that $\llbracket (q, \psi) \rrbracket = \{(q, c) \mid c \in \llbracket \psi' \rrbracket_{\geq 0} \text{ for each } \psi' \in \mathbf{groundOf}(\text{partition}(q, \psi, A))\} \subseteq \llbracket \text{constraintOf}(\text{partition}(q, \psi, A)) \rrbracket$.*

Input: A machine $M = (Q, C, T, Q_{init}, C_{init}, q_{trgt})$
Output: A feasible run covering q_{trgt} or the value **unreachable**

- 1 if $q_{init} = q_{trgt}$ then return $(q_{init}, 0_C)$;
- 2 $Sym, \Gamma := \emptyset, \text{partition}(q_{init}, \wedge_{ct \in C} (ct \geq 0), \emptyset)$;
- 3 foreach $(\gamma \in \Gamma)$ do $Sym := Sym \cup \{(\gamma, \gamma)\}$;
- 4 return $\text{exploreEagerLazy}(M, Sym, Sym, \emptyset, \epsilon_C)$;

Procedure $\text{checkReachability}(M)$, the common entry point for all variants.

Eager CMA, like the other variants, starts by passing a description of the machine to the “checkReachability” procedure. It either returns a feasible run covering q_{trgt} , or states that there are no such runs. The procedure returns directly (line 1) if initial and target states coincide. It then calls “partition” to obtain a set of well-formed grounded constraints that together capture all initial configurations. These are passed to the “exploreEagerLazy” procedure.

Explore. “ $\text{exploreEagerLazy}(M, \text{working}, \text{store}, \text{sleep}, f)$ ” results in a working list procedure that maintains three symbolic sets **working**, **store** and **sleep**. **sleep** is only relevant for the lazy variants. The partial mapping $f : C \rightarrow \text{atomsOf}(C)$ encapsulates all refinement predicates discovered so far and is therefore empty when the procedure is called from “checkReachability”. Intuitively, $f(c)$ associates with the pivot configuration c those predicates that helped eliminate a false positive when c was the minimal configuration of the constraint that made the false positive possible. We will explain how f is updated when introducing the procedure “simulate”. The symbolic set **working** is used for the grounded constraints that are yet to be visited (i.e., for which the successors are still to be computed and approximated). The **store** set is used for both those grounded constraints that have been visited and for those in **working**. The **sleep** set corresponds to those constraints that might have to be visited but for which there is an $\sqsubseteq$ -equivalent representative in **store**. In case a backtracking eliminates the representative in **store**, the corresponding grounded constraint in **sleep** has to be reconsidered. This is explained in the “backtrack” procedure of the lazy variants.

The procedure picks a pair $((\varphi, \psi), \rho)$ from **working** where φ is a member of $\text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}))$. If the initial state is reached, it calls procedure “simulate” to check the associated trace and to backtrack if needed

(lines 4-5). Otherwise, it iterates through all transitions $\mathbf{t}$ in $\mathbf{T}$ and computes an exact representation of the predecessors of the constraint. The call “ $\text{decompose}(\mathbf{q}, \psi_p, \mathbb{f}, \mathbb{b})$ ” boils down, for the eager variant, to a call to “ $\text{partition}(\mathbf{q}, \psi_p, \text{image}(\mathbb{f}))$ ”. The obtained grounded constraints are used in order to update the **working**, **store**, and **sleep** symbolic sets.

If no pair was picked at line 1, then the exploration is finished and the procedure returns **unreachable**. In fact, pairs are never removed from **store** if no target states are encountered at line 4. In addition, two pairs with $\sqsubseteq$ -equivalent constraints cannot be added to **working** (lines 10-13). For this reason, executing the first line for an infinite number of times without calling the procedure “simulate” would result in an infinite sequence of constraints that would violate Lemma 35.

<p>Input: A machine description $\mathbf{M} = (\mathcal{Q}, \mathcal{C}, \mathbf{T}, \mathbf{Q}_{init}, \mathbf{C}_{init}, \mathbf{q}_{trgt})$, a preorder $\sqsubseteq$, three symbolic sets working, store and sleep, and a partial mapping $\mathbb{f} : \mathcal{C} \rightarrow \text{atomsOf}(\mathcal{C})$</p> <p>Output: A feasible run covering $\mathbf{q}_{trgt}$ or the value unreachable</p> <pre> 1 while exists $((\varphi, \psi), \rho) \in \text{working}$ with $\varphi \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}))$ do 2 remove $((\varphi, \psi), \rho)$ from working; 3 $(\mathbf{q}, \mathbb{c}, \mathbb{b}) := \varphi$; 4 if $\mathbf{q} = \mathbf{q}_{trgt}$ then 5 return simulate($\mathbf{M}, \rho, \text{working}, \text{store}, \text{sleep}, \mathbb{f}$); 6 foreach $\mathbf{t} = (\mathbf{q}, \text{op}, \mathbf{q}')$ in $\mathbf{T}$ do 7 foreach $\psi_p \in \text{post}_{\text{op}}(\text{clo}(\mathbb{c}, \mathbb{b}))$ do 8 foreach (φ', ψ') in $\text{decompose}(\mathbf{q}', \psi_p, \mathbb{f}, \mathbb{b})$ do 9 $\rho' := \rho; \mathbf{t}; (\varphi', \psi')$; 10 if exists $((\varphi_e, \psi_e), \rho_e) \in \text{store}$ s.t. φ_e is $\sqsubseteq$-equivalent to φ' then 11 if ρ_e and ρ' are not equivalent then 12 add $((\varphi', \psi'), \rho')$ to sleep; 13 else add $((\varphi', \psi'), \rho')$ to both store and working; 14 return unreachable;</pre>
--

Procedure `exploreEagerLazy`($\mathbf{M}, \text{working}, \text{store}, \text{sleep}, \mathbb{f}$), the exploration procedure common to all variants.

Simulate. This procedure checks feasibility of a trace ρ from $\mathbf{q}_{init}$ to $\mathbf{q}_{trgt}$. The procedure incrementally builds a sequence of sets of conjuncts $\Psi_n, \dots, \Psi_0$ where each Ψ_i intuitively denotes the valuations that are backwards reachable from $\mathbf{q}_{trgt}$ after k steps of ρ (starting from $k = 0$), and are still denoted by $\text{clo}(\mathbb{c}_{(n-k)}, \mathbb{b}_{(n-k)})$. The idea is to systematically intersect (a representation of) the successors of step k with the grounded constraint that gave rise to the constraint at step $k + 1$. If the procedure finds a satisfiable Ψ_0 , then a run can be generated by construction. Such a run is then returned at line 8. Otherwise, there must have been a step where the “exact” set of conjuncts does not intersect the conjunct representing the exact part that gave rise to the corresponding constraint. In other words, the trace could be eliminated by strengthening the over-approximation at line 7 of the “`exploreEagerLazy`” procedure. In this case, (at line 6 of the “`simulate`” procedure), new refinement atoms are identified using an off-the-shelf interpolation procedure for

QF_LIA (Quantifier-Free Linear Integer Arithmetic). This information will be used differently by the eager and lazy variants when calling their respective “backtrack” procedures.

Input: A machine M , three symbolic sets **working**, **store** and **sleep**, a trace $\rho = (\varphi_0, \psi_0); \tau_1; \dots; \tau_n; (\varphi_n, \psi_n)$ with $n \geq 1$ and $\text{stateOf}(\varphi_0) = q_{init}$ and $\text{stateOf}(\varphi_n) = q_{tgt}$, and a mapping $\mathbb{f} : C \rightarrow \text{atomsOf}(C)$;

Output: A feasible run covering q_{tgt} or the value **unreachable**

```

1  $\Psi_n := \{\psi_n\};$ 
2 for  $i = (n - 1)$  to 0 do
3    $\Psi'_i := \text{pre}_{\text{op}(\tau_{i+1})}(\Psi_{i+1});$ 
4    $\Psi_i := \{(\psi_i \wedge \psi'_i) \mid \psi'_i \in \Psi'_i \text{ and } (\psi_i \wedge \psi'_i) \text{ is sat}\};$ 
5   if  $\Psi_i = \{\}$  then
6      $\mathbb{f}((\text{stateOf}(\varphi_i), \text{val}(\varphi_i))) \cup :=$ 
7        $\{\alpha \mid \alpha \in \text{atomsOf}(\pi) \text{ with } \pi \in \text{ITP}(\{\psi_i\}, \Psi'_i)\};$ 
8   return backtrack( $M, \rho, i, \text{working}, \text{store}, \text{sleep}, \mathbb{f}$ );
9 return a run starting at  $(q_{init}, c)$  for some  $c \in \Psi_0$  and following till  $q_{tgt}$ 
```

Procedure simulate($M, \rho, \text{working}, \text{store}, \text{sleep}, \mathbb{f}$), the simulation procedure common to all variants.

Input: A machine M , two symbolic sets **working** and **store** and mapping $\mathbb{f} : C \rightarrow \text{atomsOf}(C)$;

Output: A feasible run covering q_{tgt} or the value **unreachable**

```

1 store, working :=  $\emptyset, \emptyset;$ 
2  $\Gamma := \text{partition}(q_{init}, \wedge_{ct \in C} (ct \geq 0), \text{image}(\mathbb{f}));$ 
3 foreach  $(\varphi, \psi)$  in  $\Gamma$  do
4    $\text{Sym} := \text{Sym} \cup \{((\varphi, \psi), (\varphi, \psi))\};$ 
5 return explore( $M, \text{Sym}, \text{Sym}, \emptyset, \mathbb{f}$ );
```

Procedure backtrack($M, _, _, \text{working}, \text{store}, _, \mathbb{f}$), for the eager variant.

Eager backtracking throws away the explored state space (line 1) and restarts the computation from scratch using the new refinement atoms captured in $\mathbb{f}$.

Lazy backtracking. Intuitively, all three lazy approaches reuse the part of the explored state space that is not affected by the new refinements. This is done by restarting the exploration from new sets **working** and **store** that are obtained after pruning away the pivot constraint identified by the argument i passed by “simulate” together with all its descendants (identified in lines 1-6). One important aspect is that the grounded constraints that have not been added to **store** at line 11 of the “exploreEagerLazy” procedure may have been discarded for the wrong reason (i.e., there was an Ξ -equivalent constraint that needs to be pruned away now). This would jeopardize soundness. For this reason, we maintain the **sleep** set for tracking the discarded grounded constraints that have to be put back into **working** and **store** if the constraint that blocked them is pruned away (see lines 4-6). The refined pivot is added to the new sets **working** and **store** (lines 10-13). Lines 7-9 are only used

by the descendants-based approach which takes into account the box of the parent.

Input: A machine M , a trace $\rho = (\varphi_0, \psi_0); \tau_0; \dots; (\varphi_n, \psi_n)$ with $n \geq 1$ and $\text{stateOf}(\varphi_0) = q_{init}$ and $\text{stateOf}(\varphi_n) = q_{trgt}$, and a natural $i : 0 \leq i < n$, three symbolic sets **working**, **store** and **sleep**, and a mapping $\mathbb{f} : C \rightarrow \text{atomsOf}(C)$;

Output: A feasible run covering q_{trgt} or the value **unreachable**

```

1  foreach  $((\varphi, \psi), \rho) \in \text{store}$  s.t.  $(\varphi_0, \psi_0); \tau_1; \dots; (\varphi_i, \psi_i)$  is equivalent to a prefix of  $\rho$  do
2      remove, if present,  $((\varphi, \psi), \rho)$  from working, store and sleep;
3      for  $j \leftarrow i$  to  $n$  do
4          if there is still a  $((\varphi', \psi'), \rho')$  in sleep s.t.  $\varphi'$  is  $\sqsubseteq$ -equivalent to  $\varphi_j$  then
5              remove  $((\varphi', \psi'), \rho')$  from sleep;
6              add  $((\varphi', \psi'), \rho')$  to both working and store;
7  if  $i \geq 1$  then
8       $\mathbb{b}_p := \mathbb{b}_{i-1}$ 
9  else  $\mathbb{b}_p := \mathbb{b}_{true}$ ;
10 foreach  $(\varphi', \psi') \in \text{decompose}(q_i, \psi_i, \mathbb{f}, \mathbb{b}_p)$  do
11     let  $\rho' := (\varphi_0, \psi_0); \tau_1; \dots; (\varphi_{i-1}, \psi_{i-1}); \tau_i; (\varphi', \psi')$ ;
12     if there is  $((\varphi_e, \psi_e), \rho_e)$  in store s.t.  $\varphi_e$  is  $\sqsubseteq$ -equivalent to  $\varphi'$  then
13         if  $\rho_e$  and  $\rho'$  are not equivalent then
14             add  $((\varphi', \psi'), \rho')$  to sleep;
15         else add  $((\varphi', \psi'), \rho')$  to both store and working;
16 return  $\text{exploreEagerLazy}(M, \text{working}, \text{store}, \text{sleep}, \mathbb{f})$ ;
    
```

Procedure $\text{backtrack}(M, \rho, i, \text{working}, \text{store}, \text{sleep}, \mathbb{f})$, the backtrack procedure common to all lazy variants.

The main difference between the lazy variants is in the way their respective “decompose” procedures associate refinement atoms with “exact” conjuncts.

Point-based. This variant is the one that “localizes” the refinement to the greatest degree. Each time an obtained grounded conjunct is considered for approximation, it checks whether its minimal valuation has already been associated with some refinement atoms. If this is the case, it passes them when calling the “partition” procedure.

Input: A state q , a conjunct ψ and a partial mapping $\mathbb{f} : C \rightarrow \text{atomsOf}(C)$

Output: A well-formed set of grounded constraints

```

1   $A := \emptyset$ ;
2  foreach  $(c \in \text{dom}(\mathbb{f}) \text{ with } \text{val}(c) \in \text{Min}_{\sqsubseteq}(\psi))$  do  $A := A \cup \mathbb{f}(c)$ ;
3  return  $\text{partition}(q, \psi, A)$ 
    
```

Procedure $\text{decompose}(q, \psi, \mathbb{f}, -)$ of the point-based-lazy variant.

Order-based. This variant “localizes” less than the point-based variant. Each time an obtained “exact” conjunct is considered for approximation, it checks whether its minimal valuation is $\sqsubseteq$ -related to an already mapped val-

uation. The union of all corresponding atoms is passed to the “partition” procedure.

Input: A state q , a conjunct ψ and a mapping $\mathbb{f} : \mathcal{C} \rightarrow \text{atomsOf}(\mathcal{C})$
Output: A well-formed set of grounded constraints

```

1 let  $A := \emptyset$ ;
2 foreach ( $c \in \text{dom}(\mathbb{f})$ ) do
3   foreach ( $c' \in \text{Min}_{\sqsubseteq}(\psi)$ ) do
4     if ( $(c' \sqsubseteq \text{val}(c))$  or  $(\text{val}(c) \sqsubseteq c')$ ) then
5        $A := A \cup \mathbb{f}(c)$ ;
6       break;
7 return partition( $q, \psi, A$ )

```

Procedure decompose($q, \psi, \mathbb{f}, _$) of the order-based variant.

Descendants-based. This variant “localizes” less than the point-based variant, but is incomparable with the order-based one. The idea is to keep those refinement atoms that were used for the parent constraint, and that are still weaker than the current conjunct that is to be approximated.

Input: A state q , a conjunct ψ , a mapping $\mathbb{f} : \mathcal{C} \rightarrow \text{atomsOf}(\mathcal{C})$, and a box $\mathbb{b}$
Output: A well-formed set of grounded constraints

```

1 let  $A := \emptyset$ ;
2 foreach ( $c \in \text{dom}(\mathbb{f})$  with  $\text{val}(c) \in \text{Min}_{\sqsubseteq}(\psi)$ ) do  $A := A \cup \mathbb{f}(c)$  ;
3 foreach  $\alpha \in \text{dom}(\mathbb{b})$  do
4   if ( $(\mathbb{b}(\alpha) \wedge (\psi \Rightarrow \alpha))$  or  $(\neg \mathbb{b}(\alpha) \wedge (\psi \Rightarrow \neg \alpha))$ ) then
5      $A := A \cup \{\alpha\}$ ;
6 return partition( $q, \psi, A$ )

```

Procedure decompose($q, \psi, \mathbb{f}, \mathbb{b}$) of the descendants-based variant.

Finally, we state the soundness of our four exploration variants. The proof is by observing that **store** always represents, at the i^{th} iteration of the loop of procedure “exploreEagerLazy”, an over-approximation of the machine configurations obtained after i steps, combined with Lemma 35 and Lemma 36 and by well-quasi-ordering of $\sqsubseteq$ on the set of constraints for a finite number of refinement atoms.

Theorem 22. *All four exploration variants are sound. In addition, each call to the procedure “checkReachability” eventually terminates if only a finite number of calls to the procedure “simulate” are executed.*

Proof. Let working_k , store_k and sleep_k be the sets **working**, **store** and **sleep** obtained at line 1 at the k^{th} iteration of the loop in procedure “exploreEagerLazy”. We can show the following propositions:

- (a) for every valuation $\mathbb{c}$, $(q_{\text{init}}, \mathbb{c})$ belongs to $\llbracket \text{store}_k \rrbracket$,
- (b) every element $((\varphi, \psi), \rho)$ such that $\text{stateOf}(\varphi) = q_{\text{trgt}}$ is in store_k iff it is in working_k ,
- (c) for each element $((\varphi, \psi), \rho)$ of store_k such that $((\varphi, \psi), \rho) \notin \text{working}_k$ and $\varphi \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}_k))$ and for

each transition $\mathbf{t} = (\mathbf{q}, \text{op}, \mathbf{q}') \in \mathbf{T}$, the configurations in $\{(\mathbf{q}', \mathbf{c}') \mid \mathbf{c}' \in \llbracket \text{post}_{\text{op}}(\text{clo}(\text{val}(\varphi), \text{box}(\varphi))) \rrbracket\}$ are also in $\llbracket \text{store}_k \rrbracket$

For proposition (a), note that the elements covering all initial configurations will be added in the beginning to both **working** and **store** (lines 2-4 of “checkReachability”). Let an initial element be such an element. In case an initial element $((\varphi, \psi), \rho)$ is removed by “backtrack” (line 1 of the eager variant and line 2 of the lazy variant) which is the only place where elements may be removed from **store**, it will be replaced (in line 4 of the eager “backtrack” and line 6 of the lazy “backtrack”) by a set of elements obtained from “decompose” whose union still denotes ψ (Lemma 37).

For proposition (b) observe that whenever an element with the state $\mathbf{q}_{\text{trgt}}$ is added to **store**, it is also added to **working** (e.g., Line 13 of “exploreEager-Lazy”) and whenever such element is removed from **store**, it is also removed from **working** (Line 6 of “backtrack”).

The intuition of proposition (c) is as follows. The algorithm always picks an element $((\varphi, \psi), \rho) \in \text{working}_k$ such that $\varphi \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}_k))$ and computes its successors. Therefore, for any element $((\varphi', \psi'), \rho') \notin \text{working}_k$ such that $\varphi' \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}_k))$, the successors must be covered by elements in **store**_k unless “simulate” and “backtrack” are called. In case “backtrack” is not called, the only places where **store**_k and **working**_k are manipulated are at lines 2 and 13 of “exploreEager-Lazy”. At line 2, an element $((\varphi, \psi), \rho) \in \text{working}_k$ such that $\varphi \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}_k))$ is removed from **working**. At line 11, for each transition $(\text{stateOf}(\varphi), \text{op}, \text{stateOf}(\varphi')) \in \mathbf{T}$, $\text{post}_{\text{op}}(\text{clo}(\text{val}(\varphi), \text{box}(\varphi)))$ is computed, then all elements $((\varphi', \psi'), \rho, \mathbf{t}, (\varphi', \psi'))$ such that the union of the grounded constraints (φ', ψ') covers $\text{post}_{\text{op}}(\text{clo}(\text{val}(\varphi), \text{box}(\varphi)))$ are added to **working**_k and **store**_k, in case another element with constraint φ' does not already exist in **store**_k. Otherwise, the element is added to **sleep**_k. Thus, in case there is no call to “backtrack”, the successors of any element that is removed from **working**_k are denoted by the elements in **store**_k.

We show that proposition (c) holds also when “backtrack” is called. Given a trace $\rho = (\varphi_0, \psi_0); \mathbf{t}_1; \dots; \mathbf{t}_n; (\varphi_n, \psi_n)$ and $0 \leq i < n$ as the index of the element to be refined, “backtrack” removes any element $((\varphi, \psi), \rho)$ whose trace contains $(\varphi_0, \psi_0); \mathbf{t}_1; \dots; \mathbf{t}_i; (\varphi_i, \psi_i)$ from **store**_k, **working**_k, and **sleep**_k. If some of the elements removed from **store**_k had $\sqsubseteq$ -equivalent representative in **sleep**_k, the representatives will be removed from **sleep**_k and added to **store**_k and **working**_k. This is done at lines 4-6 of “backtrack”. As a result, proposition (c) holds.

Soundness. Suppose the algorithm returns **unreachable**. Then, at some iteration, there is no element $((\varphi, \psi), \rho)$ in **working** such that $\varphi \in \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}))$. Combined with the propositions (a) and (c) of

Theorem 22, we have that $\llbracket \text{store} \rrbracket$ is a fixpoint that is an over-approximation of all reachable configurations. Proposition (b) ensures that no element with the state q_{trgt} exists in **store** when the algorithm returns **unreachable**.

Termination. The procedure “checkReachability” terminates if only a finite number of calls to procedure “simulate” are executed. This relies on the fact that when the number of calls to procedure “simulate” are finite, the only source of non-termination can be the while loop in “exploreEagerLazy” if the set $\text{constraintOf}(\text{working}) \cap \text{Min}_{\sqsubseteq}(\text{constraintOf}(\text{store}))$ never becomes empty. Suppose after finishing all the calls to “simulate” an infinite sequence of constraints $A = \varphi_0, \varphi_1, \dots$ is obtained at Line 1 of the while loop while $i \geq 0$ is the iteration at which φ_i was removed from **working**. Observe that for only a finite number of constraints in the sequence where $0 \leq i < j$ we have φ_i is equivalent to φ_j with respect to $\sqsubseteq$. This is because Lines 6 and 15 of “backtrack” are the only lines in which elements are added to **store** and **working** as a result of a call to the procedure “simulate”. In both cases, the constraint φ' of an element $((\varphi', \psi'), \rho')$ that is added to **store** and **working** might be $\sqsubseteq$ -equivalent to the constraints of the elements already in those sets. However, we are sure that the procedure “backtrack” is called finitely many times because it is only called once from the procedure “simulate”. Hence, there exist only finitely many constraints in the sequence A that have $\sqsubseteq$ -equivalent constraints. Assume we extract a sequence B from the sequence A such that B contains only one representative for any set of $\sqsubseteq$ -equivalent constraints. Observe that in sequence B for any $0 \leq i < j$, $\varphi_i \not\sqsubseteq \varphi_j$. This is because 1) an element is removed from **working** only if it is minimal in **store** with respect to $\sqsubseteq$, and 2) elements are not removed from **store** when there is no call to the procedure “simulate”. This sequence violates $\mathcal{WQO}$ of $\sqsubseteq$. $\square$

7.6 Experimental Results

We have implemented our techniques in our open-source prototype tool ZAAMA. The tool and benchmarks are available¹. The tool relies on the Z3 SMT solver [104] for its internal representations and operations.

The inputs of the prototype tool are counter machine encodings of Boolean concurrent programs with broadcasts and arbitrary tests (as described in Section 7.4). We have experimented with over eighty different counter machine reachability problems. These were obtained from our prototype tool PACMAN [61], which checks local (i.e., assertion) or global (e.g., deadlock freedom) properties in concurrent programs (some inspired by [52] and [40]).

Given a property to check in a concurrent program, PACMAN proceeds in predicate abstraction iterations. For each set of tracked predicates, it creates a counter machine reachability problem. Combining PACMAN with ZAAMA

¹<https://gitlab.liu.se/live/zaama>.

Figure 7.3: Comparing execution times of eager and lazy variants on a logarithmic scale. Each axis corresponds to a variant of monotonic abstraction and the execution times of the different variants are compared against each other.

results in a nested CEGAR loop: an outer loop for generating counter machine reachability problems, and an inner loop for checking the resulting problems. About 45% of the generated counter machines are not monotonic. We tested all of those machines separately with ZAAMA in different settings for each benchmark and reported the execution times. Thus, the PACMAN overhead is not included in the reported times. Note that although 55% of the examples are monotonic, they still need refinement in forward exploration.

We also tested our benchmarks with the tool BREACH introduced in [82]. BREACH cannot take non-monotonic inputs and is inherently incapable of solving reachability problems for systems that are the main target of this chapter. Thus, we could apply it only to the monotonic benchmarks; for which, the runtime of BREACH was less than 5 seconds in each case. We consider this to be an encouraging result for adapting BREACH to non-monotonic systems. The challenge, however, is to have an under-approximation search engine for such systems and one could investigate the possibilities of developing such an engine or using acceleration tools such as FASTer [17].

We have chosen a time-out of 30 minutes for each of the variants: eager, point-based, order-based and descendants-based, both in forward and in backward exploration. We have conducted our experiments on an Intel Core i7 processor running at 2.93GHz with 8GB of memory. We report on our results in Figure 7.3 where we consider, for each setting, all possible lazy pairs. Moreover, we consider the pairs that consist of the eager approach and a lazy approach.

The forward explorations turned out to be faster than the corresponding backward ones in about 25% of the examples. We expected the forward exploration to be slower as it needs several refinement steps because it starts from the initial configurations which are typically much more constrained than the target configurations. We considered the forward exploration because it offers more opportunities to test the effect of localizing the refinement in problems that typically require more refinement steps in forward. Indeed, the figures show that the analysis times of the different variants coincide more often when checking reachability backward than checking it forward, and overall, there have been many more time-outs in forward than in backward.

Furthermore, the lazy variants were able to conclude on most of the reachability problems, in fact each of the reachability problems has been solved by at least one of the lazy variants (except for one problem in the backward exploration), when the eager variant timed out on several of them. This is an encouraging result that confirms the advantages of localizing refinement. There are some cases where the eager variant did better than all lazy ones. These correspond to cases where localization required more refinement efforts to reach a conclusion.

We also observe that the order-based approach times out in about half of the forward searches, while the point-based only times out in two cases. This goes against the initial intuition that larger valuations would profit from the refinement predicates of the smaller ones. One explanation could be that if the larger valuation were to require the same predicate as the smaller one, then adding the predicate would result in a redundant representation that should be eliminated. It therefore seems that it does not take long for the point-based to discover this redundancy while still profiting from the localization of the refinement. Instead, the order-based uses predicates even when they are not

proven to be needed resulting in finer grained symbolic elements that slow down the exploration.

It is interesting to observe that the descendants-based approach did better in the forward exploration than the point-based approach. One explanation could be that, in forward, relevant refinement interpolants sometimes correspond to weak inductive invariants that get propagated by this approach. When exploring backwards it seems that, at least for our examples the invariants corresponding to the “bad” configurations do not profit from this parent-child transmission.

7.7 Conclusion

We have introduced and discussed three different ways of localizing constrained monotonic abstraction in systems with infinite control. For this, we have targeted reachability problems for (possibly non-well-structured) counter machines obtained as abstractions of concurrent programs. Our new techniques allow us to systematically avoid trashing the explored state-space before encountering the false positives that necessitate the introduction of new refinement predicates. This allowed us to consistently improve on the existing eager exploration, both in forward and in backward explorations. Possible future works could consider combining forward and backward approximations, using the pivot configuration to make it possible to choose interpolants that are easier to generalize and assessing the feasibility of combination with new partial order techniques.

8 Verifying Safety of Parameterized Heard-Of Algorithms

We consider the problem of automatically checking safety properties of fault-tolerant distributed algorithms. We express the considered class of distributed algorithms in terms of the Heard-Of model, where arbitrarily many processes proceed in infinite rounds in the presence of failures such as message losses or message corruptions. For the considered class, we propose a sound but (in general) incomplete procedure that is guaranteed to terminate even in the presence of unbounded numbers of processes. In addition, we report on our experiments for which our approach either proved correctness or automatically found a concrete trace that violated the considered safety property.

8.1 Introduction

In general, it is difficult to prove the correctness of fault-tolerant distributed protocols. Such protocols are meant to operate in the presence of faults ranging from process crashes to message losses or corruption. We consider the parameterized case where arbitrarily many identical processes participate in running the distributed protocol. We adopt the popular Heard-Of model [30, 23]. This model uniformly describes distributed protocols in the presence of transmission-based failures, whether static or dynamic, permanent or transient. Protocols in the Heard-Of model proceed in rounds. Each process sends a message to other processes at each round, hears from some of them, and updates its state based on the received messages. Hence, at each round, a process “hears” from a set of other processes. Fault descriptions are captured

by stating constraints on the possible sets of processes and the messages each process hears from others (e.g., each process hears from at least half of the processes or at most a third of the sent messages have been corrupted).

We consider the problem of automatically establishing the correctness of safety properties for parameterized distributed protocols expressed in the Heard-Of model. The safety properties we consider concern checking state reachability, i.e., reachability of configurations where a given number of processes are in some forbidden combination of states. Observe that we do not check whether the protocols have made progress. This would require us to account for communication predicates that ensure that the processes eventually hear from enough other processes. However, we need to constrain, depending on the environment we want to capture, that messages may be lost (benign crashes or transmission failures) or altered (corruption failure). For consensus protocols, this is enough to capture all executions that violate the agreement (two processes decide on different values), validity (a value is decided although no process proposed it) or irrevocability (a decided value is revoked). The verification problem is made difficult by the parameterization in the number of processes and by the allowed faults. Parameterization requires us to verify infinite families of the protocols, one for each number of participating processes. The transmission model allows each process to receive a subset of the sent messages (benign failures) in addition to some altered messages (corrupted communication), making this information local to the processes.

Related work. Abstractions for threshold-based fault-tolerant distributed systems were introduced in [79, 86]. The work is extended to synchronous round-based semantics in [121]. These works can handle interesting fault-tolerant protocols in the presence of different faults, such as Byzantine faults. However, they require the user to encode the distributed system in terms of a threshold automaton and propose interval-based over-approximations or bounded-model-checking-based under-approximations for the parameterized verification problem. The models we consider directly target protocols expressed in the Heard-Of model with a sound over-approximation and can account for message losses (omission fault) and message alteration (corruption fault). The work in [99] has the merit of proposing cutoffs for a syntactically restricted class of consensus protocols. The class is also expressed in the Heard-Of model. While we do not provide such cutoffs, our work can afford to check correctness for richer fragments that can more faithfully capture constructs such as “the number of received messages with value v_0 is at least two thirds the number of processes” as opposed to “the number of received messages is at least two thirds the number of processes and all of them have value v_0 ”. The approach in [99] can verify Heard-Of protocols such as Paxos that we cannot verify in our current approach. Because in our current setting,

we have only many-to-many transmissions, while we need to account for one-to-many and many-to-one transmissions to capture those protocols. However, they only consider benign faults for the protocols, but our approach can handle both benign and corruption faults. To the best of our knowledge, we are the first to verify Heard-of protocols in the presence of corruption faults.

Ongoing works [90, 9] study automatizing the process of deciding satisfiability of constraints that involve arbitrarily many processes and cardinality constraints over sets of received messages with specific properties. Such constraints naturally arise when verifying fault-tolerant distributed protocols. For instance, [53, 107] consider a rich class of protocols but require the user to supply such constraints to automatically establish correctness. The work in [107] abstracts the quorum of threads in the Paxos protocols by introducing a new sort for quorum and adding an axiom to capture the fact that the intersection of two quorums is non-empty. While this abstraction is enough for verifying Paxos, it is too coarse for the protocols we consider. This is because we consider protocols in which the *size* of the intersection of quorums is essential for proving their correctness. Other approaches [66, 77, 46, 69] can tackle wider classes of systems but adopt an interactive approach to verification, while our approach is fully automatic.

Contributions. We propose a sound and automatic approach to checking safety properties. More specifically:

1. We identify a subclass of fault-tolerant distributed protocols in terms of the Heard-Of model and describe the considered safety properties.
2. We introduce a symbolic representation by which we capture cardinality constraints on multisets (formed by values of the variables or heard-of sets) using integer counters, hence avoiding the challenge of implementing quantifier elimination for theories with cardinality constraints.
3. We show how to use the symbolic representation in a sound but (in general) incomplete procedure for checking state reachability in the presence of lossy or corrupt communication.
4. We show termination of the procedure even in the presence of arbitrarily many processes.
5. We report on our experiments with correct and buggy examples.

Outline. We describe the challenges of the verification problem using an illustrative example in Section 8.2. We then introduce the class of distributed protocols and the properties we aim to verify in Section 8.3. We formalize the symbolic representations in Section 8.4 and use them in Section 8.5 in a sound

```

init:  $x, res = -1$ 
 $r \bmod 1 = 0$ :
  send  $x$ ;
  1.  $|HO| > 2n/3 \wedge |HO^1| \leq |HO^0| \leq 2n/3 \rightarrow x := 0$ 
  2.  $|HO| > 2n/3 \wedge |HO^0| < |HO^1| \leq 2n/3 \rightarrow x := 1$ 
  3.  $|HO^0| > 2n/3 \rightarrow x, res := 0, 0$ 
  4.  $|HO^1| > 2n/3 \rightarrow x, res := 1, 1$ 
  5.  $|HO| \leq 2n/3 \rightarrow skip$ 

```

Figure 8.1: One-Third-Rule consensus algorithm. An arbitrary number of processes synchronize in rounds and try to choose the same value for **res**. **HO** is the multiset of values received from other processes and $|HO^d|$ is the number of those messages that are equal to **d**.

(but in general incomplete) verification procedure for which we show termination. We describe our experiments with our open-source prototype tool that implements the procedure in Section 8.6, and we conclude in Section 8.7.

8.2 Illustrative Example

The One-Third-Rule algorithm listed in Figure 8.1 is a simple consensus protocol that can tolerate benign transmission failures such as process crashes and message losses. Each process p has two local variables x_p and res_p that range over finite domains. The values of each variable x_p range over the set $\{0, 1\}$. They are used to capture the candidate of the process p in the consensus algorithm. The values of each variable res_p range over $\{-1, 0, 1\}$ and are used to capture the decision of the process. The initial value -1 captures that the process has not decided yet. The example is formalized in the Heard-Of model where n processes operate in infinite rounds in lock-step. The goal of the protocol is for the processes to agree on one of the initial values as a common output.

In each round, a process first sends its local candidate value x_p to all other processes and receives values sent by other processes. Then, it executes one of the guarded commands that follow the send operation and whose guard is satisfied. In the original HO model [30], it is assumed that process identifiers of the processes that a process p hears from are stored in the set HO_p . We make a small modification and assume that only the received values from those processes are stored in a local multiset HO_p called the heard-of multiset of p . There are as many HO_p multisets as there are processes at each round. These are used to capture different failures uniformly (e.g., delays, losses, crashes, and corruption). For instance, if $\langle x \rangle$ is the multiset obtained by collecting the values of all variables x_p just sent by all processes, and in case of benign transmission failures (e.g., process crashes or message losses), each

HO_p will be smaller than $\langle x \rangle$ for each value, written $HO_p \leq \langle x \rangle$. For a multiset $\langle m \rangle$, we write $|\langle m \rangle|$ to mean the cardinality of $\langle m \rangle$. For instance, $|\langle x \rangle|$ is the number of processes running the algorithm (i.e., n) while $|HO_p|$ captures the total number of messages received by process p (i.e., the total number of processes that p heard from). Moreover, for any value d in the domain of the sent variables, we write $|HO_p^d|$ to mean the number of those messages that are equal to d .

In Figure 8.1, a process p that receives more messages than two-thirds of the total number of processes (i.e., $|HO_p| > 2n/3$) will update the value of its local candidate x_p with the smallest most often received value (lines 1 to 4). Besides that, if among the received messages, more than two-thirds of the number of processes have the same value (here, $|HO_p^0| > 2n/3$ or $|HO_p^1| > 2n/3$), then both variables x_p and res_p are updated to that value (lines 3 and 4). It is then said that the process has decided on the value of res_p . Observe that if a process does not receive its candidate value x_p from more than $2n/3$ processes, then it will not decide on it (lines 1 and 2). Furthermore, if a process has only heard from less than $2n/3$ processes, it will not update its local variables (line 5).

Typical safety properties for such consensus protocols include:

- Agreement: whenever two processes have reached a decision, the values they have decided on must be equal.
- Validity: if all processes propose the same initial value, then the processes which reach a decision must have decided on that initial value.
- Irrevocability: if a process decides on a value, it does not revoke its decision later.

Detecting violations of the above properties boils down to checking reachability of sets of configurations for unbounded numbers of processes. However, the correctness of the One-Third-Rule algorithm is independent of the number of processes. In fact, its correctness lies in the fact that: (1) in each round, $HO_p \leq \langle x \rangle$ for each process p , (2) only those processes can update their x which have heard from more than two thirds of the total number of processes, and (3) only those can decide which have heard the same value from more than two-thirds of the processes.

To capture an unbounded number of processes, we use constraints that group the processes based on the valuations of their local variables. Observe that there are finitely many such valuations. We then describe the bad configurations using such constraints. For instance, to check the agreement property for the One-Third-Rule algorithm, we need to check the reachability of all constraints capturing all configurations where at least two groups of processes, namely i and j , have $res_i = 0$ and $res_j = 1$. Assume this constraint had been reached after r rounds. It is not difficult to see that process groups i and

Figure 8.2: A run of the One-Third-Rule algorithm by two process groups i and j in backward analysis. The analysis starts from the configurations with processes that have decided on different values of $\mathbf{res}$. The widths of the bars model the size of the corresponding multisets. Different colors correspond to different rounds.

j could not have executed the guarded commands 3 and 4 during the same round r and assign 0 and 1 to $\mathbf{res}_i$ and $\mathbf{res}_j$ respectively. This is because they would have had to satisfy both of the guards $|\mathbf{HO}_i^0| > 2n/3$ and $|\mathbf{HO}_j^1| > 2n/3$. Combined with $\mathbf{HO}_p \leq \langle x \rangle$ for each process p (since message loss is the only fault), we get $|\langle x \rangle^0| > 2n/3$ and $|\langle x \rangle^1| > 2n/3$. This would give $|\langle x \rangle| > 4n/3$, which contradicts the assumption that the number of processes in the system is n . Thus, what remains is to look for runs in which $\mathbf{res}_i$ and $\mathbf{res}_j$ are set to different values in separate rounds.

Figure 8.2 depicts a possible backward run in which each group of processes in each round is represented by its valuation, its heard-of multiset and the weakest predicate on its local variables that needs to be satisfied to make the run possible. In this description, we do not account for corruption or du-

plication of messages and therefore assume the heard-of multisets are smaller (because of message loss) than the multiset of the sent values $\langle \mathbf{x} \rangle$. Accounting for corruption or duplication of messages is a matter of assuming other relations between $\langle \mathbf{x} \rangle$ and the heard-of multisets. A key to the correctness of the algorithm is the fact that the fraction $2n/3$ used in the guards ensures that the local heard-of multisets of the participating processes (i.e., not executing the `skip` command because they did not receive enough messages) have large intersections (in fact, larger than $n/3$ for any pair of such multisets).

We start the run without any assumption on $\langle \mathbf{x} \rangle$, therefore it satisfies `true`. If all processes in the group i and all of those in the group j had executed the commands 3 and 1 respectively during round $\mathbf{r}$ (note that each group could have also executed more commands, and we might need to split groups), one of the possible predecessors would be that the same process groups i and j existed with valuations $\mathbf{res}_i = 0$ and $\mathbf{res}_j = -1$. Moreover, the predicate $|\langle \mathbf{x} \rangle^0| > 2n/3$ needs to hold at the beginning of round $\mathbf{r}$. This is implied by the guards of the commands 3 and 1, $|\mathbf{HO}_i^0| > 2n/3$ and $|\mathbf{HO}_j| > 2n/3 \wedge |\mathbf{HO}_j^1| \leq |\mathbf{HO}_j^0| \leq 2n/3$, as well as the invariant $\mathbf{HO}_p \leq \langle \mathbf{x} \rangle$ for each process p . We could unroll the loop once more, assuming that in round $\mathbf{r} - 1$ the two process groups had executed commands 1 and 2 respectively and assigned different values to their variables $\mathbf{x}_i$ and $\mathbf{x}_j$ (this does not contradict $|\langle \mathbf{x} \rangle^0| > 2n/3$). The guards of the corresponding commands together with the invariant $\mathbf{HO}_p \leq \langle \mathbf{x} \rangle$ for each process p entail that the predicate $|\langle \mathbf{x} \rangle^0| > n/3$ held at the beginning of the round $\mathbf{r} - 1$. Further unrollings of the loop in backward analysis for any number of times will maintain $|\langle \mathbf{x} \rangle^0| > n/3$. As a result, firing command 4 in some previous iteration would have been impossible as it requires $|\mathbf{HO}^1| > 2n/3$. This command is, however, needed to reach the initial state. A systematic exploration of similar constraints allows us to conclude the impossibility of deciding on different values.

The work in [23] introduced algorithms in the Heard-Of model in which the received messages might be corrupted. One such algorithm is demonstrated in Figure 8.3. We can handle such algorithms, and the analysis is similar to the case where we have omission faults. The only difference is that the invariant in the presence of corruption faults is that no more than α messages received per round, per process, and data value will be corrupted, i.e., $|\mathbf{HO}_p^d| \leq |\langle \mathbf{x} \rangle^d| + \alpha$.

Our work proposes a sound but (in general) incomplete algorithm for checking control state reachability for an unbounded number of processes. The algorithm is guaranteed to terminate and starts from a symbolic representation of all bad configurations. It successively computes representations of over-approximations of predecessor configurations.

```

init: x, res = -1
r mod 1 = 0:
  send x;
  1.  $|HO| > T \wedge |HO^1| \leq |HO^0| \leq E \rightarrow x := 0$ 
  2.  $|HO| > T \wedge |HO^0| < |HO^1| \leq E \rightarrow x := 1$ 
  3.  $|HO^0| > E \rightarrow x, res := 0, 0$ 
  4.  $|HO^1| > E \rightarrow x, res := 1, 1$ 
  5.  $|HO| \leq T \rightarrow \text{skip}$ 

```

Figure 8.3: $\mathcal{A}_{E,T}$ consensus algorithm [23]. An arbitrary number of processes synchronize in rounds and try to choose the same value for **res**. The messages might get lost or corrupted. Per each round, process, and data value, there will be at most α corrupted messages. T is the threshold on the number of received messages and E is a high enough number of received messages with a certain value. According to [23], for correctness of the algorithm, it is sufficient that $T \geq 2(n + 2\alpha - E)$, $E \geq \frac{n}{2} + \alpha$ and $n > T \geq E$. We check correctness by adding these predicates as invariants.

8.3 Heard-of Programs

To simplify the presentation, we use a unique domain for all local variables and assume programs to proceed in infinite rounds where the state of each process is captured by the local variables. Introducing specific data domains for each variable or explicit local states in the transitions is straightforward. We use *valuations* (i.e., mapping from the set of local variables of a process to its domain) to capture the values of process variables. We define the syntax and semantics of a language to capture a class of heard-of programs. A heard-of program $\text{Pr} = (\text{B}, \text{P}, \text{Ts})$ involves:

- A set V of variables local to each process.
- A finite set $D \subset \mathbb{Z}$ of possible data values,
- An initial set of valuations **Init** sending local variables V to D ,
- A set of transitions **Tr**.

The syntax of such programs is as follows.

```

Pr    ::=  init tr1 ... tr|Tr|
init  ::=  v | v := d
tr     ::=  (r mod |Tr| = e : send v; cmd1, ... cmdK)
cmd    ::=  (grd, val1 → val2)
grd    ::=  grd ∨ grd | grd ∧ grd | true | false | atom
atom   ::=  |HOd1|| cmp |HOd2|| | |HOd| cmp c.n | |HO| cmp c.n
cmp    ::=  > | < | ≥ | ≤

```

Each process starts by setting initial values to its local variables. Then, all processes execute the transitions in a lock-step manner. Each program consists of a macro-round which is a sequence of $|\text{Tr}|$ consecutive rounds $(\mathbf{r} \bmod |\text{Tr}|) = 0, \dots, (\mathbf{r} \bmod |\text{Tr}|) = |\text{Tr}| - 1$. The program starts in round $\mathbf{r} = 0$ and at each round $\mathbf{r}$, all of the processes will execute the transition designated with $(\mathbf{r} \bmod |\text{Tr}|)$. $\mathbf{r}$ is incremented after each transition.

In each transition $(\mathbf{r} \bmod |\text{Tr}| = \mathbf{e} : \text{send } v; \text{cmd}_1, \dots, \text{cmd}_K)$, first, all processes send the value of their local variable v . After **send**, there is an implicit receive step in which the processes receive the values sent by others. Between the send and receive of the messages, an adversarial environment can choose to drop or alter messages. The received values are stored in a **HO** (heard-of) multiset that is local to each process. The impact of the environment is captured by the heard-of multiset.

After send and receive, each process $\mathbf{p}$ with heard-of multiset $\text{HO}_{\mathbf{p}}$ executes a guarded command $\text{cmd}_k = (\text{grd}_k : \text{val}_k^1 \rightarrow \text{val}_k^2)$ where $\text{HO}_{\mathbf{p}} \models \text{grd}_k$. A **grd** mainly focuses on capturing cardinality of some **HO** multiset(s). In many cases, this cardinality is compared to a fraction of the total number of processes, i.e., c.n . In order to simplify the presentation, we consider each **cmd** to be a change in the local valuation of a process. A **skip** command can easily be transformed to this format by choosing identical values for val^1 and val^2 in the command. Introducing explicit local states is simple but would not improve readability.

Configurations. Configurations of a heard-of program consist of the round number, as well as the local state of the processes, i.e., their valuations and heard-of multisets. More formally, a configuration $\mathbf{c} = (\mathbf{r}, \langle \mathbf{p}_1, \dots, \mathbf{p}_a \rangle)$ of $\text{Pr} = (\mathbf{B}, \mathbf{P}, \text{Ts})$ is a tuple in which:

- a is the number of processes in the configuration,
- $\mathbf{r}$ is the round number,
- for all i in $0 \leq i \leq a$, $\mathbf{p}_i = (\text{val}_i, \text{HO}_i)$ is a process where:
 - val_i is a mapping $\mathbf{V} \rightarrow \mathbf{D}$. In other words, the valuation val_i maps each local variable of the process to a value in the domain,
 - $\text{HO}_i : \mathbf{D} \rightarrow \mathbb{N}$ is a multiset of integer values.

Values of a configuration. For a configuration $\mathbf{c} = (\mathbf{r}, \langle \mathbf{p}_1, \dots, \mathbf{p}_a \rangle)$ and for any variable $v \in \mathbf{V}$ we define $\text{valuesOf}(\mathbf{c}, v)$ to be a multiset containing all the local values of v in all the processes. More formally, for all $\mathbf{d} \in \mathbf{D}$, $\text{valuesOf}(\mathbf{c}, v)(\mathbf{d}) = |\{ \mathbf{p}_i | \mathbf{p}_i = (\text{val}_i, \text{HO}_i) \text{ with } \text{val}_i(v) = \mathbf{d} \}|$.

Example 16. For the program in Figure 8.1, consider the following processes at round $\mathbf{r} = 0$.

- $p_1 = ((x_1 = 1, r_1 = -1), \emptyset)$
- $p_2 = ((x_2 = 1, r_2 = -1), \emptyset)$
- $p_3 = ((x_3 = 0, r_3 = -1), \emptyset)$
- $p_4 = ((x_4 = 1, r_4 = -1), \emptyset)$

The configuration $c = (0, \langle p_1, p_2, p_3, p_4 \rangle)$ captures an initial configuration. The heard-of multisets of the processes are empty initially. The values of variable x are captured by the multiset $\text{valuesOf}(c, x) = \langle 0, 1, 1, 1 \rangle$. We have that $\text{valuesOf}(c, x)(1) = 3$.

Semantics. Given a program $\text{Pr} = (B, P, \text{Ts})$, the processes start executing Tr from an initial configuration $c_{\text{init}} = (r^{\text{init}}, \langle p_1^{\text{init}}, \dots, p_a^{\text{init}} \rangle)$ where $r^{\text{init}} = 0$, and for all $1 \leq i \leq a$, $p_i^{\text{init}} = (\text{val}_i, \emptyset)$, and $\text{val}_i \in \text{Init}$. Suppose configurations c and c' can be written (up to a renaming of the processes) as $c = (r, \langle (\text{val}_1, \text{HO}_1), \dots, (\text{val}_a, \text{HO}_a) \rangle)$, $c' = (r', \langle (\text{val}'_1, \text{HO}'_1), \dots, (\text{val}'_a, \text{HO}'_a) \rangle)$, and $\text{tr} = (r \bmod |\text{Tr}| = e : \text{send } v; \text{cmd}_1, \dots, \text{cmd}_K)$ with $\text{cmd}_k = (\text{grd}_k, \text{val}_k^1 \rightarrow \text{val}_k^2)$ for each $k : 1 \leq k \leq K$. We write $c \xrightarrow{\text{tr}} c'$ to mean that $r' = r + 1$ and there is a total function $F : \{1, \dots, a\} \rightarrow \{1, \dots, K\}$ where for each $i : 1 \leq i \leq a$, $\text{val}_i = \text{val}_{F(i)}^1$, $\text{val}'_i = \text{val}_{F(i)}^2$ and $\text{HO}_i \models \text{grd}_{F(i)}$. Note that the numbers of processes in c and c' are finite, arbitrarily large and equal.

Example 17. Consider the Example 16 and the transition tr to be the transition tr in Figure 8.1. Processes 1 to 4 can take guarded commands 2, 2, 5, and 4, respectively and result in the configuration $c' = (1, \langle p'_1, p'_2, p'_3, p'_4 \rangle)$ where:

- $p'_1 = ((x_1 = 1, r_1 = -1), \langle 1, 0, 1 \rangle)$
- $p'_2 = ((x_2 = 1, r_2 = -1), \langle 1, 1, 0 \rangle)$
- $p'_3 = ((x_3 = 0, r_3 = -1), \langle 0, 1 \rangle)$
- $p'_4 = ((x_4 = 1, r_4 = 1), \langle 1, 1, 1 \rangle)$

Here, $F = \{(1, 2), (2, 2), (3, 5), (4, 4)\}$ witnesses $c \xrightarrow{\text{tr}} c'$.

8.4 Symbolic Representation

In this section, we formally define our symbolic representation and describe a corresponding entailment relation. We assume a program $\text{Pr} = (B, P, \text{Ts})$.

Constraints. A constraint φ is a tuple $(\mathbf{e}, \mathbf{gl}, \{\mathbf{val}_1, \dots, \mathbf{val}_b\})$ that denotes a possibly infinite set of configurations such that:

- An integer $\mathbf{e}$ in $\{0, \dots, |\mathbf{Tr}|-1\}$ captures the control location of the execution.
- A *global condition* $\mathbf{gl}$ in the form of a Presburger predicate with a free variable n (for the number of processes) and a set of $|\mathbf{V}| \times |\mathbf{D}|$ free variables $\# \mathbf{V} = \{\#_{\mathbf{v}^{\mathbf{d}}} \mid \mathbf{v} \in \mathbf{V} \text{ and } \mathbf{d} \in \mathbf{D}\}$, where each $\#_{\mathbf{v}^{\mathbf{d}}}$ accounts for the number of occurrences of value $\mathbf{d}$ among the variables $\mathbf{v}$ of all processes.
- A *base* formed by a set of valuations $\{\mathbf{val}_1, \dots, \mathbf{val}_b\}$. The valuations are similar to those used by the configurations and represent *groups* of processes with the same valuations.

Each valuation in the base of a constraint corresponds to one or more processes with that valuation in a denoted configuration. Besides, a constraint does not explicitly represent conditions on heard-of multisets; instead, it maintains a global condition $\mathbf{gl}$ which is a predicate on the number of occurrences of values of the local variables of all processes (i.e. global state). The intuition is that the heard-of sets ultimately originate from the global state. Hence, constraining their values (to satisfy some guarded commands) is a matter of constraining the global state and accounting for possible faults (see Section 8.5). For a predicate p that might depend on a set of integer variables $X = \{x_1, \dots, x_q\}$, we write $p(X)$ to clarify that p might have a subset of X as free variables. To simplify the presentation, we typically do not mention that a predicate might have n (for the number of processes) as a free variable. For instance, we write $\mathbf{gl}(\# \mathbf{V})$ to clarify that $\mathbf{gl}$ might have a subset of $\# \mathbf{V}$ in addition to the variable n as its free variables.

Denotations. For a constraint $\varphi = (\mathbf{e}, \mathbf{gl}^\varphi, \{\mathbf{val}_1^\varphi, \dots, \mathbf{val}_b^\varphi\})$ we write $\mathbf{c} \models \varphi$ to mean φ denotes the configuration $\mathbf{c} = (\mathbf{r}, \langle (\mathbf{val}_1^{\mathbf{c}}, \mathbf{HO}_1^{\mathbf{c}}), \dots, (\mathbf{val}_a^{\mathbf{c}}, \mathbf{HO}_a^{\mathbf{c}}) \rangle)$. Intuitively, φ should account for all local valuations in $\mathbf{c}$ (captured by a surjection from $\{1, \dots, a\}$ to $\{1, \dots, b\}$). Moreover, the predicate $\mathbf{gl}^\varphi$ must be compatible with the multiset of all local valuations of the processes. More formally:

1. $\mathbf{r} \bmod |\mathbf{Tr}| = \mathbf{e}$.
2. Replacing in the global condition $\mathbf{gl}$ each occurrence of $\#_{\mathbf{v}^{\mathbf{d}}}$ with the number of occurrences of $\mathbf{d}$ in $\mathbf{c}$ (i.e., $\text{valuesOf}(\mathbf{c}, \mathbf{v})(\mathbf{d})$) and each occurrence of n with the number of processes in $\mathbf{c}$ (i.e., a) results in a satisfiable formula.
3. There is a surjection $\mathbf{Sr} :: \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ such that for all $1 \leq i \leq a$, $\mathbf{val}_i^{\mathbf{c}} = \mathbf{val}_{\mathbf{Sr}(i)}^\varphi$

Observe that a constraint $(\mathbf{e}, \mathbf{gl}, \{\mathbf{val}_1, \dots, \mathbf{val}_b\})$ will have an empty denotation if its base requires the presence of valuations forbidden by the global condition, or if the global condition requires valuations forbidden by the base (since we require a surjection). It is safe to systematically discard such constraints in our analysis presented in Sec. 8.5.

Example 18. The configuration $\mathbf{c}'$ in Example 17 is in the denotation of the constraint $\varphi' = (0, \#x^1 > 2n/3, \{(x_1 = 1, r_1 = -1), (x_1 = 0, r_2 = -1), (x_3 = 1, r_3 = 1)\})$ with $\mathbf{Sr} = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.

Entailment. We write $\varphi_1 \sqsubseteq \varphi_2$ to mean $\varphi_1 = (\mathbf{e}, \mathbf{gl}^1, \{\mathbf{val}_1^1, \dots, \mathbf{val}_{b_1}^1\})$ is entailed by $\varphi_2 = (\mathbf{e}, \mathbf{gl}^2, \{\mathbf{val}_1^2, \dots, \mathbf{val}_{b_2}^2\})$. This will ensure that configurations denoted by φ_2 are also denoted by φ_1 . Intuitively, φ_1 and φ_2 must have the same round number modulo $|\mathbf{Tr}|$, and

- There is a bijection $Y :: \{1, \dots, b_2\} \rightarrow \{1, \dots, b_1\}$ with $\mathbf{val}_j^2 = \mathbf{val}_{Y(j)}^1$ for all $1 \leq j \leq b_2$.
- $\mathbf{gl}^2 \Rightarrow \mathbf{gl}^1$.

8.5 A Symbolic Verification Procedure

We use the constraints defined in Section 8.4 as a symbolic representation to denote sets of configurations. We adapt the working-list procedure introduced in Section 3.3 to check reachability of a $\sqsubseteq$ -minimal set Φ of target constraints by a program $\mathbf{Pr} = (\mathbf{B}, \mathbf{P}, \mathbf{Ts})$. For a bad set $\mathbf{B} = \{\mathbf{val}_1, \dots, \mathbf{val}_x\}$ of valuations, the set of target constraints $\Phi_{\mathbf{B}}$ contains each $(\mathbf{e}, \mathbf{true}, \mathbf{val}_1, \dots, \mathbf{val}_x)$ where $\mathbf{e}$ is a value in $\{0, \dots, |\mathbf{Tr}|-1\}$. In addition, it contains each constraint obtained from such a constraint by adding some unique valuations that were not in its base (since we require surjections for the denotations in Section 8.4).

The procedure computes a fixpoint using the entailment relation $\sqsubseteq$ and a predecessor computation that results, for a constraint φ and a transition $\mathbf{tr}$, in a finite set $\mathbf{pre}_{\mathbf{tr}}(\varphi)$. In fact, $\mathbf{pre}_{\mathbf{tr}}(\varphi)$ is the set of constraints that capture an over-approximation of all the configurations that might reach a configuration that is denoted by φ in one round. Figure 8.4 captures this computation and uses several sets of integer variables. The variables $\#V = \{\#v^d \mid v \in V \text{ and } d \in D\}$ (resp. $\#V' = \{\#v'^d \mid v \in V \text{ and } d \in D\}$) are used to constrain values of process variables in the successor constraint φ (resp. predecessor constraint φ'). The variables $\#HO_k = \{\#ho_k^d \mid d \in D\}$ are used to constrain values in the heard-of multisets of processes taking a guarded command $\mathbf{cmd}_k$ in $\mathbf{tr}$. The remainder of this section describes the over-approximation.

Choice of guarded commands and resulting bases. Intuitively, the set $\mathbf{I}$ in Figure 8.4 corresponds to combinations of processes in the successors (i.e.,

$$\begin{array}{c}
\varphi = (\mathbf{e}, \mathbf{gl}, \{\mathbf{val}_1, \dots, \mathbf{val}_b\}) \\
\mathbf{tr} = (\mathbf{r} \bmod |\mathbf{Tr}| = (\mathbf{e} - 1) : \mathbf{send} \ \mathbf{v}; \ \mathbf{cmd}_1, \dots, \mathbf{cmd}_K) \\
1 \leq k \leq K \implies \mathbf{cmd}_k = (\mathbf{grd}_k, \mathbf{val}_k^1 \rightarrow \mathbf{val}_k^2) \\
\mathbf{I} \subseteq \{1, \dots, K\} \times \{1, \dots, b\} \text{ st. } \mathbf{I}_{|\{1, \dots, b\}} = \{1, \dots, b\} \\
\mathbf{H} :: \{1, \dots, |\mathbf{I}|\} \rightarrow \mathbf{I} \text{ is an enumeration of } \mathbf{I} \\
1 \leq i \leq |\mathbf{I}| \wedge \mathbf{H}(i) = (k, j) \implies \mathbf{val}'_i = \mathbf{val}_k^1 \wedge \mathbf{val}_k^2 = \mathbf{val}_j \\
\Gamma = \{\gamma_k \mid k : 1 \leq k \leq K\} \text{ with} \\
\gamma_k(\#V') = \exists \# \mathbf{HO}_k. \xi(\mathbf{grd}_k)(\# \mathbf{HO}_k) \wedge \mathbf{HAX}_k(\# \mathbf{HO}_k, \#V') \\
\mathbf{gl}'(\#V') = \left(\mathbf{Inv} \wedge \bigwedge_{(k, _)\in \mathbf{I}} \gamma_k(\#V') \wedge \mathbf{PrAbs}_{[\Gamma]}(\exists \#V. \mathbf{DAX}(\#V, \#V') \wedge \mathbf{gl}(\#V)) \right) \\
\mathbf{Inv} = \left(\sum_{\mathbf{d} \in \mathbf{D}} \# \mathbf{v}^{\mathbf{d}} = n \right) \\
\varphi' = \left((\mathbf{e} - 1) \bmod |\mathbf{Tr}|, \mathbf{gl}'[\#V'/\#V], \mathbf{setOf}(\mathbf{val}'_1, \dots, \mathbf{val}'_{|\mathbf{I}|}) \right) \\
\hline
\varphi' \in \mathbf{pre}_{\mathbf{tr}}(\varphi)
\end{array}$$

Figure 8.4: Predecessor computation for a constraint φ and a transition $\mathbf{tr}$

φ) and guarded commands in the transition (i.e., $\mathbf{tr}$). Each pair $(k, j) \in \mathbf{I}$ corresponds to a group of processes with the same valuation $\mathbf{val}'_{H^{-1}((k, j))}$ in the predecessors (i.e., φ') that took the guarded command $\mathbf{cmd}_k$ in the transition $\mathbf{tr}$ resulting in a valuation $\mathbf{val}_j$ in φ . Observe that there are finitely many such combinations, and hence, finitely many such sets $\mathbf{I}$. The definition of $\mathbf{I}$ ensures that the set $\{1, \dots, b\}$ of process groups of φ is covered. In addition, two pairs (k_1, j_1) and (k_2, j_2) may result in equal valuations $\mathbf{val}'_{H^{-1}((k_1, j_1))}$ and $\mathbf{val}'_{H^{-1}((k_2, j_2))}$. We keep only one representative in φ' by making a set $\mathbf{setOf}(\mathbf{val}'_1, \dots, \mathbf{val}'_{|\mathbf{I}|})$ of the multiset $\langle \mathbf{val}'_1, \dots, \mathbf{val}'_{|\mathbf{I}|} \rangle$.

Constraints imposed by the guards. For a guarded command $\mathbf{cmd}_k$, we use the predicate $\xi(\mathbf{grd}_k)$ to encode the fact that the heard-of multisets of the predecessor configurations denoted by φ' satisfy the guard $\mathbf{grd}_k$ of $\mathbf{cmd}_k$. For this, we use an integer variable $\# \mathbf{ho}_k^{\mathbf{d}}$ for each value $\mathbf{d}$ and index $k : 1 \leq k \leq K$ to count the occurrences of $\mathbf{d}$ in the heard-of multiset of the processes taking $\mathbf{cmd}_k$. We write $\# \mathbf{HO}_k = \{\# \mathbf{ho}_k^{\mathbf{d}} \mid \mathbf{d} \in \mathbf{D}\}$ to mean the set of all such variables for all values in $\mathbf{D}$. For instance, $\mathbf{grd}_3$ is $|\mathbf{HO}^0| > 2n/3$ in Figure 8.1 and is encoded with the predicate $(\# \mathbf{ho}_3^0 > 2n/3)$. We also need to relate the constraints on the heard-of multisets to the constraints on the variables' values in the predecessor constraint φ' . Assume φ' denotes a configuration $\mathbf{c}'$ resulting, via $\mathbf{tr}$, in a configuration $\mathbf{c}$ denoted by φ . Suppose $\mathbf{tr}$ sends values of variable $\mathbf{v}$. In the case of benign failures (e.g., message losses), any heard-of multiset $\mathbf{HO}_k$ of some process that took a guarded command $\mathbf{cmd}_k$ in $\mathbf{tr}$ needs to get its values from the multiset $\mathbf{valuesOf}(\mathbf{c}', \mathbf{v})$. Recall that the multiset $\mathbf{valuesOf}(\mathbf{c}', \mathbf{v})$ is the multiset over the values of $\mathbf{v}$ in the configu-

ration c' . Therefore, we enforce $\text{HO}_k \leq \text{valuesOf}(c', v)$. This is captured by $\text{HAX}_k(\# \text{HO}_k, \# v') = \bigwedge_{d \in D} 0 \leq \# \text{ho}_k^d \leq \# v'^d$. For each guarded command cmd_k , the predicate $\gamma_k(\# v') = \exists \# \text{HO}_k. (\xi(\text{grd}_k)(\# \text{HO}_k) \wedge \text{HAX}_k(\# \text{HO}_k, \# v'))$ captures the strongest constraints imposed, in the predecessor processes, by the guard of cmd_k on values of the variable that was sent (here, v). We explain later in this section how we handle corrupt communication. These predicates only depend on the chosen guarded commands and the sent variables. We collect them in a set $\Gamma = \{\gamma_k \mid k : 1 \leq k \leq K\}$. Observe that the set Γ is finite.

Constraints imposed by the commands. Each time a process takes a chosen guarded command $\text{cmd}_k = (\text{grd}_k, \text{val}_k^1 \rightarrow \text{val}_k^2)$ with (k, j) in I for some j , it transforms its valuation from val_k^1 to val_k^2 . This affects the relation between $\text{gl}(\# v)$ and $\text{gl}'(\# v')$ as the number of occurrences of a variable with a given value depends on the proportions of the processes that take each guarded command. We first illustrate how this relation can be captured exactly by introducing auxiliary variables to represent the number of processes that took each one of the chosen guarded commands. Then, we describe how we can over-approximate this relation and avoid the introduction of those variables.

First, we introduce an integer variable δ_k , for each $k \in \{1, \dots, K\}$, to capture the number of processes in some configuration c' denoted by φ' that executed the guarded command $\text{cmd}_k = (\text{grd}_k, \text{val}_k^1 \rightarrow \text{val}_k^2)$. If $d_1 = \text{val}_k^1(v)$ and $d_2 = \text{val}_k^2(v)$, then, each process that takes the guarded command cmd_k will decrease the number of occurrences of the value d_1 among messages and increase the number of occurrences of d_2 . More precisely, for each variable v , the following relation holds:

$$\text{DAX}_e(\# v, \# v') = \left(\begin{array}{c} \exists \{ \delta_k \mid k \in \{1, \dots, K\} \} \cdot \bigwedge_{(k, _) \in I} \delta_k \geq 1 \\ \bigwedge \bigwedge_{\substack{v \in V \\ d \in D}} \# v'^d = \sum_{\substack{d = \text{val}_k^1(v) \\ (k, _) \in I}} \delta_k \wedge \# v'^d = \sum_{\substack{d = \text{val}_k^2(v) \\ (k, _) \in I}} \delta_k \end{array} \right)$$

The relation DAX_e is expensive to compute. Instead, we over-approximate it with DAX (see below) where we identify two cases in which we can relate variables in $\# v$ and $\# v'$. For each variable $v \in V$, the first case (captured with the predicate $\text{preserved}_I(v)$) is true when each chosen guarded command $(\text{grd}_k, \text{val}_k^1 \rightarrow \text{val}_k^2)$ with $(k, _) \in I$ preserves the variable v (i.e., $\text{val}_k^1(v) = \text{val}_k^2(v)$). The second case (captured with the predicate $\text{uniqueChange}_I(v, d)$) corresponds to the situation when the only allowed changes for variable v are to assign some value d to it. Formally, for all k, k' with $(k, _), (k', _) \in I$, if $\text{val}_k^1(v) \neq \text{val}_k^2(v)$ and $\text{val}_{k'}^1(v) \neq \text{val}_{k'}^2(v)$, then, $\text{val}_k^2(v) = \text{val}_{k'}^2(v) = d$. The over-approximation DAX of DAX_e is defined as:

$$\text{DAX}(\#V, \#V') = \bigwedge_{v \in V} \left(\begin{array}{l} \text{preserved}_I(v) \implies \bigwedge_{d \in D} \#v'^d = \#v^d \\ \bigwedge_{d \in D} (\text{uniqueChange}_I(v, d) \implies \#v'^d \leq \#v^d) \end{array} \right)$$

To compute $\text{gl}'(\#V')$, we start from the global condition of the successor constraint (i.e., $\text{gl}(\#V)$) and deduce constraints on V' via the relation $\text{DAX}(\#V, \#V')$. More precisely, we compute: $\pi(\#V') = \exists \#V. \text{DAX}(\#V, \#V') \wedge \text{gl}(\#V)$. In general, arbitrarily many different such predicates may be generated in the fixpoint iteration. To help termination, we use the abstraction $\text{PrAbs}_{[\Gamma]}(\pi)$ of π with respect to the predicates $\Gamma = \{\gamma_k \mid k : 1 \leq k \leq K\}$ obtained from all the guards of the commands.

Example 19. Consider the configurations c, c' and the constraint φ' in the Examples 16, 17 and 18. We have shown that $c \xrightarrow{\text{tr}} c'$ using the mapping F and $c' \models \varphi'$ using the surjection Sr . Consider now the constraint $\varphi = (0, \#x^1 > 2n/3, \{(x_1 = 1, r_1 = -1), (x_1 = 0, r_2 = -1)\})$. We define the mapping $H = \{(1, (2, 1)), (2, (5, 2)), (1, (4, 3))\}$ to show $\varphi' \in \text{pre}_{\text{tr}}(\varphi)$. Moreover, there is a surjection $\text{Sr}' = \{(1, 1), (2, 1), (3, 2), (4, 1)\}$ that witnesses $c \models \varphi$.

Corrupted communications. Corrupted communications are related to the classical Byzantine Faults in which a portion of the received messages are corrupted [23]. Note that in the classical Byzantine setting, the state of a process can also be corrupted, which is not the case in this model. All processes follow the algorithm but may receive some corrupted messages (whether accidental or malicious). We assume that in each round, for each process and each data value, no more than α (given as a fraction of n) messages received by a process may have been corrupted. This assumption is weaker than the one in [23]. We model this fault by enforcing the following invariants. DAX remains unchanged because of the assumption that states of the processes are not corrupted. It is the relation between the heard-of multisets and process variables that changes: $\text{HAX}_k = \bigwedge_{d \in D} (0 \leq \#ho_k^d \leq \#v'^d + \alpha)$. The rest of the computation of predecessors also remains unchanged.

Theorem 23. *The proposed predecessor computation method introduced in Figure 8.4 is a sound over-approximation for parameterized Heard-Of programs.*

Proof. Assume two configurations $c = (r, \langle p_1, \dots, p_a \rangle)$, $c' = (r', \langle p'_1, \dots, p'_a \rangle)$, a constraint φ , and a transition $\text{tr} = (r \bmod |\text{Tr}| = (e-1) : \text{send } v; \text{cmd}_1, \dots, \text{cmd}_K)$ with $\text{cmd}_k = (\text{grd}_k, \text{val}_k^1 \rightarrow \text{val}_k^2)$ for each $k : 1 \leq k \leq K$ as described in Figure 8.5.

The total function $F : \{1, \dots, a\} \rightarrow \{1, \dots, K\}$ witnesses $c' \xrightarrow{\text{tr}} c$. More precisely, by semantics of tr , $r' = r - 1$, for each $i : 0 \leq i \leq a$, $\text{val}_i^{c'} = \text{val}_{F(i)}^c$,

$\text{val}_i^c = \text{val}_{F(i)}^2$, and $\text{HO}_i^{c'}$ satisfies $\text{grd}_{F(i)}$ (Note that the numbers of processes in c and c' are equal).

Assume that the surjection $\text{Sr} :: \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ witnesses $c \models \varphi$. We exhibit a constraint φ' that denotes c' and is generated by the predecessor computation function, i.e. $\varphi' \in \text{pre}_{\text{tr}}(\varphi)$.

All generated e' capture r' if $c' \rightarrow c$ and $c \in \llbracket \varphi \rrbracket$. More precisely, $e = (r \bmod |\text{Tr}|)$ and $e' = ((e - 1) \bmod |\text{Tr}|)$ captures $r' = r - 1$. Observe that for $j : 1 \leq j \leq b$, Sr maps each val_j^φ to some val_i^c .

We choose I to be $\{(F(i), S(i)) \mid i : 1 \leq i \leq a\}$. Such I is valid according to Figure 8.4, because,

- By definition of F , $\text{image}(F) \subseteq \{1, \dots, K\}$, hence, $I \subseteq \{1, \dots, K\} \times \{1, \dots, b\}$
- Sr is total and surjective, thus, $I_{\{1, \dots, b\}} = \{1, \dots, b\}$

Let Sr' be a surjection $\text{Sr}' = \{1, \dots, a\} \rightarrow \{1, \dots, |I|\}$ where $\text{Sr}'(i) = H^{-1}((F(i), S(i)))$. Since F and Sr are total functions, we are sure that $\text{Sr}'(i)$ is defined for all $i : 1 \leq i \leq a$.

By definition of F and Sr , $\text{val}_i^{c'} = \text{val}_{F(i)}^1 = \text{val}_{\text{Sr}'(i)}^{\varphi'}$ and $\text{val}_i^c = \text{val}_{\text{Sr}(i)}^\varphi = \text{val}_{F(i)}^2$. Hence, each $\text{val}_i^{c'}$ is mapped to some $\text{val}_j^{\varphi'}$.

Benign faults. Since $c' \xrightarrow{\text{tr}} c$, $\text{gl}'(\#V')$ is satisfiable in the presence of benign faults. Such faults respect the axioms $\text{HAX}_k(\#HO_k, \#V') = \bigwedge_{d \in D} 0 \leq \#ho_k^d \leq \#v'^d$. Assume the following solution for $\text{gl}'(\#V')$:

- for each v and d , replace $\#v^d$ and $\#v'^d$ respectively with $\text{valuesOf}(c, v)(d)$ and $\text{valuesOf}(c', v)(d)$,
- for each i and d , replace $\#ho_{F(i)}^d$ with $|HO_i^d|$,
- replace n with a .

The proposed φ' denotes c' , because, 1) $((r - 1) \bmod |\text{Tr}| = (e - 1) \bmod |\text{Tr}|)$ (note that $e = r \bmod |\text{Tr}|$), 2) by definition of F and Sr' , for all $1 \leq i \leq a$, $\text{val}_i^{c'} = \text{val}_{F(i)}^1 = \text{val}_{\text{Sr}'(i)}^{\varphi'}$, and 3) according to the exhibited solution for $\text{gl}'(\#V')$, replacing each occurrence of $\#v^d$ in the global condition gl' by $\text{valuesOf}(c, v)(d)$ and each occurrence of n by a results in a valid formula. Because $\text{gl}'(\#V')$ is an over-approximation of the exact guard that is satisfied in c' , by existence of c' we are sure that $\text{gl}'(\#V')$ evaluates to true.

$$\begin{array}{ccc}
c' : (r', \langle p_1^{c'} \dots, p_i^{c'} = (\text{val}_i^{c'}, \text{HO}_i^{c'}) \dots, p_a^{c'} \rangle) & \xrightarrow[\begin{array}{l} (F : \{1, \dots, a\} \rightarrow \{1, \dots, K\}) \\ (\text{grd}_{F(i)} : \text{val}_{F(i)}^1 \rightarrow \text{val}_{F(i)}^2) \\ \text{val}_i^{c'} = \text{val}_{F(i)}^1, \text{val}_{F(i)}^2 = \text{val}_i^c \end{array}]{c : (r, \langle p_1^c \dots, p_i^c = (\text{val}_i^c, \text{HO}_i^c) \dots, p_a^c \rangle)} & \\
\downarrow \text{Sr'?} & & \downarrow \\
\varphi' : (e', \text{gl}', \text{setOf}(\text{val}_1^{\varphi'} \dots, \text{val}_{j'}^{\varphi'} \dots, \text{val}_{|I|}^{\varphi'})) & \xrightarrow[\begin{array}{l} H(j') = (k, j) \\ (\text{grd}_k : \text{val}_k^1 \rightarrow \text{val}_k^2) \\ \text{val}_{j'}^{\varphi'} = \text{val}_k^1, \text{val}_k^2 = \text{val}_j^{\varphi} \end{array}]{\begin{array}{l} \text{Sr} : \{1, \dots, a\} \rightarrow \{1, \dots, b\} \\ I \subseteq \{1, \dots, K\} \times \{1, \dots, b\} \\ I_{\{1, \dots, b\}} = \{1, \dots, b\} \\ H : \{1, \dots, |I|\} \rightarrow I \end{array}} & \varphi : (e, \text{gl}, \{\text{val}_1^{\varphi} \dots, \text{val}_j^{\varphi} \dots, \text{val}_b^{\varphi}\})
\end{array}$$

Figure 8.5: Illustration for proof of soundness of predecessor computation for HO programs. Given $c' \rightarrow c$ and $c \models \varphi$, soundness boils down to showing the existence of $\varphi' \in \text{pre}_{\text{tr}}(\varphi)$ for which $c' \models \varphi'$.

Corruption In the case of corruption faults, DAX is the same as benign faults, but for all $F(i)$ where $(F(i), \text{Sr}(i))$ is in I , $\text{HAX}_{F(i)} = \bigwedge_{d \in D} (0 \leq \# \text{ho}_{F(i)}^d \leq \# \mathbf{v}'^d + \alpha)$. By the existence of c' , we have that $\bigwedge_{1 \leq F(i) \leq K} (\# \text{ho}_{F(i)}^d \leq \# \mathbf{v}'^d + \alpha)$ for each value $d \in D$. The solution for $\text{gl}'(\# \mathbf{V}')$ in the case of corruption is similar to the benign case, plus, it needs to satisfy $\max_{\substack{d \in D \\ 1 \leq F(i) \leq c}} (\# \text{ho}_{F(i)}^d - \# \mathbf{v}'^d) \leq \alpha$.

The existence of c' ensures that this condition is satisfiable. This is because $\text{gl}'(\# \mathbf{V}')$ is an over-approximation of the exact guard that is satisfied in c' . $\square$

Theorem 24. *The proposed procedure terminates.*

Termination is obtained by the fact that at most a finite number of constraints might be generated. To see this, observe that the constraints consist of an integer capturing control location, a predicate (the global condition), and a set of local valuations of processes (the base). The number of control locations and that of the local valuations of the processes are finite. In addition, only finitely many global conditions are generated for a protocol. This is due to the fact that the number of the subsets of the guarded commands which are used for the predicate abstraction ($\text{PrAbs}_{[r]}$) is finite, and the invariants (Inv) and the strengthening invariants needed for corruption fault (e.g., see Figure 8.3) are fixed for each protocol.

8.6 Experimental Results

We report on the experiments with our open-source prototype tool SyncV, which is publicly available at <https://gitlab.liu.se/live/syncv> to verify a class

of HO algorithms. The experiments were conducted on an Intel Core(TM) i7 processor running at 2.9GHz with 8GB of memory. We conducted the experiments on several variations of the One-Third-Rule and $\mathcal{A}_{E,T}$ algorithms. These variations correspond to checking the correctness properties of agreement, validity, and irrevocability for correct and buggy versions of the considered algorithms and an unbounded number of processes. For each property, we verified a correct version and a buggy version. The buggy versions differ from the correct ones in the guards of their commands. For verification of the $\mathcal{A}_{E,T}$ algorithm, we strengthened our invariant Inv in Figure 8.4 with the invariants represented in Figure 8.3 that according to [23] are essential for the correctness of the algorithm.

For all correct versions, our tool reported that the program is safe, and for all buggy ones, it presented a valid trace that violates the considered property. Our implemented procedure does not eagerly concretize the local valuations of the processes. Instead, we concretize on demand. All benchmarks are available on the tool homepage.

Checking different correctness properties. We discussed in depth checking the agreement correctness property in sections 8.2 and 8.5. Checking the validity property is similar because it also uses a finite set of forbidden valuations to characterize the set of bad constraints. For checking irrevocability, one needs to see if a process can make a decision and revoke it later. To make such checks, we take into account a *history* of the changes. We do that by augmenting each process group in a constraint by a list of possible decisions as its history. This list is empty by default. A bad constraint that violates irrevocability has at least one process group with a minimum of two different values in its history.

8.7 Conclusion and Future Work

We have studied a subclass of fault-tolerant distributed algorithms in terms of the Heard-Of model and proposed a symbolic representation that uses cardinality constraints on multisets to model sets of configurations generated during the analysis of such programs. We have also introduced a sound procedure for checking state reachability to check various correctness properties, such as agreement, validity, and irrevocability for consensus programs in the presence of lossy or corrupt communications. We showed that the introduced procedure terminates even for an unbounded number of processes. To the best of our knowledge, this is the first fully-automatic approach to verifying Heard-Of protocols in the presence of corrupt communications. We reported on our experiments with the correct and buggy variations of the protocols.

program	property	safe?	result	time(m)
simple	agreement	✓	safe	2
	validity	✓	safe	~0
	irrevocability	✓	safe	1
$\frac{1}{3}$ -rule	agreement	✓	safe	19
		✗	trace	~0
	validity	✓	safe	2
		✗	trace	~0
	irrevocability	✓	safe	7
		✗	trace	~0
$\mathcal{A}_{E,T}$	agreement	✓	safe	54
		✗	trace	1
	validity	✓	safe	5
		✗	trace	~0
	irrevocability	✓	safe	21
		✗	trace	~0

Table 8.1: The results of checking agreement, validity, and irrevocability for some HO protocols with SyncV. For each algorithm, a correct and a buggy version were tested by the tool. The buggy versions differ from the correct ones by the guards of their commands. For all the correct versions our tool reported that the program is safe and for all the buggy ones, it presented a valid trace violating the considered property.

Future Work. Future work can consider improving the scalability of the tool and also extending the presented technique to more general models and more sophisticated faults such as Byzantine faults. Also, the current technique assumes symmetric processes in the sense that all of them execute the same operation in each round. One could extend the model to non-symmetric processes as in the Heard-Of examples having coordinators, for instance in *CoordUniformVoting* and *LastVoting* algorithms in [30], or the *Phase King* and *Phase Queen* algorithms introduced in [20] in which a *King* or *Queen* is distinguished in each round, or the reliable broadcast algorithm in [119]. It would also be interesting to combine the approach with abstract interpretation for loops to capture the distributed algorithms in which the number of iterations is crucial for the correctness of the algorithm, for example, the *FloodMin* algorithm in [31]. Moreover, identifying conditions for completeness of the approach, automatic refinement of the over-approximation, and combination with richer theories are interesting directions for future work.

9 Conclusion

In this thesis, we have extended model checking of parameterized systems to several classes of infinite-state systems, mostly parameterized systems. We have provided an open-source prototype tool for most of the techniques that we have developed and have presented our experimental results on several benchmarks.

First, we have proposed a gap-order-based reachability analysis for bounded phaser programs. We have shown our analysis to be exact and guaranteed to terminate when checking runtime, race, and assertion errors. We have established the undecidability of deadlock verification and explained how to turn our analysis into a sound over-approximation.

Second, we have studied parameterized reachability problems for phaser programs. We have proposed an exact verification procedure for non-atomic programs when only distinguished tasks such as `main` create phasers. The procedure can be used for answering both control and plain reachability problems. The procedure is guaranteed to terminate, even for programs that may generate arbitrarily many tasks but finitely many phasers, when checking control (e.g., assertions) reachability or when checking plain (e.g., deadlocks) reachability with bounded gaps. These results were obtained using a non-trivial symbolic representation for which termination had required showing an $\exists \leq_{\forall}$ preorder on multisets on gaps on natural numbers to be a $\mathcal{WQO}$. We believe that our general decidability results are useful to reason about synchronization constructs other than phasers. For instance, a traditional static barrier can be captured with one phaser and with bounded gaps (in fact, one). Sim-

ilarly, one phaser with one producer and arbitrarily many consumers can be used to capture futures where a “get” instruction can be modeled with a `wait`. We believe that our negative results can also be used. For instance, atomic instructions can be modeled using a `testAndSet` operation and may result in the undecidability of the reachability problem. An interesting direction for future work could be trying to find decidable subclasses of the phaser programs with parameterized phasers. One could also consider using the technique in combination with predicate abstraction and abstract interpretation in order to analyze actual source code. Moreover, one could investigate the applicability of the presented techniques for the verification of similar synchronization constructs.

Third, we have presented a technique, predicated constrained monotonic abstraction, for the automated verification of concurrent programs whose correctness depends on the synchronization between arbitrarily many processes, for example, utilizing barriers implemented using integer counters and tests. We have introduced counting predicates that mix counters referring to the number of processes satisfying some given properties and variables that are directly manipulated by the concurrent processes. We have used counting predicates in a framework that combines predicate, counter, and constrained monotonic abstraction. Our prototype implementation gave encouraging results and managed to automatically establish or refute program assertions and deadlock freedom. Future work could consider adapting PACMAN to the synchronous protocols introduced in Chapter 8 in order to perform predicate abstraction on them and count the processes that satisfy different predicates.

Fourth, we have introduced and discussed three different ways of localizing constrained monotonic abstraction in systems with infinite control. For this, we have targeted reachability problems for (possibly non-well-structured) counter machines obtained as abstractions of concurrent programs. Our new techniques allow us to avoid trashing the state-space being explored before encountering the false positives that necessitate the introduction of new refinement predicates. This allowed us to consistently improve on the existing eager exploration, both in forward and in backward explorations. Possible future works could consider combining forward and backward approximations, using the pivot configuration to make it possible to choose interpolants that are easier to generalize, assessing the feasibility of combination with partial-order techniques, or performing the lazy explorations in parallel. The latter is more suitable for the lazy variants since they do not delete the explored state-space upon finding new refinement predicates.

Finally, we have studied a subclass of fault-tolerant distributed algorithms in terms of the Heard-Of model and proposed a symbolic representation using cardinality constraints on multisets to model sets of configurations generated during the analysis of such programs. We have also introduced a sound procedure for checking state reachability to check various correctness properties for consensus programs such as agreement, validity, and irrevocability in the

presence of lossy or corrupt communications. We showed that the introduced procedure terminates even for an unbounded number of processes. To the best of our knowledge, this is the first fully-automatic approach to verifying Heard-Of protocols in the presence of corrupt communication. We reported on preliminary experiments with correct and buggy variations of the protocols. Future work can consider extending the presented technique to more general models such as [22, 21] and more sophisticated faults such as Byzantine faults. Also, one could extend the model to non-symmetric processes as in the Heard-Of protocols that have coordinators. It would also be interesting to capture distributed algorithms in which the number of iterations is crucial for the correctness of the algorithm. Moreover, identifying conditions for completeness of the approach, and automatically refining the over-approximation are interesting directions for future work.

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