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ON A CONJECTURE OF KRÁL CONCERNING THE SUBHARMONIC EXTENSION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

Stephen J. Gardiner, Dublin, Tomas Sjödin, Linköping

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Abstract. This note verifies a conjecture of Král, that a continuously differentiable function, which is subharmonic outside its critical set, is subharmonic everywhere.

Keywords: subharmonic function; extension theorem

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1. Introduction

A classical result of Radó (see Theorem 12.14 of [9]) says that if f is continuous on an open set $\Omega \subset \mathbb{C}$ and holomorphic on $\{z \in \Omega : f(z) \neq 0\}$, then f is holomorphic on all of Ω. An analogue for harmonic functions due to Král (see [6]) says that if $u\colon \Omega \to \mathbb{R}$ is C^1 on an open set $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ and harmonic on $\{x \in \Omega\colon \nabla u(x) \neq 0\}$, then u is harmonic on all of Ω. (A short proof of this result was recently given in [8].) Král conjectured in [7] that his result could be strengthened by substituting "subharmonic" for "harmonic" throughout. However, the methods of [6] and [8] are not applicable to subharmonic functions. The purpose of this note is to verify this conjecture.

2. Main result

Theorem 1. *If u is C^1 on an open set $\Omega \subset \mathbb{R}^N$ and subharmonic on $\{x \in \Omega\colon \nabla u(x) \neq 0\}$, then u is subharmonic on all of Ω.*

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The idea of the proof below comes from the theory of viscosity solutions of partial differential equations, which is expounded in [2], [3]. In fact, Theorem 1 may readily be deduced from results in [5] concerning viscosity solutions of the p-Laplace equation (cf. [4] for a generalization of Král's original result to p-harmonic functions). However, we will instead give a self-contained argument, partially inspired by [5], that uses only some basic properties of subharmonic functions. A convenient background reference is [1].

Proof. Let $\varepsilon > 0$ and B be an open ball $\{x\colon \|x - x_1\| < r\}$ such that $\bar{B} \subset \Omega$. By taking the Poisson integral of u in B and adding the polynomial

$$x \mapsto \varepsilon\Big(1 + \frac{r^2 - \|x - x_1\|^2}{2N}\Big),$$

we obtain a function $h_\varepsilon \in C(\bar{B})$ satisfying

$$\begin{cases} \Delta h_\varepsilon = -\varepsilon & \text{in } B, \\ h_\varepsilon = u + \varepsilon & \text{on } \partial B. \end{cases}$$

It will be enough to show that $h_\varepsilon \geqslant u$ in B, since we can then let ε tend to 0 to arrive at the required spherical mean value inequality for u.

The set

$$O = \{(x, y) \in \bar{B} \times \bar{B}\colon\ h_\varepsilon(x) - u(y) > \tfrac{1}{2}\varepsilon\}$$

is relatively open in $\bar{B} \times \bar{B}$ and contains $\{(x, x)\colon\ x \in \partial B\}$. Thus, the quantity $\|x - y\|^4$ is bounded away from zero on $\partial(B \times B) \setminus O$, and we may choose $c > 0$ large enough so that $w > 0$ on $\partial(B \times B)$, where

$$w(x, y) = h_\varepsilon(x) - u(y) + c\|x - y\|^4, \quad x, y \in \bar{B}.$$

We suppose, for the sake of contradiction, that the minimum value of the continuous function w on $\bar{B} \times \bar{B}$ is attained at some point $(x_0, y_0) \in B \times B$.

Setting $y = y_0$ in the inequality

$$(1) \qquad h_\varepsilon(x) - u(y) + c\|x - y\|^4 \geqslant h_\varepsilon(x_0) - u(y_0) + c\|x_0 - y_0\|^4, \quad x, y \in \bar{B},$$

we see that $h_\varepsilon \geqslant \varphi$, where

$$\varphi(x) = h_\varepsilon(x_0) + c(\|x_0 - y_0\|^4 - \|x - y_0\|^4), \quad x \in \bar{B}.$$

Further, $h_\varepsilon - \varphi$ is smooth and attains its minimum value at x_0, so

$$\frac{\partial^2(h_\varepsilon - \varphi)}{\partial x_i^2}(x_0) \geqslant 0, \quad i = 1, \ldots, N$$

and hence

$$\Delta\varphi(x_0) \leqslant \Delta h_\varepsilon(x_0) = -\varepsilon.$$

In particular, $x_0 \neq y_0$ since $\Delta\varphi(y_0) = 0$.

Similarly, setting $x = x_0$ in (1), we see that $u \leqslant \psi$, where

$$\psi(y) = u(y_0) + c(\|x_0 - y\|^4 - \|x_0 - y_0\|^4), \quad y \in \bar{B}.$$

Since $u - \psi$ is C^1 and attains its maximum value 0 at y_0, and also $x_0 \neq y_0$, we see that $\nabla u(y_0) = \nabla\psi(y_0) \neq 0$. By hypothesis, the formula

$$v(s) = w(x_0 + s, y_0 + s) = h_\varepsilon(x_0 + s) - u(y_0 + s) + c\|x_0 - y_0\|^4$$

defines a function which is superharmonic on some neighbourhood of 0 in $\mathbb{R}^N$. Since v attains a local minimum at 0, it must be constant near 0. However, this leads to the contradictory conclusion that $\Delta u = -\varepsilon < 0$ near y_0.

The theorem now follows, because

$$\min_{\bar{B}}(h_\varepsilon - u) = \min_{x\in\bar{B}} w(x,x) \geqslant \min_{\bar{B}\times\bar{B}} w = \min_{\partial(B\times B)} w \geqslant 0.$$

□

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Authors' addresses: *Stephen J. Gardiner*, School of Mathematics and Statistics, University College Dublin, Science Centre-North Belfield, Dublin 4, Ireland, e-mail: `stephen.gardiner@ucd.ie`; *Tomas Sjödin*, Department of Mathematics, Campus Valla, House B, Room 3A:681, Linköping University, 581 83, Linköping, Sweden, e-mail: `tomas.sjodin@liu.se`.