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Sensor Management for Target Tracking Applications

Per Boström-Rost

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Cover illustration: Intensity function indicating where undetected targets are likely to be found, after a sensor has passed through the region.

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Sensor Management for Target Tracking Applications

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Till min familj

Abstract

Many practical applications, such as search and rescue operations and environmental monitoring, involve the use of mobile sensor platforms. The workload of the sensor operators is becoming overwhelming, as both the number of sensors and their complexity are increasing. This thesis addresses the problem of automating sensor systems to support the operators. This is often referred to as sensor management. By planning trajectories for the sensor platforms and exploiting sensor characteristics, the accuracy of the resulting state estimates can be improved. The considered sensor management problems are formulated in the framework of stochastic optimal control, where prior knowledge, sensor models, and environment models can be incorporated. The core challenge lies in making decisions based on the predicted utility of future measurements.

In the special case of linear Gaussian measurement and motion models, the estimation performance is independent of the actual measurements. This reduces the problem of computing sensing trajectories to a deterministic optimal control problem, for which standard numerical optimization techniques can be applied. A theorem is formulated that makes it possible to reformulate a class of nonconvex optimization problems with matrix-valued variables as convex optimization problems. This theorem is then used to prove that globally optimal sensing trajectories can be computed using off-the-shelf optimization tools.

As in many other fields, nonlinearities make sensor management problems more complicated. Two approaches are derived to handle the randomness inherent in the nonlinear problem of tracking a maneuvering target using a mobile range-bearing sensor with limited field of view. The first approach uses deterministic sampling to predict several candidates of future target trajectories that are taken into account when planning the sensing trajectory. This significantly increases the tracking performance compared to a conventional approach that neglects the uncertainty in the future target trajectory. The second approach is a method to find the optimal range between the sensor and the target. Given the size of the sensor's field of view and an assumption of the maximum acceleration of the target, the optimal range is determined as the one that minimizes the tracking error while satisfying a user-defined constraint on the probability of losing track of the target.

While optimization for tracking of a single target may be difficult, planning for jointly maintaining track of discovered targets and searching for yet undetected targets is even more challenging. Conventional approaches are typically based on a traditional tracking method with separate handling of undetected targets. Here, it is shown that the Poisson multi-Bernoulli mixture (PMBM) filter provides a theoretical foundation for a unified search and track method, as it not only provides state estimates of discovered targets, but also maintains an explicit representation of where undetected targets may be located. Furthermore, in an effort to decrease the computational complexity, a version of the PMBM filter which uses a grid-based intensity to represent undetected targets is derived.

Populärvetenskaplig sammanfattning

Flygplan har varit en del av vårt samhälle i över hundra år. De första flygplanen var svårmanövrerade, vilket innebar att dåtidens piloter fick lägga en stor del av sin tid och kraft på att hålla planet i luften. Genom åren har flygplanens styrsystem förbättrats avsevärt, vilket har möjliggjort för piloterna att utföra andra uppgifter utöver att styra dem. Som en följd av det har allt fler sensorer installerats i flygplanen, vilket ger piloterna mer information om omvärlden. Även sensorerna måste dock styras, vilket kräver mycket uppmärksamhet. Samtidigt har tekniken för obemannat flyg gått snabbt framåt och det är inte längre otänkbart för en ensam operatör att ansvara för flera plattformar samtidigt. Då varje plattform kan bära flera sensorer kan arbetsbelastningen för operatören bli mycket hög. Det här gör att många sensorsystem som tidigare styrts för hand nu börjar bli alltför komplexa för att hanteras manuellt. Behovet av automatiserad sensorstyrning (eng. *sensor management*) blir därför allt större. Genom att låta datorer stötta sensoroperatörerna, antingen genom att ge förslag på hur sensorerna ska styras eller helt ta över styrningen, kan operatörerna istället fokusera på att ta beslut på högre nivå. Automatiseringen möjliggör även nya mer avancerade funktioner eftersom datorerna kan hantera stora datamängder i mycket snabbare takt än vad en människa klarar av. I den här avhandlingen studeras olika aspekter av sensorstyrning, bland annat informationsbaserad ruttplanering och hur sensorspecifika egenskaper kan utnyttjas för att förbättra prestandan vid målföljning.

Matematisk optimering används ofta för att formulera problem inom informationsbaserad ruttplanering. Optimeringsproblemen är dock i allmänhet svåra att lösa, och även om det går att beräkna en rutt för sensorplattformen är det svårt att garantera att det inte finns en annan rutt som skulle vara ännu bättre. Ett av avhandlingens bidrag gör det möjligt att omformulera optimeringsproblemen så att de bästa möjliga rutterna garanterat beräknas. Omformuleringen går även att tillämpa på planeringsproblem där sensorplattformen behöver undvika att upptäckas av andra sensorer i området.

En annan del av avhandlingen diskuterar hur osäkerheter i optimeringsproblem inom sensorstyrning kan hanteras. Scenariot som studeras är ett målföljningsscenario, där en rörlig sensor ska styras på ett sådant sätt att ett manövrerande objekt hålls inom sensorns synfält. En svårighet är då att sensorn måste förutsäga hur objektet kommer att röra sig i framtiden och två nya metoder för att hantera detta presenteras. Den ena metoden förbättrar målföljningsprestandan avsevärt genom att ta hänsyn till att målet kan utföra flera typer av manövrar och den andra gör det möjligt att optimera avståndet mellan sensor och mål för att minimera risken att tappa bort målet.

I avhandlingen undersöks även hur en grupp av sensorer ska samarbeta för att söka av ett område och hålla koll på de objekt som upptäcks. För att möjliggöra detta utvecklas en metod för att representera var upptäckta objekt kan finnas sig, som sedan används för att fördela sensorresurser mellan sökning och följning. Tekniken är användbar exempelvis vid fjäll- och sjöräddning eller för att hitta personer som gått vilse i skogen.

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Finally, I would like to thank my family for all your loving support and for always believing in me. Emma, thank you for being you and for everything you do for Oscar and me. I love you.

Linköping, March 2021
Per Boström-Rost

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Part I

Background

1

Introduction

Modern sensor systems often include several controllable operating modes and parameters. *Sensor management*, the problem of selecting the control inputs for this type of systems, is the topic of this thesis. This introductory chapter gives an overview of the concept of sensor management, lists the contributions, and outlines the content of the thesis.

Figure 1.1: Example of a sensor management application considered in the thesis. A mobile sensor with limited field of view, represented by the orange circle and sector, is used to search for and track the targets in a surveillance region. In the given situation, the sensor has to decide where to go next: it can either continue straight toward the blue region where it is likely to find new targets, or turn right to revisit the two targets that are already known to exist. The choice depends on several factors, of which the most significant is the trade-off between obtaining up-to-date information about known targets and exploring the surveillance region to find new targets.

1.1 Background and motivation

In the early days of aviation, controlling the aircraft was a challenging task for the pilots. Little time was available for anything aside from keeping the aircraft in the air. Over time, the capabilities of flight control systems have improved significantly, allowing pilots to perform more tasks than just controlling the aircraft while airborne. To this end, an increasing number of sensors are being mounted on the aircraft to provide the pilots with information about the surroundings. As a result, a major part of the workload has shifted from controlling the aircraft to controlling the sensors. In a way, the pilots have become sensor operators.

With recent advances in remotely piloted aircraft technology, the operators are no longer necessarily controlling sensors carried by a single platform. Modern sensing systems often consist of a large number of sophisticated sensors with many operating modes and functions. These can be distributed over several different platforms with varying levels of communication capabilities. As a result, an increasing number of sensing systems that traditionally have been controlled manually are now becoming too complex for a human to operate. This has led to the need for *sensor management*, which refers to the automation of sensor control systems, *i.e.*, coordination of sensors in dynamic environments in order to achieve operational objectives. The significance of sensor management becomes clearer when considering the role of the operator in sensing systems with and without sensor management [12]. As illustrated in Figure 1.2, in a system that lacks sensor management, the operator acts as the controller and sends low-level commands to the sensors. If a sensor management component is available, it computes and sends low-level commands to the sensors based on high-level commands from the operator and the current situation at hand. According to [12], sensor management thus yields the following benefits:

- *Reduced operator workload.* Since the sensor management handles the low-level commands to the sensor, the operator can focus on the operational objectives of the sensing system rather than the details of its operation.
- *More information available for decision making.* A sensor management component can use all available information when deciding which low-level commands to use, while an operator is limited to the information presented on the displays.
- *Faster control loops.* An automated inner control loop allows for faster adaptation to changing conditions than one which involves the operator.

As illustrated in Figure 1.2 and discussed in [40, 63], including a sensor management component corresponds to adding a feedback loop to the state estimation process. Sensor management is closely related to optimal control of dynamical systems. Conventional optimal control algorithms are used to select control inputs for a given system to reach a desired state [11]. In sensor management, the aim is to select sensor control inputs that improve the performance of an underlying estimation method. The desired state corresponds to having perfect knowledge of the observed system's state. In general, the performance of state estimation

Figure 1.2: Including a sensor management component in a sensor system has the potential of reducing the operator workload.

methods depends on the actual measurements that are obtained. This is a complicating factor for sensor management, as the objective function hence depends on future measurements that are unavailable at the time of planning. Due to this uncertainty, sensor management problems are typically formulated as stochastic optimal control problems [26].

An interesting subclass of sensor management is informative path planning, where the control inputs affect the movement of the sensor platforms. Informative path planning enables automatic information gathering with mobile robots and has many applications in environmental monitoring [29], that often involve the use of airborne sensors [59]. As more information become available, the trajectories for the mobile sensor platforms can be adapted to optimize the estimation performance.

Target tracking is another area in which sensor management can be applied. Target tracking is a state estimation problem where dynamic properties of objects of interest are estimated using sets of noisy sensor measurements [5, 12]. While target tracking has its historical roots in the aerospace domain, it also has a wide range of applications in other areas. Typical scenarios include estimation of the position and the velocity of airplanes near an airport, ships in a harbor, or cars on a street. In many cases, the tracking performance can be influenced by selecting appropriate control inputs for the sensors. As an example, consider the scenario in Figure 1.1 where a mobile sensor is used to track multiple targets but the sensor's field of view is too small to cover the entire surveillance region at once. Sensor management can then be used to optimize the use of the mobile sensor such that it provides accurate state estimates of the discovered targets and simultaneously searches for new targets.

1.2 Considered problem

The overall aim of this thesis is to develop methods and theory to increase the performance of sensor systems and reduce the workload of their operators. The focus is on:

- planning trajectories for sensor platforms to optimize estimation performance;
- exploiting sensor characteristics to provide more accurate state estimates of maneuvering targets; and
- improving models of where targets are likely to be found to better characterize where future search is beneficial.

1.3 Contributions

The main scientific contributions of this thesis can be divided into three categories and are presented in this section.

1.3.1 Optimality guarantees in informative path planning

Informative path planning problems for linear systems subject to Gaussian noise can be formulated as optimal control problems. Although deterministic, the problems are in general challenging to solve to global optimality as they involve nonconvex equality constraints. One of the contributions of this thesis is a theorem that can be used to reformulate seemingly intractable informative path planning problems such that they can be solved to global optimality or any user-defined level of suboptimality using off-the-shelf optimization tools. The theorem is applicable also in scenarios where the sensor platform has to avoid being tracked by an adversarial observer. These contributions have been published in Paper A and Paper B. Paper A presents the theorem and applies it in an information gathering scenario. Paper B extends the scenario to involve adversarial observers and shows that the theorem can be used also in this case.

1.3.2 Optimized tracking of maneuvering targets

The second category of contributions are within the area of single target tracking, or more specifically, within the problem of tracking a maneuvering target using a mobile sensor with limited field of view. A method to adaptively optimize the mobile sensor's trajectory is proposed in Paper C. The uncertainties inherent in the corresponding planning problem are handled by considering multiple candidate target trajectories in parallel. The proposed method is evaluated in a simulation study where it is shown to both result in more accurate state estimates and reduce the risk of losing track of the target compared to a conventional method.

A method to find the optimal tracking range for tracking a maneuvering target is proposed in Paper D. Given properties of the target and the sensor as well as an acceptable risk of losing track of the target, the tracking range is optimized to minimize the tracking error.

1.3.3 Planning for multiple target tracking

The number of targets in multiple target tracking scenarios often varies with time as targets may enter or leave the surveillance region. Paper E proposes a sensor management method based on the Poisson multi-Bernoulli mixture (PMBM) filter. It is shown that as the PMBM filter handles undetected and detected targets jointly, it allows for a unified handling of search and track.

Conventional PMBM filter implementations use Gaussian mixtures to represent the intensity of undetected targets. The final contribution of this thesis, presented in Paper F, is a version of the PMBM filter which uses a grid-based intensity function to represent where undetected targets are likely to be located. This is convenient in scenarios where there are abrupt changes in the intensity, for example if the sensor's field of view is smaller than the surveillance region.

1.4 Thesis outline

The thesis is divided into two parts. The first part contains background material and the second part is a collection of publications.

Part I: Background

The first part provides theoretical background material relevant for sensor management and the publications included in the second part of the thesis. It is organized as follows. Chapter 2 presents the state estimation problem and a number of filtering solutions. Target tracking and its relation to state estimation is discussed in Chapter 3. Chapter 4 presents a number of important concepts in mathematical optimization and Chapter 5 overview of optimal control for discrete-time systems. Chapter 6 summarizes the scientific contributions of the thesis and presents possible directions for future work.

Part II: Publications

The second part of this thesis is a collection of the papers listed below. No changes have been made to the content of the published papers. However, the typesetting has been changed in order to comply with the format of the thesis. If not otherwise stated, the author of this thesis has been the main driving force in the development of the necessary theory and in the process of writing the manuscripts. The author of this thesis also made the software implementations and designed and conducted the simulation experiments. Most of the ideas have been worked out in collaborative discussions between author of this thesis, Daniel Axehill, and Gustaf Hendeby. See below for detailed comments, where the names of the author and co-authors are abbreviated as follows: Per Boström-Rost (PBR), Daniel Axehill (DA), Gustaf Hendeby (GH) and William Dale Blair (WDB).

Paper A: On global optimization for informative path planning

P. Boström-Rost, D. Axehill, and G. Hendeby, “On global optimization for informative path planning,” *IEEE Control Systems Letters*, vol. 2, no. 4, pp. 833–838, 2018.¹

Comment: The idea of this paper originated from DA and was further developed in discussions among all authors. The manuscript was written by PBR with suggestions and corrections from the co-authors. The simulation experiments were designed and carried out by PBR.

Paper B: Informative path planning in the presence of adversarial observers

P. Boström-Rost, D. Axehill, and G. Hendeby, “Informative path planning in the presence of adversarial observers,” in *Proceedings of the 22nd International Conference on Information Fusion*, Ottawa, Canada, 2019.

Comment: The idea of this paper originated from discussions among all authors. The manuscript was written by PBR with suggestions and corrections from the co-authors. The simulation experiments were designed and carried out by PBR.

Paper C: Informative path planning for active tracking of agile targets

P. Boström-Rost, D. Axehill, and G. Hendeby, “Informative path planning for active tracking of agile targets,” in *Proceedings of IEEE Aerospace Conference*, Big Sky, MT, USA, 2019.

Comment: The idea of this paper originated from GH and was further developed in discussions among all authors. The manuscript was authored by PBR with suggestions and corrections from the co-authors. The software implementation and simulation experiments were designed and carried out by PBR.

Paper D: Optimal range and beamwidth for radar tracking of maneuvering targets using nearly constant velocity filters

P. Boström-Rost, D. Axehill, W. D. Blair, and G. Hendeby, “Optimal range and beamwidth for radar tracking of maneuvering targets using nearly constant velocity filters,” in *Proceedings of IEEE Aerospace Conference*, Big Sky, MT, USA, 2020.

Comment: The idea of this paper originated from WDB as a comment to PBR’s presentation of Paper C at the IEEE Aerospace Conference in 2019. PBR then further refined the idea and GH provided input. The manuscript was authored by PBR with input from the co-authors. The software implementation and simulation experiments were carried out by PBR.

¹The contents of this paper were also selected for presentation at the 57th IEEE Conference on Decision and Control, Miami Beach, FL, USA, 2018.

Paper E: Sensor management for search and track using the Poisson multi-Bernoulli mixture filter

P. Boström-Rost, D. Axehill, and G. Hendeby, “Sensor management for search and track using the Poisson multi-Bernoulli mixture filter,” *IEEE Transactions on Aerospace and Electronic Systems*, 2021, doi:10.1109/TAES.2021.3061802.

Comment: The idea of this paper originated from PBR and was further developed in discussions with GH. The author of this thesis authored the manuscript with input from the co-authors. The software implementation and simulation experiments were designed and carried out by PBR.

Paper F: PMBM filter with partially grid-based birth model with applications in sensor management

P. Boström-Rost, D. Axehill, and G. Hendeby, “PMBM filter with partially grid-based birth model with applications in sensor management,” 2021, arXiv:2103.10775v1.

Comment: The idea of this paper originated from PBR, who also derived the method with input from GH. The author of this thesis authored the manuscript with input from the co-authors. The software implementation and simulation experiments were designed and carried out by PBR. The manuscript has been submitted for possible publication in *IEEE Transactions on Aerospace and Electronics Systems*.

2

Bayesian state estimation

State estimation refers to the problem of extracting information about the state of a dynamical system from noisy measurements. The problem has been studied extensively, as accurate state estimates are crucial in many real-world applications of signal processing and automatic control. In recursive state estimation, the estimates are updated as new measurements are obtained. This chapter provides an overview of recursive state estimation in the Bayesian context, where the goal is to compute the posterior distribution of the state given the history of measurements and statistical models of the measurements and the observed system. For more in-depth treatments of the subject, consult, *e.g.*, [41, 46, 76].

2.1 State-space models

A *state-space model* [46, 83] is a set of equations that characterize the evolution of a dynamical system and the relation between the state of the system and available measurements. A general functional description of a state-space model with additive noise in discrete time is given by

$$x_{k+1} = f_k(x_k) + G_k w_k, \quad (2.1a)$$

$$z_k = h_k(x_k) + e_k, \quad (2.1b)$$

where the state and measurement at time k are denoted by $x_k \in \mathbf{R}^{n_x}$ and $z_k \in \mathbf{R}^{n_z}$, respectively. The function f_k in (2.1a) is referred to as the *dynamics* or the *motion model* and describes the evolution of the state variable over time. The random variable w_k corresponds to the process noise, which is used to account for the fact that the dynamics of the system are usually not perfectly known. The function h_k in (2.1b) is referred to as the *measurement model* and describes how the state relates to the measurements, and the random variable e_k represents the measurement noise.

Example 2.1: Nearly constant velocity model with position measurements

The nearly constant velocity (NCV) model [53] describes linear motion with constant velocity, which is disturbed by external forces that enter the system in terms of acceleration. In one dimension, using measurements of the position and a sampling time τ , the model is given by

$$x_{k+1} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2}\tau^2 \\ \tau \end{bmatrix} w_k \quad (2.5a)$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + e_k, \quad (2.5b)$$

where the state corresponds to the position and velocity. If the noise variables are assumed to be white and Gaussian distributed, *e.g.*, $w_k \sim \mathcal{N}(0, Q_k)$ and $e_k \sim \mathcal{N}(0, R_k)$, the NCV model is a linear Gaussian state-space model.

The general model (2.1) can be specialized by imposing constraints on the functions and distributions involved. An important special case is the linear Gaussian state-space model, where f_k and h_k are linear functions and the noise is Gaussian distributed, *i.e.*,

$$x_{k+1} = F_k x_k + G_k w_k, \quad (2.2a)$$

$$z_k = H_k x_k + e_k, \quad (2.2b)$$

where $w_k \sim \mathcal{N}(0, Q_k)$ and $e_k \sim \mathcal{N}(0, R_k)$.

State-space models can also be described in terms of conditional probability distributions. In a probabilistic state-space model, the transition density $p(x_{k+1} | x_k)$ models the dynamics of the system and the measurement likelihood function $p(z_k | x_k)$ describes the measurement model. A probabilistic representation of the state-space model in (2.1) is given by

$$p(x_{k+1} | x_k) = p_w(x_{k+1} - f_k(x_k)), \quad (2.3a)$$

$$p(z_k | x_k) = p_e(z_k - h_k(x_k)), \quad (2.3b)$$

where p_w denotes the density of the process noise and p_e denotes the density of the measurement noise. A fundamental property of a state-space model is the Markov property,

$$p(x_{k+1} | x_1, \dots, x_k) = p(x_{k+1} | x_k), \quad (2.4)$$

which implies that the state of the system at time k contains all necessary information about the past to predict its future behavior [83].

Two commonly used state-space models are given in Example 2.1 and Example 2.2. See the survey papers [53] and [52] for descriptions of more motion models and measurement models.

Example 2.2: Coordinated turn model with bearing measurements

In the two-dimensional coordinated turn model [53], the state $x = [\tilde{x}, \omega]^\top$ consists of the position and velocity $\tilde{x} = [p_1, v_1, p_2, v_2]^\top$ and turn rate ω . Combined with bearing measurements from a sensor located at the origin, it constitutes the following nonlinear state-space model,

$$x_{k+1} = F(\omega_k)x_k + w_k \quad (2.6a)$$

$$z_k = h(x_k) + e_k, \quad (2.6b)$$

where $h(x) = \arctan(p_2/p_1)$, $w_k \sim \mathcal{N}(0, Q)$, $e_k \sim \mathcal{N}(0, R)$, the state transition matrix is

$$F(\omega) = \begin{bmatrix} 1 & \frac{\sin(\omega\tau)}{\omega} & 0 & -\frac{1-\cos(\omega\tau)}{\omega} & 0 \\ 0 & \cos(\omega\tau) & 0 & -\sin(\omega\tau) & 0 \\ 0 & \frac{1-\cos(\omega\tau)}{\omega} & 1 & \frac{\sin(\omega\tau)}{\omega} & 0 \\ 0 & \sin(\omega\tau) & 0 & \cos(\omega\tau) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.7)$$

where τ is the sampling time and the covariance of the process noise is

$$Q = \begin{bmatrix} \sigma_v^2 G G^\top & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}, \quad G = I_2 \otimes \begin{bmatrix} \frac{1}{2}\tau^2 \\ \tau \end{bmatrix}, \quad (2.8)$$

where I_2 is the 2×2 identity matrix, $\otimes$ is the Kronecker product, and σ_v and σ_ω are the standard deviations of the acceleration noise and the turn rate noise, respectively.

2.2 Bayesian filtering

Recursive Bayesian estimation, or *Bayesian filtering*, is a probabilistic approach to estimate the state of a dynamic system from noisy observations. The entity of interest is the posterior density $p(x_k | z_{1:k})$, which captures all information known about the state vector x_k at time k based on the modelling and information available in the measurement sequence $z_{1:k} = (z_1, \dots, z_k)$. This section gives a brief introduction to the subject based on the probabilistic state-space model defined in (2.3). For further details, the reader is referred to [41] and [76].

Suppose that at time $k-1$, the probability density function $p(x_{k-1} | z_{1:k-1})$ captures all knowledge about the system state x_{k-1} , conditioned on the sequence of measurements received so far, $z_{1:k-1}$. As a new measurement z_k is obtained at time k , the equations of the Bayes filter [41],

$$p(x_k | z_{1:k-1}) = \int p(x_k | x_{k-1}) p(x_{k-1} | z_{1:k-1}) dx_{k-1}, \quad (2.9a)$$

$$p(x_k | z_{1:k}) = \frac{p(z_k | x_k) p(x_k | z_{1:k-1})}{\int p(z_k | x_k) p(x_k | z_{1:k-1}) dx_k}, \quad (2.9b)$$

are used to fuse the information in z_k with the information contained in the

previous density $p(x_{k-1} | z_{1:k-1})$, to yield a new posterior density $p(x_k | z_{1:k})$, also referred to as a filtering density. The first equation of the Bayes filter (2.9a) is a prediction step known as the Chapman-Kolmogorov equation and results in a predictive density $p(x_k | z_{1:k-1})$. The second equation (2.9b), known as Bayes' rule, is applied to perform a measurement update. By repeatedly applying these equations, the posterior density can be computed recursively as time progresses and new measurements become available.

2.2.1 Linear Gaussian filtering

While the Bayes filter is theoretically appealing, the posterior density can in general not be computed in closed form, and analytical solutions exist only for a few special cases [37]. A notable exception, the case of linear systems with additive white Gaussian noise, is discussed in this section.

Kalman filter

The well-known *Kalman filter*, derived in [48], provides an analytical solution to the Bayesian filtering problem in the special case of linear Gaussian systems [46, 76]. A Gaussian density function is completely parametrized by the first and second order moment, *i.e.*, the mean and the covariance. Given a Gaussian distributed state density at time $k - 1$ and the linear Gaussian model (2.2), both the predictive and the filtering densities in (2.9) are Gaussian distributed and thereby described by the corresponding means and covariances. As these are the quantities that are propagated by the Kalman filter, it yields the solution to the Bayesian filtering problem. The equations of the Kalman filter are provided in Algorithm 2.1, where the notation $\hat{x}_{k|t}$ denotes the estimate of the state x at time k using the information available in the measurements up to and including time t , *i.e.*, $\hat{x}_{k|t} = \mathbf{E}\{x_k | z_{1:t}\}$. An analogous notation is used for the covariance, $P_{k|t} = \mathbf{E}\{(x_k - \hat{x}_{k|t})(x_k - \hat{x}_{k|t})^\top | z_{1:t}\}$.

A key property of the Kalman filter is that the covariance matrices $P_{k|k-1}$ and $P_{k|k}$ are both independent of the measurements $z_{1:k}$ and depend only on the model assumptions [46]. This means that given the system model (2.2), the posterior covariance matrix at any time step k can be pre-computed before any measurements have been obtained. This property is utilized in Paper B.

Example 2.3 illustrates how the Kalman filter is used to estimate the state of a system in which the dynamics are described by the NCV model (2.5).

Information filter

An alternative formulation of the Kalman filter is the *information filter* [4, 46]. Instead of maintaining the mean and covariance as in the Kalman filter, the information filter maintains an information state and an information matrix. The information matrix is the inverse of the covariance matrix $\mathcal{I}_k = P_k^{-1}$, and the information state is defined as $\iota_k = P_k^{-1} \hat{x}_k$.

The equations of the information filter are outlined in Algorithm 2.2. Compared to the Kalman filter, the change of variables results in a shift of computational

Algorithm 2.1: Kalman filter

Input: Linear state-space model (2.2), measurement z_k , state estimate $\hat{x}_{k-1|k-1}$ with covariance $P_{k-1|k-1}$
Prediction:

$$\hat{x}_{k|k-1} = F_{k-1}\hat{x}_{k-1|k-1} \quad (2.10a)$$

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^\top + G_{k-1}Q_{k-1}G_{k-1}^\top \quad (2.10b)$$

Measurement update:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(z_k - H_k\hat{x}_{k|k-1}) \quad (2.11a)$$

$$P_{k|k} = (I - K_kH_k)P_{k|k-1} \quad (2.11b)$$

$$K_k = P_{k|k-1}H_k^\top(H_kP_{k|k-1}H_k^\top + R_k)^{-1} \quad (2.11c)$$

Output: Updated state estimate $\hat{x}_{k|k}$ and covariance $P_{k|k}$

complexity from the measurement update step to the time update step. Since information is additive, the measurement update step is cheaper in an information filter, whereas the time update step is cheaper in a Kalman filter. The information filter form also has the advantage that it allows the filter to be initiated without an initial state estimate, which corresponds to setting $\mathcal{I}_{0|0} = 0$ [37]. Furthermore, as both the time update step and measurement update step for the information matrix are independent of the actual measurement values, the posterior information matrix can be recursively computed in advance, before any measurements have been obtained [46]. This property is utilized in Paper A and Paper B.

Algorithm 2.2: Information filter

Input: Linear state-space model (2.2), measurement z_k , information state $\iota_{k-1|k-1}$, and information matrix $\mathcal{I}_{k-1|k-1}$
Prediction:

$$\mathcal{I}_{k|k-1} = (F_{k-1}\mathcal{I}_{k-1|k-1}^{-1}F_{k-1}^\top + G_{k-1}Q_{k-1}G_{k-1}^\top)^{-1} \quad (2.13a)$$

$$\iota_{k|k-1} = \mathcal{I}_{k|k-1}F_{k-1}\iota_{k-1|k-1} \quad (2.13b)$$

Measurement update:

$$\mathcal{I}_k = \mathcal{I}_{k|k-1} + H_k^\top R_k^{-1}H_k \quad (2.14a)$$

$$\iota_k = \iota_{k|k-1} + H_k^\top R_k^{-1}z_k \quad (2.14b)$$

Output: Updated information state $\iota_{k|k}$ and information matrix $\mathcal{I}_{k|k}$

Example 2.3: Nearly constant velocity Kalman filter

In this example a Kalman filter is used to estimate the state of a system, which dynamics are described by the NCV model from Example 2.1 with sampling time $\tau = 1$ s. The true state is initialized at $x_0 = [1, 1]^\top$ and simulated with process noise covariance $Q = 1 \text{ (m/s}^2\text{)}^2$. The measurement noise covariance is $R = 1 \text{ m}^2$. Using the parameters of the true model, a Kalman filter initialized with

$$\hat{x}_{0|0} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top, \quad (2.12a)$$

$$P_{0|0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.12b)$$

results in the estimated state trajectory illustrated in Figure 2.1.

Figure 2.1: Kalman filter state estimates and confidence intervals corresponding to two standard deviations.

Alpha-beta filter

Under suitable conditions, see [4, p. 211], the Kalman filter achieves steady-state and the covariance converges to a stationary value. A steady-state Kalman filter with nearly constant velocity motion model and position measurements is equivalent to an alpha-beta filter. The alpha-beta filter is a constant-gain filter that only propagates the first order moment, *i.e.*, the expected value of the state variable. This makes it computationally less demanding than the Kalman filter. The steady-state gains for an alpha-beta filter with sampling time τ are given by

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{\tau} \end{bmatrix}^\top, \quad (2.15)$$

where α and β are the optimal gains. These can be computed based on the random tracking index, a dimensionless parameter that is proportional to the ratio of the

uncertainty due to the target maneuverability and the sensor measurements [47]. Given the optimal gains, the equations of the alpha-beta filter are given by

$$\hat{x}_{k|k-1} = F\hat{x}_{k-1|k-1}, \quad (2.16a)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \begin{bmatrix} \alpha \\ \beta \\ \tau \end{bmatrix} (z_k - H\hat{x}_{k|k-1}). \quad (2.16b)$$

While the conditions for steady-state are seldom satisfied in practice, the alpha-beta filter is useful for analytical predictions of the expected tracking performance [14, 15]. In Paper D, predictions based on the alpha-beta filter are used to find the optimal tracking range and beamwidth for a radar system.

2.2.2 Nonlinear filtering

In practical applications, nonlinearities are often present in either the system dynamics or the measurement model. Approximate solutions to the Bayesian filtering recursion are then required for tractability. A commonly used idea is to approximate the true posterior distribution by a Gaussian with mean and covariance corresponding to those of $p(x_k | z_{1:k})$. This section presents a number of popular nonlinear filtering approaches.

Extended Kalman filter

The *extended Kalman filter* (EKF) [4, 41] provides an approximate solution to the Bayesian filtering problem by propagating estimates of the mean and covariance in time. In each time step, the dynamics and measurement functions are linearized at the current state estimate, and the Kalman filter equations (2.10)–(2.11) are applied to perform the time and measurement updates. In contrast to the Kalman filter, the covariance matrices computed by the EKF depend on the measurements since the nominal values used for linearization depend on the measurement values. Thus, computation of the resulting covariance matrix at design time is no longer possible. Algorithm 2.3 outlines the equations of the EKF for the general state-space model (2.1), under the assumption that the process noise w_k has zero mean and covariance Q_k and the measurement noise e_k has zero mean and covariance R_k .

Unscented Kalman filter

Unlike the EKF, the *unscented Kalman filter* (UKF) [42, 43] does not apply any linearizations. Instead, it relies on a deterministic sampling principle called the unscented transform [44] to propagate the first and second order moments of the state density through nonlinear functions. Given the density of x , the density of the transformed variable $y = \varphi(x)$, where φ is a general function, is approximated as follows. First, a set of N samples $x^{(i)}$, referred to as *sigma points*, with corresponding weights $w^{(i)}$, are carefully selected to represent the density of the state x . Each of these sigma points are then passed through the nonlinear

Algorithm 2.3: Extended Kalman filter

Input: General state-space model (2.1) with $\mathbf{E} e_k = \mathbf{E} w_k = 0$, measurement z_k , state estimate $\hat{x}_{k-1|k-1}$ with covariance $P_{k-1|k-1}$
 Prediction:

$$\hat{x}_{k|k-1} = f_{k-1}(\hat{x}_{k-1|k-1}) \quad (2.17a)$$

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^\top + G_{k-1}Q_{k-1}G_{k-1}^\top \quad (2.17b)$$

where

$$F_{k-1} = \left. \frac{\partial f_{k-1}(x)}{\partial x} \right|_{x=\hat{x}_{k-1|k-1}} \quad (2.17c)$$

and Q_{k-1} is the covariance of w_{k-1} .

Measurement update:

$$K_k = P_{k|k-1}H_k^\top(H_kP_{k|k-1}H_k^\top + R_k)^{-1} \quad (2.18a)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(z_k - h_k(\hat{x}_{k|k-1})) \quad (2.18b)$$

$$P_{k|k} = (I - K_kH_k)P_{k|k-1} \quad (2.18c)$$

where

$$H_k = \left. \frac{\partial h_k(x)}{\partial x} \right|_{x=\hat{x}_{k|k-1}} \quad (2.18d)$$

and R_k is the covariance of e_k .

Output: Updated state estimate $\hat{x}_{k|k}$ and covariance $P_{k|k}$

function as $y^{(i)} = \varphi(x^{(i)})$, and the mean μ_y and covariance Σ_y of the transformed density are estimated as

$$\hat{\mu}_y = \sum_{i=1}^N w^{(i)} \varphi(x^{(i)}), \quad (2.19a)$$

$$\hat{\Sigma}_y = \sum_{i=1}^N w^{(i)} (\varphi(x^{(i)}) - \hat{\mu}_y)(\varphi(x^{(i)}) - \hat{\mu}_y)^\top. \quad (2.19b)$$

As only the approximated mean $\hat{\mu}_y$ and the covariance $\hat{\Sigma}_y$ are known, the transformed density is often represented by a Gaussian density, *i.e.*, $p(y) \approx \mathcal{N}(y; \hat{\mu}_y, \hat{\Sigma}_y)$. Relations between the EKF and the UKF are explored in [38].

The basic idea of the unscented transform is illustrated in Figure 2.2. The same concept is utilized in Paper C, where a set of carefully selected samples are used to predict several possible future state trajectories.

Figure 2.2: Illustration of the unscented transform. Sigma points representing $x \sim \mathcal{N}(\mu_x, \Sigma_x)$ where $\mu_x = \begin{bmatrix} 50 \\ \pi/4 \end{bmatrix}$ and $\Sigma_x = \begin{bmatrix} 50 & -1 \\ -1 & \frac{\pi}{100} \end{bmatrix}$ are passed through the nonlinear function $y = \varphi(x) = \begin{bmatrix} x_1 \cos x_2 \\ x_1 \sin x_2 \end{bmatrix}$. The estimated mean and covariance ellipses are shown in blue and Monte Carlo samples from the underlying distributions are shown in gray.

Point mass filter

The *point mass filter* (PMF) [24, 49] is a grid-based method that makes use of deterministic state-space discretizations. This allows for approximating continuous densities with piecewise constant functions such that the integrals in (2.9) can be treated as sums. At time $k-1$, the discretization over the state space of x_{k-1} results in N cells, of which the i th cell is denoted $\mathcal{C}_{k-1}^{(i)}$ and has midpoint $x_{k-1}^{(i)}$. The filtering density is approximated as

$$p(x_{k-1} | z_{1:k-1}) \approx \sum_{i=1}^N w_{k-1|k-1}^{(i)} \mathcal{U}(x_{k-1}; \mathcal{C}_{k-1}^{(i)}), \quad (2.20)$$

where $\mathcal{U}(x_{k-1}; \mathcal{C}_k^{(i)})$ is the uniform distribution in x_{k-1} over the cell $\mathcal{C}^{(i)}$ and $w_{k-1|k-1}^{(i)}$ is the corresponding weight. The weights are normalized and satisfy $\sum_{i=1}^N w_{k-1|k-1}^{(i)} = 1$. In the prediction step the weights are updated according to

$$w_{k|k-1}^{(i)} = \sum_{j=1}^N w_{k-1|k-1}^{(j)} p(x_k^{(i)} | x_{k-1}^{(j)}), \quad (2.21)$$

and the measurement update step corresponds to

$$w_{k|k}^{(i)} = \frac{w_{k|k-1}^{(i)} p(z_k | x_k^{(i)})}{\sum_{i=1}^N w_{k|k-1}^{(i)} p(z_k | x_k^{(i)})}. \quad (2.22)$$

The main advantage of the PMF is its simple implementation. The disadvantage is that the complexity is quadratic in the number of grid points, which makes the filter inapplicable in higher dimensions [36].

Figure 2.3: Conceptual illustration of the probability density representations used by two different Rao-Blackwellized filters. As the RB-PMF uses a point mass density to represent the density of the nonlinear state variable x^n , the height of each Gaussian in the conditionally linear state variable x^l indicates the weight of the corresponding point mass. Instead of a deterministic grid, RB-PF represents the density in x^n using particles.

Rao-Blackwellization

The structure of the state-space model can sometimes be exploited in filtering problems. If there is a conditionally linear Gaussian substructure in the model, *Rao-Blackwellization* [13, 25, 67] can be used to evaluate parts of the filtering equations analytically even if the full posterior $p(x_k | z_{1:k})$ is intractable. For such models, consider a partitioning of the state vector into two components as

$$x_k = \begin{bmatrix} x_k^l \\ x_k^n \end{bmatrix}, \quad (2.23)$$

where x_k^l and x_k^n are used to denote the linear and the nonlinear state variables, respectively. An example of a model that allows for this partitioning is the nearly constant velocity model in Example 2.1 combined with a nonlinear measurement model, *e.g.*, measurements of range and bearing. As the motion model is linear and the nonlinear measurement model only depends on the position component, the velocity component corresponds to the linear part of the state.

The *Rao-Blackwellized particle filter* (RB-PF) [28, 39, 72] estimates the posterior density, which is factorized according to

$$p(x_k^l, x_{0:k}^n | z_{1:k}) = p(x_k^l | x_{0:k}^n, z_{1:k}) p(x_{0:k}^n | z_{1:k}), \quad (2.24)$$

where $p(x_k^l | x_{0:k}^n, z_{1:k})$ is analytically tractable, while $p(x_{0:k}^n | z_{1:k})$ is not. A particle filter [33] is used to estimate the nonlinear state density and Kalman filters, one for each particle, are used to compute the conditional density of the linear part of the state. A related approach is the *Rao-Blackwellized point mass filter* (RB-PMF) [74], which is used in Paper F. It estimates the filtering distribution

$$p(x_k^l, x_k^n | z_{1:k}) = p(x_k^l | x_k^n, z_{1:k}) p(x_k^n | z_{1:k}). \quad (2.25)$$

where a point mass filter is used to estimate the nonlinear part of the state. In contrast to (2.24), the full nonlinear state trajectory is not available in (2.25). Hence, additional approximations need to be introduced when estimating the linear part using Kalman filters. Figure 2.3 illustrates the conceptual difference between the probability density representations used by the RB-PF and the RB-PMF.

2.3 Performance evaluation

In general, the aim of sensor management is to improve the performance of the underlying state estimation method. To this end, an objective function that encodes this performance is required. This section presents a number of approaches to evaluate the performance of a state estimation method, both for the case when the true state is known and for the case when it is not.

2.3.1 Root mean square error

A standard performance metric for the estimation error is the root mean square error (RMSE) [4]. As the name suggests, it corresponds to the root of the average squared difference between the estimated state and the true state. In a Monte Carlo simulation setting, where a scenario is simulated multiple times with different noise realizations, the RMSE at each time step k is computed as

$$d_{\text{rmse},k} = \sqrt{\frac{1}{n_{\text{mc}}} \sum_{m=1}^{n_{\text{mc}}} (x_k - \hat{x}_k^m)^\top (x_k - \hat{x}_k^m)}, \quad (2.26)$$

where x_k is the true state, $\hat{x}_k^m$ is the estimated state in the m th simulation run, and n_{mc} is the number of Monte Carlo runs.

2.3.2 Uncertainty measures

The RMSE is convenient for performance evaluation of state estimation methods in cases where the true state is known, *e.g.*, in simulations. If the true state value is unknown, an indication of the estimation performance can instead be obtained by quantifying the uncertainty associated with the state estimate.

General distributions

Information theory [27, 73] provides a number of measures to quantify the uncertainty inherent in general distributions. One such measure is the differential entropy, which for a random variable x with distribution $p(x)$ is defined as

$$H(x) = -\mathbf{E} \log p(x) = -\int p(x) \log p(x) dx. \quad (2.27)$$

The conditional entropy

$$H(x|z) = -\int p(z) \int p(x|z) \log p(x|z) dx dz \quad (2.28)$$

is the entropy of a random variable x conditioned on the knowledge of another random variable z . The mutual information between the variables x and z is defined as

$$I(x; z) = H(x) - H(x | z) \quad (2.29a)$$

$$= H(z) - H(z | x) \quad (2.29b)$$

$$= \iint p(x, z) \log \frac{p(x, z)}{p(x)p(z)} dx dz, \quad (2.29c)$$

which corresponds to the reduction in uncertainty due to the other random variable and quantifies the dependency between the two variables x and y [27].

Gaussian distributions

The spread of a Gaussian distribution is completely characterized by the corresponding covariance matrix. For the Kalman filter or information filter, where the state estimate is Gaussian distributed, a measure based on the covariance matrix or information matrix can thus be used as an indication of the estimation performance. There are many different scalar performance measures that can be employed and [82] gives a thorough analysis of several alternatives. The use of scalar measures of covariance and information matrices also occurs in the field of experiment design [65], and some of the more popular criteria are:

- *A*-optimality, in which the objective is to minimize the trace of the covariance matrix,

$$\ell_A(P) = \text{tr } P = \sum_{i=1}^n \lambda_i(P), \quad (2.30a)$$

where $\lambda_i(P)$ is the i th largest eigenvalue of $P \in \mathbf{S}_+^n$. This corresponds to minimizing the expected mean square error of the state estimate [82].

- *T*-optimality, in which the objective is to minimize the negative trace of the information matrix,

$$\ell_T(\mathcal{I}) = -\text{tr } \mathcal{I} = -\sum_{i=1}^n \lambda_i(\mathcal{I}). \quad (2.30b)$$

- *D*-optimality, in which the objective is to minimize the negative determinant of the information matrix,

$$\ell_D(\mathcal{I}) = -\det \mathcal{I} = -\prod_{i=1}^n \lambda_i(\mathcal{I}). \quad (2.30c)$$

The *D*-optimality criterion has a geometric interpretation as it corresponds to minimizing the volume of the resulting confidence ellipsoid. It also has an information-theoretic interpretation. If $x \in \mathbf{R}^n$ is Gaussian distributed with covariance P , its differential entropy is given by

$$H(x) = \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det P. \quad (2.30d)$$

As the natural logarithm is a monotonically increasing function [23], the D -optimality criterion is equivalent to minimizing the differential entropy in the Gaussian case.

- E -optimality, in which the objective is to minimize the largest eigenvalue of the covariance matrix,

$$\ell_E(P) = \lambda_{\max}(P). \quad (2.30e)$$

This can be interpreted as minimizing the largest semi-axis of the confidence ellipsoid, or simply minimizing uncertainty in the most uncertain direction [82].

3

Target tracking

Target tracking is a special case of dynamic state estimation. It refers to the problem of estimating the states of one or more objects of interest, called targets, using noisy sensor observations. Complicating factors for the problem are, apart from the measurement noise, that the number of targets is both unknown and time-varying, there are misdetections, false alarms, and unknown measurement origins. This chapter provides an overview of the target tracking problem and a number of state-of-the-art target tracking algorithms.

3.1 Single and multiple target tracking

The target tracking problem can be considered as a more complicated version of the Bayesian estimation problem discussed in Chapter 2. The standard Bayesian estimation problem assumes that there exists exactly one target which generates exactly one measurement in each time step. In target tracking, where the objective is to estimate the state of all targets that are present, these assumptions are relaxed and the problem is characterized by the following properties:

- the number of targets is unknown and time-varying;
- a target generates at most one noise-corrupted measurement per time step and the detection probability is less than one;
- there are false alarms, often referred to as clutter measurements; and
- the measurement origins are unknown, *i.e.*, it is not known which measurements correspond to actual targets and which measurements are false alarms, or which target that generated which measurement.

Figure 3.1: Illustration of a multiple target tracking problem adapted from [77]. The blue circles in the lower layer illustrate states of individual targets. In the upper layer, orange circles correspond to target-derived measurements and green circles are false alarms. Note that a new target appears at time $k + 1$ and that not all targets generate measurements in each time step.

Due to these properties, which are illustrated in Figure 3.1, the classical filtering methods in Chapter 2 cannot be directly applied. A wide range of algorithms for target tracking have been developed over the years, and excellent descriptions of the subject are provided by [5, 12].

In single target tracking, it is assumed that at most one target is present in the region of interest. Suppose that at time k , a set of $N_{z,k}$ measurements is obtained and denote this set as

$$Z_k = \{z_k^1, \dots, z_k^{N_{z,k}}\}. \quad (3.1)$$

Furthermore, let $Z_{1:k}$ denote the sequence of measurement sets obtained up until time k . The objective in single target tracking is to, based on $Z_{1:k}$, determine if there is a target present and, if there is, estimate its state. This is typically done in a recursive manner, and an association problem needs to be solved in each time step to decide which of the measurements in Z_k that is target-derived, *i.e.*, corresponds to the actual target. Two commonly used methods for single target tracking are the *nearest neighbor* (NN) filter [5], in which the measurement that is, in some sense, closest to the predicted measurement is used to update the state estimate, and the *probabilistic data association* (PDA) filter [3, 5], which uses the average of all measurements weighted according to their association probabilities, *i.e.*, the likelihood of each measurement given a target state.

In the more general case of multiple target tracking, there may be more than one target present and the actual number of targets is both unknown and time-varying due to targets appearing and disappearing, as illustrated in Figure 3.1. At the core of multiple target tracking is the problem of jointly estimating the number of targets $N_{x,k}$ and their states

$$X_k = \{x_k^1, \dots, x_k^{N_{x,k}}\} \quad (3.2)$$

from the available measurements $Z_{1:k}$. The fact that there is no information about which targets generated which measurements makes the association problem in multiple target tracking considerably more complicated than in single target tracking. Classical approaches to multiple target tracking include the *global nearest neighbor* (GNN) tracker [12] and the *joint probabilistic data association* (JPDA) filter [5], which can be seen as extensions of their single target tracking equivalents. The GNN and JPDA filters both maintain a single data association hypothesis about all the measurements received over time. In contrast, the *multiple hypothesis tracker* (MHT) [5, 12, 68] propagates and maintains a set of association hypotheses, each with a different partitioning of the measurement sets according to their origin. The hypotheses are ranked according to a score function, see, *e.g.*, [2], and the one with the best score is considered to best represent the truth. Given a data association hypothesis, a standard Bayes filter can be used to estimate the states of individual targets. As the total number of hypotheses increases exponentially with time, heuristic pruning and merging of hypotheses is performed to reduce the computational cost.

The traditional solutions to multiple target tracking, such as MHT and the JPDA filter, typically consist of a data association step followed by filtering to estimate single-target state densities. The next section gives an introduction to the random finite set (RFS) based approach to multiple target tracking, which attempts to directly estimate the density of the multi-target state [57, 77].

3.2 Multi-target state estimation

The mathematical framework of finite set statistics (FISST) [57, 58] provides a probabilistic toolbox that enables a Bayesian approach to multiple target tracking. To this end, FISST generalizes many concepts from conventional vector-valued probability theory to set-valued random variables. The target tracking problem can then be formulated as an estimation problem, where the set-valued variable of interest corresponds to the states of multiple targets.

3.2.1 Random finite sets

A key concept in the FISST framework is that of random finite sets. An RFS is a random variable of which the realizations are finite sets, *i.e.*, a finite-set-valued random variable. The number of elements in an RFS is thus random, and the elements themselves are random. An RFS variable $X = \{x_1, \dots, x_n\}$ is completely characterized by a discrete cardinality distribution $\rho(n) = p(|X| = n)$, where n is a non-negative integer and $|X|$ denotes the cardinality of X , and a family of joint distributions $p_n(x_1, \dots, x_n)$, that characterize the distribution of the set's elements conditioned on the cardinality n .

In the context of multiple target tracking, where both the number of targets and their individual states are random and time-varying, the collection of target states (3.2) can conveniently be modeled as an RFS referred to as the multi-target state X_k . Similarly, an RFS Z_k can be used to model the set of measurements at time k , as both the number of measurements and their values are random [56–58].

Figure 3.2: Intensity function and an example realization of a Poisson RFS where the Poisson rate, *i.e.*, the expected number of elements, is $\mu = 5$. Note that realizations of this RFS may contain more or less than five elements, and that there are four elements in this particular realization.

Note that the same symbol is used for an RFS and its realization for notational convenience.

In finite set statistics, there are several classes of random finite sets, each having its own unique properties. Here, brief descriptions of four types commonly used in multiple target tracking are provided. For more details, see [57, 58].

Poisson RFS

A Poisson RFS, also referred to as a Poisson point process (PPP), is an RFS in which the cardinality of the set is Poisson distributed and, for each given cardinality, the elements are independent and identically distributed. The Poisson rate μ and the single-element distribution $p(x)$ form the intensity $\lambda(x)$ of the PPP as $\lambda(x) = \mu p(x)$, which completely characterizes the RFS. A high value of $\lambda(x)$ corresponds to a high rate of occurrence. The density of a PPP X is given by [69]:

$$\pi(X) = e^{-\int \lambda(x) dx} \prod_{x \in X} \lambda(x). \quad (3.3)$$

An example of a Poisson RFS is given in Figure 3.2. In multiple target tracking, PPPs are often used to model clutter measurements and the appearance of new targets.

Bernoulli RFS

The cardinality of a Bernoulli RFS is Bernoulli distributed with parameter $r \in [0, 1]$. A realization of a Bernoulli RFS is either empty, with probability $1 - r$, or, with probability r , contains a single element with density $p(x)$. The parameters r and p

Figure 3.3: Illustration of the density and three realizations of a Bernoulli RFS with existence probability $r = 0.6$ and spatial density given by $p(x) = \mathcal{N}(x; 7, 0.5)$. A realization of a Bernoulli RFS is either empty or contains a single element.

thus characterize the density of a Bernoulli RFS X , which is given by [69]:

$$\pi(X) = \begin{cases} 1 - r, & X = \emptyset, \\ rp(x), & X = \{x\}, \\ 0, & |X| > 1, \end{cases} \quad (3.4)$$

where $|X|$ denotes the cardinality of X . An example of a Bernoulli RFS is given in Figure 3.3. In multiple target tracking, a Bernoulli RFS is a convenient model for a single potential target, as it captures both the possibility that the target may or may not exist as well as the uncertainty in the target state.

Multi-Bernoulli RFS

The disjoint union of a fixed number of Bernoulli RFSs is a multi-Bernoulli (MB) RFS. Its density is defined by the parameters $\{r^i, p^i\}_{i \in \mathbb{I}}$, where $\mathbb{I}$ is an index set:

$$\pi(X) = \begin{cases} \sum_{\sqcup_{i \in \mathbb{I}} X^i = X} \prod_{i \in \mathbb{I}} \pi^i(X^i), & |X| \leq |\mathbb{I}|, \\ 0, & |X| > |\mathbb{I}|, \end{cases} \quad (3.5)$$

where the notation $X^1 \sqcup X^2 = X$ denotes disjoint union, *i.e.*, $X^1 \cup X^2 = X$ and $X^1 \cap X^2 = \emptyset$, as shown in Example 3.1. A multi-Bernoulli RFS is useful as a representation of multiple potential targets. An example with two potential targets is illustrated Figure 3.4.

Multi-Bernoulli mixture RFS

A normalized, weighted sum of MB RFSs is referred to as a multi-Bernoulli mixture (MBM) RFS. Its density can be expressed as [34]:

$$\pi(X) = \sum_{j \in \mathbb{J}} w^j \sum_{\sqcup_{i \in \mathbb{I}^j} X^i = X} \prod_{i \in \mathbb{I}^j} \pi^{j,i}(X^i) \quad (3.6)$$

and is characterized by the set of parameters $\{w^j, \{r^{j,i}, p^{j,i}\}_{i \in \mathbb{I}^j}\}_{j \in \mathbb{J}}$, where $\mathbb{J}$ is an index set for the MB components of the MBM, $\mathbb{I}^j$ is an index set for the Bernoullis of the j th MB RFS, and w^j is the weight of the j th MB. A multi-Bernoulli mixture RFS is convenient for modeling several data association hypotheses in multiple target tracking. Each MB in the mixture then corresponds to a unique hypothesis and represents a set of potential targets. In contrast to the hypotheses in MHT, each potential target in an MBM RFS has an associated parameter that represents the probability that the target is present. An example with two hypotheses is illustrated in Figure 3.5.

3.2.2 Multi-target Bayes filter

Using the FISST framework, the multiple target tracking problem can be formulated as a generalization of the Bayesian estimation problem in Chapter 2. Whereas conventional recursive Bayesian estimation is concerned with the density of a single vector-valued variable, the variable of interest is here an RFS that models the set of multiple target states. The objective is to estimate the posterior density of the multi-target state, $\pi(X_k | Z_{1:k})$, where $Z_{1:k}$ is a collection of finite sets of measurements received up until time k . This density captures all knowledge about the multi-target state, which means that it includes information about the number of targets as well as their individual states.

Multi-target state-space models

To cast the multiple target tracking problem as an RFS-based Bayesian estimation problem, a state-space model that represents the dynamics of the multi-target state and multi-target measurement model is needed. The standard dynamic model [57] for the multi-target state is defined as follows. As time progresses from k to $k+1$, each individual target $x_k \in X_k$ generates a new set $S_{k+1|k}(x_k)$, which is empty if the target disappears and contains a single element x_{k+1} with density $p(x_{k+1} | x_k)$ if the target survives. The survival probability is state-dependent and denoted

Example 3.1: Disjoint union

Consider a partitioning of the set $X = \{x^1, x^2\}$ into two sets X^1 and X^2 such that the disjoint union of X^1 and X^2 is equal to X , *i.e.*, $X^1 \uplus X^2 = X = \{x^1, x^2\}$. This gives the following possible combinations:

- $X^1 = \emptyset$ and $X^2 = \{x^1, x^2\}$;
- $X^1 = \{x^1\}$ and $X^2 = \{x^2\}$;
- $X^1 = \{x^2\}$ and $X^2 = \{x^1\}$; and
- $X^1 = \{x^1, x^2\}$ and $X^2 = \emptyset$.

If $\mathbb{I} = \{1, 2\}$, the sum in (3.5) is to be taken over these four combinations.

Figure 3.4: Illustration of the density and three realizations of a multi-Bernoulli RFS with two independent components. One with existence probability $r^1 = 0.3$ and spatial density $p^1(x) = \mathcal{N}(x; 2, 1)$, and one with existence probability $r^2 = 0.6$ and spatial density $p^2(x) = \mathcal{N}(x; 7, 0.5)$. Realizations of such a multi-Bernoulli RFS contains zero, one or two elements.

Figure 3.5: Illustration of the density and four realizations of a multi-Bernoulli mixture RFS with two independent MB components. The first MB component has weight $w^1 = 0.7$ and contains of a single Bernoulli RFS. The second MB component has weight $w^2 = 0.3$ and contains two Bernoulli RFSs.

$p_s(x_k)$. The set of targets that appear for the first time at time $k + 1$ is modeled as an RFS denoted B_{k+1} . Consequently, the multi-target state transition equation is given by

$$X_{k+1} = \bigcup_{x_k \in X_k} S_{k+1|k}(x_k) \cup B_{k+1}, \quad (3.7)$$

which can be described by a multi-target state transition density

$$p(X_{k+1} | X_k). \quad (3.8)$$

This multi-target transition density captures the underlying models of individual target motion, births and deaths, *i.e.*, target appearance and disappearance [77].

The standard multi-target measurement model [57] is defined as follows. At time k , each target $x_k \in X_k$ generates a measurement set $D_k(x_k)$. This set contains a single element with density $p(z_k | x_k)$ if the target is detected and is empty if the target is undetected. The detection probability is state-dependent and denoted $p_D(x_k)$. By representing the set of false alarms with a PPP denoted F_k , the full set of measurements Z_k generated by X_k is given by the multi-target measurement equation

$$Z_k = h(X_k) = \bigcup_{x_k \in X_k} D_k(x_k) \cup F_k, \quad (3.9)$$

which can be described by the multi-target measurement likelihood function

$$p(Z_k | X_k), \quad (3.10)$$

that gives the likelihood that the measurement set Z_k is obtained given the multi-target state X_k . This likelihood captures the underlying models of target detections, measurement noise, and false alarms [77].

Algorithm outline

Based on the multi-target state transition density (3.8) and the multi-target measurement likelihood function (3.10), the posterior density of the multi-target state can be estimated recursively via the prediction and measurement update steps of the Bayes multi-target filter [56, 57]. Assuming that the multi-target state density at time $k - 1$ is $\pi(X_{k-1} | Z_{1:k-1})$ and a set of measurements Z_k is obtained at time k , the density is propagated in time according to

$$\pi(X_k | Z_{1:k-1}) = \int p(X_k | X_{k-1}) \pi(X_{k-1} | Z_{1:k-1}) \delta X_{k-1}, \quad (3.11a)$$

$$\pi(X_k | Z_{1:k}) = \frac{p(Z_k | X_k) \pi(X_k | Z_{1:k-1})}{\int p(Z_k | X_k) \pi(X_k | Z_{1:k-1}) \delta X_k}, \quad (3.11b)$$

where the integrals are set integrals, defined in [57]. The multi-target Bayes filter is similar to the conventional Bayes filter (2.9), but the vector-valued densities and operations have been replaced by their set equivalents. In contrast to the conventional Bayes filter for linear Gaussian systems, the exact multi-target posterior is intractable to compute. Several tractable algorithms have been developed based

on different approximations. Examples of these include the probability hypothesis density (PHD) filter [56], the generalized labeled multi-Bernoulli (GLMB) filter [78, 79], and the Poisson multi-Bernoulli mixture (PMBM) filter [80], which is used in both Paper E and Paper F.

Poisson multi-Bernoulli mixture filter

The PMBM filter [31, 80] propagates the multi-target state density, which is assumed to be a PMBM: a union of a PPP component (3.3) and an MBM component (3.6). The PMBM density is a multi-target conjugate prior for the standard multi-target measurement model, which means that if the predicted density $\pi(X_k | Z_{k-1})$ is PMBM, then so is the posterior density $\pi(X_k | Z_k)$. Furthermore, with a PPP birth RFS, the PMBM form is preserved after performing the prediction step with the standard multi-target measurement model.

The MBM represents potentially detected targets and considers all possible data association hypotheses. Each measurement at each time step may be either the first detection of a new target, a new measurement of a target that has previously been detected, or a false alarm. Each measurement thus results in a new potentially detected target which is Bernoulli distributed, to account for the fact that it might not correspond to an actual target. For each potentially detected target, there are single target hypotheses which represent sequences of target-to-measurement associations. A global hypothesis, represented by an MB component in the MBM, consists of several single target hypotheses from different potentially detected targets. The unknown data associations lead to an intractably large number of components in the MBM density, and approximations are necessary for tractability, *e.g.*, using a track-oriented MHT formulation [31].

The PPP represents targets that are hypothesized to exist, but have never been detected, *e.g.*, targets that have been located in a region where the sensor system has low detection probability. The PMBM filter thus not only provides estimated states of the detected targets, but also maintains an explicit representation of where undetected targets are likely to be located.

3.3 Performance evaluation

Evaluating the performance of a target tracking method corresponds to quantifying the distance between the true set of target states X and the corresponding estimate $\hat{X}$. While target tracking has many similarities to state estimation, traditional metrics such as the RMSE cannot be directly applied. Here, the performance measure must not only capture the state estimation error, but also errors in the estimated number of targets. There may be true targets for which there are no corresponding estimates, or estimates for which there are no corresponding true targets.

3.3.1 Optimal sub-pattern assignment metric

A commonly used performance metric for multiple target tracking is the optimal sub-pattern assignment (OSPA) metric, introduced in [71]. Given a single-target metric $d(x, \hat{x})$ such as the Euclidean distance and a cut-off parameter $c > 0$, the distance between two single-target states x and $\hat{x}$ given as

$$d^{(c)}(x, \hat{x}) = \min(c, d(x, \hat{x})). \quad (3.12)$$

The OSPA metric $d_p^{(c)}(X, \hat{X})$ of order p with cut-off c between the sets of target states $X = \{x_1, \dots, x_{|X|}\}$ and $\hat{X} = \{\hat{x}_1, \dots, \hat{x}_{|\hat{X}|}\}$ is then defined as

$$\begin{aligned} d_p^{(c)}(X, \hat{X}) &= \begin{cases} \left(\frac{1}{|\hat{X}|} \min_{\pi \in \Pi_{|\hat{X}|}} \sum_{i=1}^{|X|} d^{(c)}(x_i, \hat{x}_{\pi(i)})^p + c^p (|\hat{X}| - |X|)^p \right)^{1/p} & \text{if } |X| \leq |\hat{X}| \\ \left(\frac{1}{|X|} \min_{\pi \in \Pi_{|X|}} \sum_{i=1}^{|\hat{X}|} d^{(c)}(\hat{x}_i, x_{\pi(i)})^p + c^p (|X| - |\hat{X}|)^p \right)^{1/p} & \text{if } |X| > |\hat{X}| \end{cases} \end{aligned} \quad (3.13)$$

where $c > 0$, $1 \leq p < \infty$, Π_n is the set of all permutations of the integers $\{1, \dots, n\}$ for any $n \in \mathbf{Z}_{++}$, and an element $\pi \in \Pi_n$ is a sequence $\{\pi(1), \dots, \pi(n)\}$. The OSPA metric thus assigns all single-target states in the smallest set to single-target states in the other set and quantifies the localization error based on this assignment. The remaining single-target states are accounted for by a cardinality mismatch penalty.

3.3.2 Generalized optimal sub-pattern assignment metric

The generalized OSPA (GOSPA) metric, introduced in [66], is closely related to the OSPA metric, but is based on the intuitive concepts of localization error for properly detected targets and costs for missed targets and false detections. In contrast to OSPA, it encourages trackers to have as few missed targets and false detections as possible. The GOSPA metric is defined as

$$\begin{aligned} d_p^{(c, \alpha)}(X, \hat{X}) &= \begin{cases} \left(\min_{\pi \in \Pi_{|\hat{X}|}} \sum_{i=1}^{|X|} d^{(c)}(x_i, \hat{x}_{\pi(i)})^p + \frac{c^p}{\alpha} (|\hat{X}| - |X|)^p \right)^{1/p} & \text{if } |X| \leq |\hat{X}| \\ \left(\min_{\pi \in \Pi_{|X|}} \sum_{i=1}^{|\hat{X}|} d^{(c)}(\hat{x}_i, x_{\pi(i)})^p + \frac{c^p}{\alpha} (|X| - |\hat{X}|)^p \right)^{1/p} & \text{if } |X| > |\hat{X}| \end{cases} \end{aligned} \quad (3.14)$$

where $c > 0$, $1 \leq p < \infty$, and $0 < \alpha \leq 2$. The authors of [66] argue that the choice of $\alpha = 2$ is the most appropriate one for evaluation of multiple target tracking algorithms. For this choice of α , the GOSPA metric can be expressed as an optimization problem over assignment sets rather than permutations, *i.e.*,

$$d_p^{(c, 2)}(X, \hat{X}) = \min_{\gamma \in \Gamma} \left(\sum_{(i, j) \in \gamma} (d(x_i, \hat{x}_j))^p + \frac{c^p}{2} (|X| + |\hat{X}| - 2|\gamma|)^p \right)^{1/p}, \quad (3.15)$$

where Γ is the set of all possible assignment sets γ between the index sets $\{1, \dots, |X|\}$ and $\{1, \dots, |\hat{X}|\}$. Such an assignment set satisfies $\gamma \subseteq \{1, \dots, |X|\} \times \{1, \dots, |\hat{X}|\}$, $(i, j), (i, j') \in \gamma \implies j = j'$ and $(i, j), (i', j) \in \gamma \implies i = i'$, where the last two properties ensure that every i and j gets at most one assignment [66]. Note that $|\gamma|$ is the number of properly detected targets and that $|X| - |\gamma|$ and $|\hat{X}| - |\gamma|$ represent the number of missed and false targets, respectively. As a consequence, the GOSPA metric consists of localization errors for properly detected targets and a cost $c^p/2$ for each unassigned target.

4

Mathematical optimization

Mathematical optimization is a branch of applied mathematics with applications in a vast number of fields. Given a criterion, mathematical optimization is used to find the best decision from a set of alternatives. This chapter introduces the concept of mathematical optimization and formulates a general optimization problem. Brief overviews of convex and mixed-binary optimization problems are given. The notation and explanations are inspired by [23, 64], which are extensive resources on mathematical optimization.

4.1 Problem formulation

Adopting the notation of [23], a mathematical optimization problem can be written as

$$\begin{aligned} & \underset{x}{\text{minimize}} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p, \end{aligned} \tag{4.1}$$

where $x \in \mathbf{R}^n$ is the *decision variable* and $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$ is the *objective function*. The functions $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \dots, m$, are referred to as *inequality constraint functions* and the functions $h_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \dots, p$, are called *equality constraint functions*. The *domain* of the optimization problem is the set of points for which the objective and constraint functions are defined, *i.e.*,

$$\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \tag{4.2}$$

A point $x \in \mathcal{D}$ is *feasible* if it satisfies all constraints of (4.1), and an optimization problem is said to be feasible if there is at least one feasible point. The goal is to

compute a solution x^* that minimizes the objective function while belonging to the *feasible set*, *i.e.*, the set of all feasible points. A solution can either be a globally or a locally optimal solution. If x^* is a globally optimal solution, no other feasible point z gives a smaller objective function value, *i.e.*, $f_0(z) \geq f_0(x^*)$ for any $z \in \mathcal{D}$ that satisfies $f_i(z) \leq 0$, $i = 1, \dots, m$, and $h_i(z) = 0$, $i = 1, \dots, p$. A point x^* is a locally optimal solution if no other feasible point z in a neighborhood of x^* gives a smaller objective function value than x^* . The *optimal value* p^* of the problem (4.1) is defined as

$$p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, \ i = 1, \dots, m, \ h_i(x) = 0, \ i = 1, \dots, p\}. \quad (4.3)$$

Thus, if x^* attained then $p^* = f_0(x^*)$. A feasible point x with $f_0(x) \leq p^* + \varepsilon$, where $\varepsilon > 0$ is called ε -suboptimal. Two optimization problems are said to be *equivalent* if the optimal solution to one of the problems can be trivially computed from the solution to the other problem, and vice versa. This is useful in situations where a seemingly intractable problem can be transformed into an easily solved equivalent problem, as done in [17, 19].

An instance of the general optimization problem (4.1) is referred to as a *nonlinear optimization problem* if at least one of the constraint functions or the objective function is nonlinear [23]. An important subclass of nonlinear optimization problems is that of convex optimization problems, for which locally optimal solutions are also globally optimal. This is however not the case for nonlinear optimization problems in general. For many problems, it is difficult to verify if a locally optimal solution is in fact a globally optimal solution and finding the globally optimal solution can be extremely challenging [64]. Due to the difficulty of locating and verifying globally optimal solutions, much attention has been focused towards *local optimization*, *i.e.*, computing locally optimal solutions. See, *e.g.*, [6, 10, 64] for extensive textbooks on the subject. A solution that is obtained using local optimization might depend on the initial guess that was given to the optimization algorithm and there is no guarantee that it does not exist a different feasible point that produces a lower objective function value.

4.2 Convex optimization

Convexity is an important concept in mathematical optimization. As Rockafellar noted, “the great watershed in optimization is not between linearity and nonlinearity, but convexity and nonconvexity” [70]. A fundamental property of convex optimization problems is that any locally optimal solution is also globally optimal [23]. Furthermore, there exist effective algorithms and software packages that can reliably and efficiently solve even large optimization problems, provided that they are convex, see *e.g.*, [35, 54]. These properties make convex optimization problems useful in many different areas, *i.e.*, estimation, signal processing, and finance [23].

Convex analysis and properties of convex optimization problems are thoroughly discussed in [9, 23]. To characterize a convex optimization problem, the following definitions are needed:

Definition 4.1 (Convex set). A set $\mathcal{C}$ is a *convex set* if it holds that

$$\alpha x + (1 - \alpha)y \in \mathcal{C} \quad (4.4)$$

for all $x, y \in \mathcal{C}$ and $\alpha \in [0, 1]$. _____

Definition 4.2 (Convex function). A function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a *convex function* if $\text{dom}(f)$ is a convex set and it satisfies

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad (4.5)$$

for all $x, y \in \text{dom}(f)$ and $\alpha \in [0, 1]$. _____

According to [23], an instance of the general optimization problem (4.1) is referred to as a convex optimization problem if

- the objective function f_0 is convex;
- the inequality constraint functions f_i are convex; and
- the equality constraint functions h_i are affine.

A convex optimization problem can thus be formulated as

$$\begin{aligned} & \underset{x}{\text{minimize}} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && a_i^\top x = b_i, \quad i = 1, \dots, p, \end{aligned} \quad (4.6)$$

where $x \in \mathbf{R}^n$, f_i , $i = 0, \dots, m$, are convex functions, and $a_i \in \mathbf{R}^n$ and $b_i \in \mathbf{R}$, $i = 1, \dots, p$. Next, a few important special classes of convex optimization problems are presented.

4.2.1 Quadratic and linear programming

If the objective function in the convex optimization problem (4.6) is a quadratic function and the inequality constraint functions are affine, the resulting problem is referred to as a *quadratic program* (QP) [23]. A general QP can be formulated as

$$\begin{aligned} & \underset{x}{\text{minimize}} && \frac{1}{2}x^\top Bx + q^\top x \\ & \text{subject to} && c_i^\top x \leq d_i, \quad i = 1, \dots, m \\ & && a_i^\top x = b_i, \quad i = 1, \dots, p, \end{aligned} \quad (4.7)$$

where $x \in \mathbf{R}^n$, $a_i \in \mathbf{R}^n$ and $b_i \in \mathbf{R}$, $i = 1, \dots, p$, $c_i \in \mathbf{R}^n$ and $d_i \in \mathbf{R}$, $i = 1, \dots, m$, $q \in \mathbf{R}^n$, and $B \in \mathbf{S}_+^n$, i.e., B is a positive semidefinite matrix.

4.2.2 Semidefinite programming

A semidefinite program (SDP) is an optimization problem that involves matrix inequalities [23]. Typically, a constraint that involves a matrix inequality requires all eigenvalues of a matrix-valued function of the decision variable to be non-positive, *i.e.*, the resulting matrix is constrained to be negative semidefinite. Given any two symmetric matrices X and Y , the notation $X \preceq 0$ denotes that X is negative semidefinite and $X \preceq Y$ denotes that $X - Y \preceq 0$. In general, an SDP can be formulated as

$$\begin{aligned} & \underset{x}{\text{minimize}} && f_0(x) \\ & \text{subject to} && f_i(x) \preceq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p, \end{aligned} \tag{4.8}$$

where $x \in \mathbf{R}^n$, $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$, $f_i : \mathbf{R}^n \rightarrow \mathbf{S}^{m_i}$, $i = 1, \dots, m$, and $h_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \dots, p$. Note that the general SDP formulation (4.8) coincides with the general optimization problem (4.1) if $m_i = 1$ for $i = 1, \dots, m$. An SDP is a convex optimization problem if the objective and inequality constraint functions are convex and the equality constraint functions are affine. Both Paper A and Paper B involve sensor management problems formulated as SDPs.

4.3 Mixed-binary optimization

In the preceding sections, all decision variables have been real-valued. Mathematical optimization is however not restricted to only real-valued variables. An optimization problem that involves both real-valued variables and integer-valued variables is referred to as a mixed-integer optimization problem. Any optimization problem of this type is a nonconvex problem, since the corresponding feasible set is nonconvex [62, 81].

4.3.1 Mixed-binary convex optimization problems

A subset of mixed-integer optimization is mixed-binary optimization, where all integer variables are constrained to be binary. A special case of mixed-binary problems is the mixed-binary convex problem, which is defined as

$$\underset{x, \delta}{\text{minimize}} \quad f_0(x, \delta) \tag{4.9a}$$

$$\text{subject to} \quad f_i(x, \delta) \leq 0, \quad i = 1, \dots, m \tag{4.9b}$$

$$h_i(x, \delta) = 0, \quad i = 1, \dots, p \tag{4.9c}$$

$$\delta_i \in \{0, 1\}, \quad i = 1, \dots, n_\delta, \tag{4.9d}$$

where $x \in \mathbf{R}^{n_x}$ and $\delta \in \{0, 1\}^{n_\delta}$ are the decision variables. As in a convex problem, the objective function and the inequality constraint functions f_i , $i = 0, \dots, m$, are convex and the equality constraint functions h_i , $i = 1, \dots, p$, are affine. The

property that makes mixed-binary convex problems different from general mixed-binary problems is that when the binary constraints (4.9d) are relaxed to interval constraints, *i.e.*,

$$\delta_i \in [0, 1], \quad i = 1, \dots, n_\delta, \quad (4.10)$$

the obtained problem is referred to as a *convex relaxation* of the original problem (4.9) and can be solved efficiently to global optimality. The original problem and the convex relaxation have the same objective function and the feasible set of the original problem is a subset of the feasible set of the convex relaxation. This means that it is possible to draw the following conclusions regarding the original problem by solving the relaxed one:

- the optimal value of the relaxed problem is a lower bound on the optimal value of the original problem;
- if the relaxed problem is infeasible, so is the original problem; and
- if an optimal solution to the relaxed problem is feasible in the original one, it must also be an optimal solution to the original problem.

4.3.2 Branch and bound for mixed-binary optimization

The most straightforward approach to solve a mixed-binary convex problem is to perform an exhaustive search, in which a convex problem is solved for each possible combination of the binary variables. The globally optimal solution is then the point that is feasible with respect to the constraints and gives the smallest objective function value. Since the number of convex problems that need to be solved increases exponentially with the number of binary variables, the computational cost of performing an exhaustive search quickly becomes overwhelming. To overcome this issue, several successful approaches to solve mixed-binary problems more efficiently have been proposed, see, *e.g.*, [30, 81]. One of them is known as the branch and bound method.

As illustrated in Figure 4.1, the branch and bound method for mixed-binary optimization [22] makes use of a binary search tree to represent the possible combinations of the binary variables in a structured way. As the search tree is expanded, subtrees that are guaranteed to not contain an optimal solution are successively pruned. To enable pruning, the branch and bound method maintains an upper bound on the globally optimal objective function value, and computes lower bounds on the optimal value for each subtree. As a result, the method implicitly enumerates all possible combinations of binary variables, which can substantially reduce the computational effort compared to an exhaustive search. The lower bounds are computed by solving convex relaxations of the original problems, which for mixed-binary convex problems are straightforward to obtain.

Figure 4.1: Example of a binary search tree with three binary variables. The branch and bound method implicitly performs an exhaustive search for an optimal solution. Instead of explicitly searching in all parts of the tree, the method prunes subtrees that are guaranteed not to contain an optimal solution. Such a subtree is here illustrated by the grey nodes.

5

Optimal control

Optimal control theory combines mathematical optimization and control of dynamical systems. It deals with the problem of computing optimal control inputs for a given system such that a performance criterion is minimized and constraints on the states and control inputs are satisfied. Sensor management problems can be formulated in the framework of stochastic optimal control [26]; the dynamical system corresponds to the sensor and the observed system, and the performance criterion captures the accuracy of the resulting state estimates. This chapter provides an overview of optimal control for discrete-time systems. For more detailed descriptions of optimal control theory, see, *e.g.*, [8, 11, 75].

5.1 Deterministic optimal control

Consider a dynamical system characterized by the dynamics

$$x_{k+1} = f(x_k, u_k), \quad (5.1)$$

where $x_k \in \mathbf{R}^{n_x}$ is the state of the system and $u_k \in \mathbf{R}^{n_u}$ is the control input, which is constrained to take values in the set $\mathcal{U}_k = \mathcal{U}(x_k)$ of admissible control inputs that might depend on the current state x_k . The set of feasible states at time k is denoted $\mathcal{X}_k$. The initial state of the system is assumed to be known and given by $x_0 = \bar{x}_0$.

In a finite-horizon optimal control problem, the objective function that encodes the performance criterion generally consists of two terms: a terminal cost $\ell_N(x_N)$ that penalizes deviation from a desired terminal state, and an accumulation of stage costs $\ell_k(x_k, u_k)$, which are the costs associated with the state and control input at the time indices $k = 0, \dots, N - 1$, where N is the planning horizon [11].

Given the dynamical model, constraints on the states and the control inputs, and an objective function, a discrete-time optimal control problem can be formulated

as the following optimization problem:

$$\begin{aligned}
 & \underset{u_{0:N-1}}{\text{minimize}} && \ell_N(x_N) + \sum_{k=0}^{N-1} \ell_k(x_k, u_k) \\
 & \text{subject to} && x_{k+1} = f(x_k, u_k) \\
 & && x_k \in \mathcal{X}_k, \quad u_k \in \mathcal{U}_k \\
 & && x_0 = \bar{x}_0.
 \end{aligned} \tag{5.2}$$

That is, the aim is to minimize the cost over N steps with respect to the sequence of control inputs $u_{0:N-1}$ subject to the constraints and dynamics. Solving this optimal control problem gives a sequence of admissible control inputs, which yields a state trajectory that minimizes the objective function.

Deterministic optimal control problems can be formulated in the open-loop sense since the future states can be exactly predicted given an initial state and a sequence of control inputs. Once u_k is chosen, x_{k+1} is known, and there is no advantage in waiting until the next time step to decide on u_{k+1} .

5.2 Stochastic optimal control

The dynamical system in (5.1) is deterministic; when a control input is applied at a given state, the next state is fully determined. Problem (5.2) is thus a *deterministic* optimal control problem. If the dynamical system is instead stochastic, and the succeeding state cannot be exactly determined from a given state and control input due to the presence of disturbances, the optimal control problem becomes a *stochastic* optimal control problem [8]. The dynamics of such a system can be described as

$$x_{k+1} = f(x_k, u_k, w_k), \tag{5.3}$$

where, as in (5.1), x_k is the state and $u_k \in \mathcal{U}_k$ is the control input. The disturbances are modeled as process noise w_k characterized by a probability distribution $p(w_k)$. Furthermore, instead of having perfect knowledge of the state, it has to be inferred from measurements z_k . The relation between the measurements and the states is described as

$$z_k = h(x_k, e_k), \tag{5.4}$$

where e_k is measurement noise characterized by a probability distribution $p(e_k)$.

In stochastic optimal control, the optimization is typically done in the closed-loop sense; before the controller commits to applying the next control input to the system, it is allowed to obtain a measurement of the current state. This means that rather than optimizing over control sequences, the optimization is done over *control policies* [8]. A control policy is a sequence of functions, $\{\mu_0, \dots, \mu_{N-1}\}$, where μ_k maps the information available to the controller at time k , denoted χ_k , to a control input $u_k = \mu_k(\chi_k) \in \mathcal{U}_k$. The available information at time k is summarized by the sequence of measurements until time k and applied control inputs until time $k-1$:

$$\chi_0 = \{z_0\}, \quad (5.5a)$$

$$\chi_{k+1} = \{\chi_k, u_k, z_{k+1}\}, \quad k = 0, \dots, N-1. \quad (5.5b)$$

The optimal policy thus provides the optimal control input for every possible information vector χ_k , *i.e.*, every possible sequence of measurements and control inputs.

As the dimension of the information vector χ_k increases with the time k , there is a need to find a more compact representation of the available information. This can be done using the concept of *sufficient statistics*. A function ζ_k is a sufficient statistic for χ_k if for every prior $p(\theta)$ the posterior $p(\theta | \chi_k) = p(\theta | \zeta_k(\chi_k))$, *i.e.*, all information about θ contained in χ_k is carried by $\zeta_k(\chi_k)$ [32]. Given a sufficient statistic ζ_k , the optimal control input at time k can thus, with some abuse of notation, be computed as

$$u_k = \mu_k(\chi_k) = \mu_k(\zeta_k(\chi_k)). \quad (5.6)$$

For Markovian systems, a sufficient statistic is the conditional probability distribution $p(x_k | \chi_k)$ of the state x_k given the information vector χ_k [8]. The optimal policy is in this case a function of that probability distribution, *i.e.*, $u_k = \mu_k(p(x_k | \chi_k))$. Obtaining such a policy would require a substantial amount of computations, but the optimization problem corresponding to finding the optimal policy for Markovian systems can nevertheless be formulated as

$$\begin{aligned} & \underset{\mu_0, \dots, \mu_{N-1}}{\text{minimize}} && \mathbf{E} \left\{ \ell_N(x_N) + \sum_{k=0}^{N-1} \ell_k(x_k, u_k) \right\} \\ & \text{subject to} && x_{k+1} = f(x_k, u_k, w_k) \\ & && z_k = h(x_k, e_k) \\ & && u_k = \mu_k(p(x_k | \chi_k)) \in \mathcal{U}_k. \end{aligned} \quad (5.7)$$

Note that the problem is formulated with the expected cost in the objective function. This is since the cost is a random variable due to the uncertainties in the state x_k and measurements z_k , which makes it reasonable to minimize the expected value of this variable [8]. The expectation is taken with respect to the joint distribution of all the involved random variables, *i.e.*,

$$p(x_{0:N}, z_{0:N-1}) = p(x_0) \prod_{k=0}^{N-1} p(x_{k+1} | x_k, u_k) p(z_k | x_k). \quad (5.8)$$

The stochastic optimal control problem (5.7) is often referred to as a partially observable Markov decision process (POMDP) [45, 84]. If the controller has access to the exact value of the state variable, *i.e.*, if $h(x_k, e_k) = x_k$, the problem is instead referred to as a Markov decision process (MDP).

5.2.1 Dynamic programming

Given a stochastic optimal control problem, dynamic programming [7, 8] can be applied to compute the optimal policy. Dynamic programming is based on the intuitive idea called *principle of optimality* [7], which states the following fact. If the optimal policy for the problem (5.7) is $\{\mu_0^*, \dots, \mu_{N-1}^*\}$, then $\{\mu_k^*, \dots, \mu_{N-1}^*\}$ is the optimal policy for the truncated problem starting at time $k > 0$ and running for the remaining time steps until the planning horizon is reached.

The dynamic programming algorithm starts at time step N by computing

$$J_N^*(p(x_N | \chi_N)) = \mathbf{E}\{\ell_N(x_N) \mid p(x_N | \chi_N)\}, \quad (5.9)$$

and proceeds backward in time by recursively solving

$$J_k^*(p(x_k | \chi_k)) = \min_{u_k \in \mathcal{U}_k} \mathbf{E}\left\{\ell_k(x_k, u_k) + J_{k+1}^*(p(x_{k+1} | \chi_{k+1})) \mid p(x_k | \chi_k), u_k\right\}. \quad (5.10)$$

By letting $\mu_k^*(p(x_k | \chi_k)) = u_k^*$, where u_k^* is the optimal solution to (5.10), this recursive algorithm yields the optimal policy $\{\mu_0^*, \dots, \mu_{N-1}^*\}$ [8].

5.2.2 Suboptimal approaches

In theory, dynamic programming gives the optimal solution to stochastic control problems. However, due to the *curse of dimensionality* [7], computing the optimal policy in closed form is intractable except in special cases, and approximations are necessary. This section presents two commonly used approaches.

Certainty-equivalent control

One approximate method is the certainty-equivalent control (CEC) strategy. At each time step k , the certainty-equivalent controller applies the control input that would be optimal if all stochastic quantities were replaced by their expected values. The control input $u_k = \hat{\mu}_k(\chi_k)$ at time k is computed using the following scheme:

1. Compute an estimate of x_k , *i.e.*, the mean of the conditional distribution of x_k given the information available: $\hat{x}_k = \mathbf{E}p(x_k | \chi_k)$.
2. Fix each of the process noise terms at its expected value, *i.e.*, let $\hat{w}_j = \mathbf{E}w_j$, $j = k, \dots, N-1$.
3. Compute a sequence of control inputs $\{u_k, \dots, u_{N-1}\}$ by solving the optimal control problem:

$$\begin{aligned} & \underset{u_{k:N-1}}{\text{minimize}} && \ell_N(x_N) + \sum_{j=k}^{N-1} \ell_j(x_j, u_j) \\ & \text{subject to} && x_{j+1} = f(x_j, u_j, \hat{w}_j) \\ & && x_k = \hat{x}_k, \quad u_j \in \mathcal{U}_j. \end{aligned} \quad (5.11)$$

Note that since this problem is deterministic, the optimization is performed over sequences of control inputs rather than control policies.

4. Apply the first element of the computed sequence of control inputs, u_k , to the system, and discard the remaining elements. At time $k + 1$, the process is repeated from step 1, this time with a one step shorter planning horizon.

The decision variables in the optimization problem solved in each iteration of the certainty-equivalent control loop is a sequence of control inputs to be applied from the current time step k until the end of the planning horizon N . As the time k increases toward N , the time remaining until the end of the planning horizon decreases and these sequences become shorter. For problems where the CEC strategy coincides with the optimal policy, certainty equivalence is said to hold.

Model predictive control

Another approximate method is the model predictive control (MPC) strategy [55]. It resembles the CEC strategy in the sense that it computes the control input at each time step based on the current estimate of the state x_k and expected values of future process noise realizations. The difference is that MPC uses the same planning horizon length, denoted M , in each iteration of the control loop, *i.e.*, the decision variables in the problem solved at time k is a sequence of control inputs from time k until time $k + M - 1$. The optimization problem solved at time k in the MPC loop is thus:

$$\begin{aligned}
 & \underset{u_{k:k+M-1}}{\text{minimize}} && \ell_{k+M}(x_{k+M}) + \sum_{j=k}^{k+M-1} \ell_j(x_j, u_j) \\
 & \text{subject to} && x_{j+1} = f(x_j, u_j, \hat{w}_j) \\
 & && x_k = \hat{x}_k, \quad u_j \in \mathcal{U}_j.
 \end{aligned} \tag{5.12}$$

Since this problem is based on the current state estimate, which is computed using the obtained measurements, the MPC controller operates in closed loop [55]. As the control inputs are computed by repeatedly solving optimal control problems in a receding horizon fashion, MPC is sometimes referred to as receding horizon control (RHC) [60].

5.3 Optimization-based sensor management

While the framework of optimal control is typically used to find control inputs that move a system from an initial state to a desired goal state, it is also convenient in the context of sensor management. Consider an estimation problem where the sensor configuration, and thereby the quality of the obtained measurements, can be controlled. By carefully selecting appropriate control inputs, the overall estimation performance can be improved such that more accurate state estimates are obtained. The sensor management problem of selecting these control inputs in a systematic manner can be formulated as a stochastic optimal control problem [26].

Consider as in Section 5.2 a system of interest with state x_k at time k and dynamics governed by

$$x_{k+1} = f(x_k, w_k), \tag{5.13}$$

where w_k is the process noise characterized by $p(w_k)$. This system corresponds to the observed system in the state estimation scenario. Furthermore, let s_k denote the state of the sensor system and let its dynamics be described by the deterministic model

$$s_{k+1} = g(s_k, u_k), \quad (5.14)$$

where u_k is the control input at time k . The set of feasible control inputs at time k might depend on the current sensor state, *i.e.*, $u_k \in \mathcal{U}_k = \mathcal{U}(s_k)$. The measurements depend on the states of both the system of interest and the sensor, *i.e.*, the measurement model is given by

$$z_k = h(x_k, s_k, e_k), \quad (5.15)$$

where e_k is the measurement noise characterized by $p(e_k)$.

Given the models (5.13)–(5.15), a planning horizon N , stage costs ℓ_k and terminal cost ℓ_N , a general sensor management problem can be formulated as a stochastic optimal control problem. The information available at time k is summarized by the sensor state trajectory and the obtained measurements:

$$\chi_0 = \{s_0, z_0\}, \quad (5.16a)$$

$$\chi_{k+1} = \{\chi_k, s_{k+1}, z_{k+1}\}, \quad k = 0, \dots, N-1, \quad (5.16b)$$

for which a sufficient statistic is the current state of the sensor, s_k , and the conditional probability distribution of the state of the system of interest $p(x_k | \chi_k)$. The optimal control policy is thus a sequence of functions $\{\mu_0, \dots, \mu_{N-1}\}$ that map s_k and $p(x_k | \chi_k)$ to control inputs. The optimization problem that corresponds to computing this policy can be formulated as

$$\begin{aligned} & \underset{\mu_0, \dots, \mu_{N-1}}{\text{minimize}} && \mathbf{E} \left\{ \ell_N(p(x_N | \chi_N), s_N) + \sum_{k=0}^{N-1} \ell_k(p(x_k | \chi_k), s_k, u_k) \right\} \\ & \text{subject to} && s_{k+1} = g(s_k, u_k) \\ & && x_{k+1} = f(x_k, w_k) \\ & && z_k = h(x_k, s_k, e_k) \\ & && u_k = \mu_k(p(x_k | \chi_k), s_k) \in \mathcal{U}_k \\ & && s_0 = \bar{s}_0, \end{aligned} \quad (5.17)$$

where $\bar{s}_0$ is the initial state of the mobile sensor and the expectation is taken with respect to the joint distribution of all the involved random variables. As the objective of the sensor management problem is to improve the estimation performance, the stage costs typically include a measure of the state estimation accuracy and possibly also costs related to the control effort.

In general, the problem (5.17) is intractable and approximations are necessary. The linear Gaussian case is an exception. If the following conditions are satisfied,

- the dynamics of the system of interest (5.13) are linear with additive white Gaussian noise,

- the measurement model (5.15) is linear in x_k with additive white Gaussian noise, and
- the objective function does not depend on the actual state estimate but only on the corresponding covariance matrix,

certainty equivalence holds and the stochastic problem (5.17) reduces to a deterministic one [1, 51, 61]. This case is studied in Paper A and Paper B. The problems studied in Paper C, Paper E, and Paper F are all nonlinear special cases of the general problem (5.17).

6

Concluding remarks

The first part of the thesis has given an overview of the theoretical components that form the basis for sensor management methods. This theoretical basis is utilized in the papers that constitute the second part of the thesis. In this final chapter of the first part, the main scientific contributions are summarized and the conclusions as well as some possible directions for future work are presented.

6.1 Summary of contributions

This section summarizes the main contributions of the thesis, which are divided into three categories related to sensor management.

6.1.1 Planning trajectories for mobile sensors

The first category of sensor management problems studied in the thesis is informative path planning. This is the problem of optimizing trajectories for mobile sensor platforms to maximize the utility of the obtained measurements. Informative path planning problems for linear systems subjected to Gaussian noise are studied in Paper A and Paper B. Although the considered problems seem intractable at first glance, it is shown in Paper A that they can be reformulated and solved to global optimality using standard optimization tools. In Paper B it is shown that problems that also involve adversarial observers which should be avoided can be reformulated in a similar manner. This extended problem becomes one of finding a good trade-off between stealth and informativeness when computing globally optimal sensor trajectories.

6.1.2 Tracking of maneuvering targets

The scenario studied in Paper C involves a mobile sensor that is used to track a maneuvering target. The considered problem is to control the trajectory of the sensor to maximize the expected tracking performance. The proposed solution is a receding horizon control strategy where the uncertainty in the optimization problem is handled by deterministic sampling of the distribution of future target trajectories. This means that the possibility that the targets may perform sharp maneuvers is accounted for when optimizing the sensor trajectory.

The tracking performance can also be enhanced by exploiting sensor characteristics and prior knowledge about the targets' behaviors. Paper D considers the problem of tracking a maneuvering target using a mobile radar with limited field of view. The dilemma in such scenarios is that while the risk of losing track of the target decreases with the tracking range, the tracking error increases. The contribution is a method to compute the optimal tracking range given an acceptable risk of track loss. The method can also be used to find the optimal beamwidth of the radar as a function of the tracking range.

6.1.3 Search and track

The final contributions of this thesis deal with the problem of jointly searching for and tracking an unknown and time-varying number of targets. This problem comprises a notion of both known and undiscovered targets and coordination of the available sensor resources is required to detect and track all targets. The methods proposed in Paper E and Paper F can be used to give recommendations to sensor operators about where to direct the sensors, or to deploy a team of mobile sensors that autonomously can take care of surveillance in a given area.

In Paper E, it is shown that the PMBM filter can provide a theoretical foundation for sensor management in the context of multiple target tracking. In contrast to other tracking methods, the PMBM filter does not only estimate the states of detected targets but also maintains an explicit representation of where yet undetected targets may be located. Thereby, it enables a joint approach to search and track problems.

In conventional PMBM filter implementations, Gaussian mixtures are used to represent possible locations of undetected targets. A drawback with this approach is that a large number of mixture components may be required to adequately represent the intensity of undetected targets in the case when the sensor's field of view is much smaller than the tracking region of interest. This issue is addressed in Paper F, which derives a new version of the PMBM filter that significantly reduces the computational cost compared to conventional approaches. This is achieved by replacing the Gaussian mixtures with a grid-based intensity function to represent where yet undetected targets may be located.

6.2 Conclusions

The thesis provides theoretical results on several different aspects of sensor management. In this section, the contributions are viewed from the perspective of how they relate to the problem formulation in Chapter 1.

The problem of optimizing trajectories for sensor platforms is studied in detail. As a fully automated sensor system requires the capability of deciding where to go to collect necessary information, this is an important aspect of sensor management. Even though informative path planning problems in general are challenging, it is shown that they can sometimes be solved to global optimality. While this is typically too computationally expensive to do in an online setting, it can be used to verify or estimate the performance of other methods for informative path planning.

The thesis also studies how to exploit sensor characteristics to improve tracking of maneuvering targets. Two different methods for this are developed in the thesis. The first one gives suggestions about which trajectory the sensor platform should follow to minimize state estimation errors. The second one can be used by sensor operators to calculate which distance they should maintain to the target in order not to lose track of it even if it performs a sharp maneuver.

The final aspect of sensor management studied in the thesis is that of planning for search and track. A somewhat unconventional perspective on multiple target tracking is considered, with focus on tracking of targets that have not yet been detected. The notion of tracking undetected targets may seem counterintuitive, but knowledge of where undetected targets may be located is shown to be particularly useful when planning how to deploy sensor resources in surveillance scenarios.

The methods and theory that have been developed contribute to the overall aim of the thesis; to reduce the workload of sensor operators and increase the performance of sensor systems. More generally, autonomy requires a high level of situational awareness, *i.e.*, to be well informed about the surroundings. Obtaining this is a nontrivial task that requires efficient information acquisition. Although not the primary focus of this thesis, the methods and results developed herein can also be used to increase the situational awareness of future autonomous systems.

6.3 Future work

There are several directions in which the contributions of this thesis can be extended. Although there is no theoretical limitation on the number of sensors the developed methods can be used to control, they are all centralized methods. Decentralized planning methods are the preferred choice for control of teams that include large numbers of sensors, as they both provide scalability and are better suited in scenarios with limited communication possibilities. A natural continuation is thus to investigate decentralized sensor management methods.

An interesting sensor management application is search and track in long-term surveillance scenarios. The planning methods used in such scenarios can possibly benefit from learning the targets' motion patterns, *e.g.*, using the method in [50], and exploiting this knowledge when planning where to search for new targets.

The methods developed in this thesis have mainly been evaluated in target tracking applications. It would be interesting to explore the possibility of applying the methods to applications in other domains. For example, the PMBM filter's representation of undetected targets could prove to be useful in driver assistance systems or self-driving vehicles. Consider a vehicle with onboard sensors. The sensors are used to estimate the positions of other known vehicles in the surroundings, and also where yet undetected vehicles may be located. This information can then for example be used to make informed decisions about whether it is safe to pass other vehicles.

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